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Stiff phase nucleation in a phase-transforming bar due to the collision of non-stationary waves

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Abstract We deal with a new phase nucleation in a phase-transforming bar caused by a collision of two non-stationary waves. We consider an initial stage of dynamical process in the finite bar before the moment of time when the waves emerged due to new phase nucleation reach the ends of the bar. The model of a phase-transforming bar with trilinear stress–strain relation is used. The problem is formulated as a scale-invariant initial value problem with additional restrictions in the form of several inequalities involving the problem parameters. We consider the particular limiting case where the stiffness of a new phase inclusion is much greater than the stiffness of the initial phase and obtain the asymptotic solution in the explicit form. In particular, the domains of existence of the solution in the parameter space are constructed.

Keywords Phase transitions · 1D elastodynamics · Asymptotics

1 Introduction

In this paper, we consider a new phase nucleation in a phase-transforming bar caused by a collision of two non-stationary waves. A good review of modern approaches to the phase transitions in solids can be found in the Introduction to paper [39] by Rosi et al. Briefly speaking, these are

– continuum models with a free energy density being a non-convex function of the deformation gradient (see, e.g., [1,3,16,26,32,33,35,36,38,43,44]). It is known that the problem of elastostatics for such a kind of material can have solutions with discontinuous deformation gradients [31]. In the framework of the model, the surfaces of the strain discontinuity are considered as the phase boundaries, and the domains of continuity are considered as zones occupied by different phases of the material. The solution of both statical and dynamical problems is generally non-unique. Therefore, an additional thermodynamic boundary condition [1,3,16,26,32,33,35,38,44] that establishes a relationship between the configurational (material, thermodynamical) force [26,29,34] and the speed of the phase boundary is required;

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- continuum models with a free energy density depending on both the deformation gradient, its higher gradients, and, possibly, on the gradient of the order parameter or concentration of phases (the phase-field models, see e.g. [10,11,41]). In the framework of the model, the interfaces between phases are layers of finite thickness. In particular, the strain-induced phase transformations (phase transformations occurring in the course of plastic deformations) [19,28] can be described by the phase-field approach.
- molecular dynamics models (see e.g. [27,37]);

In this paper we utilize the first approach. The stress in the bar is assumed to be a piecewise linear function of the strain containing a "negative slope segment." Thus, the small strain approximation is used in the framework of the model with the strain energy being a non-convex function of the strain. This simplest model for stress-induced martensitic phase transitions in elastic bodies is widespread in the literature [1–4,12,20,30,33,35,45]; it allows one to investigate two-dimensional and three-dimensional quasi-static problems concerning the phase transitions in finite bodies [20,45], in shells [15,16], and to get the solutions of non-stationary dynamical problems [1–3,30,33]. We use this model, because we treat analytically a non-stationary dynamic problem concerning the wave propagation in the phase-transforming bar and want to obtain the results in a clear explicit form.

The paper is organized as follows. In Sect. 2, we introduce the problem formulation. Due to non-stationary loadings applied at the ends of a phase-transforming bar, two non-stationary waves propagate from the ends of the bar to its middle point. As a result of the collision of these waves, a new phase nucleus emerges in a neighborhood of the middle point. We deal with only symmetric with respect to the middle point solutions and consider an initial stage of dynamical process in the finite bar before the moment of time when the waves emerged due to new phase nucleation reach the ends of the bar. To find the unknown value of the speed of the new phase inclusion boundary, we formulate an additional constitutive equation (the so-called thermodynamic boundary condition) in the most general form which respects the second law of thermodynamics. The solution of the problem under consideration is given in Sect. 3. It is proved that for any kind of the thermodynamic boundary condition the problem under consideration is scale-invariant; therefore, the speed of the phase boundary (as well as other quantities) does not depend on time. This allows us to reduce the problem to the algebraic equation of the 3d order for phase boundary speed with additional restrictions in the form of several inequalities involving the problem parameters. In the general case, the analytic solution of this cubic equation is quite complicated and non-informative, and it is difficult to determine analytically the number of real roots and to select those which satisfy all necessary restrictions (36)–(40) and (55) in the form of inequalities to construct the domain of existence of the solution in the parameter space. One can observe three important particular cases for which it is reasonable to obtain the solutions in an explicit form. These are the case of soft phase inclusion [25], the case of a new phase inclusion of the stiffness equal to the initial one [40], and the case of the stiff phase inclusion (where the stiffness of a new phase inclusion is much greater than the stiffness of the initial phase), which is under consideration in this paper. These three qualitatively different particular cases being considered all together give one the qualitative understanding of the overall picture. In Sect. 4 we obtain the asymptotic solution in the explicit form and construct the domains of existence of the solution in the parameter space. In "Conclusion," we summarize briefly the basic results. In "Appendix," we analyze the near-critical motion of the phase boundary. This helps us to understand the meaning of some irregular point which naturally appears when we construct the asymptotics in Sect. 4.

Our previous investigations [25,40] were motivated by classical studies by Bažant et al. [5–8]. In [23], we presented the basic formulas for all three cases for a non-dissipative system [where condition (20) is used]. The similar but not equivalent problem related with new phase nucleation caused by an impact loading applied at the end of a semi-infinite phase-transforming bar was considered by Abeyaratne and Knowles in [2,3,30]. Although this problem has different formulation in comparison with the one formulated in this paper, there are some similarities in the structure of the solutions for both problems. The detailed comparison between these two problems and corresponding methods of the solution is given in [24].

2 The problem formulation

Consider a straight elastic bar with cross-sectional area S. Let $u(x, t)$ be the longitudinal displacements of the bar cross sections. Here $x(-L < x < L)$ is the position of a cross section, t is time, $2L$ is the length of the bar. Prime and dot denote the partial derivatives with respect to x and t, respectively. The strain $\varepsilon = u'$ is assumed to be infinitesimal. We consider the pure mechanical theory; hence, any thermal effects are neglected. The material of the bar is capable of undergoing phase transitions; therefore, the stress–strain relation has to be non-monotonous and contain a "negative slope segment," i.e., the elastic potential for the material must be a non-convex function of the strain. We assume the stress σ to be a piecewise linear function of the strain ε (Fig. 1):

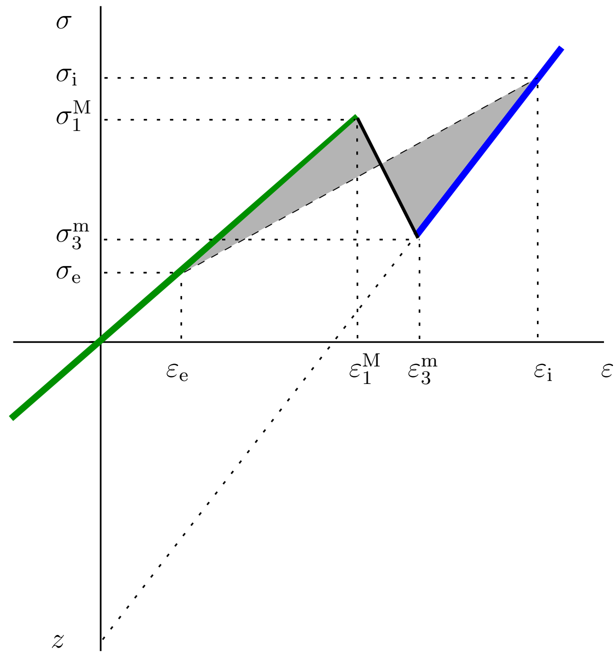

Fig. 1 The stress–strain curve for the material of a phase-transforming bar

$$\sigma(\varepsilon)=\begin{cases}E_1\varepsilon, & \varepsilon\le\varepsilon_1^{\mathrm{M}};\\ -E_2(\varepsilon-\varepsilon_1^{\mathrm{M}})+\sigma_1^{\mathrm{M}}, & \varepsilon_1^{\mathrm{M}}<\varepsilon\le\varepsilon_3^{\mathrm{m}};\\ E_3(\varepsilon-\varepsilon_3^{\mathrm{m}})+\sigma_3^{\mathrm{m}}, & \varepsilon>\varepsilon_3^{\mathrm{m}}.\end{cases}\tag{1}$$

Here $E_k>0,\ k=\overline{1,3}$ are the elastic moduli for corresponding intervals of stress–strain curve, symbols

$$\varepsilon_1^{\mathrm{M}}>0,\quad \varepsilon_3^{\mathrm{m}}>0,\quad \sigma_1^{\mathrm{M}}>0,\quad \sigma_3^{\mathrm{m}}>0\tag{2}$$

are defined in Fig. 1. The intervals $\varepsilon\le\varepsilon_1^{\mathrm{M}}$ and $\varepsilon>\varepsilon_3^{\mathrm{m}}$ correspond to the stable states of the material (phases 1 and 3, respectively). Phase 2 ($\varepsilon_1^{\mathrm{M}}<\varepsilon<\varepsilon_3^{\mathrm{m}}$), which corresponds to the incident part of the strain–stress curve, is unstable.

The governing equation for the bar is

$$\sigma'-\rho\ddot{u}=0.\tag{3}$$

Here ρ is the mass density for the material of the bar, which is assumed to be the same for both stable phases of the material. Put $c_k=\sqrt{E_k/\rho}$ ($k=1,2,3$). Equation (3) remains valid in the domains occupied by two stable phases (1 and 3) of the material of the bar. Since $\sigma'=E_k u''$, $k=1,3$, Eq. (3) can be transformed to

$$u''-c_k^2\ddot{u}=0,\quad k=1,3.\tag{4}$$

In what follows, we consider $t=-L/c_1$ as the initial moment of time and formulate zero initial conditions in the form

$$u\big|_{t=-L/c_1}=0,\quad \dot{u}\big|_{t=-L/c_1}=0.\tag{5}$$

The bar is supposed to be excited with two kinematic loadings applied at its ends $x=\mp L$:

$$u\big|_{x=\pm L}=\pm v\theta(t+L/c_1),\tag{6}$$

where $\theta(t)\equiv tH(t)$, $H(t)$ is the Heaviside function, $v>0$.

We suppose that initially the material of all cross sections of the bar is in the phase state 1. In what follows, we assume that the rate of the external loading v is such that

$$\frac{2v}{c_1} > \varepsilon_1^{\mathrm{M}}, \tag{7}$$
$$\frac{v}{c_1} < \varepsilon_1^{\mathrm{M}}. \tag{8}$$

Hence, for times $-L/c_1 < t < 0$ the problem solution is the solution of Eq. (4) together with initial conditions (5) and boundary conditions (6). One can check that the following superposition of two waves,

$$u = v\big(-\theta(t - x/c_1) + \theta(t + x/c_1)\big), \tag{9}$$

satisfies the governing Eq. (4), the boundary conditions (6), and the initial conditions (5). The corresponding expression for the strain $\varepsilon = u'$ is

$$\varepsilon = v/c_1\,(H(t - x/c_1) + H(t + x/c_1))\,. \tag{10}$$

At the moment of time $t = 0$, two waves collide. Due to the symmetry of the system with respect to the middle point of the bar, the collision of waves happens at $x = 0$. In a linearly elastic bar after such a collision, the value of the strain becomes $2v/c_1 > \varepsilon_1^{\mathrm{M}}$ inside the interval $(-c_1t, c_1t)$. One can expect that for the material with constitutive Eq. (1), at $t = 0$ a nucleus of new phase 3 emerges. Let us consider only symmetric solutions with two phase boundaries with positions $x = \pm\ell(t)$, respectively. The symmetry of the mechanical system allows one to consider only the left-hand part of the bar $-L < x < 0$ with the unique phase boundary $\ell(t) < 0$. In what follows, we consider only times $0 < t < L/c_1$ before the moment when the waves diverging from $x = 0$ reach the ends of the bar.

Let us call the part $-L < x < \ell(t)$ of the bar, where the material is in phase state 1, “the external area,” and the part $\ell(t) < x < 0$ of the bar, where the material is in phase state 3, “the internal area.” Thus, we adopt the notation $\mu_{\mathrm{e}} = \mu|_{x=\ell(t)+0}$, $\mu_{\mathrm{i}} = \mu|_{x=\ell(t)-0}$, $[\mu] = \mu_{\mathrm{e}} - \mu_{\mathrm{i}}$, $\langle\mu\rangle = (\mu_{\mathrm{e}} + \mu_{\mathrm{i}})/2$ for any arbitrary quantity $\mu(x, t)$.

If the phase boundary exists at $x = \ell$, the following boundary conditions have to be satisfied:

$$[u] = 0, \tag{11}$$
$$[\sigma] = -\rho\dot{\ell}[\dot{u}]. \tag{12}$$

Equations (11)–(12) are the Hugoniot conditions [13] for Eq. (3). These two conditions are the continuity condition and the equation for balance of momentum for the infinitesimal interval $[\ell(t)-0;\ \ell(t)+0]$, respectively. Differentiating Eq. (11) with respect to t yields

$$[\dot{u} + \dot{\ell}u'] = 0. \tag{13}$$

Thus, Eq. (12) can be written in the form

$$[\sigma] = \rho\dot{\ell}^2[\varepsilon]. \tag{14}$$

To find the unknown function $\ell(t)$, one needs to formulate an additional constitutive equation (the so-called thermodynamic boundary condition or kinetic relation). In the framework of pure mechanical theory, where the temperature and the entropy are not introduced, the second law of thermodynamics localized to the infinitesimal interval $[\ell(t)-0;\ \ell(t)+0]$ containing the phase boundary leads to the following inequality [1,3,16,26,32,44]:

$$-\mathcal{F}\dot{\ell} \ge 0, \tag{15}$$

where

$$\mathcal{F} = -S([W] - \langle\sigma\rangle[\varepsilon]) \tag{16}$$

is the configurational (thermodynamic) force,

$$SW = S\int \sigma(\varepsilon)\,\mathrm{d}\varepsilon \tag{17}$$

is the strain energy per unit length of the bar. Various types of kinetic relations were suggested in the literature (see, e.g. [1,3,9,16,32,35,44]). In [9], Berezovski et al. compare the results predicted by different theoretical models with experimental data obtained in [14,17,18]. We formulate the additional thermodynamic condition at the phase boundary, which guarantees that inequality (15) is satisfied, in the quite general form

$$\mathcal{F} = -\mathcal{F}_0(\gamma, \dot{\ell})\,\mathrm{sign}\,\dot{\ell}, \tag{18}$$

where $\mathcal{F}_0(\gamma, \dot{\ell}) \geq 0$ is a given function of the phase boundary speed $\dot{\ell}$ and a material constant γ. The following particular case of the thermodynamic condition is widespread in the literature:

$$\mathcal{F} = -\gamma\dot{\ell}, \tag{19}$$

despite the fact that in recent investigations it is emphasized that a dynamic kinetic relation should be extremely nonlinear [35, Chap. 7]. Here $\gamma > 0$ is a material constant (γ^{-1} is called the phase boundary mobility), $\mathcal{F}_0(\gamma, \dot{\ell}) = \gamma|\dot{\ell}|$. In the particular case $\mathcal{F}_0 \equiv 0$ (or $\gamma^{-1} \to +\infty$), condition (18) transforms to

$$\mathcal{F} = 0, \tag{20}$$

which ensures a dissipation-free phase boundary motion [1,3,21,22,33]. Note that we consider linear conditions (19) or (20) only as possible examples. According to Eq. (17), the geometric meaning of the non-dissipative condition (20) is as follows: The quantities $\varepsilon_e, \sigma_e, \varepsilon_i, \sigma_i$ are such that the areas of the shaded triangles in Fig. 1 are equal to each other. This is the so-called Maxwell rule [3,42]. For the scenario of the phase transformation which is under consideration in what follows, condition (15) means that the area of the left shaded triangle in Fig. 1 is less than or equal to the area of the right shaded triangle.

Note that the thermodynamic inequality (15) forbids nucleation (and subsequent expansion) of the unstable phase 2 in a vicinity of middle point $x = 0$ after the waves collision, since according to Eqs. (16) and (17) we have $\mathcal{F} < 0$ at the respective phase boundary, which contradicts with inequality (15).

3 Solution of the problem

Represent the displacements of the cross sections of the bar in the external area u_e as a superposition of two waves:

$$u_e = u_e^- + u_e^+, \tag{21}$$

where u_e^- is the incident wave and u_e^+ is the wave reflected from the phase boundary.

The expression for the incident wave u_e^- coincides with the expression for the right-running wave in the right-hand side of Eq. (9):

$$u_e^- = \frac{v}{c_1}(x - c_1 t). \tag{22}$$

We search for the reflected wave u_e^+ in the form

$$u_e^+ = u_{e1}^+\,\theta(x + c_1 t), \tag{23}$$

where the constant u_{e1}^+ must be found from the conditions at the phase boundary. The expressions for the strain and the particle velocity in the external area become

$$\varepsilon_e = \frac{v}{c_1} + u_{e1}^+, \tag{24}$$

$$\dot{u}_e = -v + c_1 u_{e1}^+, \tag{25}$$

respectively.

Analogously, we look for the displacements in the internal area of the bar in the form

$$u_i = u_{i1}^-(x - c_3 t) + u_{i1}^+(x + c_3 t). \tag{26}$$

Due to the symmetry of the system with respect to $x = 0$, in the internal area of the bar for all $t \geq 0$ one has

$$u_i(x) = -u_i(-x), \tag{27}$$

therefore,

$$u_{\mathrm{i}1}^{-} = u_{\mathrm{i}1}^{+}, \tag{28}$$
$$\dot{u}_{\mathrm{i}} = 0, \tag{29}$$
$$\varepsilon_{\mathrm{i}} = 2u_{\mathrm{i}1}^{+}. \tag{30}$$

By definition, put

$$\dot{\ell} \equiv V \tag{31}$$

and substitute Eqs. (24), (25), (29), (30) and (31) into the Hugoniot conditions (13) and (14) formulated at $x = \ell$. Taking into account the constitutive Eq. (1), we obtain a set of linear algebraic equations with respect to the quantities $u_{\mathrm{e}1}^{+}$ and $u_{\mathrm{i}1}^{+}$:

$$\begin{aligned} &-2Vu_{\mathrm{i}1}^{+} + (c_1 + V)\,u_{\mathrm{e}1}^{+} = \frac{v}{c_1}\,(c_1 - V)\,, \\ &2\left(c_3^2 - V^2\right)u_{\mathrm{i}1}^{+} - \left(c_1^2 - V^2\right)u_{\mathrm{e}1}^{+} = \frac{v}{c_1}\left(c_1^2 - V^2\right) - \frac{z}{\rho}, \end{aligned} \tag{32}$$

where

$$z \equiv \rho\big(\varepsilon_1^{\mathrm{M}}\left(c_1^2 + c_2^2\right) - \varepsilon_3^{\mathrm{m}}\left(c_2^2 + c_3^2\right)\big) = -\rho c_3^2 \varepsilon_3^{\mathrm{m}} + \sigma_3^{\mathrm{m}}. \tag{33}$$

Figure 1 illustrates the geometric meaning of the quantity z. The coefficients of the set of equations (32) are functions of the unknown quantity V. Resolving Eq. (32) and taking into account (24) and (30), one gets

$$\varepsilon_{\mathrm{e}} = \frac{2v(c_3^2 - V^2) - Vz/\rho}{(c_1 + V)\left(-c_1 V + c_3^2\right)}, \tag{34}$$
$$\varepsilon_{\mathrm{i}} = \frac{2v(c_1 - V) - z/\rho}{-c_1 V + c_3^2}. \tag{35}$$

According to the constitutive Eq. (1), the values of the strain must satisfy the following inequalities:

$$\varepsilon_{\mathrm{e}} < \varepsilon_1^{\mathrm{M}}, \tag{36}$$
$$\varepsilon_{\mathrm{i}} > \varepsilon_3^{\mathrm{m}}. \tag{37}$$

Abeyaratne and Knowles call Eqs. (36) and (37) the phase segregation conditions [3].

In order to assure that the deformation is one-to-one, we must have:

$$\varepsilon_{\mathrm{e}} > -1. \tag{38}$$

The scenario of phase transformations considered in the paper could be realized only if

$$V < 0. \tag{39}$$

Another restriction means that the phase boundary moves at a subcritical speed:

$$V > -1. \tag{40}$$

Actually, if restriction (40) is not satisfied, then the wave propagating inward the external area from the phase boundary cannot exist; hence, the set of linear algebraic equations for the strains at the phase boundary is overdetermined and generally unsolvable. The super-critical motions of the phase boundary are considered in [2,3,30]. In the latter case, which we do not consider in this paper, the phase boundary speed should be found from the solvability condition for the set of linear algebraic equations and must satisfy inequality (15). At the same time, the thermodynamic condition in the form of Eq. (18) is ignored.

In what follows, we use the dimensionless quantities $V/c_1,\ v/c_1, c_2/c_1, c_3/c_1, \sigma/\rho c_1^2, z/\rho c_1^2, \mathcal{F}/\rho S c_1^2,$ $u/L, x/L, t(c_1/L)$, assuming without loss of generality that

$$\rho = 1, \quad S = 1, \quad c_1 = 1, \quad E_1 = 1, \quad L = 1. \tag{41}$$

Now we have

$$\varepsilon_{\mathrm{e}} = \frac{2v\left(c_3^2 - V^2\right) - Vz}{(1+V)\left(c_3^2 - V\right)}, \tag{42}$$

$$\varepsilon_{\mathrm{i}} = \frac{2v(1-V) - z}{c_3^2 - V} \tag{43}$$

instead of (34) and (35), and

$$z = \varepsilon_1^{\mathrm{M}}\left(1 + c_2^2\right) - \varepsilon_3^{\mathrm{m}}\left(c_2^2 + c_3^2\right) = -c_3^2\varepsilon_3^{\mathrm{m}} + \sigma_3^{\mathrm{m}} \tag{44}$$

instead of (33). Substituting expressions (42) and (43) into the thermodynamic condition (18), one obtains the equation to determine the phase boundary speed V

$$\left(c_3^2 - 1\right)\varepsilon_{\mathrm{e}}(V)\varepsilon_{\mathrm{i}}(V) + z\big(\varepsilon_{\mathrm{i}}(V) + \varepsilon_{\mathrm{e}}(V)\big) + 2\alpha = 0, \tag{45}$$

where

$$\alpha = \frac{1}{2}(\varepsilon_3^{\mathrm{m}})^2\left(c_2^2 + c_3^2\right) - \frac{1}{2}\left(\varepsilon_1^{\mathrm{M}}\right)^2\left(1 + c_2^2\right) + \mathcal{F}_0(\gamma, V). \tag{46}$$

Hence the initial problem is reduced to determining such a root V of Eq. (45) that satisfies inequalities (36)–(40), provided that the rate of the external loading v satisfies restrictions (7) and (8). Now we see that the left-hand side of Eq. (45) does not depend on t explicitly, and the strains in the external and in the internal areas of the bar are certain constants, that indeed depend on the problem parameters. Therefore, without loss of generality we can put that in any motion

$$\dot{V} = 0, \tag{47}$$

since we formulate the problem under consideration as the Riemann problem with piecewise constant initial data (see, e.g. [1,3,36,43]). Hence, the strain in the whole bar is a self-similar function that depends on x/t only: $\varepsilon(x,t) = \varepsilon(x/t)$. Moreover, we put

$$\mathcal{F}_0(\gamma, V) = F, \qquad \dot{F} = 0, \tag{48}$$

$$F \geq 0. \tag{49}$$

Thus, instead of solving the problem with given $\mathcal{F}_0(\gamma, V)$, we solve the one-parameter family of problems with various $F \geq 0$. If such a root

$$V = \mathcal{V}_F(v) \tag{50}$$

is found, then for $0 < t < 1, x < 0$ the exact solutions for the family of problems under investigation are given by

$$u = \big(v(x-t) + (\varepsilon_{\mathrm{e}}(\mathcal{V}_F) - v)\,\theta(x+t)\big)H(\mathcal{V}_F t - x) + \varepsilon_{\mathrm{i}}(\mathcal{V}_F)xH(x - \mathcal{V}_F t), \tag{51}$$

where $\varepsilon_{\mathrm{e}}, \varepsilon_{\mathrm{i}}$ are given by Eqs. (42) and (43), respectively. The strain and the particle velocity are as follows:

$$\varepsilon = \big(v + (\varepsilon_{\mathrm{e}}(\mathcal{V}_F) - v)\,H(x+t)\big)H(\mathcal{V}_F t - x) + \varepsilon_{\mathrm{i}}(\mathcal{V}_F)H(x - \mathcal{V}_F t), \tag{52}$$

$$\dot{u} = \big(-v + (\varepsilon_{\mathrm{e}}(\mathcal{V}_F) - v)\,H(x+t)\big)H(\mathcal{V}_F t - x). \tag{53}$$

If we need to select from the family of solutions (51) the solution that corresponds to both the prescribed function $\mathcal{F}_0(\gamma, V)$ and the prescribed value of γ, we should resolve the equation

$$\mathcal{F}_0(\gamma, \mathcal{V}_F) = F \tag{54}$$

with respect to γ (see [40] for an example how to do this).

The consideration of the problem solution in the form of Eqs. (51)–(53) allows one to formulate one more restriction on the value of the phase boundary speed in addition to Eqs. (36)–(40). Indeed, the wave-field inside the internal area is caused by the moving boundaries of this area. Therefore, the non-stationary solution with constant strain and particle velocity (e.g., statics) inside the whole internal area can exist only in the case when the speed of sound in the internal area is less than the speed of the boundaries for this area. Thus,

$$-V < c_3. \tag{55}$$

It is not obvious that now we have formulated an exhaustive set of physically motivated restrictions for the proper root of the equation for the phase boundary speed. Moreover, in our previous studies not all necessary restrictions were taken into account. In [23,25,40], inequalities (38) and (55) are not taken into account. However, it is possible to prove that the solutions constructed in [23,25,40] satisfy (38) and (55), provided that assumptions (2), (7) and (8) are fulfilled.

Equation (45) can be transformed to the algebraic equation of the 3d order for phase boundary speed V. The analytic solution of this cubic equation can be found by means of symbolic calculation software MAPLE. However, in the general case the solution is quite complicated and non-informative, and it is difficult to determine analytically the number of real roots and to select those which satisfy restrictions (36)–(40) and (55).

4 Stiff inner phase ($c_3 \to +\infty$)

Put

$$c_3 = \epsilon^{-1}, \tag{56}$$

where ϵ is a formal small parameter. In what follows, we suppose that the ansatz for the phase boundary speed has the form

$$V = V_0 + O(\epsilon), \tag{57}$$

$$V_0 = O(1). \tag{58}$$

Due to Eq. (44) one has

$$z = -\varepsilon_3^{\mathrm{m}}\epsilon^{-2} + z_0, \tag{59}$$

where

$$z_0 = \sigma_3^{\mathrm{m}} = \varepsilon_1^{\mathrm{M}}\left(1 + c_2^2\right) - \varepsilon_3^{\mathrm{m}} c_2^2. \tag{60}$$

For the constant quantity α [see (46)] one obtains:

$$\alpha = \frac{1}{2}\left(\varepsilon_3^{\mathrm{m}}\right)^2 \epsilon^{-2} + \alpha_0, \tag{61}$$

$$\alpha_0 = \frac{(\varepsilon_3^{\mathrm{m}})^2}{2} c_2^2 - \frac{(\varepsilon_1^{\mathrm{M}})^2}{2}\left(1 + c_2^2\right) + F_0. \tag{62}$$

Remark 1 In what follows, we do not consider the case

$$v = \frac{\varepsilon_3^{\mathrm{m}}}{2}, \tag{63}$$

since one can prove that under condition $c_3 \to \infty$ Eq. (63) leads to $V_0 \to -1$ (see "Appendix").

Substituting expansion (57) into Eq. (42) for the strain ε_{e} one obtains:

$$\begin{aligned}\varepsilon_{\mathrm{e}} &= \frac{2v(\epsilon^{-2} - V^2) - V\left(z_0 - \varepsilon_3^{\mathrm{m}}\epsilon^{-2}\right)}{(1+V)(\epsilon^{-2} - V)} = \frac{\epsilon^{-2}\left(2v\left(1 - V^2\epsilon^2\right) - V\left(z_0\epsilon^2 - \varepsilon_3^{\mathrm{m}}\right)\right)}{\epsilon^{-2}(1+V)(1-\epsilon^2 V)} \\ &= \frac{\left(2v + V\varepsilon_3^{\mathrm{m}}\right) - \epsilon^2 V(2vV + z_0)}{1+V}(1 + V\epsilon^2 + o(\epsilon^2)) = \varepsilon_{\mathrm{e}0} + O(\epsilon^2),\end{aligned} \tag{64}$$

where

$$\varepsilon_{\mathrm{e}0} = \frac{2v + V_0\varepsilon_3^{\mathrm{m}}}{1 + V_0}. \tag{65}$$

Using formula (43) for the strain $\varepsilon_{\rm i}$, one can derive:

$$\varepsilon_{\rm i} = \frac{2v(1-V) - z_0 + \varepsilon_3^{\rm m}\epsilon^{-2}}{\epsilon^{-2} - V} = \frac{\epsilon^{-2}\left((2v(1-V)-z_0)\epsilon^2 + \varepsilon_3^{\rm m}\right)}{\epsilon^{-2}(1-\epsilon^2 V)}$$
$$= \left(\varepsilon_3^{\rm m} + \epsilon^2(2v(1-V)-z_0))(1+\epsilon^2 V + o(\epsilon)\right) = \varepsilon_3^{\rm m} + \varepsilon_{\rm i2}\epsilon^2 + o(\epsilon^2), \tag{66}$$

where

$$\varepsilon_{\rm i2} = 2v(1-V_0) - z_0 + V_0\varepsilon_3^{\rm m}. \tag{67}$$

Substitute Eqs. (64) and (66) into Eq. (45) and balance the corresponding terms of the same orders. For the coefficient before ϵ^{-2} one obtains:

$$\varepsilon_3^{\rm m}\frac{2v + V_0\varepsilon_3^{\rm m}}{1+V_0} - \varepsilon_3^{\rm m}\left(\varepsilon_3^{\rm m} + \frac{2v+V_0\varepsilon_3^{\rm m}}{1+V_0}\right) + \left(\varepsilon_3^{\rm m}\right)^2 = 0. \tag{68}$$

The above equation is satisfied identically. Obviously, the coefficient before ϵ^{-1} also equals zero identically. For the coefficient before ϵ^0 we get:

$$\frac{2v-\varepsilon_3^{\rm m}}{1+V_0}\left((2v-z_0) - V_0(2v-\varepsilon_3^{\rm m})\right) + \left(z_0 - \varepsilon_3^{\rm m}\right)\frac{2v+V_0\varepsilon_3^{\rm m}}{1+V_0} + z_0\varepsilon_3^{\rm m} + 2\alpha_0 = 0. \tag{69}$$

Provided that V_0 satisfy inequality (40), it follows from Eq. (69):

$$V_0\left(-2v^2 + 2v\varepsilon_3^{\rm m} - \varepsilon_3^{\rm m2} + z_0\varepsilon_3^{\rm m} + \alpha_0\right) + \left(2v^2 - 2v\varepsilon_3^{\rm m} + z_0\varepsilon_3^{\rm m} + \alpha_0\right) = 0. \tag{70}$$

This yields

$$V_0 = \frac{2v^2 - 2v\varepsilon_3^{\rm m} + z_0\varepsilon_3^{\rm m} + \alpha_0}{2v^2 - 2v\varepsilon_3^{\rm m} + \varepsilon_3^{\rm m2} - z_0\varepsilon_3^{\rm m} - \alpha_0} = \frac{\left(v - \frac{\varepsilon_3^{\rm m}}{2}\right)^2 - \frac{\varepsilon_3^{\rm m2}}{4} + \frac{z_0\varepsilon_3^{\rm m}}{2} + \frac{\alpha_0}{2}}{\left(v - \frac{\varepsilon_3^{\rm m}}{2}\right)^2 + \frac{\varepsilon_3^{\rm m2}}{4} - \frac{z_0\varepsilon_3^{\rm m}}{2} - \frac{\alpha_0}{2}}. \tag{71}$$

Note that since generally Eq. (45) can be transformed to the algebraic equation of the 3d order for phase boundary speed V, another two roots of (45) exist. One can show that these roots are large and do not satisfy Eqs. (57) and (58).

Now we need to verify the set of restrictions (36)–(40) and (55), where V is defined by Eqs. (57), (71), and $\varepsilon_{\rm e}$, $\varepsilon_{\rm i}$ are defined by Eqs. (64)–(67). In what follows, we assume that restrictions (7) and (8) are satisfied. Note that in the case (56) under consideration, restriction (55) is fulfilled if (40) is true.

Proposition 1 *Provided that* v *satisfies inequality*

$$v < v_{\max} \equiv \frac{\varepsilon_3^{\rm m}}{2} + \frac{1}{2}\left(\varepsilon_3^{\rm m} - \varepsilon_1^{\rm M}\right)\sqrt{1+c_2^2}, \tag{72}$$

the inequality

$$F_{\max} \equiv \frac{1}{2}\left(1+c_2^2\right)\left(\varepsilon_3^{\rm m} - \varepsilon_1^{\rm M}\right)^2 - 2\left(v - \frac{\varepsilon_3^{\rm m}}{2}\right)^2 > 0 \tag{73}$$

is true.

Proof Inequality (73) is equivalent to

$$\left|v - \frac{\varepsilon_3^{\rm m}}{2}\right| < \frac{1}{2}\left(\varepsilon_3^{\rm m} - \varepsilon_1^{\rm M}\right)\sqrt{1+c_2^2}. \tag{74}$$

For $v < \varepsilon_3^{\rm m}/2$, one can rewrite inequality (74) as follows:

$$\frac{\varepsilon_3^{\rm m}}{2} - \frac{1}{2}\left(\varepsilon_3^{\rm m} - \varepsilon_1^{\rm M}\right)\sqrt{1+c_2^2} < v. \tag{75}$$

Let us prove that

$$\frac{\varepsilon_3^{\mathrm{m}}}{2} - \frac{1}{2}\left(\varepsilon_3^{\mathrm{m}} - \varepsilon_1^{\mathrm{M}}\right)\sqrt{1 + c_2^2} < \frac{\varepsilon_1^{\mathrm{M}}}{2}. \tag{76}$$

This inequality is equivalent to the following one:

$$\left(\varepsilon_3^{\mathrm{m}} - \varepsilon_1^{\mathrm{M}}\right)\left(1 - \sqrt{1 + c_2^2}\right) < 0, \tag{77}$$

which is always true. Hence, due to restriction (7) inequality (75) is also satisfied.
For the case $v > \varepsilon_3^{\mathrm{m}}/2$ inequality (74) is equivalent to restriction (72). □

Proposition 2 *The phase boundary speed V_0 defined by Eq.* (71) *satisfies restrictions* (39) *and* (40) *simultaneously if and only if both*

– *inequality*

$$F < F_{\max}, \tag{78}$$

– *and inequality* (72)

are true. Here $F_{\max}$ is defined by Eq. (73).

Proof Taking into account Eqs. (60) and (62) on can find out that

$$\frac{\varepsilon_3^{\mathrm{m}\,2}}{4} - \frac{z_0\varepsilon_3^{\mathrm{m}}}{2} - \frac{\alpha_0}{2} = \frac{1}{4}\left(1 + c_2^2\right)\left(\varepsilon_3^{\mathrm{m}} - \varepsilon_1^{\mathrm{M}}\right)^2 - \frac{1}{2}F. \tag{79}$$

Put

$$X = \left(v - \frac{\varepsilon_3^{\mathrm{m}}}{2}\right)^2, \tag{80}$$

$$Y = \frac{1}{4}\left(1 + c_2^2\right)\left(\varepsilon_3^{\mathrm{m}} - \varepsilon_1^{\mathrm{M}}\right)^2 - \frac{1}{2}F. \tag{81}$$

Note that

$$X > 0 \tag{82}$$

according to Remark 1. Now due to Eq. (71) one has

$$V_0 = \frac{X - Y}{X + Y}, \tag{83}$$

and Eqs. (39) and (40) can be rewritten in the form of inequalities

$$\frac{X - Y}{X + Y} < 0, \tag{84}$$

$$\frac{X - Y}{X + Y} > -1, \tag{85}$$

respectively.
Take $X + Y < 0$. It follows from Eq. (85) that $X < 0$, which contradicts with Eq. (82).
Take $X + Y > 0$. Equation (85) is equivalent to inequality $X > 0$, which is true. Equation (84) reduces to

$$Y > X > 0, \tag{86}$$

which due to (80) and (81) can be equivalently transformed to (78). Note that due to Eq. (81) inequality $Y > 0$ can be written as follows:

$$F < \frac{1}{2}\left(1 + c_2^2\right)\left(\varepsilon_3^{\mathrm{m}} - \varepsilon_1^{\mathrm{M}}\right)^2. \tag{87}$$

□

Proposition 3 *Quantity*

$$F_* = \left(v - \frac{\varepsilon_3^{\mathrm{m}}}{2}\right)\left(2v - 2\varepsilon_1^{\mathrm{M}} + \varepsilon_3^{\mathrm{m}}\right) + \frac{1}{2}\left(1 + c_2^2\right)\left(\varepsilon_3^{\mathrm{m}} - \varepsilon_1^{\mathrm{M}}\right)^2 \tag{88}$$

(1) always satisfies the inequality

$$F_* > 0; \tag{89}$$

(2) and satisfies the inequality

$$F_* < F_{\max} \tag{90}$$

if and only if

$$v < v_* \equiv \frac{\varepsilon_3^{\mathrm{m}}}{2}. \tag{91}$$

Proof Let us prove (89). Taking the first term in the right-hand side of (88), one obtains

$$\begin{aligned}\left(v - \frac{\varepsilon_3^{\mathrm{m}}}{2}\right)\left(2v - 2\varepsilon_1^{\mathrm{M}} + \varepsilon_3^{\mathrm{m}}\right) &= -2\left(\frac{\varepsilon_3^{\mathrm{m}2}}{4} - v^2 - \frac{\varepsilon_3^{\mathrm{m}}\varepsilon_1^{\mathrm{M}}}{2} + \varepsilon_1^{\mathrm{M}} v\right) \\ &= -2\left(\frac{\varepsilon_3^{\mathrm{m}2}}{4} - \frac{\varepsilon_3^{\mathrm{m}}\varepsilon_1^{\mathrm{M}}}{2} + \frac{\varepsilon_1^{\mathrm{M}2}}{4} - \frac{\varepsilon_1^{\mathrm{M}2}}{4} - v^2 + \varepsilon_1^{\mathrm{M}} v\right) \\ &= 2\left(\frac{\varepsilon_1^{\mathrm{M}}}{2} - v\right)^2 - \frac{1}{2}\left(\varepsilon_3^{\mathrm{m}} - \varepsilon_1^{\mathrm{M}}\right)^2.\end{aligned} \tag{92}$$

Substituting the above expression into Eq. (88) yields the following inequality

$$F_* = 2\left(\frac{\varepsilon_1^{\mathrm{M}}}{2} - v\right)^2 + \frac{1}{2}c_2^2\left(\varepsilon_3^{\mathrm{m}} - \varepsilon_1^{\mathrm{M}}\right)^2 > 0, \tag{93}$$

which is obviously true.

Let us prove (90). Substituting Eqs. (88), (73) into (90), one gets

$$4\left(\frac{\varepsilon_3^{\mathrm{m}}}{2} - v\right)\left(v - \frac{\varepsilon_1^{\mathrm{M}}}{2}\right) > 0, \tag{94}$$

which is true according to (7) if and only if (91) holds. □

Proposition 4 *Provided that* (72) *is true, the strain* $\varepsilon_{e0}(V_0)$ *given by Eqs.* (65)*,* (71) *satisfies inequality* (36) *if and only if*

- *F satisfies the restriction*

$$F < F_*, \tag{95}$$

- *and v satisfies inequality* (91).

Proof According to Eq. (71), inequality (36) can be written in the form:

$$\frac{2v + V_0\varepsilon_3^{\mathrm{m}}}{1 + V_0} < \varepsilon_1^{\mathrm{M}}, \tag{96}$$

or, equivalently

$$V_0 < -\frac{2v - \varepsilon_1^{\mathrm{M}}}{\varepsilon_3^{\mathrm{m}} - \varepsilon_1^{\mathrm{M}}}. \tag{97}$$

Hence, taking into account Eq. (83) one gets:

$$\frac{X-Y}{X+Y} < -\frac{2v-\varepsilon_1^{\mathrm{M}}}{\varepsilon_3^{\mathrm{m}}-\varepsilon_1^{\mathrm{M}}}. \tag{98}$$

Consider the case $v < \varepsilon_3^{\mathrm{m}}/2$. Inequality (98) is equivalent to

$$X\frac{2v+\varepsilon_3^{\mathrm{m}}-2\varepsilon_1^{\mathrm{M}}}{\varepsilon_3^{\mathrm{m}}-2v} - Y < 0. \tag{99}$$

Substituting here Eqs. (80) and (81) yields

$$\frac{1}{2}\left(\frac{\varepsilon_3^{\mathrm{m}}}{2}-v\right)^2\frac{2v+\varepsilon_3^{\mathrm{m}}-2\varepsilon_1^{\mathrm{M}}}{\varepsilon_3^{\mathrm{m}}/2-v} - \frac{1}{4}\left(1+c_2^2\right)\left(\varepsilon_3^{\mathrm{m}}-\varepsilon_1^{\mathrm{M}}\right)^2 + \frac{1}{2}F < 0. \tag{100}$$

This inequality can be easily transformed to the restriction (95).

Consider now the case $\varepsilon_3^{\mathrm{m}}/2 < v < v_{\max}$. Inequality (97) leads to $V_0 < -1$, which contradicts with Proposition 2. □

Proposition 5 *Provided that* (91) *and* (95) *are true, the strain* $\varepsilon_{\mathrm{i}2}(V_0)$ *satisfies inequality*

$$\varepsilon_{\mathrm{i}2}(V_0) > 0. \tag{101}$$

Proof One can rewrite Eq. (101) as follows:

$$2v > \frac{z_0 - V_0\varepsilon_3^{\mathrm{m}}}{1-V_0}. \tag{102}$$

Adding the term $-\varepsilon_3^{\mathrm{m}}$ to the both sides of the inequality results in

$$2v - \varepsilon_3^{\mathrm{m}} > \frac{z_0 - V_0\varepsilon_3^{\mathrm{m}}}{1-V_0} - \varepsilon_3^{\mathrm{m}}. \tag{103}$$

Using Eq. (60) for z_0, we derive:

$$2\left(v-\frac{\varepsilon_3^{\mathrm{m}}}{2}\right) > -\frac{\left(\varepsilon_3^{\mathrm{m}}-\varepsilon_1^{\mathrm{M}}\right)\left(1+c_2^2\right)}{1-V_0}. \tag{104}$$

Taking into account Eqs. (91), (80), (83) one can rewrite the last inequality in the following way:

$$2\sqrt{X} < \frac{\left(\varepsilon_3^{\mathrm{m}}-\varepsilon_1^{\mathrm{M}}\right)\left(1+c_2^2\right)}{2Y}(X+Y), \tag{105}$$

or

$$f(\sqrt{X}) \equiv \frac{1}{4}\left(\varepsilon_3^{\mathrm{m}}-\varepsilon_1^{\mathrm{M}}\right)\left(1+c_2^2\right)(\sqrt{X})^2 - Y\sqrt{X} + \frac{1}{4}\left(\varepsilon_3^{\mathrm{m}}-\varepsilon_1^{\mathrm{M}}\right)\left(1+c_2^2\right)Y > 0. \tag{106}$$

In the proof of Proposition 2, it is demonstrated that inequality $Y > 0$ is equivalent to (87) and that $F_{\max}$ is less than the right-hand side of (87). Thus $Y > 0$, since $F < F_* < F_{\max}$.

Now one can see that the discriminant D of the quadratic function $f(\sqrt{X})$ is negative:

$$D = -Y\left(\frac{1}{4}c_2^2\left(1+c_2^2\right)\left(\varepsilon_3^{\mathrm{m}}-\varepsilon_1^{\mathrm{M}}\right)^2 + \frac{1}{2}F\right) < 0. \tag{107}$$

One can see that for $\sqrt{X} \to +0$

$$f(\sqrt{X}) = \frac{1}{4}\left(\varepsilon_3^{\mathrm{m}}-\varepsilon_1^{\mathrm{M}}\right)\left(1+c_2^2\right)Y > 0. \tag{108}$$

Hence, inequality (106) is satisfied for all $\sqrt{X}$. □

Note that due to Proposition 5 and Eq. (66) for enough small ϵ the strain $\varepsilon_{\rm i}$ satisfies restriction (37), provided that (91) and (95) are true.

Proposition 6 *Provided that* (91) *is true, the quantity*

$$\hat{F} = \frac{1}{2}\left(1 + c_2^2\right)\left(\varepsilon_3^{\rm m} - \varepsilon_1^{\rm M}\right)^2 - 2\left(\frac{\varepsilon_3^{\rm m}}{2} - v\right)\left(\frac{\varepsilon_3^{\rm m}}{2} + v + 1\right) \tag{109}$$

(1) always satisfies the inequality

$$\hat{F} < F_*; \tag{110}$$

(2) and satisfies the inequality

$$\hat{F} > 0 \tag{111}$$

if and only if

$$v > \hat{v} \equiv -\frac{1}{2}\left(1 - \sqrt{\left(\varepsilon_3^{\rm m} + 1\right)^2 - \left(1 + c_2^2\right)\left(\varepsilon_3^{\rm m} - \varepsilon_1^{\rm M}\right)^2}\right). \tag{112}$$

(3) The quantity $\hat{v}$ *is such that*

$$\hat{v} > \frac{\varepsilon_1^{\rm M}}{2}, \tag{113}$$

$$\hat{v} < v_*. \tag{114}$$

Proof To prove (110), we substitute Eqs. (88) and (109) there. This yields

$$\left(\frac{\varepsilon_3^{\rm m}}{2} - v\right)\left(1 + \varepsilon_1^{\rm M}\right) > 0, \tag{115}$$

which is true due to (91).

Let us prove (111). Equation (109) can be rewritten as follows:

$$\hat{F} = \frac{1}{2}\left(1 + c_2^2\right)\left(\varepsilon_3^{\rm m} - \varepsilon_1^{\rm M}\right)^2 - \frac{1}{2}\left(\varepsilon_3^{\rm m} + 1\right)^2 + \frac{1}{2}(2v + 1)^2. \tag{116}$$

Substituting this into Eq. (111) results in

$$(2v + 1)^2 > \left(\varepsilon_3^{\rm m} + 1\right)^2 - \left(1 + c_2^2\right)\left(\varepsilon_3^{\rm m} - \varepsilon_1^{\rm M}\right)^2. \tag{117}$$

Now we want to show that the right-hand side of the last inequality is positive:

$$\left(\varepsilon_3^{\rm m} + 1\right)^2 - \left(1 + c_2^2\right)\left(\varepsilon_3^{\rm m} - \varepsilon_1^{\rm M}\right)^2 > 0, \tag{118}$$

or equivalently

$$c_2^2 - \frac{\varepsilon_1^{\rm M}}{\varepsilon_3^{\rm m} - \varepsilon_1^{\rm M}} < \frac{1 + 2\varepsilon_3^{\rm m} + \varepsilon_1^{\rm M}\varepsilon_3^{\rm m}}{\left(\varepsilon_3^{\rm m} - \varepsilon_1^{\rm M}\right)^2}. \tag{119}$$

On the other hand, from Eqs. (1) and (2) it follows that

$$c_2^2 - \frac{\varepsilon_1^{\rm M}}{\varepsilon_3^{\rm m} - \varepsilon_1^{\rm M}} < 0. \tag{120}$$

The right-hand side of (119) is positive, thus (119) is true. Now it is possible to transform inequality (117) to the form

$$2v + 1 > \sqrt{\left(\varepsilon_3^{\mathrm{m}} + 1\right)^2 - \left(1 + c_2^2\right)\left(\varepsilon_3^{\mathrm{m}} - \varepsilon_1^{\mathrm{M}}\right)^2}, \tag{121}$$

which is equivalent to (112).

Let us prove (113). Due to Eq. (112) this inequality is equivalent to

$$\sqrt{\left(\varepsilon_3^{\mathrm{m}} + 1\right)^2 - \left(1 + c_2^2\right)\left(\varepsilon_3^{\mathrm{m}} - \varepsilon_1^{\mathrm{M}}\right)^2} > \varepsilon_1^{\mathrm{M}} + 1 \tag{122}$$

or

$$\left(1 + c_2^2\right)\left(\varepsilon_3^{\mathrm{m}} - \varepsilon_1^{\mathrm{M}}\right)^2 < \left(\varepsilon_3^{\mathrm{m}} + 1\right)^2 - \left(\varepsilon_1^{\mathrm{M}} + 1\right)^2. \tag{123}$$

Taking into account Eq. (120), the left-hand side of the last inequality can be estimated as follows:

$$(1 + c_2^2)\left(\varepsilon_3^{\mathrm{m}} - \varepsilon_1^{\mathrm{M}}\right)^2 < \left(1 + \frac{\varepsilon_1^{\mathrm{M}}}{\varepsilon_3^{\mathrm{m}} - \varepsilon_1^{\mathrm{M}}}\right)\left(\varepsilon_3^{\mathrm{m}} - \varepsilon_1^{\mathrm{M}}\right)^2. \tag{124}$$

Thus, it is enough to prove that

$$\left(1 + \frac{\varepsilon_1^{\mathrm{M}}}{\varepsilon_3^{\mathrm{m}} - \varepsilon_1^{\mathrm{M}}}\right)\left(\varepsilon_3^{\mathrm{m}} - \varepsilon_1^{\mathrm{M}}\right)^2 < \left(\varepsilon_3^{\mathrm{m}} + 1\right)^2 - \left(\varepsilon_1^{\mathrm{M}} + 1\right)^2, \tag{125}$$

or, equivalently:

$$\left(\varepsilon_1^{\mathrm{M}} + 2\right)\left(\varepsilon_3^{\mathrm{m}} - \varepsilon_1^{\mathrm{M}}\right) > 0, \tag{126}$$

which is true.

Let us prove (114). This inequality is equivalent to

$$-\frac{1}{2}\left(1 - \sqrt{\left(\varepsilon_3^{\mathrm{m}} + 1\right)^2 - \left(1 + c_2^2\right)\left(\varepsilon_3^{\mathrm{m}} - \varepsilon_1^{\mathrm{M}}\right)^2}\right) < \frac{\varepsilon_3^{\mathrm{m}}}{2}, \tag{127}$$

or

$$\left(1 + c_2^2\right)\left(\varepsilon_3^{\mathrm{m}} - \varepsilon_1^{\mathrm{M}}\right)^2 > 0 \tag{128}$$

which is true. □

Proposition 7 *Provided that (91) is true, the strain $\varepsilon_{e0}(V_0)$ satisfies inequality (38) if and only if F satisfies the restriction*

$$F > \hat{F}. \tag{129}$$

Proof According to Eq. (64), inequality (38) can be rewritten in the form:

$$\frac{2v + V_0 \varepsilon_3^{\mathrm{m}}}{1 + V_0} > -1. \tag{130}$$

Substituting here Eq. (83) yields

$$(2v + 1) + \frac{X - Y}{X + Y}\left(\varepsilon_3^{\mathrm{m}} + 1\right) > 0 \tag{131}$$

or, equivalently

$$X\left(2v + \varepsilon_3^{\mathrm{m}} + 2\right) - 2Y\left(\frac{\varepsilon_3^{\mathrm{m}}}{2} - v\right) > 0. \tag{132}$$

The last inequality is equivalent to Eq. (129). □

Taking into account inequality (49), we can summarize the statements proved in Propositions 1–7 as the following theorem:

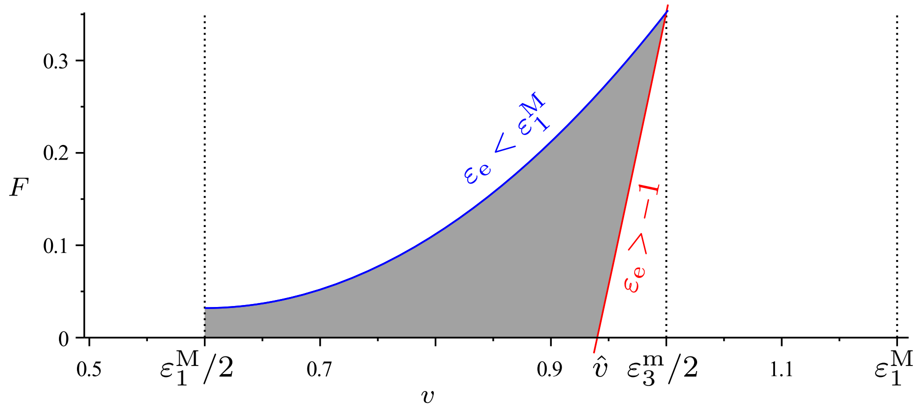

Fig. 2 The domain of existence for the solution in the Case 1. Here $\epsilon = 0.1$, $c_2 = 0.32$, $\varepsilon_1^M = 1.2$, $\varepsilon_3^m = 2$

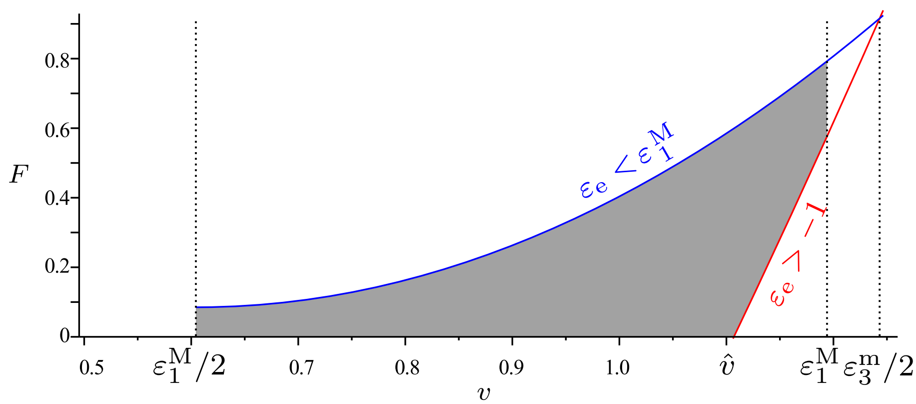

Fig. 3 The domain of existence for the solution in the Case 2. Here $\epsilon = 0.1$, $c_2 = 0.32$, $\varepsilon_1^M = 1.2$, $\varepsilon_3^m = 2.5$

Theorem 1 *Provided that v satisfies restrictions (7), (8), for small enough ϵ the root of Eq. (45) defined by Eqs. (57), (71) satisfies restrictions (36)–(40) and (55) simultaneously if and only if*

$$v < \hat{v} \quad \text{and} \quad 0 \le F < F_*, \tag{133}$$

or

$$\hat{v} < v < v_* \quad \text{and} \quad \hat{F} < F < F_*. \tag{134}$$

Here $\hat{v}$ is given by (112) and satisfies inequalities (113) and (114), $\hat{F} > 0$ is given by Eq. (109), F_* is defined by Eq. (88).

Let us plot the domain of existence for the solution in the plane of parameters $(v; F)$. Since restriction (8) must be taken into account together with (133) or (134), one can observe three possible situations:

(1) $\frac{\varepsilon_1^M}{2} < \hat{v} < v_* < \varepsilon_1^M$. In this case, the domain of existence is shown in Fig. 2.

(2) For the case $\frac{\varepsilon_1^M}{2} < \hat{v} < \varepsilon_1^M < v_*$ the domain of existence is presented in Fig. 3.

(3) For the case $\frac{\varepsilon_1^M}{2} < \varepsilon_1^M < \hat{v} < v_*$, one can see the domain of existence in Fig. 4.

5 Conclusion

A new phase nucleation in a phase-transforming bar caused by a collision of two non-stationary waves is under consideration. To find the unknown value of the speed of the new phase inclusion boundary, we formulate an

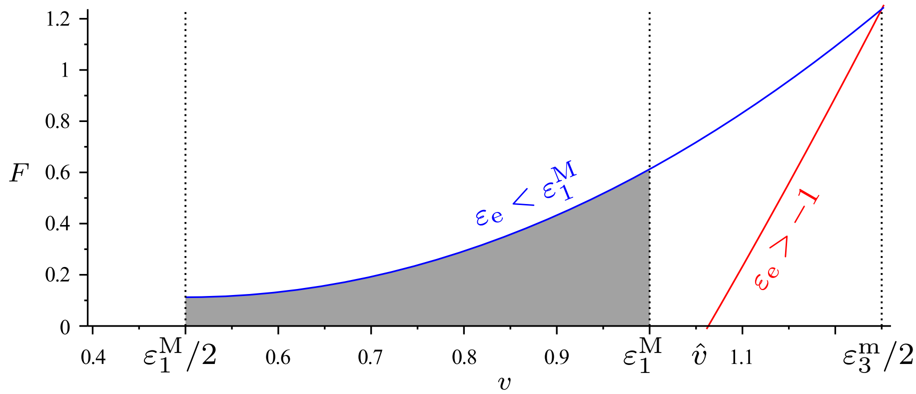

Fig. 4 The domain of existence for the solution in the Case 3. Here $\epsilon = 0.1$, $c_2 = 0.32$, $\varepsilon_1^M = 1$, $\varepsilon_3^m = 2.5$

additional constitutive equation (the so-called thermodynamic boundary condition) in the most general form (18) which respects the second law of thermodynamics. Since the non-stationary problem under investigation is scale-invariant for all $\mathcal{F}_0(\gamma, \dot{\ell})$ in Eq. (18), the value F of the configurational force at the phase boundary, as well as the phase boundary speed $V \equiv \dot{\ell}$, do not depend on time (but indeed do depend on the problem parameters). In particular, F and V do depend on the choice of function $\mathcal{F}_0(\gamma, \dot{\ell})$ in (18). Thus, instead of solving the problem with given $\mathcal{F}_0(\gamma, V)$, we solve the one-parameter family of problems with various $F \geq 0$. The expression for the phase boundary speed is obtained as function (50) such that it satisfies the set of restrictions (36)–(40), (55), provided that the rate of the external loading v satisfies restrictions (7), (8). If such a value of the phase boundary speed is found, then the exact expression for the displacement field for the one-parameter family of problems under consideration is given by Eq. (51). The obtained solution remains valid before the moment when the waves diverging from the central point of the bar reach its ends. If we need to select from the family of solutions (51) the solution that corresponds to both the prescribed function $\mathcal{F}_0(\gamma, V)$ and the prescribed value of γ, we should resolve Eq. (54) with respect to γ (see our previous paper [40] for an example how to do this).

In this paper, we analytically consider the particular limiting case where the stiffness of a new phase inclusion in the phase-transforming bar is much greater than the stiffness of the initial phase. Two other important particular cases, namely the case of a soft phase inclusion and the case of a phase inclusion of the stiffness equal to the initial one, we considered in our previous studies [25,40], respectively. It may be noted that all these solutions are developed only for different particular cases and do not allow one to understand the properties of the solution in general case for arbitrary parameters.

As well as in the case of equal stiffnesses considered in [40], in the case of a stiff new phase one can observe values of problem parameters for which the solution does not exist (see Figs. 2, 3, 4). Note that in the case of a soft new phase inclusion considered in [25] the solution always exists. Unlike the case considered in [40], in the case of a stiff new phase inclusion one can observe the nonexistence of the solution under some values of the problem parameters even for zero dissipation at the phase boundary. The nonexistence of the solution can be explained either by the fact that the trilinear model of the material is poor or, possibly, by a different scenario of a phase transformation. The latter one is the subject for our future work.

Appendix: The motion of the phase boundary at a near-critical speed

Assume that the phase boundary speed V is subcritical and close to its critical value:

$$V = -1 + \delta^2, \tag{135}$$

where $\delta > 0$ is a small parameter. Let us try to find the possible value for the rate of loading

$$v = v_0 + O(\delta) \tag{136}$$

such that the problem solution can possess this property. According to (42) and (43) one has

$$\varepsilon_{\mathrm{e}} = \varepsilon_{\mathrm{e}(-2)}\delta^{-2} + O(\delta^{-1}) \equiv \frac{2v_0\left(c_3^2-1\right)+z}{c_3^2+1}\delta^{-2} + O(\delta^{-1}), \tag{137}$$

$$\varepsilon_{\mathrm{i}} = \varepsilon_{\mathrm{i}(0)} + O(\delta) \equiv \frac{4v_0 - z}{c_3^2+1} + O(\delta). \tag{138}$$

Substituting expansions (137) and (138) into Eq. (45) and taking coefficients at δ^{-2}, one gets:

$$\left(c_3^2-1\right)\varepsilon_{\mathrm{i}(0)}\varepsilon_{\mathrm{e}(-2)} + z\varepsilon_{\mathrm{e}(-2)} = 0. \tag{139}$$

From the above equation it follows that

$$\varepsilon_{\mathrm{i}(0)} = -\frac{z}{c_3^2-1}. \tag{140}$$

Taking into account Eq. (138), one can see that

$$v_0 = -\frac{z}{2\left(c_3^2-1\right)}. \tag{141}$$

Consider the case $c_3 \to \infty$. Substituting Eq. (44) into (141) now yields

$$v_0 \to \frac{\varepsilon_3^{\mathrm{m}}}{2}. \tag{142}$$

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