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Determination of stiffness modifications to keep certain natural frequencies of a system unchanged after mass modifications

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Abstract Structural modifications in the form of mass, stiffness and damping to a structure change the dynamic properties of that structure. However, in some cases, after modifications are made to the structures, certain specified natural frequencies of the structure are desired to remain unchanged. This study is interested in the determination of necessary stiffness modifications in order to keep a certain number of natural frequencies of the system unchanged despite mass attachments. In particular, two methods based on the Sherman–Morrison formula are developed in order to determine the spring coefficients needed to keep one and more than one natural frequency of the structures unchanged. The developed methods directly use the Frequency Response Functions of the unmodified system relating the modification coordinates only and they need neither a physical model nor a modal model. The numerical simulations show that they are very effective. However, due to the nature of the inverse problem, any solution or practical realistic solution may be not found. The existence of the solution depends on also the modification coordinates chosen. A simple sensitivity approach demonstrated by a 3D graph is proposed to be able to choose a suitable modification.

Keywords Structural modification · Frequency response function · Natural frequency · Sherman–Morrison formula

1 Introduction

In engineering, it may be necessary to physically modify an existing structure. The physical modifications are in the form of mass, stiffness and damping. Since the dynamic properties of the structure are dependent on these physical parameters the dynamic properties can be changed after modifications are made to the structure. The modifications are usually made to attain the desired dynamic properties of the existing structure. Masses are not only directly added onto a coordinate of the structure, but also can be added with the springs. If a mass is added by means of a spring, an additional degree of freedom is introduced to the system. In other cases, springs are added either between a couple of generalized coordinates or between a generalized coordinate and the ground. Generally, the structural modification has been performed in two ways: (i) the direct structural modification which, deals with how the physical modifications affect the dynamics of the structure and (ii) the inverse structural modification, which deals with the determination of the necessary modifications to satisfy the desired dynamic properties of the structure. The methods for structural modifications are based on the use of

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modal properties or the direct use of FRFs. Modal properties are derived from a finite element (FE) solution or experimental modal analysis (EMA). FRFs are measured experimentally or calculated theoretically. Inherently, both EMA and FE solutions form an incomplete set of natural frequencies and mode shapes. Bucher and Braun [1] developed a theory to show how the necessary mass and stiffness modifications can be computed using only modal test results, even when only a partial set of eigen-solutions are available from tests. Sivan and Ram [2] developed a method for determining mass and stiffness modifications to achieve desired natural frequencies by using modal analysis. The difficulties arising from truncated data provided by modal analysis were overcome by an optimization procedure. Chang and Park [3] proposed a method based on FRF sensitivity analysis called component receptance sensitivity method for modal updating. The suggested method was applied to estimation of spring stiffness values in a spring supported plate. Li and He [4] presented an approach for the determination of mass and spring modifications that needed to satisfy the desired dynamic characteristics of undamped systems. The method considers one natural frequency and mode shape, and uses FRFs relating the modification coordinates only. Park and Park [5,6] proposed some methods based on FRF based-substructure-coupling concept for the identification of structural parameters.

In order to avoid resonance, one way is to shift the troublesome natural frequency of the system to a safe value. Therefore, the problem of shifting natural frequencies has been studied by many researchers [7–16]. For a design aim, it may be needed to add a number of masses or springs, which represent the modification system, at the specified locations on an existing structure. Once these modifications are made to the structure, one or more of the natural frequencies of the structure may shift to a frequency value such that it may be close to a harmonic forcing frequency and consequently the modified structure vibrates at resonance. Therefore, it is necessary to avoid this situation to maintain the stability of the system. This can be achieved by keeping these troublesome natural frequencies unchanged after modifications. In this case, the main problem is to determine the necessary modifications. Gürgöze and Inceoğlu [17] examined the problem of determining the stiffness coefficient of the grounded spring to be placed at a specified position so that the fundamental frequency of the bending beam exposed to various supporting conditions did not change despite the addition of a mass at a predefined position. Mermertaş and Gürgöze [18] studied the same problem of the rectangular plates by using the impedance coupling method. Then, Çakar [19] investigated this problem for more general structures. He developed a new method based on the Sherman–Morrison (SM) formula in order to determine the necessary spring coefficient. His method can be applied not only to beams or plates, but also to real engineering structures that include damping.

This paper deals with the specific aforementioned inverse modification problem and it is an extension of the work proposed by Çakar [19], where only one natural frequency was kept unchanged using one grounded spring. However, this current paper aims to preserve more than one natural frequency by using multiple springs, where the necessary springs are added between two generalized coordinates instead of a grounded one. This is an important feature because some practical structures, such as airplane and space structures, cannot be modified by grounded springs. Two efficient methods are developed for the calculation of the necessary spring coefficients. The first method is used to preserve one natural frequency and it is an exact method. The second method is used to preserve more than one frequency and it results in a set of dependent nonlinear equations which can be solved numerically.

2 Theory of structural modification based on SM formula

The method proposed in this paper is based on the SM identity [20]. This identity allows a direct inversion of a modified matrix $[A^*]$ by using the inverse of the initial matrix $[A]$ which already exists and the modification matrix $[\Delta A]$ which can be expressed as a product of two vectors such as $\{u\}\{v\}^{\mathrm{T}}$:

$$[A^*]^{-1} = [A]^{-1} - \frac{([A]^{-1}\{u\})(\{v\}^T[A]^{-1})}{1+\{v\}^T[A]^{-1}\{u\}} \tag{1}$$

If an analogy is built up between the structural dynamics and the SM formula, the equations of motion for a dynamic system are given by:

$$\left([K] - \omega^2[M] + i\,[D]\right)\{q\} = [Z]\{q\} = \{f\} \tag{2}$$

where $\{q\}$ is a vector of response amplitudes; $\{f\}$ is a vector of harmonic loads; $[K]$, $[M]$ and $[D]$ are the initial stiffness, mass and structural damping matrices of the system, respectively; $[Z]$ is the frequency (ω) dependent dynamic stiffness matrix and $i = \sqrt{-1}$.

Expressing the dynamic stiffness matrix of the modified structure as $[Z^*] = [Z] + [\Delta Z]$ where $[\Delta Z] = \{u\}\{v\}^{\mathrm{T}}$, and using the well-known relationship between the dynamic stiffness and the receptance, $[Z]^{-1} = [\alpha]$, the receptances of the modified structure can be computed using the SM formula as follows:

$$[\alpha^*] = [Z^*]^{-1} = [\alpha] - \frac{([\alpha]\{u\})\left(\{v\}^T[\alpha]\right)}{1 + \{v\}^T[\alpha]\{u\}} \tag{3}$$

For the rank one modification, the elements of vectors $\{u\}$ and $\{v\}$ corresponding to the modification coordinates for (a) mass modification m_i^* at coordinate i, (b) a grounded spring modification k_{i0}^* at coordinate i, (c) a spring modification k_{ij}^* between coordinates i and j, respectively, as follows;

$$\{u\} = \{\cdots \ 1 \ \cdots\}^T, \quad \{v\} = \left\{\cdots \ -\omega^2 m_i^* \ \cdots\right\}^T \tag{4a}$$

$$\{u\} = \{\cdots \ 1 \ \cdots\}^T, \quad \{v\} = \left\{\cdots \ k_{i0}^* \ \cdots\right\}^T \tag{4b}$$

$$\{u\} = \{\cdots \ 1 \ \cdots \ -1 \ \cdots\}^T, \quad \{v\} = \left\{\cdots \ k_{ij}^* \ \cdots \ -k_{ij}^* \ \cdots\right\}^T \tag{4c}$$

where all the elements of the modification vectors $\{u\}$ and $\{v\}$ are zero except the elements corresponding to the modification coordinates. Although the receptance matrix $[\alpha]$ in Eq. (3) contains all the FRFs that are not practical to measure in a modal test, the elements of the receptance matrix can only be taken into account at active coordinates, i.e. excitation, response and modification coordinates [21]. However, Cakar and Sanliturk [22] expressed an explicit formula for the calculation of any receptance of the modified structure and used it to remove transducer mass effect from measured FRFs as follows:

$$\alpha_{pq}^* = \frac{\alpha_{pq} + v_r(\alpha_{rr}\alpha_{pq} - \alpha_{pr}\alpha_{rq})}{1 + v_r\alpha_{rr}} \tag{5}$$

where p, q and r are response, excitation and modification coordinates, respectively, and $u_r = 1$, $v_r = \omega^2 m$ for the negative mass modification at coordinate r.

For the three types of modifications given in Eqs. (4a–4c), the receptance of the modified system α_{ij}^* can be calculated, respectively, as follows:

$$\alpha_{ij}^* = \frac{\alpha_{ij}}{1\text{-}\omega^2 m_i^*\alpha_{ii}} \tag{6}$$

$$\alpha_{ij}^* = \frac{\alpha_{ij}}{1 + k_{i0}^*\alpha_{ii}} \tag{7}$$

$$\alpha_{ij}^* = \frac{\alpha_{ij} + k_{ij}^*\left(\alpha_{ii}\alpha_{jj} - \alpha_{ij}^2\right)}{1 + k_{ij}^*\left(\alpha_{ii} - 2\alpha_{ij} + \alpha_{jj}\right)}. \tag{8}$$

3 Modifying a structure with a mass and calculation of the necessary spring coefficients to keep natural frequencies unchanged

One of the modification types which is mostly encountered in practical applications is the mass modification. Modifying an existing structure for a design aim may be needed, e.g. for attaining new capabilities and properties for the structure. These modifications mostly introduce additional mass and/or stiffness and consequently the natural frequencies of the system shift. The present study considers the undesirable shifting of a certain number of natural frequencies after modifications and specifically focuses on the preservation of one or more natural frequencies of the system by the stiffness modifications after a known mass modification. In other words, it is aimed that a certain number of the natural frequencies of the structures will not be changed after the modifications. The main problem is the determination of the necessary stiffness modifications.

In order to clarify the developed methods, an undamped mass-spring system with n degree of freedoms (DOFs) in Fig. 1 is considered. In the figure, mass m_i^* represents the modification made at coordinate i for a design aim. In the rest of paper, the system without additional mass m_i^* is referred to as the original or unmodified system.

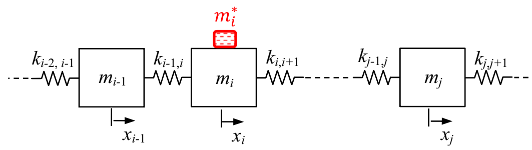

Fig. 1 *n* DOFs mass-spring system modified by a mass

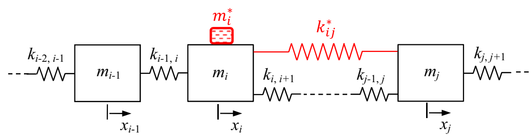

Fig. 2 *n* DOFs mass-spring system modified by a mass and a spring

The transfer FRF or special receptance α_{ij}^* of the system with additional mass can be calculated by means of Eq. (6). It should be noted that more than one modification can be made sequentially, although one mass modification is considered for the sake of brevity. On the other hand, the modifications may be in the form of springs. In this case, the receptance of the modified system can be calculated by using Eq. (7) or (8) depending on the type of spring modification. Once the mass is added to the system, the natural frequencies tend to decrease, but shifting a certain number of natural frequencies is not desired. In order to keep these frequencies unchanged after mass modification, the springs can be added between some predefined coordinates, which is the aim of this study. The main challenge is determining the coefficients of the added springs.

In the following sections, two methods are developed for the calculation of the necessary stiffness. The first method is to preserve a single natural frequency and the second is to preserve multiple natural frequencies for the system after adding one or more known masses. The proposed method takes advantage of that the structural modification equation based on the SM formula, i.e. the modifications can sequentially be made by using Eqs. (6–8).

3.1 Preservation of a single natural frequency

A spring k_{ij}^* can be added between coordinates i and j which are arbitrarily chosen in order to compensate the shifting effect of the mass m_i^* located at coordinate i on the natural frequency, say ω_{s1}, as shown in Fig. 2.

In this case, the receptance of the new system, α_{ij}^{**} can be obtained by using Eq. (8), sequentially as follows:

$$\alpha_{ij}^{**} = \frac{\alpha_{ij}^* + k_{ij}^*\left(\alpha_{ii}^*\alpha_{jj}^* - \alpha_{ij}^{*2}\right)}{1 + k_{ij}^*\left(\alpha_{ii}^* - 2\alpha_{ij}^* + \alpha_{jj}^*\right)} \tag{9}$$

where the receptance of the new system, α_{ij}^{**} consists of both mass m_i^* and k_{ij}^* spring modifications.

It is desired that any one particular natural frequency of the original system, e.g. ω_{s1} should not change after the mass and spring modifications. In other words, the new system should have a natural frequency at ω_{s1}. This requires that the denominator of Eq. (9) at this frequency value go to zero for undamped systems. By equating the denominator of Eq. (9) to zero, the spring coefficient k_{ij}^* can be obtained as follows:

$$k_{ij}^* = \frac{1}{2\alpha_{ij}^* - \alpha_{ii}^* - \alpha_{jj}^*} \tag{10}$$

Equation (10) is exact and it is clear that the driving point and transfer receptances related to the modification coordinates are sufficient for the calculation of the necessary spring coefficient. Note that Eq. (10) is identical to the result given in [4] where it was found using a different method. Moreover, if a grounded spring modification is considered at coordinate i, the necessary spring coefficient can be calculated as:

$$k_{i0}^* = -\frac{1}{\alpha_{ii}^*} \tag{11}$$

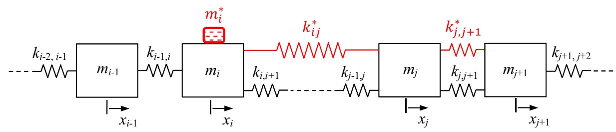

Fig. 3 *n* DOFs mass-spring system with added mass and two springs

3.2 Preservation of multiple natural frequencies

In this section, multiple natural frequencies are considered to be kept unchanged despite mass attachment. For the sake of simplicity, one mass modification and the preservation of only two natural frequencies are considered here, without the loss of generality. In this study, the number of spring modifications is chosen equal to the number of natural frequencies desired to remain unchanged, in order to avoid over-determined or under-determined system of equations. Therefore, consider a second spring modification $k^*_{j,j+1}$ between coordinates j and $j+1$ in addition to spring k^*_{ij} as seen in Fig. 3.

In this case, the transfer receptance related to coordinates i and j of the new system can be calculated from Eq. (9) by writing $i=j$ and $j=j+1$ as follows:

$$\alpha^{***}_{j,j+1}=\frac{\alpha^{**}_{j,j+1}+k^*_{j,j+1}\left(\alpha^{**}_{jj}\alpha^{**}_{j+1,j+1}-{\alpha^{**}_{j,j+1}}^2\right)}{1+k^*_{j,j+1}\left(\alpha^{**}_{jj}-2\alpha^{**}_{j,j+1}+\alpha^{**}_{j+1,j+1}\right)} \tag{12}$$

where one of the superscripts (*) of the receptances shows mass modification and the remaining number of superscripts (*) shows the number of spring modifications. For example, a receptance with three stars, e.g. $\alpha^{***}_{j,j+1}$ shows that one mass and then two spring modifications were made. $\alpha^{***}_{j,j+1}$ is dependent on both unknown parameters k^*_{ij} and $k^*_{i,j+1}$, since the receptances with two stars, i.e. $\alpha^{**}_{ij}(i,j=1,2,\cdots,n)$, are dependent on the spring modification k^*_{ij}. Therefore at least two equations are needed to find these two unknown modification parameters. In a similar manner, equating the denominator of Eq. (12) to zero for the frequency values ω_{s1} and ω_{s2}, one by one, two nonlinear equations can be obtained in the form as,

$$f_1\left(k^*_{ij},k^*_{j,j+1}\right)=1+k^*_{j,j+1}\left(\alpha^{**}_{jj}(\omega)-2\alpha^{**}_{j,j+1}(\omega)+\alpha^{**}_{j+1,j+1}(\omega)\right)=0 \quad for \quad \omega=\omega_{s1} \tag{13a}$$

$$f_2\left(k^*_{ij},k^*_{j,j+1}\right)=1+k^*_{j,j+1}\left(\alpha^{**}_{jj}(\omega)-2\alpha^{**}_{j,j+1}(\omega)+\alpha^{**}_{j+1,j+1}(\omega)\right)=0 \quad for \quad \omega=\omega_{s2} \tag{13b}$$

The values of the spring coefficients k^*_{ij} and $k^*_{i,j+1}$ can be calculated by solving the above nonlinear equation set numerically. Consequently, any particular two natural frequencies of the original structure, after the mass and spring modifications, can be kept unchanged when these calculated stiffness modifications are applied to the related coordinates.

It should be noted that, if the number of natural frequencies desired to keep constant is n, the number of spring modifications should also be n in order to be able to apply the proposed method. In this case, Eq. (8) is kept going sequentially and new equations are obtained similar to Eq. (12). Then, the additional equations are acquired in addition to Eqs. (13a) and (13b) by equating the denominator of the obtained new equations to zero. On the other hand, the number of the natural frequencies desired to keep constant is usually smaller than the number of modification coordinates for practical application. This means that there is different choice for modification coordinates. In this case, different solutions can be obtained depending on choosing modification coordinates.

4 Verification of the proposed methods

This section presents two numerical simulations demonstrating the validity and efficiency of the proposed methods. For the simulations, a six degree of freedom model consisting of six concentrated masses attached to nine translational springs is considered as shown in Fig. 4, [4,10–12]. The value of all the masses is 1 kg and the springs' coefficients are 1 N/m. In Fig. 4, the mass m^* represents a modification made for a design aim, and the springs with superscript (*) represent the modifications that will be made to preserve the natural frequencies, which will be determined by using the proposed method. Firstly, the eigenvalues and eigenvectors of the original system are obtained by solving a symmetric eigenvalue problem and six natural frequencies of the original system are given in the first line of Table 1.

A point mass (m^*) of 0.5 kg is located at coordinate 2 as seen in Fig. 4 for a design aim. The natural frequencies of the modified system can be obtained by either solving the eigenvalue problem or by analysing the receptances of the modified system, which can be calculated by means of Eq. (6) using receptances of the original system. The natural frequencies ω_a^* of the system, modified by mass m^*, are given in the second line of Table 1. The changes of the natural frequencies of the modified system relative to the original system are also given in the third line of Table 1. The transfer receptance between coordinates 2 and 3 of both the original and modified systems are compared in Fig. 5. As expected, the natural frequencies of the modified system tend to decrease. As shown in Table 1, the biggest changes occur at the fifth (9.2%), second (5.5%) and third (1.95%) modes, respectively. Hereinafter, two simulations will be carried out for the verification of the developed methods.

4.1 Example 1: single natural frequency case

For the first demonstration, only the second natural frequency that is more affected by mass modification is considered. The second mode's frequency shifts from 0.145 to 0.137 Hz after mass modification. It is desired that this frequency do not change despite the mass attachment. To satisfy this desire, a stiffness modification, k_{34}^*, is considered between coordinates 3 and 4 as seen in Fig. 4. The required stiffness modification can be calculated directly by using Eq. (10) so that $k_{34}^* = 1.722$ N/m. After modifying the spring k_4 with k_{34}^*, the natural frequencies of the new system are calculated and given in the second line of Table 2. Note that, the desired and achieved natural frequencies were written in bold numbers in the tables.

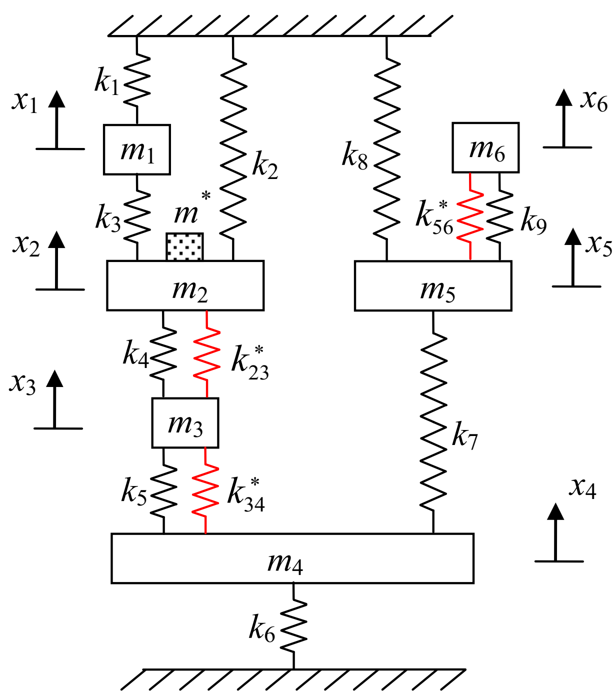

Fig. 4 Six DOFs mass-spring system with added mass and springs

Table 1 Natural frequencies (Hz) of the original (ω_r) and mass modified (ω_r^*) systems

Modes (r)	1	2	3	4	5	6
ω_r	0.109	0.145	0.205	0.261	0.314	0.337
ω_r^*	0.108	0.137	0.201	0.256	0.285	0.334
Rel. change (%)	0.9	5.5	1.95	1.92	9.2	0.89

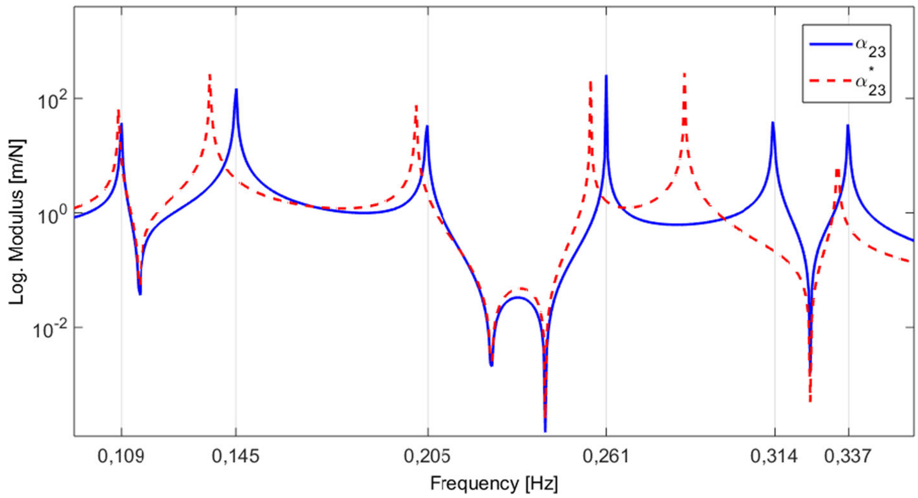

Fig. 5 Comparison of the receptances of original and modified systems

Table 2 Calculated stiffness modifications (N/m) and natural frequencies (Hz) for different modification coordinates

Modif. coords.	Calc. stiffness	Natural frequencies (Hz)						Error norm
		1	**2**	3	4	5	6	
None	None	0.109	**0.145**	0.205	0.261	0.314	0.337	
3–4	1.722	0.108	**0.145**	0.201	0.268	0.299	0.427	27.35
		(0.9%)	(0%)	(1.9%)	(2.7%)	(4.8%)	(27%)	
2–6	0.099	0.111	**0.145**	0.201	0.256	0.287	0.33	9.17
		(1.8%)	(0 %)	(2.0%)	(1.9%)	(8.5%)	(1.1%)	

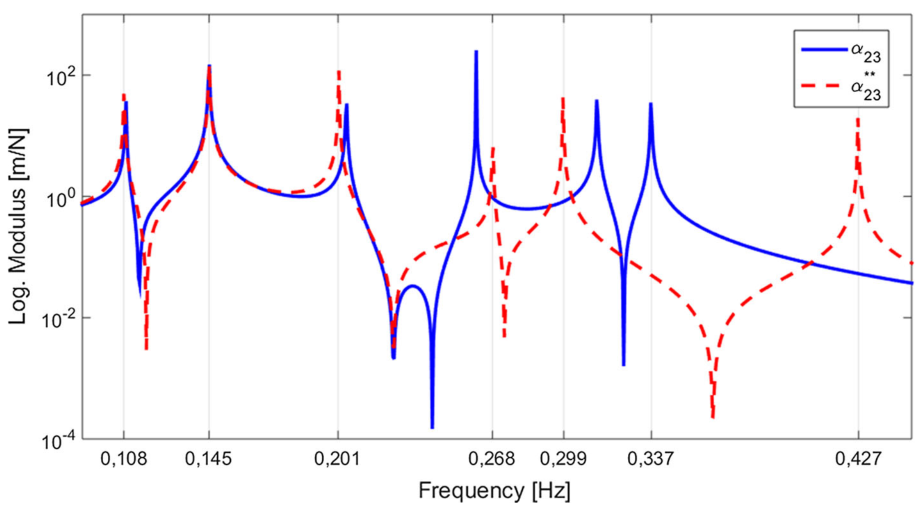

Fig. 6 Comparison of the receptances of original and modified systems

As seen in Table 2, the second natural frequency of the new system is 0.145 Hz; it is exactly the same as that of the original system. As expected, other natural frequencies change too, but they do not coincide with their original values. The relative change of modes is given in parenthesis. Also the receptance of the modified system is calculated and compared with those of the original in Fig. 6.

As seen in this figure, the second natural frequencies of the modified and original systems align with each other. Consequently, the value of the second natural frequency of the original system is preserved despite mass attachment by using the proposed method. It should be noted that although this method is exact, different stiffness modifications can be obtained depending on the modification coordinates chosen. For this system, fifteen different spring modifications can be made between two coordinates except grounded ones. However, not only reliable solutions cannot be found but also any solution may not be found for some coordinates due to nature of the inverse problem. This strictly depends on the relative motion of modification coordinates. If the relative motion between two coordinates is zero or near to zero for a mode, then the natural frequency of this mode is not sensitive to the stiffness modification between these coordinates. The sensitivities of the modes can be used to decide where the modification will be made. The sensitivities can be calculated by dividing the change of natural frequency to a reasonable change of stiffness modification (i.e. $\Delta\omega_r/\Delta k_{ij}$). In this study, the sensitivity of the modes demonstrated by a 3D graph is proposed. For our example, the normalized sensitivities of the natural frequencies to the stiffness modification (about 25%) for all of the possible modification coordinates are calculated and demonstrated as shown in Fig. 7. In the figure, the diagonal elements correspond to the grounded spring modification coordinates and they are nullified because the grounded spring modifications were not considered in this study. Also note that the symmetric elements were not included for clarity.

As an example, it can be seen in Fig. 7 that the second and fourth modes are not sensitive to the stiffness modification between the coordinates 2 and 3. Therefore, the preservation of the second and fourth modes is not achieved with the stiffness modification between coordinates 2 and 3. On the other hand, the second mode is very sensitive to the modification between coordinates 2 and 6. For this case, the necessary stiffness modification is determined as $k_{26}^* = 0.099$ N/m and the calculated natural frequencies are given in the last line of Table 2. It is seen that the result is very satisfactory for this case. The necessary stiffness modification obtained for this case is small compared to the case of modification between coordinates 3 and 4. Because the sensitivity of the second mode to the modification between coordinates 2 and 6 is bigger than that of the modification between coordinates 3 and 4.

For this special inverse problem, it is desired that while the natural frequency of the mode in question is preserved, other frequencies remain close to the original values. In this manner, it can be said that the modification coordinate, which has the minimum norm of the relative change, may be the best choice. In the last column of Table 2, Euclidian norms of the relative changes for two cases were given. It is seen that the modification between coordinates 2–6 is the best choice for this example.

4.2 Example 2: multiple natural frequencies case

For the second demonstration, second, third and fifth natural frequencies, which are more effected by mass modification, are chosen. Three spring modifications are required in order to keep these frequencies unchanged after modification. The stiffness modification coordinates 2–3, 3–4 and 5–6, respectively, are considered as seen in Fig. 4. As can be seen in Fig. 7, the natural frequencies chosen are sensitive to at least one of the chosen modification coordinates. The necessary spring modifications k_{23}^*, k_{34}^* and k_{56}^* are calculated by solving a set of three nonlinear equations similar to Eqs. (13a) and (13b). For the solution of the system of nonlinear equations, a MATLAB code was implemented and the solve function was used. Inıtial values were set to zero and the obtained stiffness modifications are given in Table 3. After applying these modifications to the system, the natural frequencies of the final system and the relative difference to the original values are given in Table 3.

As seen in Table 3, all three of the predefined natural frequencies of the final system are exactly equal to those of the original system. The receptances of the modified system are calculated, then a point and a transfer receptance are compared with those of the original system in Figs. 8 and 9, respectively.

As shown in Figs. 8 and 9, all of the natural frequencies in question of the modified and original systems align with each other. Consequently, three natural frequencies of the original system are effectively preserved simultaneously by using the proposed method. For the same problem, different modification coordinates were chosen. The obtained stiffness modifications and the natural frequencies of the final system are given in Table 3. It is seen that the solution does not converge for some chosen modification coordinates, even though they are sensitive to the stiffness modification (see Fig. 7). Also it is noted, the modification which has minimum error norm may be the best choice.

When considering the real structures there is a large number of locations to modify the structure. However, the optimum modification case can be chosen amongst the predefined modification coordinates which have a solution. The criterion is that while the natural frequencies in question are preserved others do not change considerably after the modifications. In this way, an objective function can be described as follows:

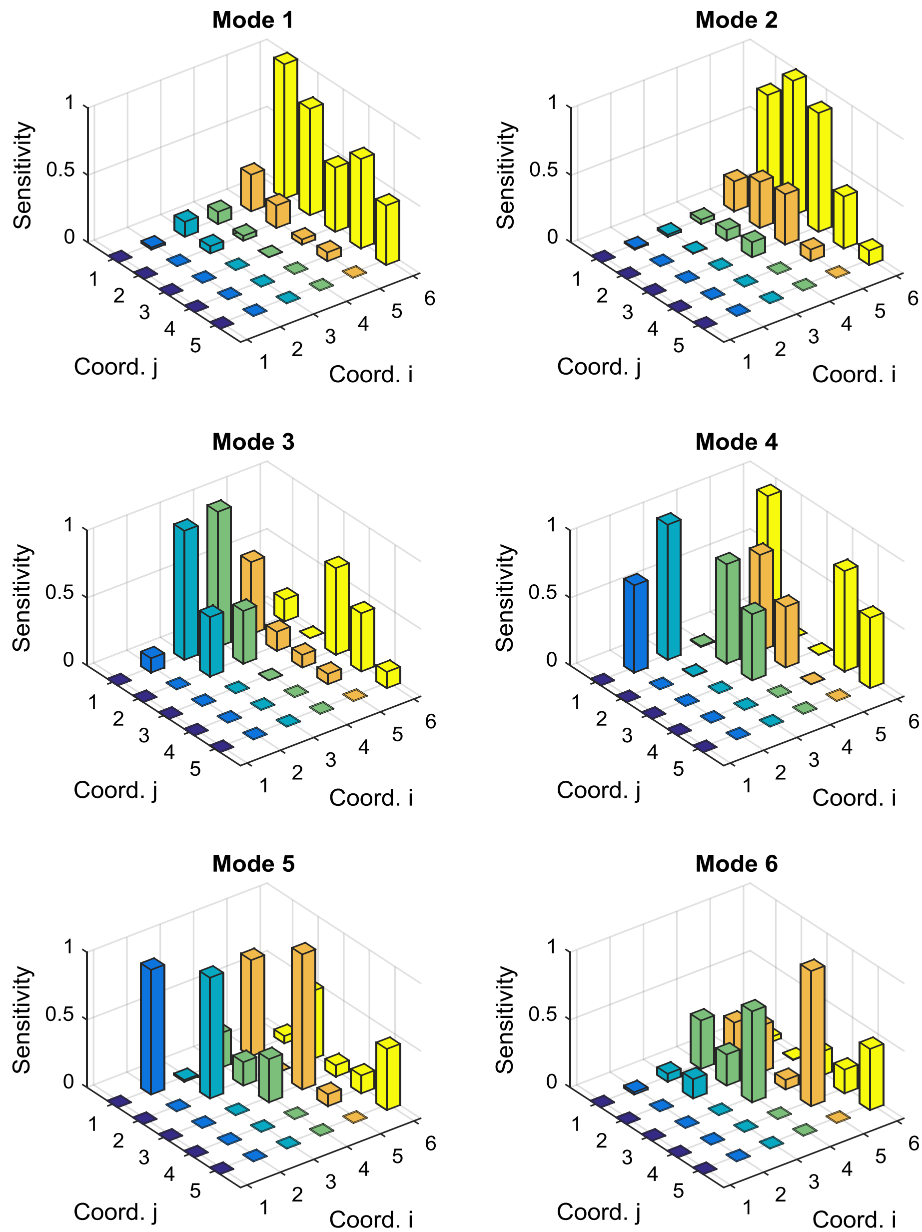

Fig. 7 The normalized sensitivities of the modes to the stiffness modifications for different coordinates

$$J_q = \left(\sum_{r=1}^{N} \left(\omega_r^{**} - \omega_r \right)^2 \right)^{1/2} \qquad (r = 1, 2, \ldots, N; q = 1, 2, \ldots, L) \tag{14}$$

where q represents a possible modification case, J_q is the Euclidian norm of the natural frequencies change for the modification case q, and ω_r and ω_r^{**} are the natural frequencies of the original and final systems, respectively. It can be noticed from Eq. (14) that the modification case having the smallest J_q may be the optimum one. If the numerical model of the structure does not exist, the modification coordinates are chosen and the needed FRFs for these predetermined modification coordinates are measured. The necessary stiffness modifications are calculated using the proposed method for all of the modification cases. Then, the best modification case can be chosen amongst them according to the procedure mentioned above.

It should be noted that, although the addition of a small mass on to a real structure is relatively easy, a large mass introduces the rotational inertia. Moreover, the modification may be like a beam. In such cases, it is essential to measure the rotational type of FRFs which is hard in practice. On the other hand, the spring

Table 3 Calculated stiffness modifications (N/m) and natural frequencies (Hz) for different modification coordinates

Modif. coords.	Obtained modifs.	Natural frequencies (Hz)						Error norm
		1	**2**	**3**	4	**5**	6	
None	None	0.109	**0.145**	**0.205**	0.261	**0.314**	0.337	–
2–3	0.0228							10.62
3–4	0.6522	0.112	**0.145**	**0.205**	0.269	**0.314**	0.370	
5–6	0.3898	(2.8%)	**(0.0%)**	**(0.0%)**	(3.1%)	**(0.0%)**	(9.8%)	
1–5	1.441							14.27
2–4	−0.353	0.110	**0.145**	**0.205**	0.248	**0.314**	0.382	
5–6	−0.113	(0.9%)	**(0.0%)**	**(0.0%)**	(5.0%)	**(0.0%)**	(13.4%)	
1–2	Not converged	–	–	–	–	–	–	–
1–5								
2–6								
1–5	Not converged	–	–	–	–	–	–	–
2–3								
3–6								

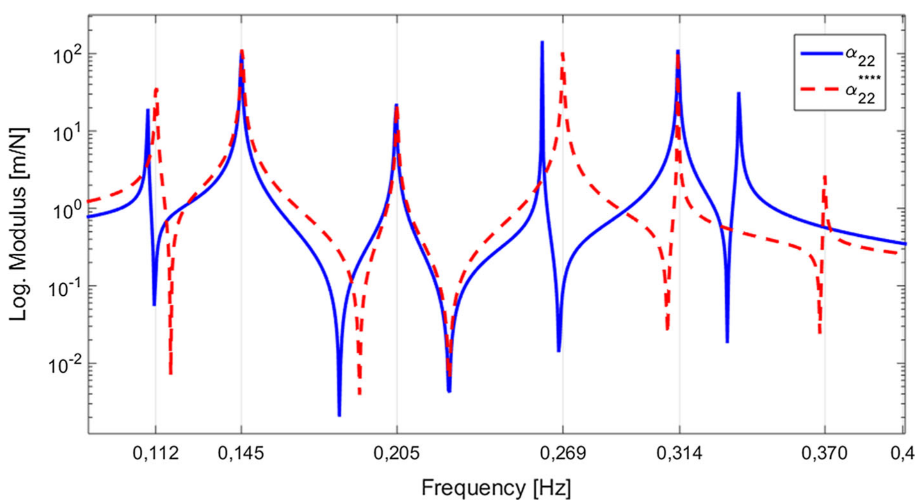

Fig. 8 Comparison of the point receptances of original and modified systems

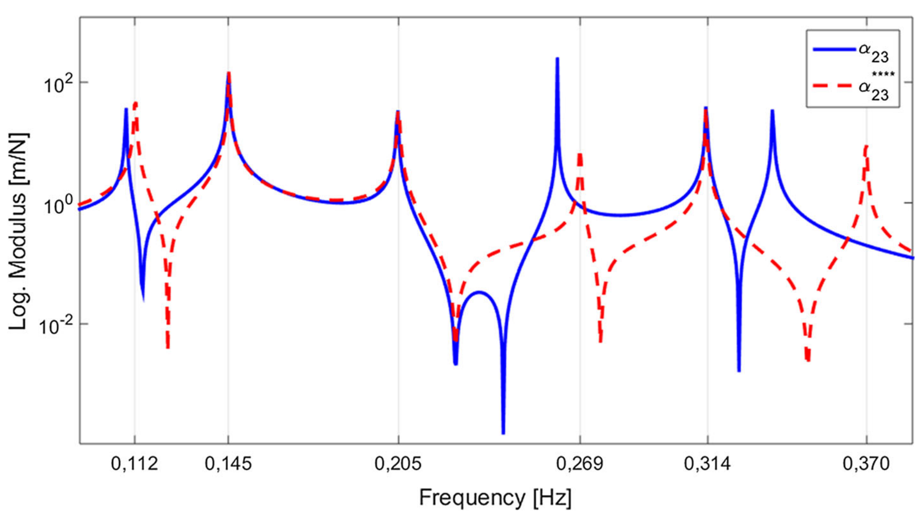

Fig. 9 Comparison of the transfer receptances of original and modified systems

modification cannot be often made between two coordinates of a real structure. Instead, the structure can be divided into elements per the finite element method and the stiffness modifications can be made by changing the cross section of the elements chosen for the modification. However, it may be difficult to realize this on an existing structure. Readers are referred to, e.g. [4,5] and [7] for the modification examples of the continuous system.

5 Conclusions

A type of inverse modification problem is considered. When a system is modified for a design aim, its natural frequencies shift such that one or more of them may be overlapped with the frequencies of the harmonic excitation forces. In order to prevent shifting of the troublesome natural frequencies of a system after mass modification, a number of springs can be added between a certain couples of coordinates in the system. The determination of the necessary stiffness modifications is the challenge of inverse modification. In this study, two methods based on the SM formula were developed for the determination of the necessary spring coefficients. The proposed method is precise for the preservation of a single natural frequency. In the proposed method, for the preservation of multiple natural frequencies, after sequential modifications equal to the number of preserved natural frequencies a set of nonlinear equations are obtained and then this set of equations is solved numerically. One of the important features of the structural modification based on the SM formula is that it allows for sequential modifications when more than one modification is required. The proposed methods utilize this feature. The methods use the receptances of the original system related to modification coordinates only and it is unnecessary to know the system's physical parameters or modal parameters, therefore, the methods are suitable for practical applications. The efficiency and accuracy of the suggested methods are demonstrated by the numerical simulations. The results show that the methods are very effective.

It is also shown that, due to the nature of the inverse problem the solution is not unique and the solution may not be converged in some cases. The number of natural frequencies desired to keep unchanged is usually smaller than the number of modification coordinates. For the choosing of proper modification coordinates amongst the predefined ones, a sensitivity approach demonstrated by a 3D graph is proposed in this study. Nevertheless, this does not guarantee a solution exists. On the other hand, the best modification case can be chosen by a proposed objective function, amongst the alternative modification cases which have a solution.

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