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Nonlinear bending of size-dependent circular microplates based on the modified couple stress theory

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Abstract The present study proposes a nonclassical Kirchhoff plate model for the axisymmetrically nonlinear bending analysis of circular microplates under uniformly distributed transverse loads. The governing differential equations are derived from the principle of minimum total potential energy based on the modified couple stress theory and von Kármán geometrically nonlinear theory in terms of the deflection and radial membrane force, with only one material length scale parameter to capture the size-dependent behavior. The governing equations are firstly discretized to a set of nonlinear algebraic equations by the orthogonal collocation point method, and then solved numerically by the Newton–Raphson iteration method to obtain the size-dependent solutions for deflections and radial membrane forces. The influences of length scale parameter on the bending behaviors of microplates are discussed in detail for immovable clamped and simply supported edge conditions. The numerical results indicate that the microplates modeled by the modified couple stress theory causes more stiffness than modeled by the classical continuum plate theory, such that for plates with small thickness to material length scale ratio, the difference between the results of these two theories is significantly large, but it becomes decreasing or even diminishing with increasing thickness to length scale ratio.

Keywords Circular microplate · Geometrically nonlinear bending · Modified couple stress theory · Von Kármán theory · Orthogonal collocation point method

1 Introduction

Recently, micron-scale structures, such as deformable beams and plates with extremely small overall dimensions, which are generally of the order of microns or sub-microns, have found tremendous applications in the areas of biosensors, micro-/nano- electromechanical systems, and also atomic force microscopes. In such structures, the size-dependent effect of material makes an important role in the deformation behavior and has been proven experimentally [1,2]. However, classical continuum elasticity, which is a scale-free theory, is inadequate to predict the size effects. Therefore, the utilization of nonclassical continuum theories containing internal material length scale parameters is inevitable. There are various nonclassical continuum theories, which can capture size effects such as strain gradient theories [3,4], nonlocal elasticity theories [5,6], and couple stress theories [7–10]. Among these size-dependent theories, the couple stress theory has been commonly used in the theoretical investigations of deformable microbeams [11–17] and microplates [18–24]. The present research endeavors to use the couple stress theory to analyze the nonlinear bending of size-dependent circular microplates.

The classical couple stress theory, originated by the Cosserat brothers [25], Toupin [7], Mindlin, and Tiersten [8], and Koiter [9], has been developed to describe the size-dependent effects. They used two higher-

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order material length scale parameters in addition to the two classical Lame constants for isotropic elastic material in its constitutive equation. Yang et al. [10] reduced the two independent higher-order material length scale parameters to only one and formed a modified couple stress theory. This feature makes the modified couple stress theory easier to use. Although there exist some doubts about the modified couple stress theory of Yang et al. [10], see, for example, the comments from Lazopoulos [3], it has been developed and extensively used in many aspects such as bending, buckling and post-buckling, and vibration in recent years to investigate the mechanical behavior of the structures at small scale.

Review of the literature indicates that in most of the studies with regard to the modified couple stress theory, a great deal of attention has been focused on the mechanical problems arising in microbeams [11–17]. On the contrary, limited attention appears related to the problem of microplate due to the more complex nature of resulting governing equations, especially when geometric nonlinearity is considered. Tsiatas [18] developed a geometrically linear Kirchhoff plate model in terms of the deflection for the static analysis of isotropic microplates with arbitrary shape. Jomehzadeh [19] presented a variation formulation for free vibration analysis of both rectangular and circular Kirchhoff microplates under the assumption of small deformation, the corresponding analytical solution of natural frequency is obtained. Asghari [20] derived a size-dependent geometrically nonlinear couple stress-based microplate model for uniform flat microplates with arbitrary shapes in terms of the stress resultants and kinematic functions, respectively, but failed to give a numerical result. Taking both bending and stretching deformations into consideration, Ma et al. [21] established a microstructure-dependent nonclassical Mindlin plate model via a variational formulation based on Hamilton's principle and obtained the analytical solutions for the static bending and free vibration problems of a simply supported plate. Ke et al. [22] derived a size-dependent microplate model for the free vibration analysis based on the Mindlin plate theory incorporating the effects of transverse shear deformation, rotary inertia, and size effect. Chen et al. [23] developed a model for the static bending of composite laminated Reddy plates. Reddy et al. [24] proposed a general third-order plate theory with microstructure-dependent length scale parameter and the bending–extensional coupling through the von Kármán nonlinear strains.

In most practical circumstances, microscale plate-like elements commonly sustain relatively large deformations where the deflections are of order of the plate thickness. The geometrically induced nonlinearity causes the mechanical and vibrational properties of microscale devices to be changed significantly which has experimentally been observed [16,20]. The infinitesimal deformation and scale-free model then is invalid. Hence, studying the geometric nonlinearity considering the effect of microscale is evidently essential for a microplate. Although many studies related to the nonlinearity of macroscale plates using classical Von Kármán plate models have been published [26,27], the size-dependent nonlinear analysis for microplates based on the modified couple stress theory seems to be insufficient.

For such micro-objects as thin-walled microplate and microshell, it is quite natural and reasonable to adopt the two-dimensional models of the mechanics of structures. As mentioned above, the known plate and shell models are related to the names of Kirchhoff, Love, Cosserat, Von Kármán, Timoshenko, Reissner, Mindlin, Koiter, and Reddy among others. For instance, in the literature are known various models of plates [28] and shells [29] related to the Cosserat model. Let us mention here the recent review and bibliography in Ref. [30], where many references to other papers can be found.

In this study, the axisymmetrically nonlinear bending behavior of a microplate is studied. The plate material is assumed to be size-dependent according to the modified couple stress theory, and the deformation is nonlinear in terms of Von Kármán theory. The governing differential equations of circular microplates are formulated from the principle of minimum total potential energy and are discretized by the orthogonal collocation point method. This treatment reduces the governing equations to a pair of nonlinear algebraic equations, which are accomplished numerically by the Newton–Raphson iteration method. Several numerical results for circular microplates with immovably clamped and simply supported boundary conditions are presented in both tabular and graphical forms to illustrate the size-dependent bending behavior.

2 Governing equations of microplates

The microscale circular plate under consideration is treated as an elastica and is assumed to be thin, and deformable. It has a radius a, and constant thickness h. The cylindrical coordinate (r, θ, z) is chosen such that the origin of the coordinate is at the center of the middle surface of the plate, which is coincide with the $r\theta$-plane. It is assumed that the plate is bent under the distributed transverse load $q(r)$, the deformation complies with Von Kármán theory, and the material obeys the modified couple stress theory.

2.1 Modified couple stress theory

With reference to the modified couple stress theory presented by Yang et al. [10], the strain energy $\prod$ in an isotropic linear elastic material occupying region Ω can be written as

$$\prod = \frac{1}{2} \int_{\Omega} (\boldsymbol{\sigma} : \boldsymbol{\varepsilon} + \boldsymbol{m} : \boldsymbol{\chi}) \mathrm{d}\Omega \tag{1}$$

where the strain tensor $\boldsymbol{\varepsilon}$, the Cauchy (classical) stress tensor $\boldsymbol{\sigma}$ (conjugated to $\boldsymbol{\varepsilon}$), the symmetric curvature tensor $\boldsymbol{\chi}$, and the deviatoric part of the couple stress tensor $\boldsymbol{m}$ (conjugated to $\boldsymbol{\chi}$), are, respectively, defined by

$$\boldsymbol{\varepsilon} = \frac{1}{2} \left[\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{T}} \right] \tag{2}$$

$$\boldsymbol{\chi} = \frac{1}{2} \left[\nabla \boldsymbol{\theta} + (\nabla \boldsymbol{\theta})^{\mathrm{T}} \right] \tag{3}$$

$$\boldsymbol{\sigma} = \lambda \mathrm{tr}\,(\boldsymbol{\varepsilon}) \boldsymbol{I} + 2\mu \boldsymbol{\varepsilon} \tag{4}$$

$$\boldsymbol{m} = 2l^2 \mu \boldsymbol{\chi} \tag{5}$$

with λ and μ being Lame's constants, and l a material length scale parameter, which is mathematically the square root of the ratio of the modulus of curvature to the modulus of shear and is physically a property characterizing the effect of couple stress [15], $\boldsymbol{I}$ the unit tensor. The rotation vector $\boldsymbol{\theta}$ is related to the displacement vector $\boldsymbol{u}$ by

$$\boldsymbol{\theta} = \frac{1}{2} \mathrm{curl} \boldsymbol{u} \tag{6}$$

The two main advantages of the modified couple stress theory over the classical couple stress theory are the inclusion of a symmetric couple stress tensor and the involvement of only one length scale parameter in addition to the conventional Lame's constants [11,15].

The state of plane stress is described by the stress (4) and couple stress (5) tensors, which, after the appropriate replacement of the Lame's constants by the modulus of elasticity E and the Poisson's ratio ν, take the following form [18]

$$\sigma_{\alpha\beta} = \frac{E}{1-\nu^2} \left[\nu \varepsilon_{kk} \delta_{\alpha\beta} + (1-\nu)\, \varepsilon_{\alpha\beta} \right], m_{\alpha\beta} = 2Gl^2 \chi_{\alpha\beta} \tag{7}$$

where the modulus of elasticity E and Poisson ratio ν have replaced the Lame's constants λ and μ using the relations $\lambda = E\nu/(1+\nu)(1-2\nu)$ and $\mu = G = E/2(1+\nu)$. Here, G is the shear modulus, δ_{ij} denotes the Kronecker delta.

2.2 Strain–displacement relations and stress resultants

For the axisymmetric deformation of a Kirchhoff circular plate, the displacements (u_r, u_θ, u_z) can be expressed in terms of the displacements of a point on the middle surface of the plate as [26]

$$u_r\,(r,z) = u\,(r) - z w_{,r}\,(r)\,, u_\theta\,(r,z) = 0, u_z\,(r,z) = w\,(r) \tag{8}$$

where $r \in [0, a]$, $u(r)$ and $w(r)$ are the radial and transverse displacements of the point on the middle surface of the plate, respectively. Hereinbelow, a subscript comma denotes the differentiation with respect to radial coordinate.

The nonzero Von Kármán nonlinear strain components for large axisymmetric deformation of circular plates take the form [27]

$$\varepsilon_{rr}\,(r,z) = u_{,r} + \frac{1}{2} w_{,r}^2 - z w_{,rr}, \quad \varepsilon_{\theta\theta}\,(r,z) = \frac{u}{r} - z\frac{1}{r} w_{,r} \tag{9}$$

In view of the displacement field in Eq. (8), the only nonvanishing component of the rotation vector and the corresponding symmetric curvature tensor can be obtained from Eqs. (6) and (3) and changed in polar coordinate as

$$\theta_\theta = -w_{,r}, \chi_{r\theta} = \frac{1}{2} \left(\frac{1}{r} w_{,r} - w_{,rr} \right) \tag{10}$$

According to the constitutive Eq. (7), the membrane forces N_{rr} and $N_{\theta\theta}$, bending moments M_{rr} and $M_{\theta\theta}$, and couple moment $Y_{r\theta}$ are defined, respectively, by

$$(N_{rr}, N_{\theta\theta}, M_{rr}, M_{\theta\theta}, Y_{r\theta}) = \int_{-h/2}^{h/2} (\sigma_{rr}, \sigma_{\theta\theta}, z\sigma_{rr}, z\sigma_{\theta\theta}, m_{r\theta})\mathrm{d}z \tag{11}$$

which in terms of displacements are written

$$N_{rr} = C\left(u_{,r} + \frac{1}{2}w_{,r}^2 + \nu\frac{u}{r}\right), N_{\theta\theta} = C\left[\nu\left(u_{,r} + \frac{1}{2}w_{,r}^2\right) + \frac{u}{r}\right] \tag{12}$$

$$M_{rr} = -D\left(w_{,rr} + \nu\frac{1}{r}w_{,r}\right), M_{\theta\theta} = -D\left(\frac{1}{r}w_{,r} + \nu w_{,rr}\right) \tag{13}$$

$$Y_{r\theta} = -Gl^2h\left(w_{,rr} - \frac{1}{r}w_{,r}\right) \tag{14}$$

in which $C = Eh/(1-\nu^2)$, $D = Eh^3/[12(1-\nu^2)]$ signify the extensional rigidity, and flexural rigidity of the microplate, respectively.

2.3 Nonlinear governing equation

The equilibrium equations as well as the related boundary conditions of a microplate can be formulated through the application of the principle of stationary (minimum) total potential energy, which is symbolically written, in the absence of body force and body couple, as

$$\delta\left(\prod + V_q\right) = 0 \tag{15}$$

where $\prod$ is the strain energy in the plate, and V_q is potential energy of the external transverse distributed loading of the plate which is equal to

$$V_q = -\int_0^{2\pi}\int_0^a qwr\mathrm{d}\theta\mathrm{d}r = -2\pi\int_0^a qwr\mathrm{d}r \tag{16}$$

with its variation reads

$$\delta V_q = -2\pi\int_0^a rq\delta w\mathrm{d}r \tag{17}$$

From Eqs. (1) and (7), and Eqs. (9)–(14), the total strain energy in the plate takes the form

$$\begin{aligned}\prod &= \frac{1}{2}\int_V (\sigma_{rr}\varepsilon_{rr} + \sigma_{\theta\theta}\varepsilon_{\theta\theta} + 2m_{r\theta}\chi_{r\theta})\mathrm{d}V \\ &= \pi\int_0^a \left[N_{rr}\left(u_{,r} + \frac{1}{2}w_{,r}^2\right) + N_{\theta\theta}\frac{u}{r}\right]r\mathrm{d}r - \pi\int_0^a \left(M_{rr}w_{,rr} + M_{\theta\theta}\frac{1}{r}w_{,r}\right)r\mathrm{d}r \\ &\quad - \pi\int_0^a Y_{r\theta}\left(w_{,rr} - \frac{1}{r}w_{,r}\right)r\mathrm{d}r\end{aligned} \tag{18}$$

The three integrations expressed in Eq. (18) imply the classical membrane strain energy due to stretching and strain energy due to bending, and the couple stress term, respectively.

Carrying out variation for $\prod$ and integrating by parts with respect to r, yield, after grouping terms by δw and δu, the following expression

$$\frac{1}{2\pi}\delta\prod = \int_0^a \left[N_{\theta\theta} - (rN_{rr})_{,r}\right]\delta u \mathrm{d}r + \int_0^a \left[-(r\overline{m}_{rr})_{,rr} + \overline{m}_{r\theta,r} - \left(rN_{rr}w_{,r}\right)_{,r} - rq\right]\delta w \mathrm{d}r$$
$$+ rQ_r\delta w|_{r=0,a} - r\,\overline{m}_{rr}\delta w_{,r}\Big|_{r=0,a} + rN_{rr}\delta u|_{r=0,a} \tag{19}$$

here

$$\overline{m}_{rr} = M_{rr} + Y_{r\theta}, \overline{m}_{r\theta} = M_{\theta\theta} - Y_{r\theta}, Q_r = N_{rr}w_{,r} + \overline{m}_{rr,r} + \frac{1}{r}\left(\overline{m}_{rr} - \overline{m}_{r\theta}\right)$$

By applying Eqs. (17) and (19) to Eq. (15) and also by noting the arbitrariness of δu and δw, the following nonlinear ordinary differential equations

$$N_{\theta\theta} - (rN_{rr})_{,r} = 0 \tag{20}$$
$$(r\overline{m}_{rr})_{,rr} - \overline{m}_{r\theta,r} + \left(rN_{rr}w_{,r}\right)_{,r} + rq = 0 \tag{21}$$

together with the boundary conditions

$$rN_{rr}|_{r=0,a} = 0, \text{ or } \delta u|_{r=0,a} = 0 \tag{22}$$
$$rQ_r|_{r=0,a} = 0, \text{ or } \delta w|_{r=0,a} = 0 \tag{23}$$
$$r\,\overline{m}_{rr}|_{r=0,a} = 0, \text{ or } \delta w_{,r}\Big|_{r=0,a} = 0 \tag{24}$$

can be obtained and can further be expressed as usual in terms of w and N_{rr} by considering Eqs. (12)–(14) as

$$r\left[\frac{1}{r}(r^2N_{rr})_{,r}\right]_{,r} = -\frac{1}{2}Ehw_{,r}^2 \tag{25}$$
$$(D + Gl^2h)\nabla^4 w - \frac{1}{r}\left(rN_{rr}w_{,r}\right)_{,r} = q \tag{26}$$

Here, ∇^4 is an ordinary differential operator defined by

$$\nabla^4 = \frac{\mathrm{d}^4}{\mathrm{d}r^4} + \frac{2}{r}\frac{\mathrm{d}^3}{\mathrm{d}r^3} - \frac{1}{r^2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r^3}\frac{\mathrm{d}}{\mathrm{d}r}$$

For the purpose of convenience, the following coordinate transformations and nondimensional notations are introduced

$$R = \frac{r}{a}, W = \frac{w}{h}, S = \frac{arN_{rr}}{Eh^3}, Q = \frac{a^4q}{Eh^4}, \kappa = \frac{Gl^2h}{D} = 6\,(1-\nu)\,\frac{l^2}{h^2}$$

With these quantities, the field equations can be transformed into the following nondimensional forms

$$R\left[\frac{1}{R}(RS)_{,R}\right]_{,R} = -\frac{1}{2}W_{,R}^2 \tag{27}$$
$$\frac{1+\kappa}{12(1-\nu^2)}\nabla^4 W - \frac{1}{R}(SW_{,R})_{,R} = Q \tag{28}$$

The dimensionless boundary conditions, take a circular plate with immovably supported outer edge for example, require that

$$W_{,R}\,(0) = 0, S\,(0) = 0, \lim_{R\to 0}\left(W_{,RRR} + \frac{1}{R}W_{,RR} - \frac{1}{R^2}W_{,R}\right) = 0 \tag{29}$$
$$W\,(1) = 0, \zeta\,(1+\kappa)\,W_{,RR}\,(1) + (\nu-\kappa)\,W_{,R}\,(1) = 0, S_{,R}(1) - \nu S(1) = 0 \tag{30}$$

The boundary conditions for simply supported and clamped edges are obtained by taking the parameter $\zeta = 1$ and $\zeta = 0$, respectively.

There exists a singularity in the last condition of Eq. (29) when R tends to zero, noting that the combination of the second and third terms in the left-hand side of this condition takes the indefinite form 0/0, so to avoid this singularity, L'Hospitale's rule is introduced [31]

$$\lim_{R\to 0}\left(W_{,RRR}+\frac{1}{R}W_{,RR}-\frac{1}{R^2}W_{,R}\right)=\lim_{R\to 0}\left(W_{,RRR}+\frac{RW_{,RR}-W_{,R}}{R^2}\right)=\frac{3}{2}W_{,RRR}=0 \tag{31}$$

The nonclassical governing Eqs. (27)–(30) characterize an ordinary differential equation two-point boundary value problem, which is complicated due to the nonlinearity and coupling. An exact solution is at present unknown. In what follows, the nonlinear bending problem is studied by the orthogonal collocation point method [26] and Newton–Raphson iteration method [32].

3 Method of solution

The governing equations are firstly discretized to a set of nonlinear algebraic equations by the orthogonal collocation point method. The N collocation points ρ_i $(i=1,2,\ldots,N)$ are taken at the zeros of Chebyshev polynomial

$$\rho_i=\frac{1}{2}\left\{1+\cos\left[\frac{(2i-1)\,\pi}{2N}\right]\right\},\quad i=1,2,\ldots,N \tag{32}$$

Within the domain of definition (0,1), the dimensionless deflection and radial membrane force are expanded as power series in ρ

$$W(\rho)=\sum_{j=1}^{N+4}\rho^{j-1}W_j,\; S(\rho)=\sum_{j=1}^{N+2}\rho^{j-1}S_j \tag{33}$$

in terms of the unknown coefficients W_j and S_j. Here, W and S are expanded in polynomials of $N+4$ and $N+2$ terms, since these have to satisfy, respectively, 4 and 2 boundary conditions.

Substitution of Eq. (33) into Eqs. (27)–(31) yields the following $2N+6$ quadratic nonlinear algebraic collocation equations

$$\sum_{j=1}^{N+2}\left(j^2-2j\right)\rho_i^{j-2}S_j+\frac{1}{2}\sum_{j=1}^{N+4}\sum_{k=1}^{N+4}(j-1)\,(k-1)\,\rho_i^{j+k-4}W_jW_k=0,\quad i=1,2,\ldots,N \tag{34}$$

$$\begin{aligned}&\frac{1+\kappa}{12(1-\nu^2)}\sum_{j=1}^{N+4}(j-1)^2\,(j-3)^2\,\rho_i^{j-5}W_j\\&-\sum_{j=1}^{N+4}\sum_{k=1}^{N+2}(j-1)\,(j+k-3)\,\rho_i^{j+k-5}W_jS_k-Q=0\end{aligned},\quad i=1,2,\ldots,N \tag{35}$$

$$\sum_{j=1}^{N+4}W_j=0,\; W_2=0,\; S_1=0 \tag{36}$$

$$\sum_{j=1}^{N+4}(j-1)\left[\zeta\,(1+\kappa)\,(j-2)+(\nu-\kappa)\right]W_j=0 \tag{37}$$

$$\sum_{j=1}^{N+2}(j-1-\nu)\,S_j=0,\; W_4=0 \tag{38}$$

The above-discretized equations are solved iteratively by the Newton–Raphson method. The load is applied incrementally in small steps, and for a given value, the iteration procedures are to be continued until the required accuracy is reached.

4 Numerical results and discussions

On the basis of the preceding analyses, detailed studies about the size-dependent nonlinear bending are carried out for thin circular microplates with the outer edge clamped or simply supported. For each loading increment step, the numerical iteration lasts until the error norm at successive iterations becomes less than 10^{-6}, then continue the calculation of the next step. To facilitate illustrating the microstructural effect, the thickness h of the plate takes to be equal to a fraction of the material length scale parameter l.

First of all, a convergence study to decide the choice of the number of collocation points N is presented in Table 1. In this regard, the dimensionless central deflection of a immovably clamped microplate subjected to a uniform load $Q=25$ with varying number of collocation points in the numerical procedure is checked. It is seen that the results become closer to each other as N increases and those with $N = 10$ and 12 are identical. Hence, the collocation point for convergence and acceptable accuracy of the results is selected to be $N = 12$ in all subsequent calculations.

Since there exists no nonclassical numerical result, one has to resort to the classical solution for comparison. The dimensionless nonlinear central deflection $W(0)$ under six values of uniform transverse pressure load Q is presented in Table 2 for a immovably clamped circular microplate with various cases of including or not including the length scale effect, and is compared with the classical solution obtained in Ref. [26]. The agreement is very well. It can be inferred that the effects of the material length scale parameter are to make the plate behave stiffer, and the nonclassical microplate theory predicts the defection closer to that of the classical plate theory when the thickness to length scale ratio h/l becomes larger. Such a variable tendency has been observed in the bending problem of a beam [15] and will be quantitatively shown in what follows.

The nondimensional central deflection $W(0)$ and radial membrane force at the outer edge $S(1)$ of a immoveable clamped microplate are plotted against the transverse pressure load Q in Fig. 1 as a function of the thickness to material length scale ratio h/l. The curves can be considered as the bending equilibrium path of the plate. For a given value of h/l, $W(0)$ and $S(1)$ increase with the increasing of Q, and vice versa. It can be found that the thickness to material length scale ratio h/l influences the bending equilibrium behavior of the microplate appreciably, and the modified couple stress theory models the plates stiffer than does the classical plate theory. The influence of the thickness to length scale ratio on both central deflection and radial membrane force is pronounced for small values, but less pronounced or even negligible for large ones, keeping all other plate parameters fixed.

Numerical simulations are now carried out for a circular microplate subjected to two outer edge constraints for six sets of thickness to material length scale ratio. The dimensionless deflection of a microplate under uniformly distributed load $Q = 1.0$ is computed by the present nonlinear nonclassical model and is depicted in Fig. 2. It comes to a conclusion that the classical plate theory predicts a large deflection than the modified couple stress theory, both for a clamped and for a simply supported circular microplate. It can also be concluded

Table 1 Convergence of the central deflection for a clamped circular microplate with $\nu = 0.3$ under uniform load $Q = 25$

N	$h/l = 15$	$h/l = 10$	$h/l = 5$	$h/l = 4$	$h/l = 3$	$h/l = 2$	$h/l = 1$
6	1.64562	1.63956	1.60655	1.58154	1.52716	1.37344	0.77190
7	1.64634	1.64024	1.60707	1.58197	1.52745	1.37354	0.77189
8	1.64642	1.64032	1.60712	1.58201	1.52747	1.37355	0.77189
10	1.64643	1.64033	1.60713	1.58201	1.52748	1.37355	0.77189
12	1.64643	1.64033	1.60713	1.58201	1.52748	1.37355	0.77189

Table 2 Effect of the thickness to length scale ratio on the dimensionless nonlinear central deflection under various transverse load for a clamped circular microplate with $\nu = 0.3$

Q	Classical results		$h/l = 15$	$h/l = 10$	$h/l = 5$	$h/l = 4$	$h/l = 3$	$h/l = 2$	$h/l = 1$
5	0.680†	0.6796	0.6725	0.6638	0.6187	0.5874	0.5269	0.3992	0.1636
10	1.052†	1.0520	1.3045	1.0370	0.9929	0.9606	0.8937	0.7286	0.3245
15	1.302†	1.3021	1.2962	1.2888	1.2489	1.2192	1.1560	0.9882	0.4805
20	1.494†	1.4938	1.4885	1.4818	1.4456	1.4184	1.3598	1.1980	0.6300
25	1.651†	1.6513	1.6464	1.6403	1.6071	1.5820	1.5275	1.3736	0.7719
30	1.786†	1.7862	1.7817	1.7760	1.7452	1.7219	1.6709	1.5246	0.9060

†Values that are taken from Ref. [26]

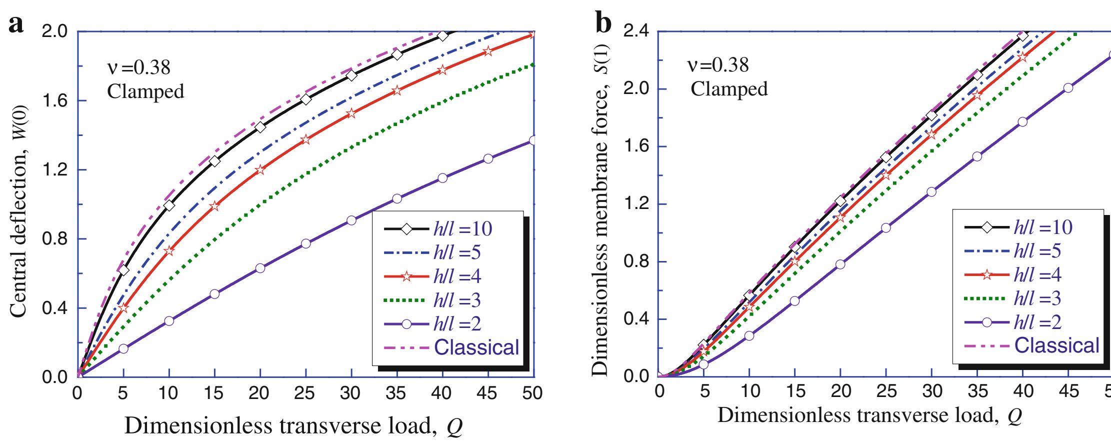

Fig. 1 Characteristic curves between the uniformly distributed transverse loads and **a** Central deflections, and **b** radial membrane force along the outer edge for a clamped circular microplate under various thickness to length scale ratios with $\nu = 0.38$

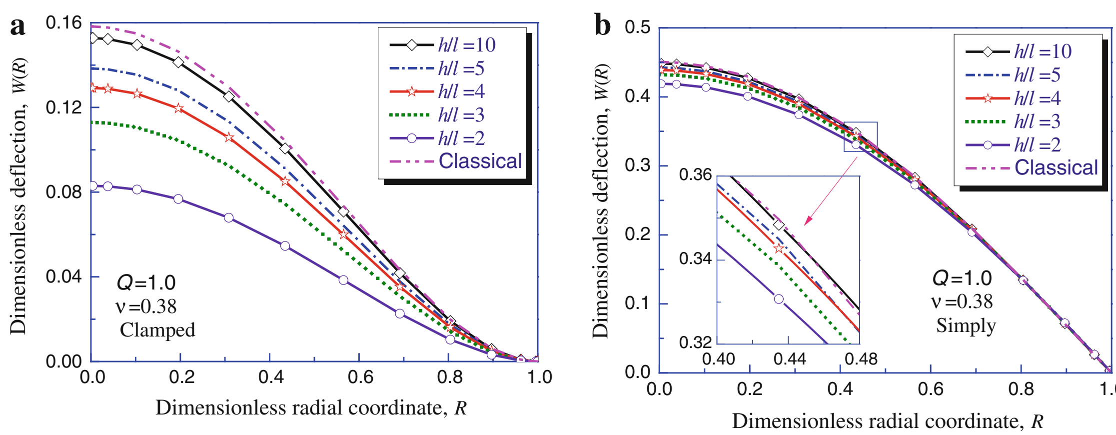

Fig. 2 Deflection curves under various thickness to length scale ratios for **a** Clamped and **b** simply supported circular microplate with $\nu = 0.38$ and $Q = 1.0$

that the thickness to length scale ratio nearly has small effect on the curve shape for the simply supported microplate, but it is relatively large for the clamped one.

Figure 3 is plotted for the relationship of the dimensionless radial membrane force versus radial coordinate R of a plate under some specific values of thickness to length scale ratio h/l. It can be seen that, both for the two boundary conditions, the radial membrane force is increased nonlinearly along the radial position. The radial membrane forces estimated by the proposed model are always smaller than those by the classical theory. Also, the microscale effect influences the clamped plate larger compared to the simply supported one, and the difference between the nonclassical and classical model reduces as an increasing thickness to length scale ratio, indicating that the size effect is only significant for small values of h/l. This conclusion is akin to what has been observed in the deflection curve plotted in Fig. 2.

From the above analyses, one sees that both for the deflection and membrane force, the difference between the results calculated by the current modified couple stress theory and that by the classical plate theory is pronounced and the size effect must be taken into account only when the thickness to length scale ratio is small, the value of present model approaches that of the classical model with the increase in the thickness to length scale ratio. For clamped and simply supported microscale plate under uniformly distributed transverse load, such a variable tendency is shown in Fig. 4, which delineates how the nonlinear maximum central deflection predicted by the classical and nonclassical models changes with the thickness to length scale ratio. It can be observed that for larger values of h/l, the results predicted by the two theories are approximately identical. It is also notable that for all values of the thickness to length scale ratios h/l, the central deflections

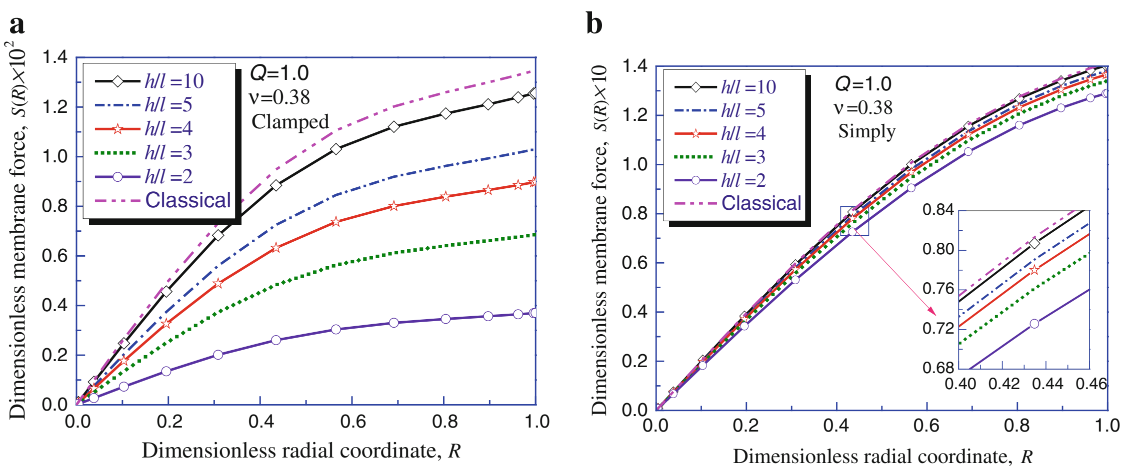

Fig. 3 Variation in the radial membrane force with the radial position under various thickness to length scale ratios for **a** Clamped and **b** simply supported circular microplate with $\nu = 0.38$ and $Q = 1.0$

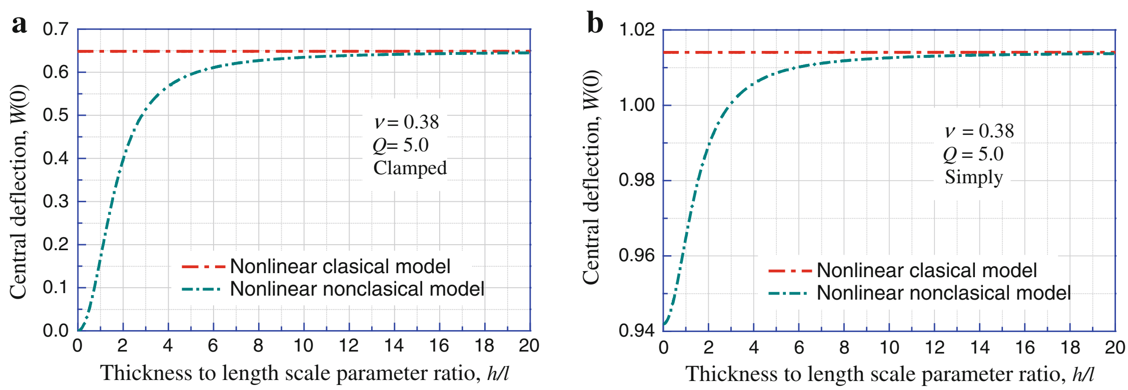

Fig. 4 Dimensionless central deflection versus thickness to length scale ratio for **a** Clamped and **b** simply supported circular microplate with $\nu = 0.38$ and $Q = 5.0$

predicted by the modified couple stress theory are smaller than those by the classical theory, which reveals again that the modified couple stress-theory-based plate is stiffer than the classical-theory-based one.

5 Conclusions

The size-dependent geometrically nonlinear mathematical model for bending of a microplate is formulated by the principle of minimum total potential energy based on the modified couple stress theory and von Kármán nonlinear theory. The model is size-dependent with an additional material length scale parameter to capture the size effect. The orthogonal collocation point method in conjunction with the Newton–Raphson iteration method is introduced to solve the nonlinear governing differential equations.

The intrinsic size dependence of the material increases the stiffness and hence decreases the deflection and membrane force of the microplate. However, the size effect is significant only when the thickness to material length scale ratio is relatively small, it diminishes as the ratio increases. It is also observed that the size effect on a simply supported plate is less sensible than a clamped one.

The current established governing equations will be reduced to the classical Von Kármán nonlinear bending theory for a circular plate when $l = 0$ (i.e. $\kappa = 0$).

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