

K. A. Lazopoulos · A. K. Lazopoulos

# On fractional bending of beams

Received: 28 June 2015 / Accepted: 12 November 2015 / Published online: 27 November 2015  
© Springer-Verlag Berlin Heidelberg 2015

**Abstract** Clarifying the geometry of the fractional tangent space of a curve, fractional differential geometry of curves has already been revisited, Lazopoulos and Lazopoulos (2015), defining also the curvature vector. Fractional bending of a beam is introduced, applying Euler–Bernoulli bending principle. The proposed theory is implemented to the bending deformation of a cantilever beam under continuously distributed loading.

**Keywords** Fractional tangent space · Fractional differential geometry · Curvature vector · Fractional bending · Euler–Bernoulli bending principle · A cantilever beam

## 1 Introduction

Fractional derivative models account for long-range (non-local) dependence of phenomena and result in better description of their behavior. Various material models based upon fractional time derivatives have been presented, for describing their viscoelastic interaction (Refs. [4, 5]). Lazopoulos [6] has proposed a model for lifting Noll's axiom of local action (Truesdell [7]). Fractional calculus, originated by Leibniz [1], Liouville [2], and Riemann [3], has recently applied to modern advances in physics engineering. Carpinteri et al. [8] have also proposed a fractional approach to non-local mechanics. Applications in various physical areas may be also found in various books Refs. [9–12]. Tarasov [13] has also presented a book including fractional mathematics and its applications to various physics areas.

Nevertheless, the clear picture of the fractional tangent space of a manifold is missing. References [14, 15] are concerned with fractional geometry in manifolds, with extensive information concerning fractional differential geometry topics. However, a clear picture of the fractional differential geometry is missing, since the picture of the fractional tangent space of a manifold has not been clarified.

Lazopoulos and Lazopoulos [16] have presented a work clarifying the picture of the virtual tangent space of a curve and helping in the accurate description of the fractional differential of curves. Likewise, the fractional curvature vectors and the fractional normal plane were defined.

The interesting aspect of fractional calculus is that it expresses the contribution of the non-local strain in the constitutive equations when the microstructure, characterized by microcracks, microdefects, micrononuniformities, etc., influences the deformation geometry of the material. It is obvious that conventional theories can express the local influence at a point, based upon Noll's local axiom for deformations of simple materi-

---

K. A. Lazopoulos  
14 Theatrou Str., 19009 Rafina, Greece

A. K. Lazopoulos (✉)  
Mathematical Sciences Department, Hellenic Army Academy, 16673 Vari, Greece  
E-mail: Orfeakos74@gmail.com; Ofeakos74@gmail.com  
Tel.: 302294075816

als. Sometimes, models recall fractal patterns that do not accept derivatives of integer order. Hence, another mathematical tool for this case should be recalled, and that tool is fractional calculus.

In the present work, outlining the procedure for defining the fractional curvature of a curve, the fractional curvature of plane curves is defined. In addition for fractional beam small deflection, the fractional curvature is approximated. Introducing that curvature approximation to Bernoulli's bending principle, fractional bending theory of beams is established. This is the novelty of this article compared to [16]. The nonlinear fractional curvature formula [16] is approximated for small deformations. Therefore, linearizing the curvature formula, Bernoulli's bending principle is invoked, just to describe bending in linear fractional elasticity. Universal equation of the fractional deflection beam curve is also presented. Implementation of the proposed theory to the fractional cantilever beam under continuously distributed loading is presented. The present fractional bending theory is quite general and may be used for describing vibrations of the beam (may be viscoelastic), buckling, and optimization problems.

## 2 Basic properties of fractional calculus

Fractional calculus has become lately a branch of pure mathematics with many applications in physics and engineering (Tarasov [13,17]). Fractional calculus is originated by Leibniz, looking for the possibility of defining the derivative  $\frac{d^n g}{dx^n}$  when  $n = \frac{1}{2}$ . The detailed properties of fractional derivatives may be found in Kilbas et al. [9], Podlubny [10], and Samko et al. [11]. Many definitions of fractional derivatives exist. They are not local, contrary to the conventional ones. Those derivatives exhibit some advantages over the others. Nevertheless, they all share one common property of non-locality.

Starting from Cauchy formula for the  $n$ -fold integral of a primitive function  $f(x)$

$${}_a I_x^n f(x) = \frac{1}{(n-1)!} \int_a^x (x-s)^{n-1} f(s) ds, \quad x > 0, n \in N \quad (1)$$

and

$${}_x I_b^n f(x) = \frac{1}{(n-1)!} \int_x^b (s-x)^{n-1} f(s) ds, \quad x > 0, n \in N \quad (2)$$

We define the left and right fractional integrals of  $f$  as:

$${}_a I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(s)}{(x-s)^{1-\alpha}} ds \quad (3)$$

$${}_x I_b^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(s)}{(s-x)^{1-\alpha}} ds \quad (4)$$

In Eqs. (3, 4), we assume that  $\alpha$  is the order of fractional integrals with  $0 < \alpha \leq 1$ , considering  $\Gamma(x) = (x-1)!$  with  $\Gamma(\alpha)$  Euler's Gamma function.

Thus, the left and right Riemann–Liouville (R–L) derivatives are defined by:

$${}_a D_x^\alpha f(x) = \frac{d}{dx} ({}_a I_x^{1-\alpha} f(x)) \quad (5)$$

and

$${}_x D_b^\alpha f(x) = -\frac{d}{dx} ({}_x I_b^{1-\alpha} f(x)) \quad (6)$$

Nevertheless, the fact that the R–L derivatives of a constant  $c$  are not zero, imposed the need for Caputo's derivative that is more friendly in the description of physical systems, although it is more restrictive. In fact, Caputo derivatives are defined by:

$${}^C D_x^a f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{f'(s)}{(x-s)^\alpha} ds \tag{7}$$

and

$${}^C D_b^a f(x) = -\frac{1}{\Gamma(1-\alpha)} \int_x^b \frac{f'(s)}{(s-x)^\alpha} ds \tag{8}$$

Evaluating Caputo’s derivatives for functions of the type

$$f(x) = (x - a)^n \text{ or } f(x) = (b - x)^n,$$

we get:

$${}^C D_x^a (x - a)^\nu = \frac{\Gamma(\nu + 1)}{\Gamma(-\alpha + \nu + 1)} (x - a)^{\nu-\alpha} \tag{9}$$

and for the corresponding right Caputo’s derivative:

$${}^C D_b^a (b - x)^\nu = \frac{\Gamma(\nu + 1)}{\Gamma(-\alpha + \nu + 1)} (b - x)^{\nu-\alpha}$$

Likewise, Caputo’s derivatives are zero for constant functions:

$$f(t) = c. \tag{10}$$

### 3 The geometry of fractional differential

It is reminded that the  $a$ -fractional differential of a function  $f(x)$  is defined by [18]:

$$d^a f(x) = {}^C D_x^a f(x)(dx)^\alpha \tag{11}$$

Since  ${}^C D_x^a x \neq 1$ , the  $a$ -fractional differential of the variable  $x$  is:

$$d^a x = {}^C D_x^a x(dx)^\alpha \tag{12}$$

Hence,

$$d^a f(x) = \frac{{}^C D_x^a f(x)}{{}^C D_x^a x} d^a x \tag{13}$$

It is evident that  $d^a f(x)$  is a nonlinear function of  $dx$ , although it is a linear function of  $d^a x$ . That fact makes possible the consideration of the virtual fractional tangent space that we propose.

In fact, in Fig. 1, the function  $y = f(x)$  has been drawn with the corresponding first differential space at a point  $x$ .

It is well known that the differential space is not tangent to the function at  $x_0$ , but intersects the figure  $y = f(x)$  at least at one point  $x_0$ . Further considering a point  $x_0 + dx$  in the neighborhood of  $x_0$ , we have

$$\Delta f = f(x_0 + \Delta x) - f(x_0) \tag{14}$$

Now, magnifying the neighborhood of  $x_0$ , we may define a space  $(d^a x, \Delta f)$  (see Fig. 2).

According to Eq. (3), the first differential  $d^a f(x)$  is a linear function of  $d^a x$  and tangent space of  $\Delta f$  at the point  $x_0$ . This space, we introduce, is called virtual tangent space, that is updated at any point  $x$ . Nevertheless, we may consider the virtual fractional tangent space and the virtual normal at any point. Hence, we are able to work combining some kind of fractional differential geometry with the fractional field theory. It is evident that when  $\alpha = 1$ , the virtual tangent space coincides with the tangent space.

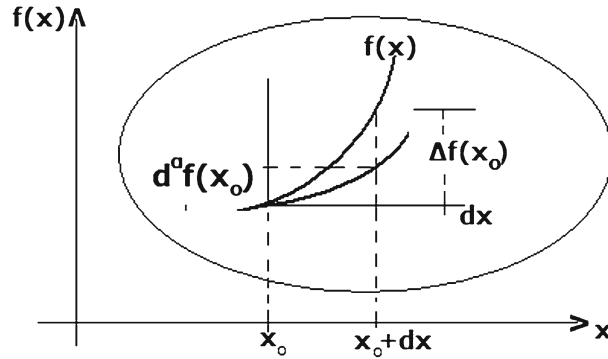


Fig. 1 The nonlinear differential of  $f(x)$

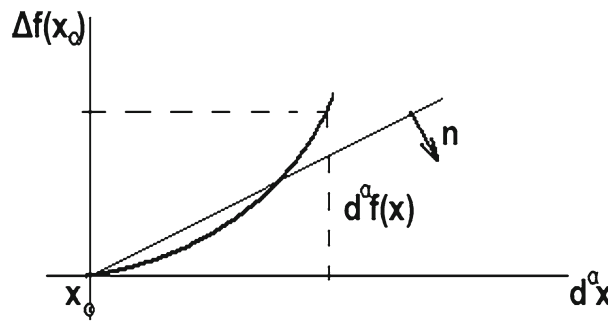


Fig. 2 The virtual tangent space of the  $f(x)$

**4 The fractional arc length**

Let  $y = f(x)$  be a function which may be non-differentiable but has a fractional derivative of order  $\alpha$ ,  $0 < \alpha < 1$ . The fractional differential of  $y = f(x)$  in the virtual differential space is defined by:

$$d^a y = \frac{{}_0^C D_x^a f(x)}{{}_0^C D_x^a x} d^a x \tag{15}$$

Therefore, the arc length is defined by:

$$s_1(x, a) = {}_0 I_x^a [(d^a y)^2 + (d^a x)^2]^{1/2} = {}_0 I_x^a \left[ \left( \frac{{}_0^C D_x^a f(x)}{{}_0^C D_x^a x} \right)^2 + 1 \right]^{1/2} d^a x \tag{16}$$

Furthermore, for parametric curves of the type:

$$y = f(t), \quad x = g(t) \tag{17}$$

The fractional  $\alpha$ -differentials are defined by:

$$\begin{aligned} d^a x &= \frac{{}_0^C D_t^a g(t)}{{}_0 D_t^a t} d^a t \\ d^a y &= \frac{{}_0^C D_t^a f(t)}{{}_0 D_t^a t} d^a t, \end{aligned} \tag{18}$$

and the fractional differential of the arc length is expressed by:

$$d^a s = \sqrt{(d^a y)^2 + (d^a x)^2} = \left[ \left( \frac{{}_0^C D_t^a f(t)}{{}_0^C D_t^a t} \right)^2 + \left( \frac{{}_0^C D_t^a g(t)}{{}_0^C D_t^a t} \right)^2 \right]^{1/2} d^a t \tag{19}$$

and

$$s = {}_0I_x^a d^a s = {}_0I_t^a \left[ \left( \frac{{}_C D_t^a f(t)}{{}_C D_x^a t} \right)^2 + \left( \frac{{}_C D_t^a g(t)}{{}_C D_x^a t} \right)^2 \right]^{\frac{1}{2}} \quad (20)$$

### 5 The fractional tangent space

Let  $\mathbf{r} = \mathbf{r}(s)$  be a natural representation of a curve  $C$  where  $s$  is the  $\alpha$ -fractional length of the curve. Since the velocity of a moving material point on the curve  $r(s)$  defines the tangent space, the fractional tangent space of the curve  $\mathbf{r} = \mathbf{r}(s)$  is defined by the first derivative:

$$\mathbf{r}_1 = \frac{d^a \mathbf{r}}{d^a s} = \frac{{}_C D_s^a \mathbf{r}}{{}_C D_s^a s} \quad (21)$$

Since

$$d^a |\mathbf{r}| = d^a s \quad (22)$$

The length  $|\mathbf{r}_1|$  of the fractional tangent vector is unity.

The tangent space lines of the curve  $\mathbf{r} = \mathbf{r}(s)$  at the point  $\mathbf{r}_0 = \mathbf{r}(s_0)$  are defined by:

$$\mathbf{r} = \mathbf{r}_0 + k \mathbf{t}_0 \quad 0 < k < \infty \quad (23)$$

where  $\mathbf{t}_0 = \mathbf{t}(s_0)$  is the unit tangent vector at  $\mathbf{r}$ .

The plane through  $\mathbf{r}_0$  orthogonal to the tangent line at  $\vec{r}_0$  is called the normal plane to the curve  $C$  at  $s_0$ . The points  $\mathbf{y}$  of that orthogonal plane are defined by:

$$(\mathbf{y} - \mathbf{r}_0) \cdot \mathbf{t}(s_0) = (\mathbf{y} - \mathbf{r}_0) \cdot \mathbf{r}_1(s_0) = 0 \quad (24)$$

### 6 Fractional curvature of curves

Considering the fractional tangent vector:

$$\mathbf{t} = \mathbf{r}_1(s) = \frac{{}_C D_s^a \mathbf{r}}{{}_C D_s^a s}$$

we may consider its fractional derivative:

$$\mathbf{r}_2(s) = \frac{d^a \mathbf{t}}{d^a s} = \frac{{}_C D_s^a \mathbf{t}}{{}_C D_s^a s} = \mathbf{t}_1(s) \quad (25)$$

The vector  $\mathbf{t}_1(s)$  is called the fractional curvature vector on  $C$  at the point  $r(s)$  and is denoted by  $\boldsymbol{\kappa} = \boldsymbol{\kappa}(s) = \mathbf{t}_1(s)$ .

Since  $\mathbf{t}$  is a unit vector

$$\mathbf{t} \cdot \mathbf{t} = 1 \quad (26)$$

Restricted to fractional derivatives that yield zero for a constant function, such as Caputo's derivatives, the curvature vector on  $C$   $\mathbf{t}_1(s)$  is orthogonal to  $\mathbf{t}$  and parallel to the normal plane. The magnitude of the fractional curvature vector:

$$\kappa = |\boldsymbol{\kappa}(s)| \quad (27)$$

is called the fractional curvature of  $C$  at  $\mathbf{r}(s)$ . The reciprocal of the curvature  $\kappa$  is the fractional radius of curvature at  $\mathbf{r}(s)$ :

$$\rho = \frac{1}{\kappa} = \frac{1}{|\boldsymbol{\kappa}(s)|} \quad (28)$$

## 7 The fractional radius of curvature of a curve

Following Porteous [19], for the fractional curvature of a plane curve  $\mathbf{r}$ , we study at each point  $\mathbf{r}(t)$  of the curve how closely the curve approximates there to a parameterized circle. Now in the virtual space at a point  $\mathbf{r}(t_0)$ , the circle with center  $\mathbf{c}$  and radius  $\rho$  consists of all  $\mathbf{r}(t)$  in the virtual differential space such that:

$$(\mathbf{r} - \mathbf{c}) \cdot (\mathbf{r} - \mathbf{c}) = \rho^2 \quad (29)$$

Further, Eq. (29) yields:

$$\mathbf{c} \cdot \mathbf{r} - \frac{1}{2} \mathbf{r} \cdot \mathbf{r} = \frac{1}{2} (\mathbf{c} \cdot \mathbf{c} - \rho^2) \quad (30)$$

with the right-hand side been constant.

Therefore, the derivation of the function

$$V(\mathbf{c}): t \rightarrow \mathbf{c} \cdot \mathbf{r}(t) - \frac{1}{2} \mathbf{r}(t) \cdot \mathbf{r}(t) \quad (31)$$

Hence,

$$V(\mathbf{c})_1 = (\mathbf{c} - \mathbf{r}(t)) \cdot \mathbf{r}_1(t) = 0 \quad (32)$$

$$V(\mathbf{c})_2 = (\mathbf{c} - \mathbf{r}(t)) \cdot \mathbf{r}_2(t) - \mathbf{r}_1(t) \cdot \vec{\mathbf{r}}_1(t) = 0 \quad (33)$$

Suppose that  $r$  is a parametric curve with  $\mathbf{r}(t)$  in the virtual tangent space. Then,  $V(\mathbf{c})_1(t) = 0$  when the vector  $\mathbf{c} - \mathbf{r}(t)$  in the virtual space is orthogonal to the virtual tangent vector  $\mathbf{r}_1(t) = 0$ . Indeed, when the point  $\mathbf{c}$  in the virtual space lies on the normal to  $\mathbf{r}(t)$  at  $t$ , that is the line in the virtual space through  $\mathbf{r}(t)$  orthogonal to the tangent line in the virtual space.

When  $\mathbf{r}_2(t)$  is not linearly dependent upon  $\mathbf{r}_1(t)$ , there will be a unique point  $\mathbf{c} \neq \mathbf{r}(t)$  on the normal line such that also  $V(\mathbf{c})_2(t) = 0$ .

Considering plane curves,

$$\mathbf{r}(x) = x\mathbf{i} + y(x)\mathbf{j} \quad (34)$$

Eqs. (32, 33), defining the fractional centers of curvature  $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j}$ , become

$$\begin{aligned} (c_1 - x) + (c_2 - y(x)) D_1 y(x) &= 0 \\ (c_2 - y(x)) D_2 y(x) - (1 + D_1 y(x)^2) &= 0. \end{aligned} \quad (35)$$

Since the fractional radius of curvature is defined by,

$$\rho^\alpha = \rho_1^\alpha \mathbf{i} + \rho_2^\alpha \mathbf{j} = (c_1 - x)\mathbf{i} + (c_2 - y(x))\mathbf{j} \quad (36)$$

the components of the fractional curvature are given by,

$$\begin{aligned} \rho_1^\alpha &= -\frac{1 + D_1 y(x)^2}{D_2 y(x)} D_1 y(x) \\ \rho_2^\alpha &= \frac{1 + D_1 y(x)^2}{D_2 y(x)} \end{aligned} \quad (37)$$

Further, for the case of  $|y(x)| \ll 1$ , which we consider in linear bending, Eq. (19) yields

$$d^a s = d^a x + o(d^a x)^2 \quad (38)$$

yields

$$D_1(\cdot) = \frac{{}_0^C D_x^\alpha(\cdot)}{{}_0^C D_x^\alpha x}, \quad D_2(\cdot) = \frac{{}_0^C D_x^\alpha \left( \frac{{}_0^C D_x^\alpha(\cdot)}{{}_0^C D_x^\alpha x} \right)}{{}_0^C D_x^\alpha x} \quad (39)$$

with

$$\rho^\alpha = |\mathbf{r}^\alpha| \approx \frac{1}{D_2 y(x)} \quad (40)$$

### 8 Bending of fractional beams

Let us consider a fractional beam with its source point  $(x, y, z) = (0, 0, 0)$ . That means that the  $u$  fractional derivatives of any function concerning the beam are defined by

$${}_0^C D_u^\alpha f(u) = \frac{1}{\Gamma(1-\alpha)} \int_0^u \frac{f'(s)}{(u-s)^\alpha} ds$$

where  $u$  might be one of the variables  $(x, y, z)$ .

Recalling conventional bending, we consider a beam with its main principal axis along the  $x$ -direction and its height along the  $y$ -direction. Then, according to the conventional theory, the strain

$$\varepsilon_{xx} = -\frac{y}{R} \tag{41}$$

where  $R$  is the curvature of the unstressed elastic curve. Further, the longitudinal stress

$$\sigma_{xx} = E\varepsilon_{xx} = -\frac{Ey}{R} \tag{42}$$

with  $E$  Young's elastic modulus. Then, the bending moment is equal to

$$M = -b \int_{-h/2}^{h/2} y\sigma_{xx} dy \tag{43}$$

Hence,

$$M = \frac{EI}{R} \tag{44}$$

with  $I$  the moment of inertia of the cross section and

$$\sigma_{xx} = -\frac{M}{I}y \tag{45}$$

for fractional bending, the fractional strain,

$$\varepsilon_{xx}^\alpha = -\frac{y}{\rho^\alpha} \tag{46}$$

where the fractional curvature is defined by

$$\frac{1}{\rho^\alpha} = D_2 w(x) = \frac{1}{{}_0^C D_x^\alpha} {}_0^C D_x^\alpha \left( \frac{{}_0^C D_x^\alpha w(x)}{{}_0^C D_x^\alpha x} \right) \tag{47}$$

where  $w(x)$  denotes the elastic line of the beam.

Likewise, the fractional bending moment is expressed by:

$$M = -2 \int_0^{h/2} \sigma_{xx}^\alpha y d^\alpha y = \frac{EI^\alpha}{\rho^\alpha} \tag{48}$$

with the fractional stress (see Lazopoulos and Lazopoulos [16]),

$$\sigma_{xx}^\alpha = -\frac{M}{I^\alpha}y \tag{49}$$

and

$$I^\alpha = \frac{2b}{\Gamma(\alpha)\Gamma(2-\alpha)} \int_0^{h/2} \frac{y^{3-\alpha} dy}{(\frac{h}{2}-y)^{1-\alpha}} = \frac{(3-\alpha)(2-\alpha)bh^3\pi Csc(\alpha\pi)}{24\Gamma(\alpha)\Gamma(-\alpha)} \tag{50}$$

Hence, the fractional bending of beam formula is revisited and expressed by:

$$M = EI^\alpha D_2 w(x) = \frac{EI^\alpha}{{}_0^C D_x^\alpha} {}_0^C D_x^\alpha \left( \frac{{}_0^C D_x^\alpha w(x)}{{}_0^C D_x^\alpha x} \right) \tag{51}$$

Therefore, the deflection curve  $W(x)$  is defined by,

$$w(x) = \int_0^x {}_0^C D_x^\alpha s \left( \int_0^s \frac{M(t)}{EI^\alpha} {}_0^C D_t^\alpha t dt^\alpha \right) ds^\alpha + c_1 x + c_2 \tag{52}$$

In conventional integration, the deflection curve is defined by,

$$w(x) = \int_0^x \frac{s^{1-\alpha}}{\Gamma(2-\alpha)} \left( \int_0^s \frac{M(t)}{EI^\alpha} \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \frac{1}{\Gamma(\alpha)(s-t)^{1-\alpha}} dt \right) \frac{ds}{(x-s)^{1-\alpha}} + c_1 x + c_2 \tag{53}$$

In the above equations,  $\alpha$  is the fractional order corresponding to the non-smooth beam deformation due to the microcracks, microvoids, and microdefects generally;  $b$  is the width and  $h$  the height of the beam.

**9 The fractional cantilever beam under continuously distributed loading**

Let us consider the cantilever beam of Fig. 3, with isotropic fractional linear dimension  $\alpha$ , under the continuously distributed loading  $q$ . The vertical force reaction  $\mathbf{A}$  at the clamped end is equal to,

$$\mathbf{A} = \mathbf{Q}_0 = \frac{1}{\Gamma(\alpha)} \int_0^L q d^\alpha x = \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)} \int_0^L \frac{qs^{1-\alpha}}{(x-s)^{1-\alpha}} ds = Lq. \tag{54}$$

The moment reaction at the clamped end is equal to

$$M_0 = - \int_0^L qx d^\alpha x = - \frac{q}{\Gamma(\alpha)\Gamma(2-\alpha)} \int_0^L \frac{s^{2-\alpha} ds}{(L-s)^{1-\alpha}} = \frac{(\alpha-2)L^2\pi q Csc(\alpha\pi)}{2\Gamma(\alpha)}. \tag{55}$$

Further, the bending moment at a point  $x$  of the fractional beam is defined by

$$\begin{aligned} M &= -M_0 + Q_0x - \int_0^x qx d^\alpha x = -M_0 + Q_0x - \int_0^x \frac{q}{\Gamma(\alpha)\Gamma(2-\alpha)} \frac{(x-s)s^{1-\alpha}}{(x-s)^{1-\alpha}} ds \\ &= - \frac{(\alpha-2)L^2\pi q Csc(\alpha\pi)}{2\Gamma(\alpha)} + Lqx - \frac{\alpha\pi qx^2 Csc(\alpha\pi)}{\Gamma(\alpha)\Gamma(1-\alpha)} \end{aligned} \tag{56}$$

The figures that follow bending moment diagrams will be shown for various values of the fractional dimension  $\alpha$  and for beam length  $L = 1$ . Thus, Fig. 4 shows the bending moment diagram for the conventional case  $\alpha = 1$  (Figs. 5, 6).

It is clear that reducing the fractional dimension, the bending moment increases.

The fractional moment of inertia of the cross section is, Eq. (50),

$$I^\alpha = \frac{(3-\alpha)(2-\alpha)bh^3\pi^2 Csc(\alpha\pi)^2}{24\Gamma(1-\alpha)^2\Gamma(\alpha)^2} \tag{57}$$



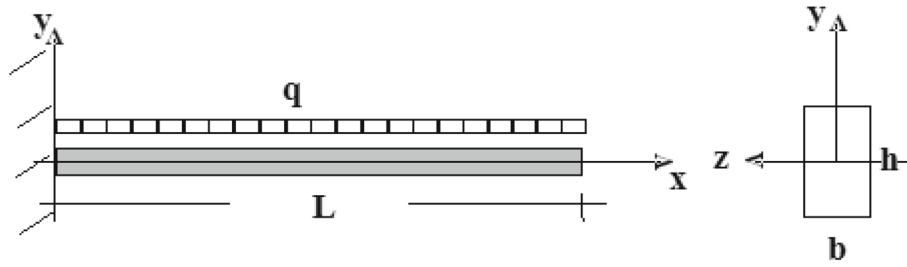


Fig. 3 The geometry of the cantilever beam

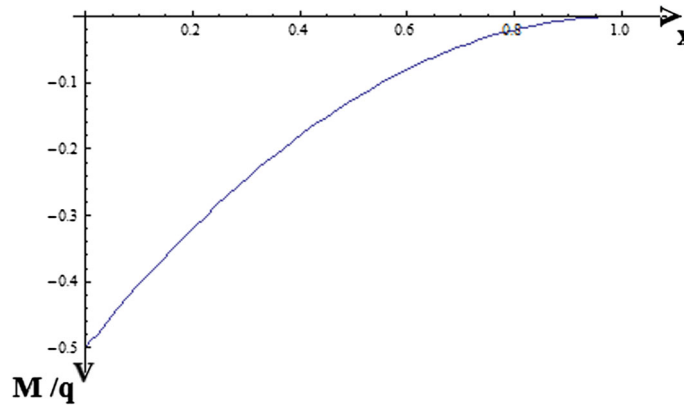


Fig. 4 Bending moment diagram for the conventional case  $\alpha = 1$

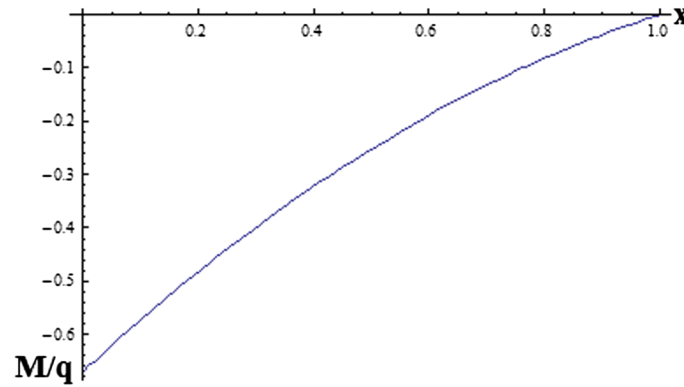


Fig. 5 Bending moment diagram for the case  $\alpha = 0.66$

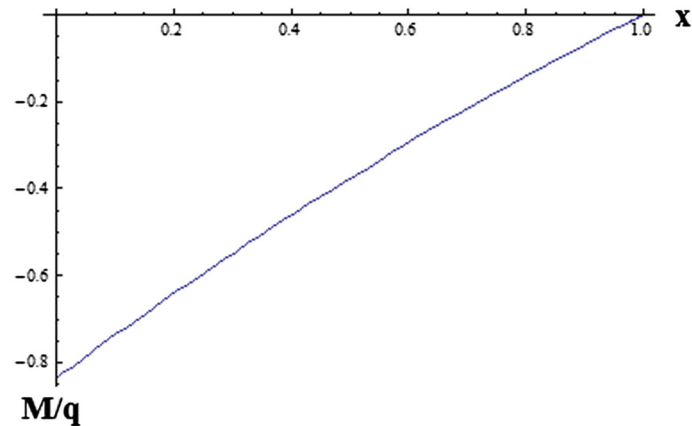
For the conventional case with the fractional dimension  $\alpha = 1$ ,  $I^1 = \frac{bh^3}{12}$ . Hence,

$$\frac{I^\alpha}{I^1} = \frac{(3 - \alpha)(2 - \alpha)\pi^2 Csc(\alpha\pi)^2}{2\Gamma(1 - \alpha)^2 \Gamma(\alpha)^2} \tag{58}$$

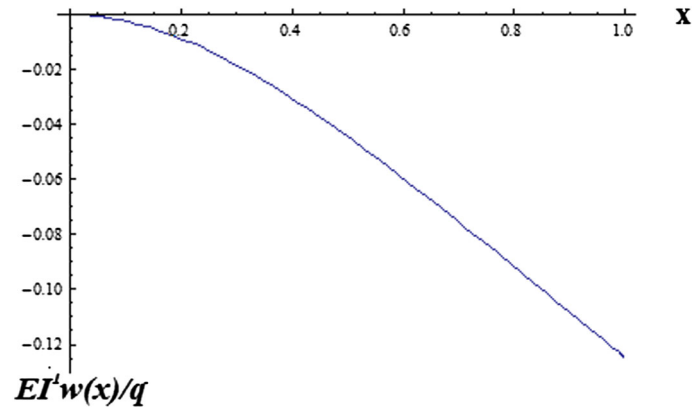
With the help of Mathematica computerized algebra pack, the flexure curve is defined by Eq. (53). Since the boundary conditions for the clamped end ( $x = 0$ ) require  $c_1 = 0$ ,  $c_2 = 0$ , Eq. (53) becomes,

$$w(x) = \frac{q}{EI^\alpha} \frac{(2 - \alpha)(1 - \alpha)^2 \pi^2 x^2 Csc(\alpha\pi)^2 (-72L^2 \Gamma(3 - \alpha) + (24L\Gamma(4 - \alpha) + (-3 + \alpha)\alpha x \Gamma(5 - \alpha)))}{288\Gamma(2 - \alpha)^3 \Gamma(\alpha)^2} \tag{59}$$

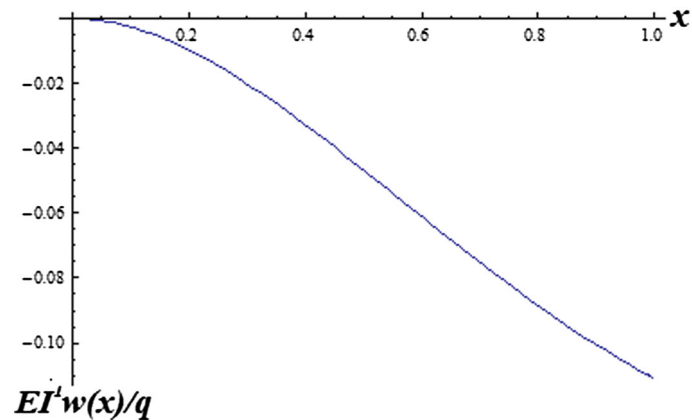
With the help of Mathematica, the flexure curve for various isotropic fractional dimensions has been drawn. In Fig. 7, the flexure curve in the conventional case  $\alpha = 1$  has been drawn. Further, in Fig. 8, the flexure curve with fractional dimension  $\alpha = 0.66$  has been shown. In addition, Fig. 9. corresponds to the case with  $\alpha = 0.33$ .



**Fig. 6** Bending moment diagram for the case  $\alpha = 0.33$



**Fig. 7** The flexure curve of the fractional cantilever beam for  $\alpha = 1$



**Fig. 8** The flexure curve of the fractional cantilever beam for  $\alpha = 0.66$

It is evident from Figs. 7, 8, and 9 that increasing the isotropic fractional dimension  $\alpha$  the beam is more stiff. Moreover, small  $\alpha$  (e.g.,  $\alpha = 0.33$ ) represents a beam with many microvoids and dislocations which cause a beam to curl at the end. The higher the fractional dimension  $\alpha$ , the less dislocations and microvoids in the beam which results to a more “linear” beam.

For the non-isotropic fractional case with axial fractional dimension different from the fractional dimensions in the other directions, we might expect different results. In the case it is studied in the remaining section, the beam with fractional axial dimension is different one; however, in the other directions  $y, z$ , the fractional

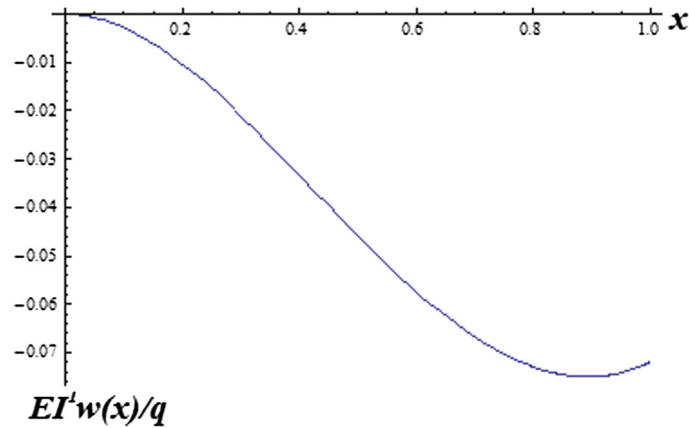


Fig. 9 The flexure curve of the fractional cantilever beam for  $\alpha = 0.33$

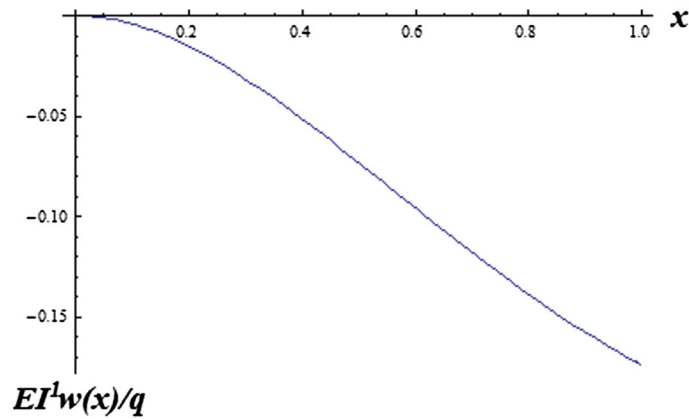


Fig. 10 The flexure curve of the fractional cantilever beam for  $\alpha = 0.66$

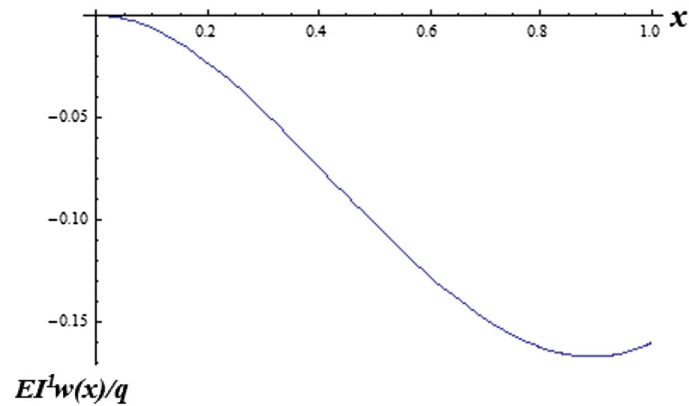


Fig. 11 The flexure curve of the fractional cantilever beam for  $\alpha = 0.33$

dimension is the conventional one ( $\alpha = 1$ ). In this case,  $I^\alpha = I^1$  in Eq. (59). Figure 10 indicates the flexural curve for axial-only fractional dimension  $\alpha = 0.66$ .

Finally, Fig. 11 shows the flexural curve for axial-only fractional dimension  $\alpha = 0.33$ .

Figure 12 compares the flexural curves of the beam for various values of the axial fractional dimensions.

Figure 13 shows the flexural curves of the beam for the isotropic case and the axial case with fractional dimension  $\alpha = 0.66$ . It is evident that the axial case fractional beam is stiffer than the isotropic case.

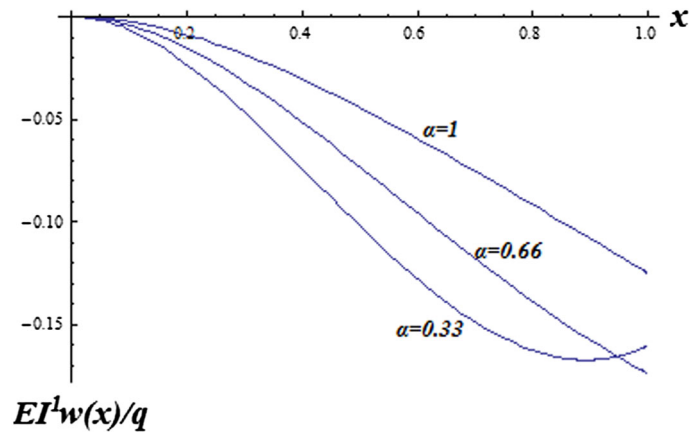


Fig. 12 The flexural curves for various axial fractional dimensions

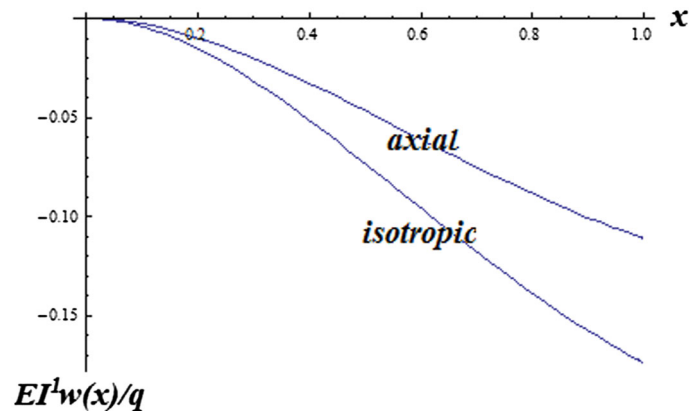


Fig. 13 The flexural curves of the beam for the isotropic and axial fractional cases

## 10 Conclusion—further research

Taking into consideration the fractional differential geometry of the curves (Lazopoulos and Lazopoulos [16]), fractional bending theory of beams is established based upon Bernoulli's principle. Isotropic and axial fractional dimension cases are studied and compared. For fractional order  $\alpha = 1$ , conventional beam bending theory is shown up. It is proven that the higher the fractional order  $\alpha$ , the stiffer the beam is. Further research includes numerical procedures, vibration, and stability analyses.

## References

1. Leibnitz, G.W.: Letter to G. A. L'Hospital. Leibniz Mathematische Schriften **2**, 301–302 (1849)
2. Liouville, J.: Sur le calcul des differentielles a indices quelconques. J. Ec. Polytech. **13**, 71–162 (1832)
3. Riemann, B.: Versuch einer allgemeinen auffassung der integration and differentiation. In: Gesammelte Werke, vol. 62 (1876)
4. Atanackovic, T.M., Stankovic, B.: Generalized wave equation in nonlocal elasticity. Acta Mech. **208**(1–2), 1–10 (2009)
5. Beyer, H., Kempfle, S.: Definition of physically consistent damping laws with fractional derivatives. ZAMM **75**(8), 623–635 (1995)
6. Lazopoulos, K.A.: Nonlocal continuum mechanics and fractional calculus. Mech. Res. Commun. **33**, 753–757 (2006)
7. Truesdell, C.: A First Course in Rational Continuum Mechanics, vol. 1. Academic Press, New York, San Francisco, London (1977)
8. Carpinteri, A., Cornetti, P., Sapora, A.: A fractional calculus approach to non-local elasticity. Eur. Phys. J. Spec. Top. **193**, 193–204 (2011)
9. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)

10. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional integrals and derivatives: theory and applications. Gordon and Breach, Amsterdam (1993)
11. Podlubny, I.: Fractional Differential Equations (An Introduction to Fractional Derivatives, Fractional Differential Equations, Some Methods of Their Solution and Some of Their Applications). Academic Press, San Diego, Boston, New York, London, Tokyo, Toronto (1999)
12. Oldham, K.B., Spanier, J.: The Fractional Calculus. Academic Press, New York, London (1974)
13. Tarasov, V.E.: Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media. Springer, Berlin (2010)
14. Calcani, G.: Geometry of fractional spaces. Adv. Theor. Math. Phys. **16**, 549–644 (2012)
15. Jumarie, Guy: Riemann–Christoffel tensor in differential geometry of fractional order. Application to fractal space-time. Fractals **21**(1), 1350004 (2013)
16. Lazopoulos K.A., Lazopoulos A.K.: On the geometry of curves. Adv. Theor. Math. Phys. (2015) (submitted)
17. Tarasov, V.E.: Fractional vector calculus and fractional Maxwell’s equations. Ann. Phys. **323**, 2756–2778 (2008)
18. Adda, F.B.: The differentiability in the fractional calculus. Nonlinear Anal. **47**, 5423–5428 (2001)
19. Porteous, I.R.: Geometric Differentiation. University Press, Cambridge (1994)