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Electrically and magnetically induced Maxwell stresses in a magneto-electro-elastic medium with periodic limited permeable cracks

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Abstract In this work, the Maxwell stresses arising due to an electric and a magnetic field both at the crack faces and infinity are taken into account to determine and analyse some main fracture parameters for a periodic system of cracks in a magneto-electro-elastic material. A plane strain problem is formulated and analysed. At the crack faces, the limited permeable electromagnetic boundary conditions are assumed. The material is subjected to a relatively weak mechanical and a strong electric and magnetic loading applied at infinity. The solution of the problem is obtained in a closed form using a complex function theory. Formulas for stresses, magnetic induction and electric displacement vector, elastic displacements, magnetic and electric potential jumps at the interface as well as the intensity factors at the crack tips are presented as relatively simple analytical expressions. The system of two cubic equations is obtained for the electric displacement and magnetic induction in the crack regions. A case of a single limited permeable crack in a magneto-electro-elastic medium is studied as well, and the results related to this case and to the periodic crack set are compared.

Keywords Magneto-electro-elastic material · Periodic limited permeable crack · Analytical solutions · Maxwell stresses

1 Introduction

Multi-physics materials are often used in modern technologies such as smart structures, active control and others. They include piezoelectric materials, magneto-electro-elastic materials, shape memory alloys, dielectric elastomers and hydrogels (e.g. [1,21]). These materials have the capability to serve as sensors, actuators, transducers, vibration absorber and so on. Some of these materials are very brittle and can have single or multiple cracks sensitive to different physical fields. For example, in magneto-electro-elastic materials, a premature failure can be caused by mechanical stresses coupled with an electrical or/and magnetic fields. Usually, appearing and developing of cracks is initiated by the initial imperfect bonding or during the service life. Therefore, it is very important to study cracks in such active materials under coupled action of mechanical

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stresses, electric and magnetic fields. Among the important aspects of studying cracks in such multi-physics materials are the electric and magnetic boundary conditions on the crack faces and infinity. These factors have already attracted attention in the scientific literature. The crack face conditions taking into account the properties of the crack medium (air, water and so on) were suggested by Parton and Kudryavtsev [18] and Hao and Shen [7]. They were analysed and further developed in papers by McMeeking [17], Gruebner et al. [6], Wang and May [26] and others.

Studies of systems of cracks and periodic sets of cracks in piezoelectric and magneto-electro-elastic solids with different boundary condition along the crack faces were performed by, for example, Gao et al. [5], Zhou et al [31], Wang and Mai [27], Zhong [32], Kozinov et al. [11], Bhargava and Jangaid [2], Viun et al. [25] and others.

The overwhelming majority of previous investigations were based on the traditional assumption that the crack surfaces are traction-free. However, if the adjacent bodies (e.g. a crack face and a crack filler) exhibit different physical properties, then the electric fields induce forces at the interfaces. In electrostatics, these forces are called as Coulomb forces [20] while within the framework of a continuum field theory the term "Maxwell stresses" is frequently used [29]. Since the associated forces are called "Maxwell stresses" in magneto statics as well, this term will be used in the present paper.

In Landis [12], the magnitude of the Coulomb traction in piezoelectric materials was evaluated. In this work, the energetically consistent boundary conditions are applied to the Griffith crack problem in a piezoelectric solid. After that, a crack in piezoelectric ceramic under the nontraction-free condition was studied in Li and Chen [14], Ricoeur and Kuna [20] and axisymmetric problem of a penny-shaped crack was studied in Li et al. [16]. The limited permeable electric condition with an account of electric traction for the crack moving along the interface of a piezoelectric bi-material is studied by Lapusta et al. [13]. Coulomb traction for a semi-permeable interface crack between dielectric and piezoelectric material is considered in Li and Chen [15]. The effects of Maxwell stresses on the stress and electric displacement intensity factors are studied in Zhang and Wang [29] for the crack in piezoelectric material. In this work, the Maxwell stress on the crack faces and on the infinity was used. The results are presented for the case, when the crack and the surrounding space at infinity are filled with different dielectric medium. In Hasebe [9], an infinite thin plate with an elliptical hole in magneto-elastic material is considered and Maxwell stress induced by the magnetic field is introduced.

In an "infinite" magneto-electro-elastic material subjected to a simultaneous action of mechanical, magnetic and electric loads, the Maxwell stresses due to both electric and magnetic fields will appear at the crack surfaces and "infinity". These stresses can have an important effect on the fracture behaviour of cracked magneto-electro-elastic bodies, as mentioned in some of the above citations. However, such an analysis of a periodic system of interacting cracks has not been considered neither in the recent authors' paper [25] related to this subject nor in any other paper. Therefore, in the present paper, we account for these Maxwell stresses induced by both electric and magnetic fields in a problem for a periodic set of limited electrically and magnetically permeable cracks in a magneto-electro-elastic material. Also, the Maxwell stresses at infinity are accounted for and a particular case of single crack is analysed as well.

2 Formulation of the problem

Consider an infinite magneto-electro-elastic medium with a periodic set of cracks. The cracks have a length $2a$ and a period h, as depicted in Fig. 1. At the crack faces, limited permeable electric and magnetic boundary conditions are assumed. Also, a mechanical stress $\sigma_{33} = \sigma^{\infty}$, a magnetic induction $B_3 = B^{\infty}$ and an electric displacement $D_3 = D^{\infty}$ are applied at infinity.

Similarly to the works of Hao and Shen [7] and Wang and Mai [27] for a single crack in a homogeneous piezoelectric material and in magneto-electro-elastic material, respectively, we assume that the electric flux and magnetic induction at the crack regions $(-a, a)$ have a constant but unknown values D_0 and B_0.

The Maxwell stress σ_0^M caused by an electric displacement and a magnetic induction on the crack has the following form ([9,20,23]):

$$\sigma_0^M = \frac{1}{2}\frac{D_0^2}{\varepsilon_a}\left(1 - \frac{\varepsilon_a}{\varepsilon_{33}}\right) + \frac{1}{2}\frac{B_0^2}{\gamma_a}\left(1 - \frac{\gamma_a}{\gamma_{33}}\right) \tag{1}$$

where ε_a and γ_a are the electric and magnetic permeability of the crack medium, respectively; ε_{33} and γ_{33} are the electric and magnetic permeability of the material in the direction x_3, respectively. Note that this equation for Maxwell stress includes the unknown values of electric displacement D_0 and magnetic induction B_0 in the crack regions.

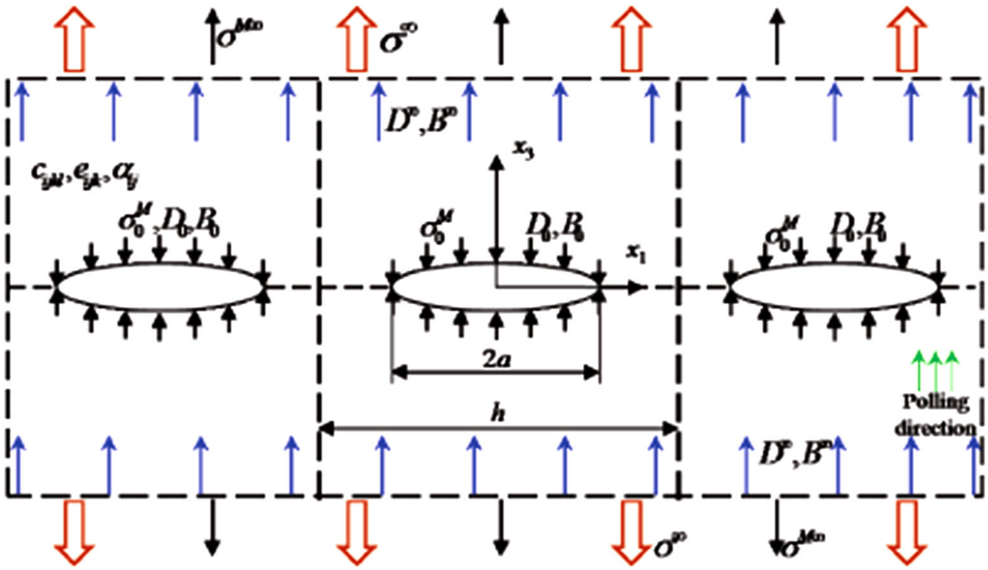

Fig. 1 Periodic set of cracks in magneto-electro-elastic space

For a strip $|x_1| \in (-h, h)$ in case of limited permeable cracks $|x_1| \in (-a, a)$ in a homogeneous magneto-electro-elastic medium, we get the following formulation:

$$\sigma_{13}(x_1, 0) = 0, |x_1| < a; \tag{2}$$

$$\sigma_{33}(x_1, 0) = \sigma_0^M, D_3(x_1, 0) = D_0, B_3(x_1, 0) = B_0, |x_1| < a, \tag{3}$$

Here, as mentioned before, D_0 and B_0 are unknown electric and magnetic fluxes through the crack regions.

The Maxwell stress $\sigma^{M\infty}$ at infinity (for $x_3 \to \infty$) caused by external electric and magnetic fields can be presented in the following form:

$$\sigma^{M\infty} = \frac{1}{2}\frac{(D^\infty)^2}{\varepsilon_{\mathrm{sp}}}\left(1 - \frac{\varepsilon_{\mathrm{sp}}}{\varepsilon_{33}}\right) + \frac{1}{2}\frac{(B^\infty)^2}{\gamma_{\mathrm{sp}}}\left(1 - \frac{\gamma_{\mathrm{sp}}}{\gamma_{33}}\right) \tag{4}$$

where $\varepsilon_{\mathrm{sp}}$ and μ_{sp} are electric and magnetic permeability of the surrounding space. It should be noted that the load $\sigma^{M\infty}$ can be found directly from Eq. (4). At the same time, σ_0^M is coupled with known external loads (σ^∞, D^∞, B^∞), known external Maxwell stress $\sigma^{M\infty}$ and unknown constants D_0 and B_0 (electric and magnetic fluxes through the crack regions).

The conditions of mechanical, magnetic and electric continuity have the following form:

$$\sigma_{i3}^+(x_1, 0) = \sigma_{i3}^-(x_1, 0), u_i^+(x_1, 0) = u_i^-(x_1, 0), \tag{5a}$$

$$D_3^+(x_1, 0) = D_3^-(x_1, 0), \phi^+(x_1, 0) = \phi^-(x_1, 0), \tag{5b}$$

$$B_3^+(x_1, 0) = B_3^-(x_1, 0), \psi^+(x_1, 0) = \psi^-(x_1, 0) \tag{5c}$$

for $x_1 \notin (-a,\ a)$, $i = 1, 3$ where $u_k, \psi, \phi, \sigma_{ij}, D_i$ and B_i are elastic displacements, magnetic potential, electric potential, stresses, electric displacements and magnetic induction components, respectively.

3 Problem solution

Similarly to Viun et al. [25], the electro-magneto-mechanical quantities are introduced in the form:

$$\begin{cases} \sigma_{33}(x_1,0) = G_{33}\left[W_3^+(x_1)+W_3^-(x_1)\right] + G_{34}\left[W_4^+(x_1)+W_4^-(x_1)\right] + G_{35}\left[W_5^+(x_1)+W_5^-(x_1)\right] \\ D_3(x_1,0) = G_{43}\left[W_3^+(x_1)+W_3^-(x_1)\right] + G_{44}\left[W_4^+(x_1)+W_4^-(x_1)\right] + G_{45}\left[W_5^+(x_1)+W_5^-(x_1)\right] \\ B_3(x_1,0) = G_{53}\left[W_3^+(x_1)+W_3^-(x_1)\right] + G_{54}\left[W_4^+(x_1)+W_4^-(x_1)\right] + G_{55}\left[W_5^+(x_1)+W_5^-(x_1)\right] \end{cases}, \tag{6}$$

$$\begin{cases} \langle u_3'(x_1,0)\rangle = W_3^+(x_1) - W_3^-(x_1) \\ \langle \phi'(x_1,0)\rangle = W_4^+(x_1) - W_4^-(x_1) \\ \langle \psi'(x_1,0)\rangle = W_5^+(x_1) - W_5^-(x_1) \end{cases}, \tag{7}$$

where $\mathbf{G} = \left[G_{ij}\right]\big|_{i,j=3,4,5}$ is a material characteristics matrix $\langle f(x_1)\rangle = f^+(x_1) - f^-(x_1)$ is the jump of the function $f(x_1)$ across x_1-axis. $W_i(z)\ i = 3,4,5$ are functions, which under the conditions (5) are analytic in the whole plane with a cut along the crack segment.

Satisfying the boundary conditions (2), (3) by means of (6), (7), one gets the problem of linear relationship for electro-magneto-mechanical fields:

$$\begin{cases} G_{33}\left[W_3^+(x_1)+W_3^-(x_1)\right] + G_{34}\left[W_4^+(x_1)+W_4^-(x_1)\right] + G_{35}\left[W_5^+(x_1)+W_5^-(x_1)\right] = \sigma_0^M, \\ G_{43}\left[W_3^+(x_1)+W_3^-(x_1)\right] + G_{44}\left[W_4^+(x_1)+W_4^-(x_1)\right] + G_{45}\left[W_5^+(x_1)+W_5^-(x_1)\right] = D_0, \quad |x_1| \le a, \\ G_{53}\left[W_3^+(x_1)+W_3^-(x_1)\right] + G_{54}\left[W_4^+(x_1)+W_4^-(x_1)\right] + G_{55}\left[W_5^+(x_1)+W_5^-(x_1)\right] = B_0. \end{cases} \tag{8}$$

Taking into consideration the Maxwell stress at the interface between the magneto-electro-elastic media and another dielectric medium at infinity, defined by means of Eqs. (4) and using (6) for $x_1 \to \infty$, the conditions at infinity attain the following form:

$$\left[W_{i+2}^+(x_1) + W_{i+2}^-(x_1)\right]\big|_{x_1\to\infty} = 0.5\sum_{n=1}^{3}\Omega_{in}F_n^\infty,\ i = 1,2,3, \tag{9}$$

where $\mathbf{F}^\infty = \left[\sigma^\infty + \sigma^{M\infty}, D^\infty, B^\infty\right]^T, \mathbf{\Omega} = \mathbf{G}^{-1}$.

The conditions of the single valuedness of the mechanical displacement and the absence of the total electric and magnetic charges in the crack regions are:

$$\int_{-a}^{a}\left[W_i^+(x_1) - W_i^-(x_1)\right]\mathrm{d}x_1 = 0,\ i = 3,4,5. \tag{10}$$

Solving the problem of linear relationship (8) (Gakhov [4]) with the additional conditions (9) and (10), we obtain:

$$W_{i+2}(z) = \frac{0.5\left[\sum_{n=1}^{3}\Omega_{in}\left(F_n^\infty - F_n\right)\right]\sin\left(\pi\frac{z}{h}\right)}{\sqrt{\sin\left(\pi\frac{z-a}{h}\right)\sin\left(\pi\frac{z+a}{h}\right)}} + 0.5\sum_{n=1}^{3}\Omega_{in}F_n, \quad i = 1,2,3 \tag{11}$$

where $\mathbf{F} = \left[\sigma_0^M, D_0, B_0\right]^T$.

Using Eqs. (6) and (11), we get the following expressions for electro-magneto-mechanical fields on $x_1 \notin (-a,\ a)$:

$$\begin{cases} \sigma_{33}(x_1,0) = \left(\sigma^\infty + \sigma^{M\infty} - \sigma_0^M\right)H(x_1) + \sigma_0^M, \\ D_3(x_1,0) = \left(D^\infty - D_0\right)H(x_1) + D_0, \\ B_3(x_1,0) = \left(B^\infty - B_0\right)H(x_1) + B_0, \end{cases} \tag{12}$$

where $H(x_1) = \sin\left(\pi\frac{x_1}{h}\right)\left[\sin\left(\pi\frac{x_1-a}{h}\right)\sin\left(\pi\frac{x_1+a}{h}\right)\right]^{-0.5}$.

Introducing the intensity factors (IFs) vector of electro-magneto-mechanical quantities:

$$K_1 = \lim_{x_1 \to a+0} \sqrt{2\pi\,(x_1 - a)}\sigma_{33}(x_1, 0)$$
$$K_4 = \lim_{x_1 \to a+0} \sqrt{2\pi\,(x_1 - a)}D_3(x_1, 0)\,,$$
$$K_5 = \lim_{x_1 \to a+0} \sqrt{2\pi\,(x_1 - a)}B_3(x_1, 0)$$

one gets the expression:

$$\mathbf{K} = \left(\mathbf{F}^{\infty} - \mathbf{F}\right)\sqrt{h\tan\left(\pi\frac{a}{h}\right)}, \tag{13}$$

where $\mathbf{K} = [K_1, K_4, K_5]^T$.

Using the electro-magneto-mechanical quantities in the form (6) and the equations of linear relationship in the form (8), the jumps of the normal mechanical displacement, electrical and magnetic potentials on $|x_1| \le a$ can be presented in the form:

$$\begin{cases} \left\langle u_3'(x_1, 0)\right\rangle = 4\sum_{n=1}^{3}\Lambda_{1n}\left(F_n^{\infty} - F_n\right)H(x_1) \\ \left\langle \phi'(x_1, 0)\right\rangle = 4\sum_{n=1}^{3}\Lambda_{2n}\left(F_n^{\infty} - F_n\right)H(x_1) \quad , \quad |x_1| \le a, \\ \left\langle \psi'(x_1, 0)\right\rangle = 4\sum_{n=1}^{3}\Lambda_{3n}\left(F_n^{\infty} - F_n\right)H(x_1) \end{cases}$$

where $\boldsymbol{\Lambda} = i\mathbf{G}^{-1}/4$. Performing the integration of the last equation, we arrive at the formula

$$\begin{cases} \langle u_3(x_1, 0)\rangle = 4\sum_{n=1}^{3}\Lambda_{1n}\left(F_n^{\infty} - F_n\right)H_1(x_1) \\ \langle \phi(x_1, 0)\rangle = 4\sum_{n=1}^{3}\Lambda_{2n}\left(F_n^{\infty} - F_n\right)H_1(x_1) \quad , \quad |x_1| \le a, \\ \langle \psi(x_1, 0)\rangle = 4\sum_{n=1}^{3}\Lambda_{3n}\left(F_n^{\infty} - F_n\right)H_1(x_1) \end{cases} \tag{14}$$

where $H_1(x_1) = \frac{h}{\pi}\ln\left[\cos\left(\pi\frac{x_1}{h}\right) + \sqrt{\sin^2\left(\pi\frac{a}{h}\right) - \sin^2\left(\pi\frac{x_1}{h}\right)}\right] - \frac{h}{\pi}\ln\left[\cos\left(\pi\frac{a}{h}\right)\right]$.

The electric and magnetic conditions in the crack region $|x_1| \le a$ are defined in the form of Hao and Shen [7], Wang and Mai [27]

$$D_0 = -\varepsilon_a\frac{\langle \phi(x_1, 0)\rangle}{\langle u_3(x_1, 0)\rangle}, \quad B_0 = -\gamma_a\frac{\langle \psi(x_1, 0)\rangle}{\langle u_3(x_1, 0)\rangle} \tag{15}$$

By using the crack opening $\langle u_3(x_1, 0)\rangle$, electric $\langle \phi(x_1, 0)\rangle$ and magnetic $\langle \psi(x_1, 0)\rangle$ potential jumps in the form (14), Eqs. (15) can be presented in the form:

$$D_0 = -\varepsilon_a\frac{\sum_{n=1}^{3}\Lambda_{2n}\left(F_n^{\infty} - F_n\right)}{\sum_{n=1}^{3}\Lambda_{1n}\left(F_n^{\infty} - F_n\right)}, \quad B_0 = -\gamma_a\frac{\sum_{n=1}^{3}\Lambda_{3n}\left(F_n^{\infty} - F_n\right)}{\sum_{n=1}^{3}\Lambda_{1n}\left(F_n^{\infty} - F_n\right)}. \tag{16}$$

Equations (16) provide a system of two transcendental equations with respect to the electric displacement D_0 and magnetic induction B_0 in the crack region which can be solved numerically. The electro-magneto-mechanical parameters (12)–(14) can be defined only after solution of this system. After determining of D_0 and B_0, the Maxwell stress σ_0^M can be found by using of Eqs. (1). Note that, in the model neglecting Maxwell stresses (see, e.g. [25]), Eqs. (16) are second-degree equations since the function $\Theta(D_0, B_0)$ is equal to zero. In the present work, in the contrast to [25], the electric displacement D_0 and magnetic induction B_0 in the crack region are presented by cubic Eqs. (16):

$$D_0 = -\varepsilon_a\frac{\Lambda_{21}\left[\sigma^{\infty} + 0.5\Theta(D_0, B_0)\right] + \Lambda_{22}(D^{\infty} - D_0) + \Lambda_{23}(B^{\infty} - B_0)}{\Lambda_{11}[\sigma^{\infty}0.5\Theta(D_0, B_0)] + \Lambda_{12}(D^{\infty} - D_0) + \Lambda_{13}(B^{\infty} - B_0)},$$
$$B_0 = -\gamma_a\frac{\Lambda_{31}\left[\sigma^{\infty} + 0.5\Theta(D_0, B_0)\right] + \Lambda_{32}(D^{\infty} - D_0) + \Lambda_{33}(B^{\infty} - B_0)}{\Lambda_{11}[\sigma^{\infty}0.5\Theta(D_0, B_0)] + \Lambda_{12}(D^{\infty} - D_0) + \Lambda_{13}(B^{\infty} - B_0)};$$

where

$$\Theta(D_0, B_0) = \left(D^{\infty}\right)^2\left(1 - \varepsilon_{sp}/\varepsilon_{33}\right)\varepsilon_{sp}^{-1} + \left(B^{\infty}\right)^2\left(1 - \gamma_{sp}/\gamma_{33}\right)\gamma_{sp}^{-1}$$
$$-D_0^2(1 - \varepsilon_a/\varepsilon_{33})\varepsilon_a^{-1} - B_0^2(1 - \gamma_a/\gamma_{33})\gamma_a^{-1}.$$

The energy release rate (ERR) G at the crack tips can be obtained by using the crack closure integral. According to Wang and Mai [27], the total ERR is the sum of mechanical, electric and magnetic components and can be expressed as:

$$\begin{aligned} G = & \lim_{\Delta l \to 0} \frac{1}{2\Delta l} \int_a^{a+\Delta l} \sigma_{33}(x_1, 0) \langle u_3 (x_1 - \Delta l, 0)\rangle \mathrm{d}x_1 \\ & + \lim_{\Delta l \to 0} \frac{1}{2\Delta l} \int_a^{a+\Delta l} D_3(x_1, 0) \langle \phi (x_1 - \Delta l, 0)\rangle \mathrm{d}x_1 \\ & + \lim_{\Delta l \to 0} \frac{1}{2\Delta l} \int_a^{a+\Delta l} B_3(x_1, 0) \langle \psi (x_1 - \Delta l, 0)\rangle \mathrm{d}x_1 . \end{aligned} \tag{17}$$

This expression can be written in the following vector form:

$$G = \lim_{\Delta l \to 0} \frac{1}{2\Delta l} \int_a^{a+\Delta l} \mathbf{F}^T (x_1) \; \mathbf{U} (x_1 - \Delta l) \mathrm{d}x_1 . \tag{18}$$

Performing the integration (18), one gets

$$G = \mathbf{K}^T \mathbf{\Lambda} \boldsymbol{K}, \tag{19}$$

In a similar way, using the presentations of electro-magneto-mechanical quantities (6), (7), the solution for a single electrically and magnetically limited permeable crack in magneto-electro-elastic material can be found. The jumps of mechanical displacement, electric and magnetic potentials as well as the IFs can be written in the form:

$$\begin{cases} \langle u_3 (x_1, 0)\rangle = 4 \sum_{n=1}^{3} \Lambda_{3n} \left(F_n^\infty - F_n\right) \sqrt{a^2 - x_1^2} \\ \langle \phi (x_1, 0)\rangle = 4 \sum_{n=1}^{3} \Lambda_{4n} \left(F_n^\infty - F_n\right) \sqrt{a^2 - x_1^2} \\ \langle \psi (x_1, 0)\rangle = 4 \sum_{n=1}^{3} \Lambda_{5n} \left(F_n^\infty - F_n\right) \sqrt{a^2 - x_1^2} \end{cases}, \tag{20}$$

$$\mathbf{K} = \left(\mathbf{F}^\infty - \mathbf{F}\right) \sqrt{\pi a}. \tag{21}$$

Using Eqs. (15) and (20), it is easy to show that the set of equations for definition of electric displacement D_0 and magnetic induction B_0 on the cracks $|x_1| \leq a$ in this case coincides with (16).

4 Numerical results

Using the obtained analytical solutions, let us analyse the influence of the external mechanical, magnetic and electric fields on the mechanical stress, magneto-electric fluxes and on their intensity factors for different electric and magnetic permeability values. In calculations, we consider $BaTiO_3{-}CoFe_2O_4$ material whose effective properties for different volume fractions V_f of the piezoelectric material $BaTiO_3$ are presented in Table 2 of Appendix.

Table 1 Mechanical stress in the crack regions ($|x_1| < a$), calculated for different external electromagnetic loads

	$10^{-5}(\sigma_0^M)$, Pa					
	Crack filled by water			Crack filled by air		
B^∞ \ D^∞	0.001	0.005	0.01	0.001	0.005	0.01
-0.500	1.064	1.137	1.374	1.412	13.704	55.139
-0.357	0.571	0.644	0.881	0.919	13.212	54.651
-0.167	0.231	0.304	0.541	0.579	12.873	54.318
-0.024	0.043	0.116	0.353	0.391	12.687	54.143
0.024	0.007	0.080	0.317	0.355	12.654	54.124
0.167	0.124	0.197	0.434	0.472	12.774	54.262
0.357	0.393	0.466	0.703	0.741	13.047	54.556
0.500	0.814	0.887	1.124	1.162	13.474	55.006

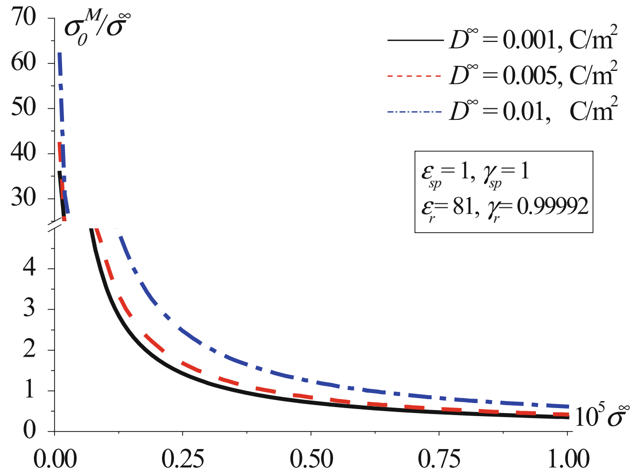

Fig. 2 Maxwell stress, calculated for different electro-mechanical load and different environments of crack and space

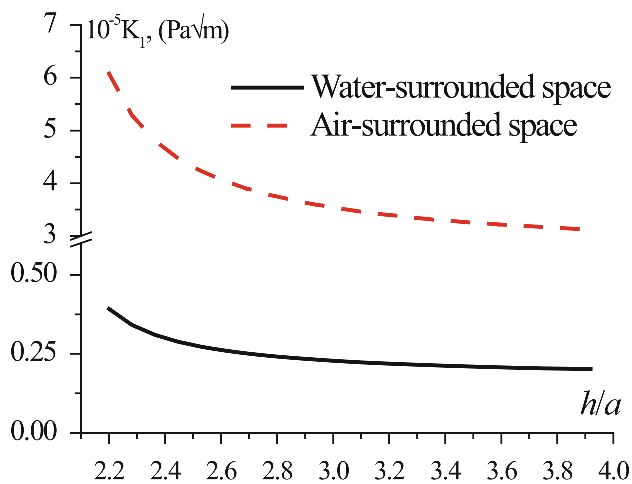

Fig. 3 Stress intensity factor K_1, calculated for different environments of crack and space

First, we consider the dependence of the constant mechanical stress σ_0^M in the crack region on the crack filler parameters. The results presented in Table 1 were obtained for $V_f = 0.1$, $\sigma^\infty = 10$ MPa and Eq. (1) was used. It was assumed that the space was surrounded by water ($\varepsilon_{sp} = 81\varepsilon_0 \quad N/V^2$, $\gamma_{sp} = 0.99992\gamma_0 \quad N/A^2$) and the crack was filled by water or air ($\varepsilon_a = \varepsilon_0 \equiv 8.85 \times 10^{-12} \quad N/V^2$, $\gamma_a = \gamma_0 \equiv 4\pi \times 10^{-7} \quad N/A^2$). The results demonstrate that the Maxwell stress for the case of the crack filled by water is smaller than for the case of the crack filled by air.

It should be noted that σ_0^M is coupled with known external loads (σ^∞, D^∞, B^∞), known external Maxwell stress $\sigma^{M\infty}$ and with unknown electric and magnetic fluxes through the crack regions (D_0, B_0). In Fig. 2, the coefficient σ_0^M/σ^∞ is drawn for a case of the water-filled cracks and the air-surrounded space. The results were obtained for $V_f = 0.1$, $B^\infty = 0.3$T, D^∞ was equal 0.001, 0.005 and 0.01 C/m^2, and the stress σ^∞ varied from 0.001 to 0.1 MPa. We observe that when the applied mechanical stress σ^∞ is small, the normalized Maxwell stress is larger than σ^∞. Also, increasing of D^∞ leads to increasing of σ_0^M and the intersection point of σ_0^M/σ^∞ with line $\sigma_0^M/\sigma^\infty = 1$ moves to the right.

Figure 3 presents the values of the stress intensities factor K_1 at the crack tip. One can see that for the cracks filled by water the SIF for the air-surrounded space is larger than for the water-surrounded space. In calculations, Eq. (13) was used and the results were obtained for $\sigma^\infty = 0.1$ MPa, $D^\infty = 0.0051$ C/m^2, $B^\infty = 0.3$ T. The solid line corresponds to the water-filled cracks and the water-surrounded space, and the dashed line corresponds

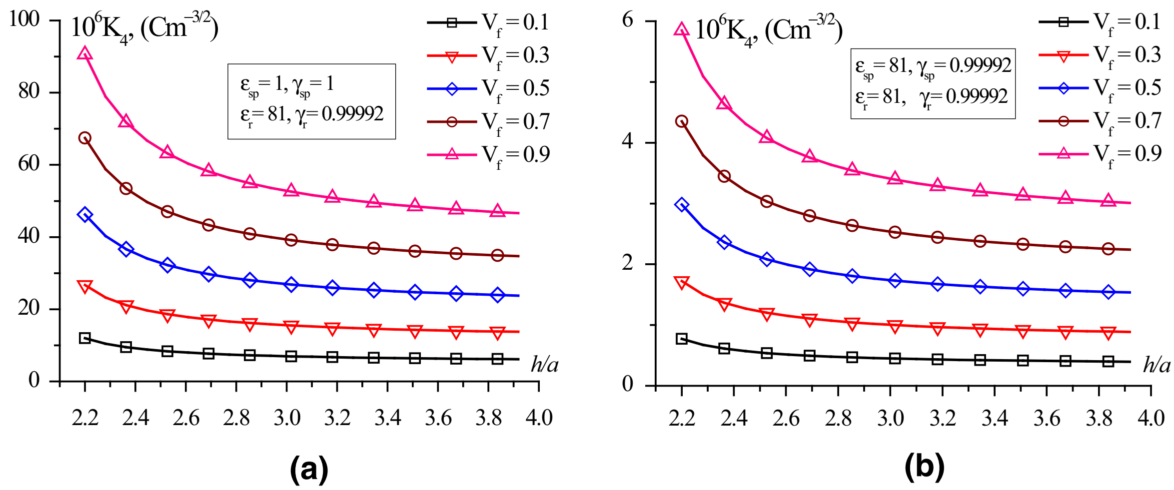

Fig. 4 Electric displacement intensity factor K_4, calculated for different surroundings of the space and the crack filled with water. **a** For water-filled cracks and the air-surrounded space. **b** For water-filled cracks and water-surrounded space

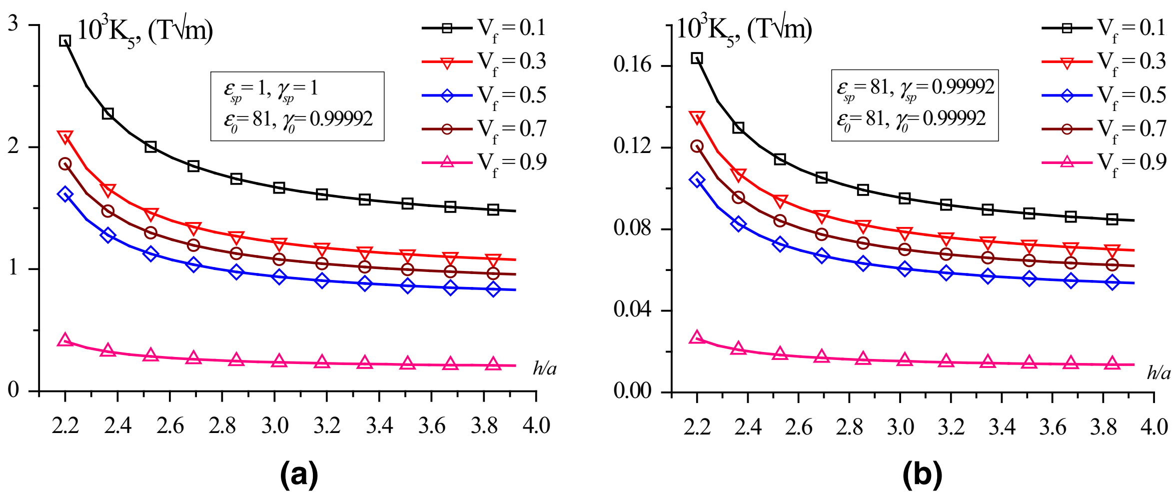

Fig. 5 Magnetic induction intensity factor K_5 calculated for different surrounding of the space and the crack filled with water. **a** For water-filled cracks and the air-surrounded space. **b** For water-filled cracks and water-surrounded space

to the water-filled cracks and the air-surrounded space. Note that according to Eq. (13), the stress intensity factor K_1 depends only on the length of cracks a, the external mechanical load σ^{∞} and the period h.

The intensity factors for different values V_{f} are presented in Figs. 4a and 5a in a case of the water-filled cracks and the air-surrounded space, and in Figs. 4b and 5b in a water-filled cracks and water-surrounded space case. From these figures, one can see that decreasing of the crack length $2a$ with respect to the period h leads to decreasing of analysed intensity factors and tending them to the associated quantities for a single crack. Besides, the increasing of volume fraction V_{f} leads to decreasing of the magnetic intensity factors K_5 and increasing of the electric intensity factors K_4. From Figs. 4 and 5, one can see that results for the water-filled cracks and water-surrounded space case are much smaller than the associated results for the water-filled cracks and air-surrounded space case.

The results in Fig. 6 show the variation of the maximum crack opening in its centre for different values of V_{f} and D^{∞} for $h/a = 2.28$. The cracks are filled with water, and the space is surrounded by air. To underline the effect of Maxwell stresses, let us compare some results obtained in the present work and in [25]. Figure 6a shows some results without the effect of Maxwell stresses obtained according to Viun et al. [25]. Figures 6b, c show different possibilities of accounting for Maxwell stresses. First, these stresses were taken into account only on the crack faces (Fig. 6b) and, then, on the crack faces and at infinity (Fig. 6c). From the presented results, one can see that the influence of Maxwell stresses on the crack opening $\langle u_3(0,0)\rangle$ is rather important

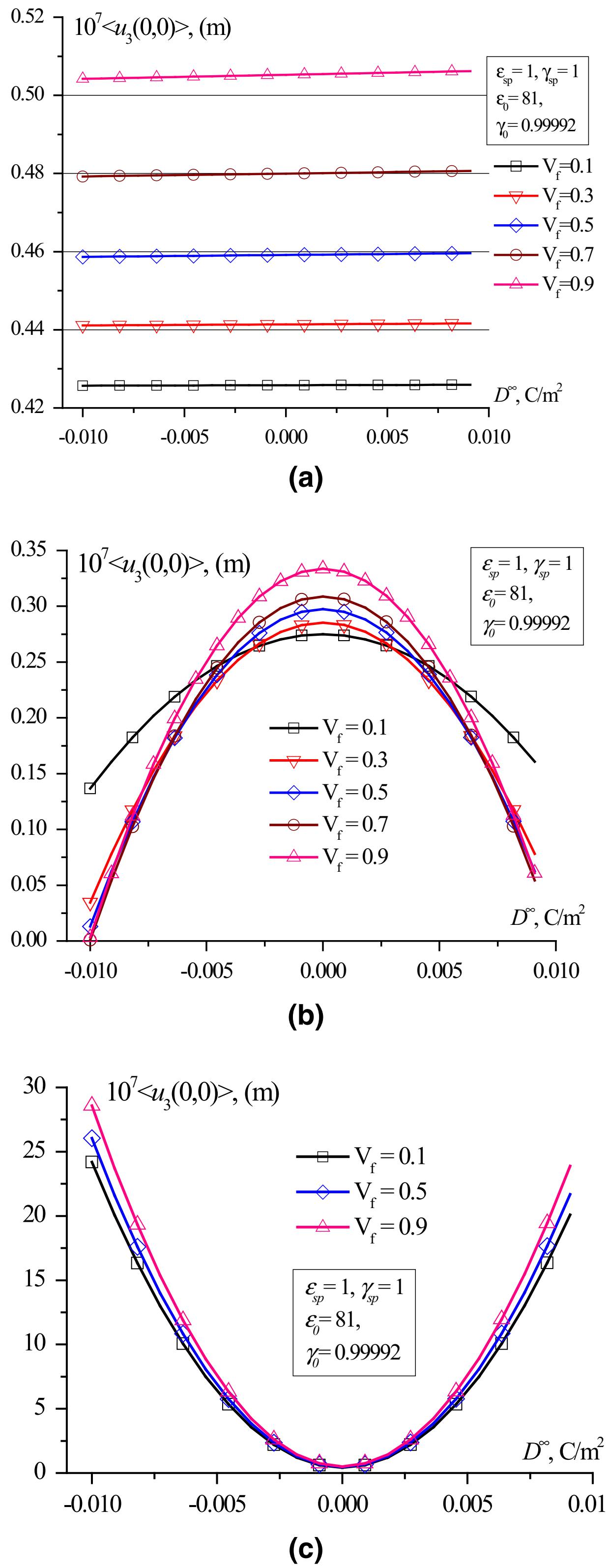

Fig. 6 Crack opening $< u_3(0, 0) >$ obtained for different electric load D^{∞}. **a** Without accounting for Maxwell stresses. **b** Maxwell stresses on the crack faces. **c** Maxwell stresses on the crack faces and infinity

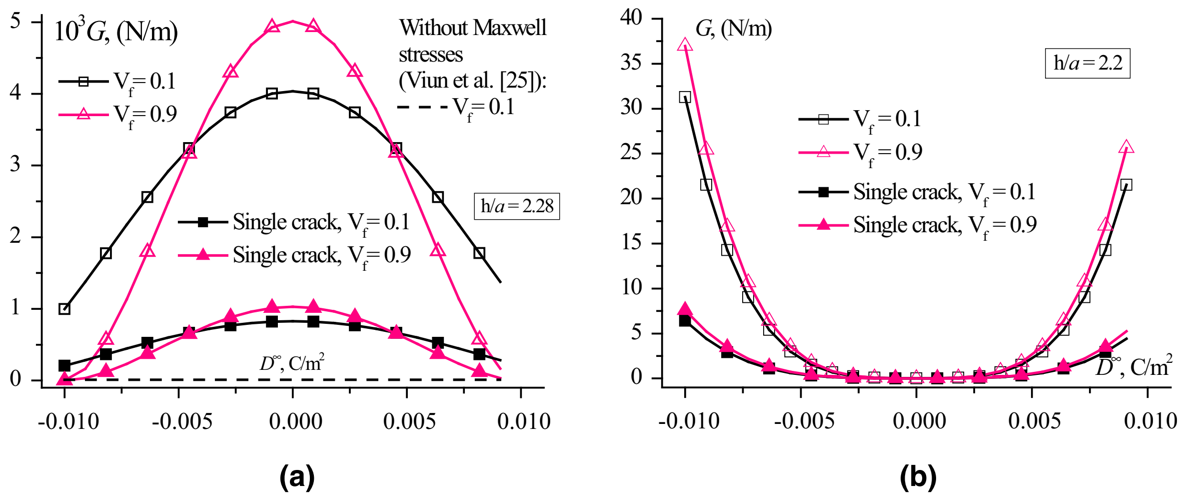

Fig. 7 ERR obtained for the case of Maxwell stresses taking into account on the crack faces (**a**) and Maxwell stresses taking into account on the crack faces and infinity (**b**)

for the considered cases of cracks that are filled with water and the space that is surrounded by air. Also we note that different ways of accounting for the Maxwell stresses lead to different character of dependencies of the crack opening on the electric field D^∞:

1. When Maxwell stresses are taken into account on the crack faces and infinity the crack opening $\langle u_3\,(0,0)\rangle$ increases with increasing of electric field D^∞ (by module);
2. When Maxwell stresses are taken into account only on the crack faces, the crack opening decreases with increasing of electric field D^∞ (by module);
3. Without accounting for Maxwell stresses at all the crack opening increases linearly with increasing of electric field D^∞.

The dependency of the energy release rate on D^∞ is illustrated in Fig. 7 for two cases, when Maxwell stresses are taken into account on the crack faces and infinity (b), and when Maxwell stresses are taken into account on the crack faces only (a); the cracks are filled by water, and the space is surrounded by air. The results are obtained for $V_\mathrm{f} = 0.1, 0.9$ and $\sigma^\infty = 0.1$ MPa, $B^\infty = 0.3$ T. The lines with shaded markers corresponded to the results for a single crack. It can be seen from the results in Fig. 7 that an increase in the external electric load in absolute value leads to a decrease in ERR when the Maxwell stresses are accounted for only on the crack faces (Fig. 7 a) and it leads to an opposite effect, i.e. to an increase in ERR when the Maxwell stresses are accounted for on the crack faces and infinity (see Fig. 7b). These results demonstrate that accounting for the Maxwell stresses is an important issue in predicting fracture behaviour of magneto-electro-elastic materials. Figure 7a also presents, for comparison, a dashed line presenting results obtained without Maxwell stresses (model [25]). It is clearly seen that the Maxwell stresses influence significantly the results, and therefore, it is important to take them into account.

5 Conclusion

In this work, an infinite magneto-electro-elastic space with a periodic set of cracks is considered under a uniformly distributed relatively weak tensile stress as well as under a strong magnetic induction and an electric displacement. At the crack faces, the limited permeable electric and magnetic conditions are adopted. The Maxwell stresses on the crack faces and on the infinity are taken into account. The plane strain conditions in plane perpendicular to the crack fronts are considered. The Maxwell stresses on the crack faces σ_0^M are coupled with prescribed external stresses σ^∞, $\sigma^{M\infty}$, electromagnetic fluxes D^∞, B^∞ at infinity and also with electric and magnetic fluxes D_0, B_0, which are constant in the crack regions. The magnitudes of the normalized Maxwell stress also depend on the permeability of the magneto-electro-elastic material (Eq. 1) or surrounding space (Eq. 4).

Between important outcomes following from the obtained results, we especially emphasize that the magnitude of the Maxwell stresses on the crack surfaces demonstrates a nonlinear dependence on the applied electromagnetic loading. It can be explained by the fact that the traction is determined from the cubic Eq. (16)

and the quadratic Eq. (1). The growth of the applied electric loading leads to increase in the Maxwell stresses (see Table 1) while the increasing of the magnetic load leads to decreasing of the Maxwell stresses. Also, one can see that the mechanical stress in the crack regions for the air-filled crack is larger than for the water-filled crack, and with increasing of the electric load the distinctions of the results obtained for two crack fillers increase.

When the applied mechanical loading is small, the Maxwell stresses are relatively large and should be taken into consideration (see Fig. 2). But, when the applied tensile loading becomes larger, the Maxwell stresses become small with respect to this loading, and it could be neglected in a reasonable way. Also, the increasing of the electric load leads to the increasing of the Maxwell stress in the crack areas. The significance of consideration of the surrounded materials property is presented in Fig. 2, where one can see the difference between the results related to air- and water-surrounded spaces.

The dependence of the stress intensity factor on the electromagnetic properties of a surrounding space is presented in Fig. 3. The intensity factors versus the period and the crack length h/a for different volume fractions of piezoelectric $BaTiO_3$ are presented in Figs. 3, 4 and 5. It was established that the decreasing of the crack length with respect to the crack period leads to decreasing of the analysed intensity factors (K_4, K_5) and tending them to the results for a single crack. In the same time, the increasing of volume fraction V_f leads to decreasing of the magnetic intensity factors K_5 and increasing of the electric intensity factors K_4. In Figs. 4 and 5, one can see that these intensity factors for the water-filled cracks and water-surrounded space case are much smaller than the associated results for the water-filled cracks and air-surrounded space case.

A comparison of results presented in Fig. 6a–c shows that the Maxwell stresses can significantly affect the values of the crack opening for relatively weak mechanical and strong electric and magnetic loadings. Also, the crack opening more substantially depends on the electric load with accounting of Maxwell stresses than for the Maxwell stresses-free case.

The value of the ERR for different external electric loads is drawn in Fig. 7. One can see that increasing of D^{∞} in absolute magnitudes leads to decreasing of ERR for the case when Maxwell stresses are taken into account on the crack faces and to increasing of ERR when Maxwell stresses are taken into account on both the crack faces and infinity. Also, this figure represents the ERR for the Maxwell stresses-free case demonstrating its significant difference from the previous results and emphasizing the importance of Maxwell stresses accounting.

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7 Appendix

Table 2 Effective properties of $BaTiO_3 - CoFe_2O_4$ material for different V_f (Sih and Song [22])

Properties	V_f				
	0.1	0.3	0.5	0.7	0.9
c_{11} (GPa)	274	250.0	226.0	202.0	178.0
c_{13} (GPa)	161	142.7	124	105.7	87.2
c_{33} (GPa)	259	237.3	216	194.2	172.8
c_{44} (GPa)	45	44.6	44	43.7	43.2
e_{31} (C/m^2)	−4.4	−1.32	−2.2	−3.08	−3.96
e_{33} (C/m^2)	1.86	5.58	9.3	13.02	16.74
e_{15} (C/m^2)	1.16	3.48	5.8	8.12	10.44
$\alpha_{11}(\times 10^{-10}$ C^2/Nm^2)	11.9	34.2	56.4	78.6	100.9
$\alpha_{33}(\times 10^{-10}$ C^2/Nm^2)	13.4	38.5	63.5	88.5	113.5
h_{31} (N/Am)	522.3	406.2	290.2	174.1	58.03
h_{33} (N/Am)	629.7	489.8	350.0	209.9	69.97
h_{15} (N/Am)	495.0	385.0	275.0	165.0	55.00
$d_{11}(\times 10^{-6}$ Ns^2/C^2)	531.5	414.5	297.0	180.5	63.5
$d_{33}(\times 10^{-6}$ Ns^2/C^2)	142.3	112.9	83.5	541.0	24.7

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