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SPECIAL

Lei Zhang · Jizeng Wang · You-He Zhou

Wavelet solution for large deflection bending problems of thin rectangular plates

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Abstract In this study, we introduce a modified wavelet Galerkin method proposed recently by us to analyze the large deflection bending problems of thin rectangular plates, which are governed by the well-known von Kármán equations, consisting of two coupled fourth-order two-dimensional nonlinear partial differential equations. This adopted wavelet method is established based on a modified wavelet approximation scheme to interval-bounded $\mathbf{L}^2$-functions, in which each series-expansion coefficient can be explicitly expressed by a single-point sampling of the functions, and corresponding boundary values and derivatives can be embedded in the modified scaling function bases. By incorporating this approximation scheme into the conventional Galerkin method, the resulting algorithm can make the solution of the von Kármán equations become very effective and accurate, as demonstrated by the case studies that the wavelet solutions on the deflection–load relations have better accuracy and less consumed computing time than that of other numerical methods including the finite element method and the meshless method.

Keywords Modified wavelet Galerkin method · Large deflection · Von Kármán equations · Thin rectangular plate · Nonlinear problems

1 Introduction

In 1910, von Kármán [1] derived the governing equations for the large deflection problem of a thin rectangular elastic plate, which includes two coupled fourth-order two-dimensional strong nonlinear differential equations (PDEs) expressed in terms of the transverse displacement and Airy stress function. Exact solution to these von Kármán equations can be rarely achieved except for the axisymmetric bending of circular plates [1]. Although approximate solutions in terms of trigonometric series for the equations under certain boundary conditions have been given by Levy [2], such approximations can generally lead to complicated parametric formulas [3]. On the other hand, many existing numerical methods on the solution of nonlinear PDEs usually take the von Kármán equations as their benchmark testing problems [1–11], which include but not limited to the homotopy analysis method [4], the meshless method [5,6], the boundary element method [7], the finite element method [8,9], the perturbation method [10], and the differential quadrature method [11] etc. These conventional methods can be effectively applied to the solution of nonlinear boundary value problems under certain conditions. However, as has been pointed out by Liu et al [12], most of them share a common weakness on that their numerical error and/or computation cost sensitively depend on the nonlinear intensity of the equations. Especially, it is known

L. Zhang · J. Wang (✉) · Y.-H. Zhou (✉)
Key Laboratory of Mechanics on Disaster and Environment in Western China, Ministry of Education,
College of Civil Engineering and Mechanics, Lanzhou University, Lanzhou 730000, Gansu, China
E-mail: jzwang@lzu.edu.cn

Y.-H. Zhou
E-mail: zhouyh@lzu.edu.cn

that the FEM is considered one of the most effective methods for structure analysis. However, when using the FEM to solve nonlinear engineering problems, the effectiveness and the primarily concerned computational cost of the method becomes significantly relying on the choice of appropriate element size and discretization. And even more seriously, in the analysis of plate and shell structures, using FEM may lead to the so-called shear locking phenomena under certain conditions.

Wavelets [12–24] represent a newly developed powerful mathematical tool, which can decompose/construct a function space in a quite unique way that differs from conventional Fourier theory. Based on the wavelet theory, various numerical methods have been developed to solve the PDEs that arise in many different areas of science and engineering, examples including, but not limited to the wavelet Galerkin method [13–18], the wavelet finite element method [19,20] and the so-called adaptive wavelet method [21,22]. These wavelet-based methods have achieved great success in improving computing accuracy and efficiency, and controlling spatial and time resolutions by using wavelet functions of different length scales. However, for general strong nonlinear PDEs, these conventional wavelet-based Galerkin methods still expose the same critical drawbacks as those traditional ones referring to perturbations, iterations, and series expansions. This is because the existence of strong nonlinear terms of the unknown functions in the PDEs can make almost all solutions obtained by these quantitative techniques significantly depend on the truncated parts of the corresponding series expansions. In addition, typical difficulties arose when using conventional wavelet Galerkin method to solve nonlinear PDEs, for example, Liu et al. [16] have applied this method to solve large deflection problems of mechanical beams and rubber films. In their treatment, one needs to calculate the integral of eightfold products of relevant scaling function and its derivatives by using numerical quadrature approaches, which is usually a formidable task due to the strong oscillation nature of the derivatives of the scaling function. Similar trouble has also been encountered during the solution of several other nonlinear differential equations as shown in [14].

Recently, we have developed a new wavelet approximation scheme [23–25] on interval-bounded functions based on techniques of boundary extension and Coiflet-type wavelet expansion. When applying this modified wavelet approximation to the Galerkin method, the boundary extension treatment can eliminate the undesired oscillating error near boundary points due to function value jump. In addition, the interpolating property of this approximation approach can avoid numerically calculating the integral of multi-fold products of scaling functions and their derivatives, and make the nonlinear terms of unknown functions in nonlinear PDEs to be expressed in linear combinations of single-point samplings of composites of the unknown functions at corresponding nodal points, which is crucial for the successful employment of the method. By solving the one-dimensional and two-dimensional Bratu equations with second- order derivatives, numerical results have demonstrated that such a new wavelet algorithm has a much better accuracy [12,24] than many other numerical methods. And the resulting numerical error is almost independent of the nonlinear intensity of the equations, in contrary to most other methods whose numerical accuracy usually decays very fast as the nonlinear intensity becomes strong.

In spite of the above progresses on justifying the new wavelet method based on the Bratu-type equations, it is still unclear that whether the method is also efficient in solving coupled nonlinear PDEs with multi-unknown functions, high-order derivatives and multi-spatial dimensions. To answer this question, in the present study, we first introduce a wavelet approximation scheme for the two-dimensionally bounded functions and their high-order derivatives based on the similar techniques of boundary extension and Coiflet-type wavelet expansion and then combine this approximation scheme with Galerkin method to solve the von Kármán equations with fourth-order derivatives and two coupled unknown functions, and eventually, we evaluate the accuracy and efficiency of the proposed method by comparing the wavelet results to those obtained by several other methods including the FEM.

2 Wavelet approximation of an interval-bounded $\mathbf{L}^2$-function

Based on our previous work [25], for a function $f(x) \in \mathbf{L}^2[0, 1]$, we have

$$f(x) \approx \mathbf{P}_x^n f(x) = \sum_{k=0}^{2^n} f\left(\frac{k}{2^n}\right) \varphi_{n,k}(x) \qquad x \in [0, 1] \tag{1}$$

in which

$$\varphi_{n,k}(x) = \begin{cases} \sum_{i=2-3N+M_1}^{-1} T_{0,k}\left(\frac{i}{2^n}, \boldsymbol{\beta}_0\right)\phi(2^n x - i + M_1) + \phi(2^n x - k + M_1) & k \in [0, 3] \\ \phi(2^n x - k + M_1) & k \in [4,\ 2^n - 4] \\ \sum_{i=2^n+1}^{2^n-1+M_1} T_{1,2^n-k}\left(\frac{i}{2^n}, \boldsymbol{\beta}_1\right)\phi(2^n x - i + M_1) + \phi(2^n x - k + M_1) & k \in [2^n - 3,\ 2^n] \end{cases} \tag{2}$$

and $\phi(x)$ is the generalized Coiflet-type orthogonal scaling function developed by Wang [25], Supp$[\phi(x)] = [0, 3N - 1]$, and

$$T_{0,k}(x, \boldsymbol{\beta}_0) = \sum_{i=0}^{3} \beta_{0,i} \frac{\alpha_{0,i,k}}{i!} x^i, \quad T_{1,k}(x, \boldsymbol{\beta}_1) = \sum_{i=0}^{3} \beta_{1,i} \frac{\alpha_{1,i,k}}{i!} (x-1)^i$$
$$\boldsymbol{\beta}_0 = \{\beta_{0,i} = 1\},\ \boldsymbol{\beta}_1 = \{\beta_{1,i} = 1\},\ i = 0, 1, 2, 3 \tag{3}$$

where $\zeta_0 = \left\{2^{-in}\alpha_{0,i,k}\right\}$ and $\zeta_1 = \left\{2^{-in}\alpha_{1,i,k}\right\}$, $i, k = 0, 1, 2, 3$, are given by Wang [25] as

$$\zeta_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -11/6 & 3 & -3/2 & 1/3 \\ 2 & -5 & 4 & -1 \\ -1 & 3 & -3 & 1 \end{bmatrix}, \ \zeta_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 11/6 & -3 & 3/2 & -1/3 \\ 2 & -5 & 4 & -1 \\ 1 & -3 & 3 & -1 \end{bmatrix}. \tag{4}$$

The modified wavelet approximation of the interval-bounded function in Eqs. (1–3) can easily satisfy homogeneous boundary conditions by setting corresponding elements of $\boldsymbol{\beta}_0$ and $\boldsymbol{\beta}_1$ as zeros [25,26]. For example, $\partial^i f/\partial x^i|_{x=0} = 0$ can be realized by just setting $\beta_{0,i} = 0$, and all other elements of $\boldsymbol{\beta}_0$ and $\boldsymbol{\beta}_1$ remain unchanged.

In addition, this approximation scheme has a very interesting interpolating property: for any composite functions of $f(x)$ satisfying $\mathbf{N}[f(x)] \in \mathbf{L}^2[0, 1]$, by treating $\mathbf{N}[f(x)]$ as a new function and applying Eq. (1), we have

$$\mathbf{N}[f(x)] \approx \sum_{k=0}^{2^n} \mathbf{N}\left[f\left(\frac{k}{2^n}\right)\right]\varphi_{n,k}(x), \qquad x \in [0, 1]. \tag{5}$$

This interpolating property of the approximation scheme is very useful when using Galerkin-type method to solve nonlinear PDEs. The Fourier or Fourier-like bases usually have no such characteristic.

Assuming that the function $f(x)$ is smooth enough, we have its ith-order derivative to be approximated according to Eq. (1) as

$$\frac{d^i}{\mathrm{d}x^i} f(x) \approx \frac{d^i}{\mathrm{d}x^i} \mathbf{P}_x^n f(x) = \sum_{k=0}^{2^n} f\left(\frac{k}{2^n}\right) \frac{d^i}{\mathrm{d}x^i} \varphi_{n,k}(x) \qquad x \in [0, 1]. \tag{6}$$

The error estimation of the above equation has been derived by Liu et al. [27] as

$$\frac{d^i f(x)}{\mathrm{d}x^i} = \frac{d^i \mathbf{P}_x^n f(x)}{\mathrm{d}x^i} + O[2^{-n(N-i)}]. \tag{7}$$

Following the theory of multi-resolution analysis [28], a set of scaling bases for a two-dimensional space can be directly extended by the tensor products of one-dimensional wavelet bases. For $f(x, y) \in \mathbf{L}^2[0, 1]^2$, we thus simply have

$$f(x, y) \approx \mathbf{P}_x^n \mathbf{P}_y^m f(x, y) = \sum_{k=0}^{2^n} \sum_{l=0}^{2^m} f\left(\frac{k}{2^n}, \frac{l}{2^m}\right) \varphi_{n,k}(x) \varphi_{m,l}(y) \tag{8}$$

where $\varphi_{n,k}(x)$ and $\varphi_{m,l}(y)$ are given by Eq. (2). And if this function is smooth enough, we further have

$$\frac{\partial^{i+j}}{\partial x^i \partial y^j} f(x, y) \approx \frac{\partial^{i+j}}{\partial x^i \partial y^j} \mathbf{P}_x^n \mathbf{P}_y^m f(x, y) = \sum_{k=0}^{2^n} \sum_{l=0}^{2^m} f\left(\frac{k}{2^n}, \frac{l}{2^m}\right) \frac{\partial^i}{\partial x^i} \varphi_{n,k}(x) \frac{\partial^j}{\partial y^j} \varphi_{m,l}(y). \tag{9}$$

Similar to Eq. (5), we note that the interpolating property of the approximation scheme in Eq. (8) for the two-dimensional functions is still valid, i.e.

$$\mathbf{N}\left[f(x,y)\right] \approx \sum_{k=0}^{2^n}\sum_{l=0}^{2^m} \mathbf{N}\left[f\left(\frac{k}{2^n}, \frac{l}{2^m}\right)\right]\varphi_{n,k}(x)\varphi_{m,l}(y). \tag{10}$$

The accuracy of the approximation Eq. (8) depends on the number of vanishing moment N of the scaling function, $\phi(x)$, adopted in Eq. (2) and the decomposition level n, which can be derived based on Eq. (7) as following:

$$\frac{\partial^i f(x,y)}{\partial x^i} = \frac{\partial^i \mathbf{P}_x^n f(x,y)}{\partial x^i} + O[2^{-n(N-i)}], \tag{11}$$

$$\frac{\partial^j}{\partial y^j}\left[\frac{\partial^i \mathbf{P}_x^n f(x,y)}{\partial x^i}\right] = \frac{\partial^j}{\partial y^j}\mathbf{P}_y^m\left[\frac{\partial^i \mathbf{P}_x^n f(x,y)}{\partial x^i}\right] + O\left[2^{-m(N-j)}\right]. \tag{12}$$

By combining Eqs. (11) and (12), we have

$$\begin{aligned}\frac{\partial^{i+j}}{\partial x^i \partial y^j} f(x,y) &= \frac{\partial^{i+j}}{\partial x^i \partial y^j}\left[\mathbf{P}_x^n \mathbf{P}_y^m f(x,y)\right] + O\left[2^{-n(N-i)}\right] + O\left[2^{-m(N-j)}\right] \\ &= \frac{\partial^{i+j}}{\partial x^i \partial y^j}\left[\mathbf{P}_x^n \mathbf{P}_y^m f(x,y)\right] + O\left[2^{-\min[n(N-i),m(N-j)]}\right].\end{aligned} \tag{13}$$

3 Solution of the von Kármán equations

The large deformation problem of a thin rectangular plate under distributed load can be described by the von Kármán equations [1] as,

$$\begin{aligned} D\nabla^4 w = q(x,y) + h\mathbf{H}(w,f) \quad x \in [0,a] \\ \nabla^4 f = -\tfrac{E}{2}\mathbf{H}(w,w) \quad y \in [0,b] \end{aligned} \tag{14}$$

where $w(x,y)$ is the deflection of the plate, $f(x,y)$ the stress function, h the thickness of the plate, E the Young modulus, $D = Eh^3/12(1-\nu^2)$ the bending stiffness, ν the Poisson ratio, ∇^4 the biharmonic operator, and the operator $\mathbf{H}(w,f)$ is defined as

$$\mathbf{H}(w,f) = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 f}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 f}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}. \tag{15}$$

In this study, we assume that the plate is in-plane movable clamped or in-plane movable simply supported, so that the corresponding boundary conditions can be given by

$$w = \frac{\partial w}{\partial x} = 0,\ N_x = N_{xy} = 0,\ \text{at } x = 0, a, \quad w = \frac{\partial w}{\partial y} = 0,\ N_y = N_{xy} = 0,\ \text{at } y = 0, b \tag{16a}$$

$$w = \frac{\partial^2 w}{\partial x^2} = 0,\ N_x = v = 0,\ \text{at } x = 0, a\ , \quad w = \frac{\partial^2 w}{\partial y^2} = 0,\ N_y = u = 0,\ \text{at } y = 0, b \tag{16b}$$

where N_x, N_y and N_{xy} are in-plane membrane forces, u and v are displacements along x and y axes. Following a simple derivation, Eq. (16) can be equivalently replaced by [29]

$$w = \frac{\partial w}{\partial x} = 0,\ f = \frac{\partial f}{\partial x} = 0,\ \text{at } x = 0, a\ , \quad w = \frac{\partial w}{\partial y} = 0,\ f = \frac{\partial f}{\partial y} = 0,\ \text{at } y = 0, b. \tag{17a}$$

$$w = \frac{\partial^2 w}{\partial x^2} = 0,\ f = \frac{\partial^2 f}{\partial x^2} = 0,\ \text{at } x = 0, a\ , \quad w = \frac{\partial^2 w}{\partial y^2} = 0,\ f = \frac{\partial^2 f}{\partial y^2} = 0,\ \text{at } y = 0, b. \tag{17b}$$

We introduce the following dimensionless variables

$$\overline{w} = \frac{w}{h}, \quad \overline{f} = \frac{f}{Eh^2}, \quad \overline{x} = \frac{x}{a}, \quad \overline{y} = \frac{y}{b}, \quad \beta = \frac{a}{b}, \quad \overline{q}(\overline{x}, \quad \overline{y}) = \frac{q(x,y)a^4}{Eh^4}. \tag{18}$$

By using Eq. (18), then Eqs. (14) and (17b) can be rewritten as

$$\begin{array}{l}\overline{\nabla}^4\overline{w}=\overline{q}(\overline{x},\overline{y})\lambda+\beta^2\lambda\mathbf{H}\left(\overline{w},\overline{f}\right)\quad \overline{x}\in[0,1]\\ \overline{\nabla}^4\overline{f}=-\frac{\beta^2}{2}\mathbf{H}\left(\overline{w},\overline{w}\right)\quad \overline{y}\in[0,1]\end{array}, \tag{19}$$

and

$$\begin{aligned}\overline{w}|_{\overline{x}=0,1}&=\left.\frac{\partial^2\overline{w}}{\partial\overline{x}^2}\right|_{\overline{x}=0,1}=0,\ \ \overline{f}|_{\overline{x}=0,1}=\left.\frac{\partial^2\overline{f}}{\partial\overline{x}^2}\right|_{\overline{x}=0,1}=0,\\ \overline{w}|_{\overline{y}=0,1}&=\left.\frac{\partial^2\overline{w}}{\partial\overline{y}^2}\right|_{\overline{y}=0,1}=0,\ \ \overline{f}|_{\overline{y}=0,1}=\left.\frac{\partial^2\overline{f}}{\partial\overline{y}^2}\right|_{\overline{y}=0,1}=0\end{aligned} \tag{20}$$

where $\lambda=12\left(1-\nu^2\right)$, and operator $\overline{\nabla}^4=\partial^4/\partial\overline{x}^4+2\beta^2\partial^4/\partial\overline{x}^2\partial\overline{y}^2+\beta^4\partial^4/\partial\overline{y}^4$. For simplicity, we will drop the bar over each dimensionless quantity except the operator $\overline{\nabla}^4$ in the following sections.

We further define

$$\begin{array}{lll}w_1(x,y)=w_{xx}(x,y), & w_2(x,y)=w_{xy}(x,y), & w_3(x,y)=w_{yy}(x,y),\\ f_1(x,y)=f_{xx}(x,y), & f_2(x,y)=f_{xy}(x,y), & f_3(x,y)=f_{yy}(x,y).\end{array} \tag{21}$$

Equation (14) can be accordingly rewritten as

$$\begin{array}{c}\overline{\nabla}^4 w=\beta^2\lambda(f_1w_3+f_3w_1-2f_2w_2)+\lambda q(x,y)\\ \overline{\nabla}^4 f=\beta^2(w_2^2-w_1w_3).\end{array} \tag{22}$$

When using the modified wavelet Galerkin method to solve Eq. (22), we need to apply Eq. (8) to approximate various functions in Eqs (21) and (22). Then, we have the following modified scaling function series expansions

$$\begin{aligned}&w(x,y)\approx\sum_{i=1}^{2^n-1}\sum_{j=1}^{2^n-1}w\left(\frac{i}{2^n},\frac{j}{2^n}\right)\Phi_{n,i}(x)\Phi_{n,j}(y),\ \ f(x,y)\approx\sum_{i=1}^{2^n-1}\sum_{j=1}^{2^n-1}f\left(\frac{i}{2^n},\frac{j}{2^n}\right)\Phi_{n,i}(x)\Phi_{n,j}(y),\\ &w_r(x,y)\approx\sum_{i=0}^{2^n}\sum_{j=0}^{2^n}w_r\left(\frac{i}{2^n},\frac{j}{2^n}\right)\varphi_{n,i}(x)\varphi_{n,j}(y),\ \ f_s(x,y)\approx\sum_{i=0}^{2^n}\sum_{j=0}^{2^n}f_s\left(\frac{i}{2^n},\frac{j}{2^n}\right)\varphi_{n,i}(x)\varphi_{n,j}(y),\\ &w_rf_s\approx\sum_{i=0}^{2^n}\sum_{j=0}^{2^n}w_r\left(\frac{i}{2^n},\frac{j}{2^n}\right)f_s\left(\frac{i}{2^n},\frac{j}{2^n}\right)\varphi_{n,i}(x)\varphi_{n,j}(y),\\ &w_rw_s\approx\sum_{i=0}^{2^n}\sum_{j=0}^{2^n}w_r\left(\frac{i}{2^n},\frac{j}{2^n}\right)w_s\left(\frac{i}{2^n},\frac{j}{2^n}\right)\varphi_{n,i}(x)\varphi_{n,j}(y),\\ &q(x,y)\approx\sum_{i=0}^{2^n}\sum_{j=0}^{2^n}q_{ij}\varphi_{n,i}(x)\varphi_{n,j}(y)\end{aligned} \tag{23}$$

where $\Phi_{n,k}(x)$ is $\varphi_{n,k}(x)$ to be specified by assigning zeros to $\beta_{0,0}$, $\beta_{0,2}$, $\beta_{1,0}$, $\beta_{1,2}$ in Eq. (2), and all other elements of $\boldsymbol{\beta}_0$ and $\boldsymbol{\beta}_1$ remain unchanged, subscripts r and s are a combination of the numbers of 1, 2 and 3.

Substituting the expansions in Eq. (23) into Eq. (21), and apply Galerkin method, it yields

$$\mathbf{A}_1\mathbf{W}_r=\mathbf{B}_r\mathbf{W},\ \mathbf{A}_1\mathbf{F}_r=\mathbf{B}_r\mathbf{F} \tag{24}$$

where matrices $\mathbf{A}_1=\{a_{1,kl,ij}=\Omega_{kl,ij}\}$, $\mathbf{B}_1=\{b_{1,kl,ij}=\Upsilon_{kl,ij}^{20,00}\}$, $\mathbf{B}_2=\{b_{2,kl,ij}=\Upsilon_{kl,ij}^{11,00}\}$, $\mathbf{B}_3=\{b_{3,kl,ij}=\Upsilon_{kl,ij}^{02,00}\}$, vectors $\mathbf{W}_r=\{w_{r,ij}=w_r(i/2^n,j/2^n)\}$, $\mathbf{W}=\{w_{ij}=w(i/2^n,j/2^n)\}$, $\mathbf{F}=\{f_{ij}=f(i/2^n,j/2^n)\}$, $\mathbf{F}_r=\{f_{r,ij}=f_r(i/2^m,j/2^m)\}$, $r=1,2,3$, $\Omega_{kl,ij}=\int_0^1\varphi_{n,i}(x)\varphi_{n,k}(x)\mathrm{d}x\int_0^1\varphi_{n,j}(y)\varphi_{n,l}(y)\mathrm{d}y$, $\Upsilon_{kl,ij}^{de,00}=\int_0^1\Phi_{n,i}^{(d)}(x)\varphi_{n,k}(x)\mathrm{d}x\int_0^1\Phi_{n,j}^{(e)}(y)\varphi_{n,l}(y)\mathrm{d}y$.

Table 1 Coefficients p_k

k	p_k	k	p_k	k	p_k
0	−2.392638657280051E−03	6	6.459945432939942E−01	12	1.238869565706006E−02
1	−4.932601854180402E−03	7	1.116266213257999E+00	13	−1.583178039255944E−02
2	2.714039971139949E−02	8	5.381890557079980E−01	14	−2.717178600539990E−03
3	3.064755594619984E−02	9	−9.961543386239989E−02	15	2.886948664020020E−03
4	−1.393102370707997E−01	10	−7.992313943479994E−02	16	6.304993947079994E−04
5	−8.060653071779983E−02	11	5.149146293240031E−02	17	−3.058339735960013E−04

Substituting the wavelet series expansions in Eq. (23) into Eq. (22), we have

$$\sum_{i=1}^{2^n-1}\sum_{j=1}^{2^n-1} w_{ij}\overline{\nabla}^4\left[\Phi_{n,i}(x)\Phi_{n,j}(y)\right] = \lambda\sum_{i=1}^{2^n}\sum_{j=1}^{2^n}\left[\beta^2(f_{1,ij}w_{3,ij} - 2f_{2,ij}w_{2,ij} + f_{3,ij}w_{1,ij}) + q_{ij}\right]\varphi_{n,i}(x)\varphi_{n,j}(y)$$
$$\sum_{i=1}^{2^n-1}\sum_{j=1}^{2^n-1} f_{ij}\overline{\nabla}^4\left[\Phi_{n,i}(x)\Phi_{n,j}(y)\right] = \beta^2\sum_{i=0}^{2^n}\sum_{j=0}^{2^n}(w_{2,ij}^2 - w_{1,ij}w_{3,ij})\varphi_{n,i}(x)\varphi_{n,j}(y), \quad (25)$$

Multiplying both sides of Eq. (25) by the scaling bases $\Phi_{n,k}(x)\Phi_{n,l}(y)$, $k, l = 1, 2, \ldots, 2^n - 1$, respectively, and performing integration over the domain $[0, 1]^2$, we have

$$\begin{aligned}\mathbf{AW} &= \lambda\mathbf{A}_2[\beta^2\text{diag}(\mathbf{A}_1^{-1}\mathbf{B}_1\mathbf{F})(\mathbf{A}_1^{-1}\mathbf{B}_3\mathbf{W}) - 2\beta^2\text{diag}(\mathbf{A}_1^{-1}\mathbf{B}_2\mathbf{F})(\mathbf{A}_1^{-1}\mathbf{B}_2\mathbf{W})\\ &\quad + \beta^2\text{diag}(\mathbf{A}_1^{-1}\mathbf{B}_3\mathbf{F})(\mathbf{A}_1^{-1}\mathbf{B}_1\mathbf{W}) + \mathbf{Q}],\\ \mathbf{AF} &= \beta^2\mathbf{A}_2[\text{diag}(\mathbf{A}_1^{-1}\mathbf{B}_2\mathbf{W})(\mathbf{A}_1^{-1}\mathbf{B}_2\mathbf{W}) - \text{diag}(\mathbf{A}_1^{-1}\mathbf{B}_1\mathbf{W})(\mathbf{A}_1^{-1}\mathbf{B}_3\mathbf{W})]\end{aligned} \quad (26)$$

in which $\mathbf{A} = \{a_{kl,ij} = \Gamma_{kl,ij}^{40,00} + 2\beta^2\Gamma_{kl,ij}^{22,00} + \beta^4\Gamma_{kl,ij}^{04,00}\}$, $\mathbf{A}_2 = \{a_{2,kl,ij} = \Upsilon_{ij,kl}^{00,00}\}$, $\mathbf{Q} = \{q_{ij} = q(i/2^n, j/2^n)\}$, $\Gamma_{kl,ij}^{de,00} = \int_0^1 \Phi_{n,i}^{(d)}(x)\Phi_{n,k}(x)\mathrm{d}x \int_0^1 \Phi_{n,j}^{(e)}(y)\Phi_{n,l}(y)\mathrm{d}y$, and diag($\mathbf{U}$) represents the diagonal matrix with diagonal elements to be the elements of vector $\mathbf{U}$. The generalized connect coefficients $\Gamma_{kl,ij}^{de,00}$, $\Upsilon_{kl,ij}^{de,00}$ and $\Omega_{kl,ij}$ can be obtained exactly by using the procedure suggested by Wang [25] associated with lower pass filter coefficients of Coiflet-type wavelet listed in Table 1 and the expression of the modified scaling basis given in Eq. (2).

Equation (26) is a set of nonlinear algebraic equations with $2 \times (2^n - 1)^2$ unknown variables. By employing the classical Newton iteration method to solve the nonlinear algebraic equations (26), we can obtain the nodal values of unknown functions $w(i/2^n, j/2^n)$ and $f(i/2^n, j/2^n)$, $i, j = 1, 2, \ldots, 2^n - 1$, which can be used to reconstruct $w(x, y)$ and $f(x, y)$ in terms of Eq. (8).

4 Numerical examples

In order to evaluate the accuracy and convergence of the proposed wavelet-based Galerkin method, we first consider a similar equation with the same boundary conditions as the von Kármán equations in Eqs. (20) and (22)

$$\begin{aligned}\overline{\nabla}^4 w &= \beta^2\lambda(f_1w_3 + f_3w_1 - 2f_2w_2) + \lambda q(x, y)\\ \overline{\nabla}^4 f &= \beta^2(w_2^2 - w_1w_3) + h(x, y)\end{aligned} \quad (27)$$

where

$$\begin{aligned}q(x, y) &= \frac{w_0\pi^4}{\lambda}(1 + 2\beta^2 + \beta^4)\sin(\pi x)\sin(\pi y) + w_0 f_0\beta^2\pi^4[\cos(2\pi x) + \cos(2\pi y)]\\ h(x, y) &= f_0\pi^4(1 + 2\beta^2 + \beta^4)\sin(\pi x)\sin(\pi y) - \frac{w_0^2\beta^2\pi^4}{2}[[\cos(2\pi x) + \cos(2\pi y)]]\end{aligned} \quad (28)$$

For this equation, the analytic solutions can be given by

$$w(x, y) = w_0\sin\pi x\sin\pi y,\ f(x, y) = f_0\sin\pi x\sin\pi y \quad (29)$$

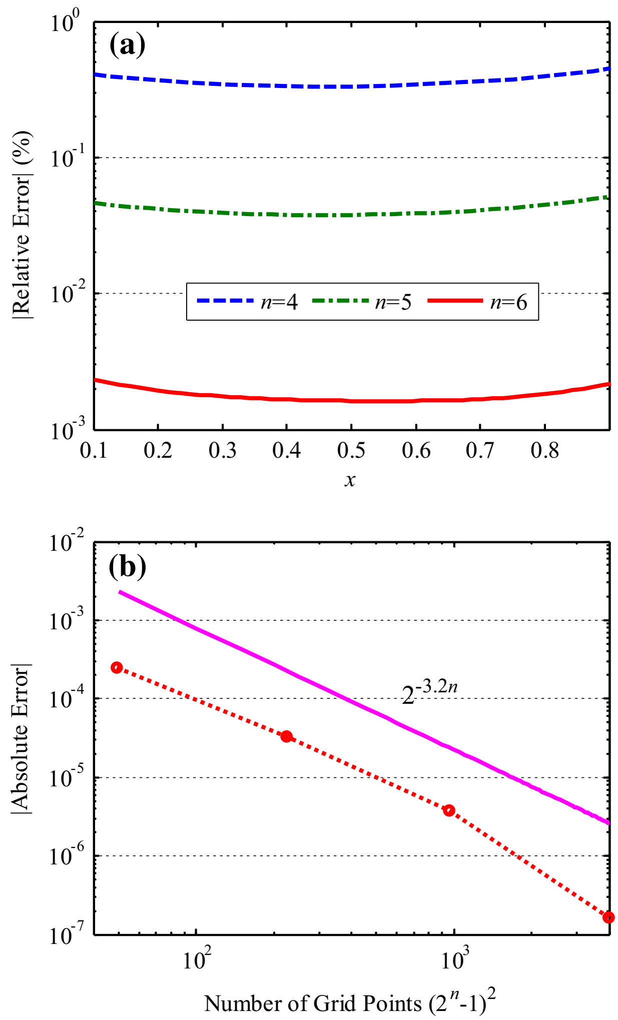

Fig. 1 Errors of the wavelet-based numerical solutions of Eq. (27) with $w_0 = f_0 = 0.01$ under the resolution level $n = 4$, 5 and 6, respectively (corresponding to the number of grid points $(2^n - 1)^2$): **a** the relative error of the solution to $w(x, 0.5)$ and **b** the absolute error of the solution to $w(0.5, 0.5)$ as a function of the number of grid points

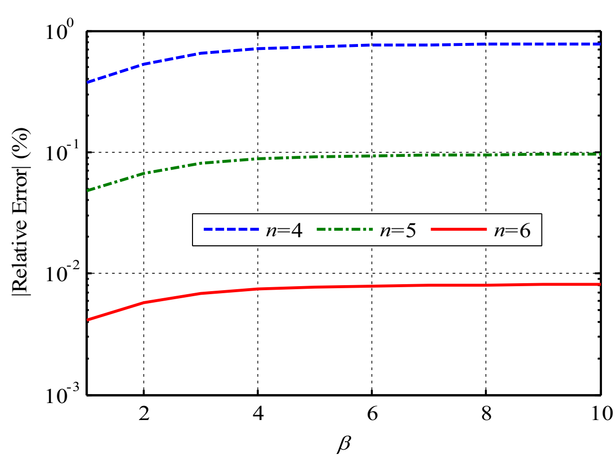

Fig. 2 Absolute values of the numerical solution of Eq. (27) to $w(0.5, 0.5)$ as a function of the plate aspect ratio β under the resolution level $n = 4$, 5 and 6, where $w_0 = f_0 = 0.01$

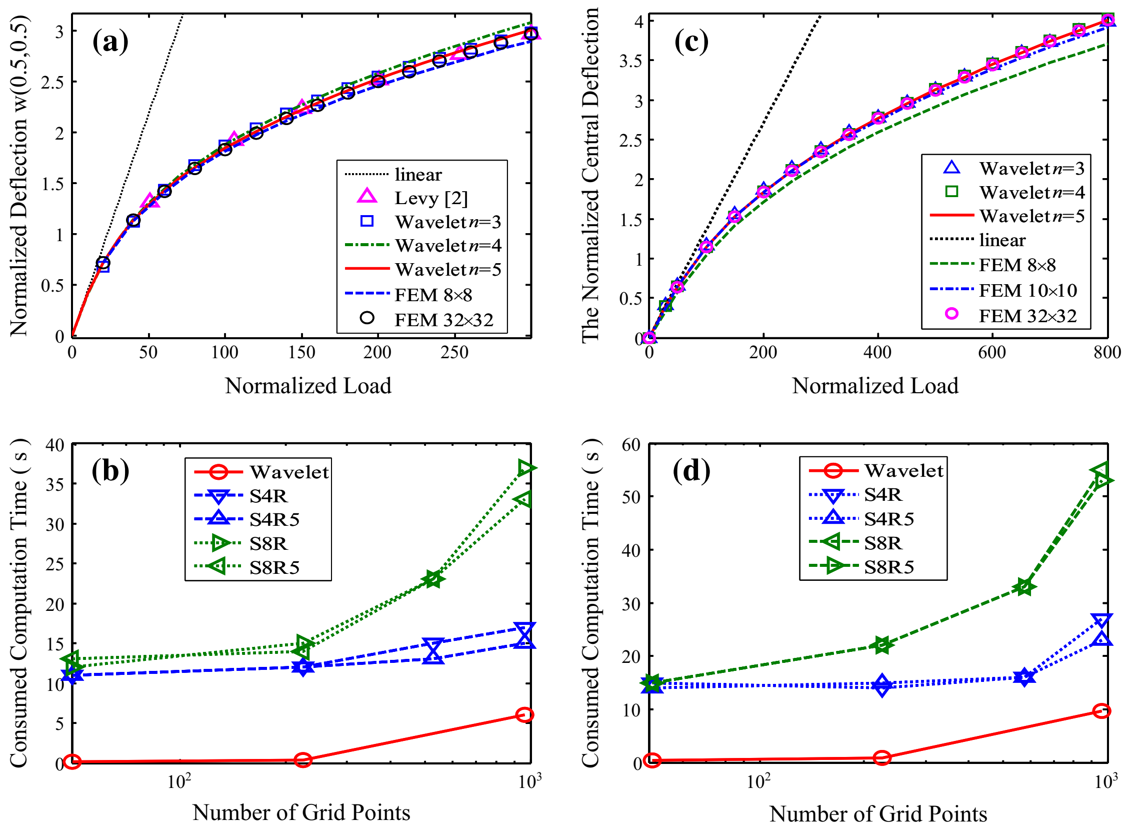

Fig. 3 The large deflection bending of a square plate under uniform load: **a** and **c** the normalized central deflection of the plate as a function of the normalized load under simply supported and clamped boundary respectively, for the wavelet method, $n = 3, 4, 5$, and for the FEM, mesh type, S8R, meshes 8×8, 32×32; **b** and **d** the computation time as a function of the number of grid points by using the same PC for the wavelet method, and the FEM with different mesh types, and mesh sizes 8×8, 16×16, 24×24 and 32×32, respectively, for the simply supported and clamped plate respectively

By using the proposed wavelet method, we can obtain the numerical solutions of Eqs. (27) and (28). Figure 1 shows the relative error of the numerical solution to $w(x, 0.5)$ under the resolution level $n = 4, 5$ and 6, respectively. It can be seen from Fig. 1a that the relative error decays very fast along with n, which approaches about 0.004 % when $n = 6$. Figure 1b further shows the absolute error of the numerical solution to $w(0.5, 0.5)$ as a function of the number of grid points $(2^n - 1)^2$. It can be observed from Fig. 1b that such an absolute error decreases by $2^{-3.2n} C_0$ as the resolution level n increases, where C_0 is a positive constant. Figure 2 plots the relative error of the wavelet-based numerical solution to $w(0.5, 0.5)$ as a function of the plate aspect ratio β. It can be seen from Fig. 2 that the relative error becomes almost independent of β as β exceeds 4. And this relative error keeps below 0.01 % when the resolution level $n = 6$.

The FEM is usually considered as one of the most effective methods for structure analysis. There are even many sophisticated commercial softwares which use the FEM to solve complicated engineering problems. The Abaqus is one of the most successful such software which is suitable for finite element analysis of structures. In order to further evaluate the accuracy and computing efficiency of the proposed wavelet method, we consider the deformation of a square plate under uniformly distributed load. We use the Abaqus to perform FEM analysis to this problem. As the numerical results may depend on the choice of element types, we adopt four kinds of general shell elements [30], S4R, S4R5, S8R and S8R5 in the FEM computation. By taking $\beta = 1$, $\lambda = 12 \times (1 - 0.316^2)$, Fig. 3a, c show the normalized central deflection of the plate as a function of the normalized load under simply supported and clamped boundary respectively, which is obtained by using the wavelet method under resolution levels $n = 3, 4, 5$, and the FEM with S8R under meshes 8×8, 32×32. Figure 3b, d plot the computation time as a function of the number of grid points by using the same PC for the wavelet method, and the FEM with different mesh types, respectively. It can be seen from Fig. 3a, c that the wavelet solution at $n = 5$ approaches the FEM results for 32×32 elements, indicating that both of the wavelet method and the FEM method can solve the large deflection bending problem even when the central deflection to thickness ratio

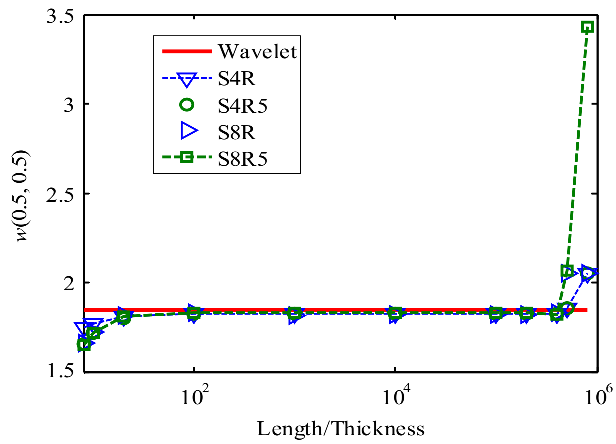

Fig. 4 Comparison of the normalized central deflections obtained by using the FEM with mesh 8 × 8, and the wavelet method with $n = 3$

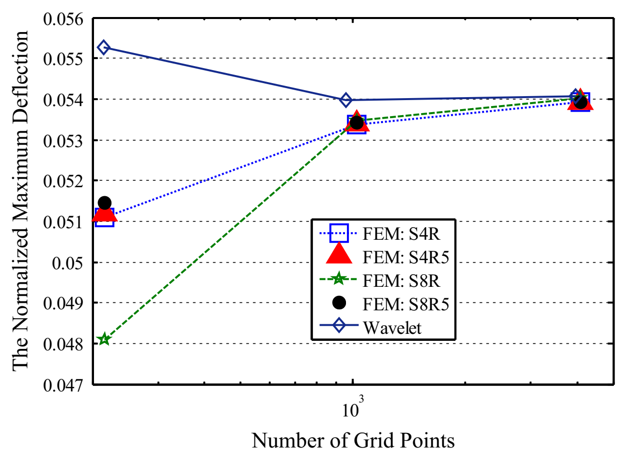

Fig. 5 The normalized maximum deflection of the simply supported plate under normalized load, $q(x, y) = 500\sin(4\pi x)\sin(4\pi y)$, as a function of the number of grid points

Table 2 Central deflection of a square plate under uniformly distributed load

q	FEM (32 × 32)	Wavelet (15 × 15)	RPIM [6] (11 × 11)	RBF [5] (36+81)
10	0.41289	0.4062	0.42892	0.43636
20	0.72269	0.7204	0.74655	0.75331
30	0.95377	0.9592	0.98039	1.00341
40	1.13770	1.1506	1.15196	–
50	1.29092	1.3111	1.32353	–

exceeds 3 in Fig. 3a and exceeds 4 in Fig. 3c. However, it also can be seen from Fig. 3c that the wavelet solution is already sufficient accurate under resolution level $n = 3$, while the FEM solution is inaccurate with S8R under meshes 8×8 for the in-plane movable clamped plate. And from Fig. 3b, d, we can see that with the same accuracy level, the computing time of the FEM method with different mesh types is much large than that of the wavelet method, i.e., under the same amount of grid points, the wavelet method is much faster than the FEM.

We may also question the reliability of the numerical results obtained by using different methods. Figure 4 illustrates the central displacement of the simply supported plate as a function of the length-thickness ratio. It can be seen from Fig. 4 that there is large discrepancy between the FEM results and the wavelets results, which is caused by the so-called shear locking numerical problems of the FEM [6].

When the external load has a high spatial frequency, for example $q(x, y) = 500\sin(4\pi x)\sin(4\pi y)$, we can see from Fig. 5 that, the numerical result on the maximum deflection based on the proposed wavelet method converges faster than those by the FEM.

Table 2 shows the central deflections of the simply supported plate obtained by the proposed wavelet method, the FEM, and two different meshless methods [5] and [6] under different number of grid points. It can be seen from Table 2 that results by using different methods are close to each other. And the results based on the wavelet method are closer to the results of the FEM than that of the meshless methods.

5 Conclusion

In this paper, we have introduced a wavelet approximation scheme for the two-dimensionally bounded $\mathbf{L}^2$ functions based on the Coiflet-type scaling function expansions. By combining this scheme with the Galerkin method, we have constructed a modified wavelet Galerkin method for the solution of strong nonlinear PDEs with high- order derivatives, multi-spatial dimensions and multi-coupled-unknown functions. The solution of the von Kármán equations by using the proposed wavelet method have demonstrated that this wavelet algorithm has a convergence rate of $\sim 2^{-3.2n}$, and shows a very high computing efficiency comparing to the FEM methods with different mesh types. As our new wavelet approximation scheme on the nonlinear terms is a complete expansion in a Riesz basis of a closed linear subspace of $\mathbf{L}^2[0, 1]^2$, we expect that the proposed Galerkin method has the so-called closed property. This unique property makes the proposed method be able to successfully deal with strong nonlinear problems, such as general geometrically nonlinear problems in mechanics.

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