Arch Appl Mech (2014) 84:1135–1143
DOI 10.1007/s00419-014-0871-9

ORIGINAL

S. M. Mousavi · J. Paavola

Analysis of plate in second strain gradient elasticity

Received: 30 July 2013 / Accepted: 7 May 2014 / Published online: 25 May 2014

Abstract The bending analysis of a thin rectangular plate is carried out in the framework of the second gradient elasticity. In contrast to the classical plate theory, the gradient elasticity can capture the size effects by introducing internal length. In second gradient elasticity model, two internal lengths are present, and the potential energy function is assumed to be quadratic function in terms of strain, first- and second-order gradient strain. Second gradient theory captures the size effects of a structure with high strain gradients more effectively rather than first strain gradient elasticity. Adopting the Kirchhoff's theory of plate, the plane stress dimension reduction is applied to the stress field, and the governing equation and possible boundary conditions are derived in a variational approach. The governing partial differential equation can be simplified to the first gradient or classical elasticity by setting first or both internal lengths equal to zero, respectively. The clamped and simply supported boundary conditions are derived from the variational equations. As an example, static, stability and free vibration analyses of a simply supported rectangular plate are presented analytically.

Keywords Second strain gradient elasticity · Rectangular plate · Kirchhoff's theory · Bending analysis

1 Introduction

Plates are one of the most commonly used structures in different applications. Due to the specific geometrical properties, the analysis of these structures is usually carried out with the aid of dimension reduction assumptions. The Kirchhoff theory of plates is a quite popular model for thin plates. The problem of Kirchhoff plate in the framework of classical elasticity is dealt extensively in the literature [1].

In the classical elasticity, the lack of characteristic length results in a formulation independent of the scale of the structures. This issue has been dealt with in the gradient elasticity framework. For instance, in the strain gradient elasticity theory, the strain energy is generalized and is not simply a function of strain but also depend on the gradient of strain. Thus, an internal length will appear in the constitutive equations. In the first strain gradient elasticity, the strain energy is assumed to be a quadratic function in terms of strain and first-order gradient strain, while in the second strain gradient elasticity, the strain energy is a function of strain, first- and second-order gradient strain.

Recently, the analysis of the gradient elastic plates has been the topic of some investigations. Papargyri-Beskou and Beskos [2], using the equilibrium equations, derived the sixth-order governing equation of gradient elastic flexural Kirchhoff plates. As was expected, this approach presents no information about the modeling of the boundary conditions. Later, Papargyri-Beskou et al. [3] presented a variational formulation of the same problem and derived the governing equation as well as possible boundary conditions. Lazopoulos [4] derived the governing plate equation with its boundary conditions through a variational method. A new Kirchhoff

S. M. Mousavi (✉) · J. Paavola
Department of Civil and Structural Engineering, Aalto University, Box 12100, 00076 Aalto, Finland
E-mail: mahmoud.mousavi@aalto.fi
Tel.: +358503503184

plate model based on a modified couple stress theory was reported by Tsiatas [5]. It should be noted that the resulting boundary value problem is of the fourth order instead of existing gradient theories which is of the sixth order.

The above-mentioned formulations considered one material length scale parameter. Wang et al. [6] developed a size-dependent Kirchhoff micro-plate model based on the strain gradient elasticity theory. Their model contains three material length scale parameters to capture the size effect. They have solved the problem of a simply supported micro-plate. Furthermore, Ashoori Movassagh and Mahmoodi [7] presented a Kirchhoff micro-plate model based on the modified strain gradient elasticity theory. Their analysis is general and can be reduced to the modified couple stress plate model or classical plate model once two or all material length scale parameters in the theory are set zero, respectively.

Second gradient elasticity can capture the size effects more effectively than the first gradient theory [8,9]. Recently, second gradient elasticity has been the topic of investigations in the generalized continuum mechanics. Dell'isola et al. [8] investigated the Generalized hooke's law for isotropic second gradient materials. Lazar et al. [9] analyzed the dislocations in the second strain gradient elasticity. These studies motivate the analysis of plates in the second gradient elasticity.

Similar to the classical elasticity, the analytical solution is limited to special cases such as simply supported micro-plates. Some of the numerical techniques have been extended to the problem of gradient elasticity. In the territory of the Kirchhoff micro-plate, Tsiatas [5] applied the method of fundamental solutions (which is a boundary-type meshless method) to a plate model in the modified couple stress theory. The extended Kantorovich method is applied successfully to a micro-plate model based on the modified strain gradient elasticity theory [7]. Ahmadi et al. [10] presented the static deflection analysis of flexural simply supported sectorial micro-plate using p-version finite element method. Furthermore, a higher continuity finite element method was used by Ahmadi and Farahmand [11] for the static deflection analysis of flexural rectangular micro-plate.

The boundary element method has also been proved to be capable for the static and dynamic analysis of strain gradient elastic solids and structures [12,13]. Fischer et al. [14] extended the concept of isogeometric analysis toward the numerical solution of the problem of gradient elasticity in two dimensions. Differential quadrature method (DQM) is another technique which is used by Wang and Wang [15] to determine the dynamic behavior of a micro-cantilever plate. They applied the DQM to the governing equation obtained by Tsiatas [5].

In the present article, the bending analysis of the Kirchhoff plate is carried out in a simplified second gradient elasticity formulation. The variational approach provides the governing differential equation as well as possible boundary conditions of the second gradient elasticity. By setting first or both internal lengths equal to zero, this general case can be simplified to the first gradient or classical elasticity, respectively. In the case of a simply supported rectangular plate, the analytical solution is provided for three boundary value problems dealing with the static, stability and free vibration of the plate.

2 Second strain gradient elasticity

For a linear elastic solid, the potential energy function, W, is assumed to be quadratic function in terms of strain, first-order gradient strain and second-order gradient strain [16]

$$W = W\left(\varepsilon_{ij}, \partial_k \varepsilon_{ij}, \partial_l \partial_k \varepsilon_{ij}\right) \tag{1}$$

while in a compatible situation (defect-free), the elastic strain is a function of gradient of the displacement as

$$\varepsilon_{ij} = \varepsilon_{ji} = \frac{1}{2}\left(u_{j,i} + u_{i,j}\right) \tag{2}$$

while $u_{i,j}$ is the gradient of the displacement field. In the gradient elasticity, the general stress tensors (including the classical and hyper-stress components) are defined as

$$\sigma_{ij} := \frac{\partial W}{\partial \varepsilon_{ij}}, \quad \tau_{ijk} := \frac{\partial W}{\partial \varepsilon_{ij,k}}, \quad \tau_{ijkl} := \frac{\partial W}{\partial \varepsilon_{ij,kl}} \tag{3}$$

while $\varepsilon_{ij,k} = \partial_k \varepsilon_{ij}$ and $\varepsilon_{ij,kl} = \partial_l \partial_k \varepsilon_{ij}$.

A simplified second strain gradient model can be defined in the following form [9]

$$W = \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl} + \frac{1}{2}l^2 C_{ijmn}\varepsilon_{mn,k}\varepsilon_{ij,k} + \frac{1}{2}l'^4 C_{ijmn}\varepsilon_{mn,kl}\varepsilon_{ij,kl}, \tag{4}$$

where l and l' are internal lengths, and C_{ijkl} is the stiffness tensor of an isotropic material

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}\right). \tag{5}$$

where λ and μ are Lame's constants. Thus, the potential energy function for an isotropic material will be

$$W = \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{kk} + \mu\varepsilon_{ij}\varepsilon_{ij} + l^2\left(\frac{1}{2}\lambda\varepsilon_{ii,k}\varepsilon_{jj,k} + \mu\varepsilon_{ij,k}\varepsilon_{ij,k}\right) + l'^4\left(\frac{1}{2}\lambda\varepsilon_{jj,kl}\varepsilon_{ii,kl} + \mu\varepsilon_{ij,kl}\varepsilon_{ij,kl}\right) \tag{6}$$

Consequently, the general stress tensors (3) are simplified to

$$\begin{aligned}\sigma_{ij} &= \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} \\ \tau_{ijk} &= l^2\left(\lambda\delta_{ij}\varepsilon_{mm,k} + 2\mu\varepsilon_{ij,k}\right) \\ \tau_{ijkl} &= l'^4\left(\lambda\delta_{ij}\varepsilon_{mm,kl} + 2\mu\varepsilon_{ij,kl}\right)\end{aligned} \tag{7}$$

For the sake of simplicity, dimension reduction is useful in the gradient elasticity analysis of thin plate structures. Similar to the classical elasticity where Hooke's law can be simplified for plane stress assumption, the general stress tensors in-plane stress can be assumed as

$$\begin{aligned}\sigma_{\alpha\beta} &= \lambda'\delta_{\alpha\beta}\varepsilon_{\gamma\gamma} + 2\mu\varepsilon_{\alpha\beta}, \sigma_{zz} = \lambda\varepsilon_{\gamma\gamma} \quad && \alpha, \beta, \gamma = x, y \\ \tau_{\alpha\beta k} &= l^2\sigma_{\alpha\beta,k}, \tau_{zzk} = l^2\sigma_{zz,k} && k = x, y, z \\ \tau_{\alpha\beta km} &= l'^4\sigma_{\alpha\beta,km}, \tau_{zzkm} = l'^4\sigma_{zz,km} && k, l = x, y, z\end{aligned} \tag{8}$$

while $\lambda' = \nu E/(1-\nu^2)$ and $\mu = E/2(1+\nu)$.

3 Governing equations of the Kirchhoff's plate model

An initially flat thin rectangular plate of constant thickness h, length a and width b is considered (Fig. 1). The plate is made of homogeneous and isotropic elastic material. A lateral load $q = q(x, y)$ is applied on the plate's upper flat surface. Adopting the Kirchhoff's theory of plates, the plate can be geometrically described by its mid-plane occupying the two-dimensional domain Ω bounded by the curve Γ. Following the Kirchhoff's theory, the displacements of the plate are

$$\begin{aligned}u_\alpha(x, y, z) &= -zw_{,\alpha} \quad \alpha = x, y \\ u_z(x, y, z) &= w(x, y)\end{aligned} \tag{9}$$

where u_x, u_y and u_z represent the x, y and z components of the displacement vector, respectively.

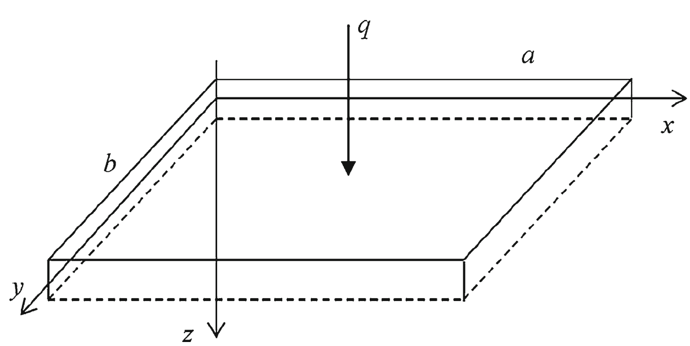

Fig. 1 *Thin plate*: geometrically described by its mid-plane Ω in the (x, y) plane bounded by the curve Γ

Considering Eq. (9), the Kirchhoff's theory assumptions result in the following nonzero strain tensor components

$$\varepsilon_{\alpha\beta} = -z w_{,\alpha\beta} \tag{10}$$

The nonzero stress components can be determined by applying (10) to (8) ("Appendix A").

The governing equation of the plate can be derived through the variational method. In the framework of the second strain gradient elasticity, the first variation of the strain energy is defined by,

$$\delta U = \int_V \left(\sigma_{ij}\delta\varepsilon_{ij} + \tau_{ijk}\delta\varepsilon_{ij,k} + \tau_{ijkl}\delta\varepsilon_{ij,kl}\right) \mathrm{d}v \tag{11}$$

while V is the region occupied by the plate. Using the strain (10) and general stress components (8), the variation of the strain energy for a Kirchhoff's plate is

$$\delta U = -\int_V \left(z\sigma_{\alpha\beta}\delta w_{,\alpha\beta} + zl^2\sigma_{\alpha\beta,\gamma}\delta w_{,\alpha\beta\gamma} + l^2\sigma_{\alpha\beta,z}\delta w_{,\alpha\beta} + zl'^4\sigma_{\alpha\beta,\gamma\kappa}\delta w_{,\alpha\beta\gamma\kappa} + 2l'^4\sigma_{\alpha\beta,\gamma z}\delta w_{,\alpha\beta\gamma}\right) \mathrm{d}v \tag{12}$$

To apply the dimension reduction for the Kirchhoff's theory of plates, definition of the general bending moments is useful. The variation of the strain energy in terms of bending moments is

$$\delta U = -\int_\Omega \left(M_{\alpha\beta}\delta w_{,\alpha\beta} + l^2\bar{N}_{\alpha\beta}\delta w_{,\alpha\beta} + l^2 M_{\alpha\beta,\gamma}\delta w_{,\alpha\beta\gamma} + 2l'^4\bar{N}_{\alpha\beta,\gamma}\delta w_{,\alpha\beta\gamma} + l'^4 M_{\alpha\beta,\gamma\kappa}\delta w_{,\alpha\beta\gamma\kappa}\right) \mathrm{d}a \tag{13}$$

while Ω is the mid-plane of the plate in the (x, y) plane and the general bending moments are defined as

$$\bar{N}_{\alpha\beta} = \int_{-h/2}^{h/2} \sigma_{\alpha\beta,z}\mathrm{d}z, \quad M_{\alpha\beta} = \int_{-h/2}^{h/2} z\sigma_{\alpha\beta}\mathrm{d}z \tag{14}$$

In view of the stress components given in "Appendix A", the general bending moments in term of deflection take the form

$$\bar{N}_{\alpha\beta} = h\left(-\lambda'\delta_{\alpha\beta}w_{,\gamma\gamma} - 2\mu w_{,\alpha\beta}\right), \quad M_{\alpha\beta} = \frac{h^3}{12}\left(-\lambda'\delta_{\alpha\beta}w_{,\gamma\gamma} - 2\mu w_{,\alpha\beta}\right) \tag{15}$$

in which $D = Eh^3/12(1-\nu^2)$. Applying the Green's theorem to (13) yields

$$\begin{aligned}
\delta U = &\int_\Omega \left[-\left(1 - l^2\nabla^2\right)M_{\alpha\beta,\alpha\beta} - \left(l^2 - 2l'^4\nabla^2\right)\bar{N}_{\alpha\beta,\alpha\beta} - l'^4\nabla^4 M_{\alpha\beta,\alpha\beta}\right]\delta w \mathrm{d}a \\
&+ \int_\Gamma \left[\left(1 - l^2\nabla^2\right)M_{\alpha\beta,\alpha} + \left(l^2 - 2l'^4\nabla^2\right)\bar{N}_{\alpha\beta,\alpha} + l'^4\nabla^2 M_{\alpha\kappa,\alpha\beta\kappa}\right] n_\beta \delta w \mathrm{d}s \\
&+ \int_\Gamma \left[-\left(1 - l^2\nabla^2\right)M_{\alpha\beta} - \left(l^2 - 2l'^4\nabla^2\right)\bar{N}_{\alpha\beta} - l'^4 M_{\gamma\kappa,\alpha\beta\gamma\kappa}\right] n_\alpha \delta w_{,\beta}\mathrm{d}s \\
&+ \int_\Gamma \left[\left(-l^2 M_{\alpha\beta,\gamma} - 2l'^4\bar{N}_{\alpha\beta,\gamma} + l'^4 M_{\kappa\gamma,\alpha\beta\kappa}\right) n_\gamma \delta w_{,\alpha\beta}\right]\mathrm{d}s - \int_\Gamma \left[l'^4 M_{\alpha\beta,\gamma\kappa} n_\alpha \delta w_{,\beta\gamma\kappa}\right]\mathrm{d}s
\end{aligned} \tag{16}$$

while $\nabla^2 = \partial_{\gamma\gamma}$, $\nabla^4 = \partial_{\gamma\gamma\kappa\kappa}$.

Moreover, the variation of the work of the external force including the lateral load is,

$$\delta W = \int_\Omega q\delta w \mathrm{d}\Omega \tag{17}$$

The principle of virtual work reads

$$\delta W = \delta U \tag{18}$$

Considering the variation of the strain energy (16) and the variation of the work of the external force (17), principle of virtual work (18) yields

$$\int_{\Omega} \left[-\left(1 - l^2\nabla^2\right) M_{\alpha\beta,\alpha\beta} - \left(l^2 - 2l'^4\nabla^2\right) \bar{N}_{\alpha\beta,\alpha\beta} - l'^4\nabla^4 M_{\alpha\beta,\alpha\beta} - q \right] \delta w \mathrm{d}a = 0 \tag{19a}$$

and

$$\begin{cases} \int_{\Gamma} \left[\left(1 - l^2\nabla^2\right) M_{\alpha\beta,\alpha} + \left(l^2 - 2l'^4\nabla^2\right) \bar{N}_{\alpha\beta,\alpha} + l'^4\nabla^2 M_{\alpha\kappa,\alpha\beta\kappa} \right] n_\beta \delta w \mathrm{d}s = 0 \\ \int_{\Gamma} \left[-\left(1 - l^2\nabla^2\right) M_{\alpha\beta} - \left(l^2 - 2l'^4\nabla^2\right) \bar{N}_{\alpha\beta} - l'^4 M_{\gamma\kappa,\alpha\beta\gamma\kappa} \right] n_\alpha \delta w_{,\beta} \mathrm{d}s = 0 \\ \int_{\Gamma} \left[\left(-l^2 M_{\alpha\beta,\gamma} - 2l'^4 \bar{N}_{\alpha\beta,\gamma} + l'^4 M_{\kappa\gamma,\alpha\beta\kappa}\right) n_\gamma \delta w_{,\alpha\beta} \right] \mathrm{d}s = 0 \\ \int_{\Gamma} \left[l'^4 M_{\alpha\beta,\gamma\kappa} n_\alpha \delta w_{,\beta\gamma\kappa} \right] \mathrm{d}s = 0 \end{cases} \tag{19b}$$

Due to the fundamental lemma of calculus of variation, the variational equation (19a) results in the governing equilibrium equation,

$$-\left(1 - l^2\nabla^2\right) M_{\alpha\beta,\alpha\beta} - \left(l^2 - 2l'^4\nabla^2\right) \bar{N}_{\alpha\beta,\alpha\beta} - l'^4\nabla^4 M_{\alpha\beta,\alpha\beta} = q \tag{20a}$$

Taking into account the definition of the moments (15), the total differential order of the governing differential equation (20a) in terms of the displacement of plate is 8. Therefore, four boundary conditions for each edge of plate are expected. The variational boundary equation (19b), in view of the fundamental lemma of calculus of variation, gives the four expected consistent boundary conditions for the rectangular plate

$$\begin{cases} \left[\left(1 - l^2\nabla^2\right) M_{\alpha\beta,\alpha} + \left(l^2 - 2l'^4\nabla^2\right) \bar{N}_{\alpha\beta,\alpha} + l'^4\nabla^2 M_{\alpha\kappa,\alpha\beta\kappa} \right] n_\beta = 0 & \text{or} \quad \delta w = 0 \\ \left[-\left(1 - l^2\nabla^2\right) M_{\alpha\beta} - \left(l^2 - 2l'^4\nabla^2\right) \bar{N}_{\alpha\beta} - l'^4 M_{\gamma\kappa,\alpha\beta\gamma\kappa} \right] n_\alpha = 0 & \text{or} \quad \delta w_{,\beta} = 0 \\ \left(-l^2 M_{\alpha\beta,\gamma} - 2l'^4 \bar{N}_{\alpha\beta,\gamma} + l'^4 M_{\kappa\gamma,\alpha\beta\kappa}\right) n_\gamma = 0 & \text{or} \quad \delta w_{,\alpha\beta} = 0 \\ M_{\alpha\beta,\gamma\kappa} n_\alpha = 0 & \text{or} \quad \delta w_{,\beta\gamma\kappa} = 0 \end{cases} \tag{20b}$$

These equations list all the possible boundary conditions of the second gradient elastic plate which refer to either prescribed boundary deformations or prescribed boundary actions (in terms of general bending moments). For any specific types of boundaries, appropriate conditions can be selected among the above-mentioned conditions.

In the case of a simply supported plate, the conditions for the boundaries of the plate are selected as

$$\begin{cases} \delta w = 0 \\ \left[-\left(1 - l^2\nabla^2\right) M_{\alpha\beta} - \left(l^2 - 2l'^4\nabla^2\right) \bar{N}_{\alpha\beta} - l'^4 M_{\gamma\kappa,\alpha\beta\gamma\kappa} \right] n_\alpha = 0 \\ \left(-l^2 M_{\alpha\beta,\gamma} - 2l'^4 \bar{N}_{\alpha\beta,\gamma} + l'^4 M_{\kappa\gamma,\alpha\beta\kappa}\right) n_\gamma = 0 \\ M_{\alpha\beta,\gamma\kappa} n_\alpha = 0 \end{cases} \tag{21}$$

while for a rectangular plate, Eq. (21) reach

$$\begin{aligned} &\left\{ \begin{aligned} & w = 0 \\ & -\left(1 - l^2\nabla^2\right) M_{xx} - \left(l^2 - 2l'^4\nabla^2\right) \bar{N}_{xx} - l'^4 M_{\gamma\kappa,xx\gamma\kappa} = 0 \\ & -l^2 M_{xx,x} - 2l'^4 \bar{N}_{xx,x} + l'^4 M_{\kappa x,xx\kappa} = 0 \\ & M_{xx,xx} = 0 \end{aligned} \right\} \quad \text{at} \quad x = 0, a \\ &\left\{ \begin{aligned} & w = 0 \\ & -\left(1 - l^2\nabla^2\right) M_{yy} - \left(l^2 - 2l'^4\nabla^2\right) \bar{N}_{yy} - l'^4 M_{\gamma\kappa,yy\gamma\kappa} = 0 \\ & -l^2 M_{yy,y} - 2l'^4 \bar{N}_{yy,y} + l'^4 M_{\kappa y,yy\kappa} = 0 \\ & M_{yy,yy} = 0 \end{aligned} \right\} \quad \text{at} \quad y = 0, b \end{aligned} \tag{22}$$

Setting $l = l' = 0$ or $l' = 0$, the classical boundary conditions and the first gradient elasticity boundary conditions are achieved, respectively.

In the case of a clamped plate, the boundary conditions are selected as

$$\begin{cases} \delta w = 0 \\ \delta w_{,\beta} = 0 \\ \left(-l^2 M_{\alpha\beta,\gamma} - 2l'^4 \bar{N}_{\alpha\beta,\gamma} + l'^4 M_{\kappa\gamma,\alpha\beta\kappa}\right) n_\gamma = 0 \\ l'^4 M_{\alpha\beta,\gamma\kappa} n_\alpha = 0 \end{cases} \tag{23}$$

which for a rectangular plate are simplified to

$$\begin{aligned} &\left\{\begin{array}{l} w = 0 \\ w_{,x} = 0 \\ -l^2 M_{xx,x} - 2l'^4 \bar{N}_{xx,x} + l'^4 M_{\kappa x,xx\kappa} = 0 \\ M_{xx,xx} = 0 \end{array}\right\} \quad \text{at} \quad x = 0, a \\ &\left\{\begin{array}{l} w = 0 \\ w_{,y} = 0 \\ -l^2 M_{yy,y} - 2l'^4 \bar{N}_{yy,y} + l'^4 M_{\kappa y,yy\kappa} = 0 \\ M_{yy,yy} = 0 \end{array}\right\} \quad \text{at} \quad y = 0, b \end{aligned} \tag{24}$$

Using Eq. (15), the governing differential equation (20a) can be written in term of the deflection in the following form

$$\left(1 + 12\frac{l^2}{h^2}\right)\nabla^4 w - \left(l^2 + 24\frac{l'^4}{h^2}\right)\nabla^6 w + l'^4 \nabla^8 w = \frac{q}{D} \tag{25}$$

where ∇^4, ∇^6 and ∇^8 are given explicitly by

$$\begin{aligned} \nabla^4 &= \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \\ \nabla^6 &= \frac{\partial^6}{\partial x^6} + 3\frac{\partial^6}{\partial x^4 \partial y^2} + 3\frac{\partial^6}{\partial x^2 \partial y^4} + \frac{\partial^6}{\partial y^6} \\ \nabla^8 &= \frac{\partial^8}{\partial x^8} + 4\frac{\partial^8}{\partial x^6 \partial y^2} + 6\frac{\partial^8}{\partial x^4 \partial y^4} + 4\frac{\partial^8}{\partial x^2 \partial y^6} + \frac{\partial^8}{\partial y^8} \end{aligned} \tag{26}$$

and $D = h^3(\lambda' + 2\mu)/12 = Eh^3/[12(1 - \nu^2)]$.

It is observed that for $l = l' = 0$, the Eq. (25) leads to the classical Kirchhoff plate governing equation [1],

$$D\nabla^4 w = q \tag{27}$$

and for $l' = 0$, it leads to the equation for the first strain gradient theory [4],

$$\left(1 + 12\frac{l^2}{h^2}\right) D\nabla^4 w - l^2 D\nabla^6 w = q. \tag{28}$$

Due to different constitutive equations, the above governing equation is different from the one derived by Papargyri-Beskou et al. [3].

For a straightforward stability analysis, considering finite strain will directly lead to the proper governing equation. However, the current static analysis can be applied for stability analysis by augmenting Eq. (25) with the terms originated from the in-plane forces. The effect of the in-plane forces can be modeled similar to the classical elasticity [17]. Consider a plate under the in-plane constant normal compressive (P_x, P_y) and shear forces (P_{xy}). The governing equation of equilibrium (25) for this case takes the form

$$\left(1+12\frac{l^2}{h^2}\right)\nabla^4 w-\left(l^2+24\frac{l'^4}{h^2}\right)\nabla^6 w+l'^4\nabla^8 w+\frac{P_x}{D}\frac{\partial^2 w}{\partial x^2}+2\frac{P_{xy}}{D}\frac{\partial^2 w}{\partial x\partial x}+\frac{P_y}{D}\frac{\partial^2 w}{\partial y^2}=\frac{q}{D} \quad (29)$$

Once the plate is under dynamic loading $q(x, y, t)$, it undergoes deflection $w = w(x, y, t)$, where t denotes time. Similar to the classical elasticity, an inertial force $\rho h\partial^2 w/\partial t^2$ appears in the equilibrium equation, where ρ is the mass per unit volume of the plate [18]. Thus, the governing equation of motion of a second gradient elastic flexural plate takes the form

$$\left(1+12\frac{l^2}{h^2}\right)\nabla^4 w-\left(l^2+24\frac{l'^4}{h^2}\right)\nabla^6 w+l'^4\nabla^8 w+\frac{\rho h}{D}\frac{\partial^2 w}{\partial t^2}=\frac{q(x,y,t)}{D} \quad (30)$$

It should be mentioned that by using the Hamilton' principle, the motion equation for the dynamic analysis of plate can be derived directly.

4 Simply supported rectangular plate

Among the plates with different boundary conditions, simply supported rectangular plate can be solved analytically with the aid of Fourier series as in classical elasticity [1]. In this section, the static, stability and dynamic analyses of the simply supported rectangular plate are presented.

4.1 Static analysis of simply supported rectangular plate

The governing equation (25) and the boundary conditions (22) describe the behavior of a rectangular plate with simply supported boundaries. In this case, a solution is assumed of the form

$$w(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}w_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad (31)$$

which fulfills the boundary conditions (22). The applied static load q can also be expressed in a similar sinusoidal series form,

$$q(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}q_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad (32)$$

while for a general loading, q_{mn} is

$$q_{mn}=\frac{4}{ab}\int_0^b\int_0^a q(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\mathrm{d}x\mathrm{d}y. \quad (33)$$

Substitution of (31, 32) into (25) yields an expression for w_{mn} of the form

$$w_{mn}=q_{mn}/\left\{DA_{mn}^2\left[\left(1+12l^2/h^2\right)+\left(l^2+24l'^4/h^2\right)A_{mn}+l'^4A_{mn}^2\right]\right\} \quad (34)$$

while $A_{mn} = (m\pi/a)^2 + (n\pi/b)^2$. In the case of the classical elasticity, ($l = l' = 0$), the expression is simplified to $w_{mn}^{cl} = q_{mn}/(DA_{mn}^2)$ [1], and in the case of the first gradient elasticity ($l' = 0$), it is reduced to [4]

$$w_{mn}=q_{mn}/\left\{DA_{mn}^2\left[\left(1+12l^2/h^2\right)+l^2A_{mn}\right]\right\} \quad (35)$$

To shed more light on the plate deformation, the second strain gradient elasticity can be written with respect to classical case as

$$\frac{w_{mn}}{w_{mn}^{cl}} = \frac{1}{\left(1+12l^2/h^2\right)+\left(l^2+24l'^4/h^2\right)A_{mn}+l'^4A_{mn}^2} \tag{36}$$

For a square plate ($a = b$) under a simple loading, such as$q(x, y) = q_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$, Eq. (36) yields

$$\frac{w_{mn}}{w_{mn}^{cl}} = \frac{1}{1+12l^2/h^2+2\left(1+24l'^4/l^2h^2\right)(\pi l/a)^2+4\left(\pi l'/a\right)^4} \tag{37}$$

It is observed that increasing the first and second gradient lengths reduces the plate's deflection. Equation (37) depicts the plate's deflection with geometrical quantities (a, h) and for any combination of internal lengths (l, l').

4.2 Stability analysis of simply supported rectangular plate

Equation (29) belongs to the stability analysis of a plate under general in-plane loadings. As an example, consider a plate under in-plane loading along x-axis direction (P_x). Thus, the governing equation of the plate (29) is reduced to

$$\left(1+12\frac{l^2}{h^2}\right)\nabla^4 w-\left(l^2+24\frac{l'^4}{h^2}\right)\nabla^6 w+l'^4\nabla^8 w+\frac{P_x}{D}\frac{\partial^2 w}{\partial x^2}=0 \tag{38}$$

Once again, the solution (31) fulfills the boundary conditions. Thus, for a square plate ($a = b$), substitution of (31) into (38) produces following expression for P_x,

$$P_x = DA_{mn}^2\left[\left(1+12l^2/h^2\right)+\left(l^2+24l'^4/h^2\right)A_{mn}+l'^4A_{mn}^2\right]\Big/(m\pi/a)^2 \tag{39}$$

Minimizing the value of P_x over (m, n), the critical load for a square plate of side a takes the form

$$P_{cr} = 4D\left(\pi/a\right)^4\left[\left(1+12l^2/h^2\right)+2\left(l^2+24l'^4/h^2\right)(\pi/a)^2+4l'^4\left(\pi/a\right)^4\right]\Big/(\pi/a)^2 \tag{40}$$

while this expression (40) is reduced to the classical critical load value for $l = l' = 0$ [17].

4.3 Free vibration of simply supported rectangular plate

The governing equation of motion (30) for the free flexural vibrations of a simply supported rectangular plate reads

$$\left(1+12\frac{l^2}{h^2}\right)\nabla^4 w-\left(l^2+24\frac{l'^4}{h^2}\right)\nabla^6 w+l'^4\nabla^8 w+\frac{\rho h}{D}\frac{\partial^2 w}{\partial t^2}=0 \tag{41}$$

Equation (41) with conditions (22) has a solution of the form

$$w\left(x, y, t\right)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\sin\left(\omega t\right) \tag{42}$$

while ω is the vibrational frequency. Substitution of (42) in motion equation (41) gives the expression for vibrational frequency as

$$\omega^2 = A_{mn}\sqrt{D\left[\left(1+12l^2/h^2\right)+\left(l^2+24l'^4/h^2\right)A_{mn}+l'^4A_{mn}^2\right]/\rho h} \tag{43}$$

The frequency of the classical plate is obtained by substitution of $l = l' = 0$ in Eq. (43) [18].

5 Conclusion

The Kirchhoff's theory of plate is formulated in the second strain gradient elasticity. The potential energy function is assumed to be quadratic function in terms of strain, first-order gradient strain and second-order gradient strain. The governing equation and the possible boundary conditions are derived in a variational approach. The application of the second strain gradient elasticity results in scale sensitivity of the model with first and second gradient lengths. Furthermore, the simply supported and clamped boundary conditions are determined from variational equations. The governing equation is extended to analyze the stability and dynamic problems. The static, stability and free vibration analyses of a micro-plate with simply supported boundaries are carried out analytically. The results depict that the deflection decreases with increasing values of first and second internal lengths. Once the internal lengths are set zero, the solution in the classical elasticity is obtained.

Appendix A

The nonzero stress components of second gradient elasticity for the Kirchhoff's theory of plate are

$$\begin{aligned}
\sigma_{\alpha\beta} &= -\lambda' \delta_{\alpha\beta} z w_{,\gamma\gamma} - 2\mu z w_{,\alpha\beta}, \sigma_{zz} = -\lambda z w_{,\gamma\gamma} && \alpha, \beta, \gamma = x, y \\
\tau_{\alpha\beta k} &= -l^2 \delta_{\alpha\beta} z w_{,\gamma\gamma k} - 2l^2 \mu z w_{,\alpha\beta k}, \tau_{zzk} = -l^2 \lambda z w_{,\gamma\gamma k} && k = x, y, z \\
\tau_{\alpha\beta km} &= -l'^4 \delta_{\alpha\beta} z w_{,\gamma\gamma km} - 2l'^4 \mu z w_{,\alpha\beta km}, \tau_{zzkm} = l'^4 \lambda z w_{,\gamma\gamma km} && k, l = x, y, z
\end{aligned} \tag{A1}$$

References

1. Timoshenko, S., Woinowsky-Krieger, S.: Theory of Plates and Shells, 2nd edn. McGraw-Hill, New York (1959)
2. Papargyri-Beskou, S., Beskos, D.E.: Static, stability and dynamic analysis of gradient elastic flexural Kirchhoff plates. Arch. Appl. Mech. **78**, 625–635 (2008)
3. Papargyri-Beskou, S., Giannakopoulos, A.E., Beskos, D.E.: Variational analysis of gradient elastic flexural plates under static loading. Int. J. Solids Struct. **47**, 2755–2766 (2010)
4. Lazopoulos, K.A.: On bending of strain gradient elastic micro-plates. Mech. Res. Commun. **36**, 777–783 (2009)
5. Tsiatas, G.C.: A new Kirchhoff plate model based on a modified couple stress theory. Int. J. Solids Struct. **46**, 2757–2764 (2009)
6. Wang, B., Zhou, S., Zhao, J., Chen, X.: A size-dependent Kirchhoff micro-plate model based on strain gradient elasticity theory. Eur. J. Mech. A Solids **30**, 517–524 (2011)
7. Ashoori Movassagh, A., Mahmoodi, M.J.: A micro-scale modeling of Kirchhoff plate based on modified strain-gradient elasticity theory. Eur. J. Mech. A Solids **40**, 50–59 (2013)
8. Dell'isola, F., Sciarra, G., vidoli, S.: Generalized Hooke's law for isotropic second gradient materials. Proc. R. Soc. A **465**, 2177–2196 (2009)
9. Lazar, M., Maugin, G.A., Aifantis, E.C.: Dislocations in second strain gradient elasticity. Int. J. Solids Struct **43**, 1787–1817 (2006)
10. Ahmadi, A.R., Farahmand, H., Arabnejad, S.: Static deflection analysis of flexural simply supported sectorial micro-plate using p-version finite-element method. Int. J. Multiscale Comput. Eng. **9**, 193–200 (2011)
11. Ahmadi, A.R., Farahmand, H.: Static deflection analysis of flexural rectangular micro-plate using higher continuity finite-element method. Mech. Ind. **13**, 261–269 (2012)
12. Tsinopoulos, S.V., Polyzos, D., Beskos, D.E.: Static and dynamic BEM analysis of strain gradient elastic solids and structures. Comput. Model. Eng. Sci. **86**, 113–144 (2012)
13. Tsepoura, K.G., Polyzos, D.: Static and harmonic BEM solutions of gradient elasticity problems with axisymmetry. Comput. Mech. **32**, 89–103 (2003)
14. Fischer, P., Klassen, M., Mergheim, J., Steinmann, P., Müller, R.: Isogeometric analysis of 2D gradient elasticity. Comput. Mech. **47**, 325–334 (2011)
15. Wang, X., Wang, F.: Size-dependent dynamic behavior of a microcantilever plate. J. Nanomater. ID 891347 (2012)
16. Mindlin, R.D.: Second gradient of strain and surface-tension in linear elasticity. Int. J. Solids Struct. **1**, 417–438 (1965)
17. Chajes, A.: Principles of Structural Stability Theory. Prentice-Hall, Englewood Cliffs (1975)
18. Graff, K.F.: Wave Motion in Elastic Solids. Ohio State University Press, Columbus (1975)