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**ORIGINAL**

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# New exact solution of Euler's equations (*rigid body dynamics*) in the case of rotation over the fixed point

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**Abstract** A new exact solution of Euler's equations (rigid body dynamics) is presented here. All the components of angular velocity of rigid body for such a solution differ from both the cases of symmetric rigid rotor (*which has two equal moments of inertia: Lagrange's or Kovalevskaya's case*), and from the *Euler's case* when all the applied torques are zero, or from other well-known particular cases. The key features are the next: the center of mass of rigid body is assumed to be located at meridional plane along the main principal axis of inertia of rigid body, besides, the principal moments of inertia are assumed to satisfy to a simple algebraic equality. Also, there is a restriction at choosing of initial conditions. Such a solution is also proved to satisfy to Euler–Poinsot equations, including invariants of motion and additional Euler's invariant (*square of the vector of angular momentum is a constant*). So, such a solution is a generalization of Euler's case.

**Keywords** Euler's equations (rigid body dynamics) · Poinsot's equations · Applied torques · The principal moment of inertia · Principal axis

## 1 Introduction, equations of motion

Euler's equations (dynamics of a rigid body rotation) are known to be one of the famous problems in classical mechanics, besides we should especially note that a lot of great scientists have been trying to solve such a problem during last 300 years.

Despite the fact that initial system of ODE has a simple presentation, only a few exact solutions have been obtained until up to now [1–4]:

– The case of symmetric rigid rotor {two principal moments of inertia are equal to each other: 1) *Lagrange's* case, or 2) *Kovalevskaya's* case};
– The *Euler's* case when all the applied torques are zero (*torque-free precession of the rotation axis of rigid rotor*);
– other well-known but particular cases [5].

Let us consider the system of ordinary differential equations for the dynamics of a rigid body rotation, at given initial conditions. In accordance with [1–3], Euler's equations describe the rotation of a rigid body in a frame of reference fixed in the rotating body for the case of rotation over the fixed point as below:

$$\begin{cases} I_1 \frac{d\Omega_1}{dt} + (I_3 - I_2) \cdot \Omega_2 \cdot \Omega_3 = P\,(\gamma_2 c - \gamma_3 b)\,, \\ I_2 \frac{d\Omega_2}{dt} + (I_1 - I_3) \cdot \Omega_3 \cdot \Omega_1 = P\,(\gamma_3 a - \gamma_1 c)\,, \\ I_3 \frac{d\Omega_3}{dt} + (I_2 - I_1) \cdot \Omega_1 \cdot \Omega_2 = P\,(\gamma_1 b - \gamma_2 a)\,, \end{cases} \tag{1.1}$$

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where $I_i \neq 0$ are the principal moments of inertia ($i$ = 1, 2, 3) and $\Omega_i$ are the components of the *angular velocity vector* along the proper principal axis; $\gamma_i$ are the components of the weight of mass P and a, b, c are the appropriate coordinates of the center of mass in a frame of reference fixed in the rotating body (*in regard to the absolute system of coordinates* X, Y, Z).
Poinsot's equations for the components of the weight in a frame of reference fixed in the rotating body (*in regard to the absolute system of coordinates* X, Y, Z) should be presented as below [1–3]:

$$\begin{cases} \frac{d\gamma_1}{dt} = \Omega_3\gamma_2 - \Omega_2\gamma_3, \\ \frac{d\gamma_2}{dt} = \Omega_1\gamma_3 - \Omega_3\gamma_1, \\ \frac{d\gamma_3}{dt} = \Omega_2\gamma_1 - \Omega_1\gamma_2, \end{cases} \tag{1.2}$$

besides, we should present the invariants of motion (*integrals of motion*) as below

$$\begin{cases} \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, \\ I_1 \cdot \Omega_1 \cdot \gamma_1 + I_2 \cdot \Omega_2 \cdot \gamma_2 + I_3 \cdot \Omega_3 \cdot \gamma_3 = \text{ const } = C_0, \\ \frac{1}{2}\left(I_1 \cdot \Omega_1^2 + I_2 \cdot \Omega_2^2 + I_3 \cdot \Omega_3^2\right) + P(a\gamma_1 + b\gamma_2 + c\gamma_3) = \text{ const } = C_1. \end{cases} \tag{1.3}$$

## 2 Euler's invariant of motion (square of the vector of angular momentum)

As in *Euler's case* [1], we assume that the proper invariant of motion (*integral of motion*) is valid for the system of Eqs. (1.1)–(1.2) as below:

$$I_1^2 \cdot \Omega_1^2 + I_2^2 \cdot \Omega_2^2 + I_3^2 \cdot \Omega_3^2 = C_0^2 \tag{2.1}$$

It means that the scalar square of the vector of angular momentum is equal to the square of constant $C_0 \neq 0$.
In such a case, we obtain from the 1st and 2nd equations of system (1.3) and (2.1):

$$\gamma_3 = \frac{I_3 \cdot \Omega_3}{C_0}, \qquad \gamma_2 = \frac{I_2 \cdot \Omega_2}{C_0}, \qquad \gamma_1 = \frac{I_1 \cdot \Omega_1}{C_0} \tag{2.2}$$

Then, by the proper linear combination of the 3rd equation of system (1.3), and equalities (2.1), (2.2), we obtain the equality below

$$a \cdot I_1 \cdot \Omega_1 = \left\{ C_2 + \left[ \frac{(I_2 - I_1) \cdot I_2 \cdot \Omega_2^2 - (I_1 - I_3) \cdot I_3 \cdot \Omega_3^2}{C_3} \right] - b \cdot I_2 \cdot \Omega_2 - c \cdot I_3 \cdot \Omega_3 \right\} \tag{2.3}$$

where we designate (*just for simplicity of presentation of* Eq. (2.3)):

$$C_2 = C_0 \cdot \left( \frac{C_1}{P} - \frac{C_0^2}{2P \cdot I_1} \right), \qquad C_3 = \left( \frac{2P \cdot I_1}{C_0} \right).$$

Let us search for solutions, which should satisfy to the invariant (2.3) above, which is the linear combination of invariant (1.3), Euler's invariant (2.1) and (2.2). Euler's classical solution of Euler–Poinsot equations $a = b = c = 0$ is a simplest case of (2.3).

## 3 Exact solution, the case a = 0

Let us choose ($I_2 > I_1 > I_3$, $c \neq 0$)

$$\left(\frac{b}{c}\right)^2 = \frac{(I_2 - I_1)I_3}{(I_1 - I_3)I_2} \tag{3.1}$$

besides, let us choose $a = 0$, $C_2 = 0$ in (2.3). The condition $C_2 = 0$ also means the choosing as below:

$$C_0^2 = 2I_1 \cdot C_1$$

In such a case, Eq. (2.3) could be represented as below:

$$bI_2\Omega_2 + cI_3\Omega_3 = \left(\sqrt{\frac{(I_2 - I_1)I_2}{C_3}} \cdot \Omega_2 + \sqrt{\frac{(I_1 - I_3)I_3}{C_3}} \cdot \Omega_3\right) \cdot \left(\sqrt{\frac{(I_2 - I_1)I_2}{C_3}} \cdot \Omega_2 - \sqrt{\frac{(I_1 - I_3)I_3}{C_3}} \cdot \Omega_3\right),$$

thus, the equality (2.3) is valid under conditions (3.1), and the last algebraic equation could be reduced as below:

$$\sqrt{\frac{(I_2 - I_1)I_2}{C_3}} \cdot \Omega_2 + \sqrt{\frac{(I_1 - I_3)I_3}{C_3}} \cdot \Omega_3 = 0, \quad \Rightarrow \quad \Omega_3 = -\left(\frac{bI_2}{cI_3}\right) \cdot \Omega_2$$

Taking into consideration Eq. (2.1), we obtain

$$\Omega_2 = \left(\frac{c}{I_2}\right) \cdot \sqrt{\frac{C_0^2 - I_1^2 \cdot \Omega_1^2}{b^2 + c^2}}, \quad \Omega_3 = -\left(\frac{b}{I_3}\right) \cdot \sqrt{\frac{C_0^2 - I_1^2 \cdot \Omega_1^2}{b^2 + c^2}} \tag{3.2}$$

Then, substituting the equalities (3.2) to the system of Eq. (1.1), we could conclude that each of them is valid under conditions (3.1). Besides, each of Eq. (1.1) should be reduced to the single ODE below:

$$\frac{d\Omega_1}{\mathrm{d}t} = A(C_0^2 - I_1^2 \cdot \Omega_1^2) + B\sqrt{C_0^2 - I_1^2 \cdot \Omega_1^2} \quad \Rightarrow \int \frac{\mathrm{d}\Omega_1}{A(C_0^2 - I_1^2 \cdot \Omega_1^2) + B\sqrt{C_0^2 - I_1^2 \cdot \Omega_1^2}} = \int \mathrm{d}t \tag{3.3}$$

here ($\mathrm{I}_2 > \mathrm{I}_1 > \mathrm{I}_3, c \neq 0$):

$$A = \left(\frac{I_3 - I_1}{I_3 I_1^2}\right) \cdot \frac{b}{c}, \quad B = \frac{P}{C_0} \cdot \frac{\sqrt{b^2 + c^2}}{I_1}.$$

So, the final solution of Eq. (3.3) for $\Omega_1$ is given by the proper *elliptical integral* [6].
Also, let us note that due to choosing of Euler's invariant (2.1), the proper restriction is valid for (3.2)–(3.3):

$$C_0^2 - I_1^2 \cdot \Omega_1^2 > 0.$$

## 4 Exact solution, the case $a = c = 0$

Let us choose the case $a = c = 0$ in (2.3), $I_1 \neq I_3$:

$$\Omega_3^2 = \left\{\frac{C_2 \cdot C_3 + (I_2 - I_1) \cdot I_2 \cdot \Omega_2^2 - b \cdot I_2 \cdot C_3 \cdot \Omega_2}{(I_1 - I_3) \cdot I_3}\right\} \tag{4.1}$$

Besides, from Eq. (2.1), we obtain

$$\Omega_1^2 = \left\{\frac{C_0^2 - 2I_3 \cdot C_1}{I_1 \cdot (I_1 - I_3)}\right\} + \frac{2P \cdot I_3 \cdot b \cdot I_2}{C_0 \cdot I_1 \cdot (I_1 - I_3)} \cdot \Omega_2 - \frac{I_2}{I_1} \cdot \frac{(I_2 - I_3)}{(I_1 - I_3)} \cdot \Omega_2^2 \tag{4.2}$$

Then, substituting the equalities (4.1)–(4.2) to the system of Eq. (1.1), we could conclude that each of them is valid. Besides, each of Eq. (1.1) should be reduced to the single ODE below:

$$\int \frac{\mathrm{d}\Omega_2}{\sqrt{f_1(\Omega_2, \Omega_2^2)} \cdot \sqrt{f_2(\Omega_2, \Omega_2^2)}} = \int \mathrm{d}t, \tag{4.3}$$

where ($I_1 \neq I_3$)

$$f_1 = \left( \left\{ \frac{2I_1 C_1 - C_0^2}{I_3 \cdot I_2} \right\} - \frac{2P \cdot I_1 \cdot b}{C_0 \cdot I_3} \cdot \Omega_2 + \frac{(I_2 - I_1) \cdot \Omega_2^2}{I_3} \right)$$

$$f_2 = \left( \left\{ \frac{C_0^2 - 2I_3 \cdot C_1}{I_1 \cdot I_2} \right\} + \frac{2P \cdot I_3 b}{C_0 \cdot I_1} \cdot \Omega_2 - \frac{(I_2 - I_3)}{I_1} \cdot \Omega_2^2 \right),$$

$$f_1(\Omega_2, \Omega_2^2) \cdot f_2(\Omega_2, \Omega_2^2) > 0.$$

So, the final solution of Eq. (4.3) for $\Omega_2$ is given by the proper *elliptical integral* [6].

## 5 Discussions

As for the physical meaning of new exact solutions of Euler–Poinsot equations, we should especially note that such a solutions are the generalization of previous classical, well-known Euler solution (*torque-free precession of the rotation axis of rigid rotor*).

But for our case, the center of mass of rigid body should not be located at the fixed point of body rotation (in Euler case, the center of mass coincides with the fixed point of rotation). Otherwise, it should be located at a proper point of meridional plane along the main principal axis of inertia of rigid body.

To relate such new results to the physical aspects of gyrodynamics, we also should note that all the components of new solutions are proved to be given by the proper elliptical integrals. The elliptical integrals were proposed as the analytical generalization of inverse periodic functions [6].

Thus, if we obtain the re-inverse dependence of a solution from time-parameter, we could present all the components of solutions as a set of quasi-periodic cycles.

As for the practical or engineering application of solutions of such a type, it is well-known fact that all the analytical results of the theory of rigid body rotation are used to explore the attitude stabilization of satellites in orbit or in autopilots of aircrafts.

## 6 Conclusion

We have obtained absolutely new exact solutions (3.2)–(3.3) of Euler's Eq. (1.1), combined with the solutions (2.2) of Poinsot's Eq. (1.2), which are determined by the choosing of additional Euler's invariant of motion (*the scalar square of the vector of angular momentum is equal to the proper constant*).

All the components of angular velocity of rigid body for such a solution differ from both the cases of symmetric rigid rotor ($I_1 = I_2$, $a = b = 0$ — *for Lagrange's case or* $I_1 = I_2 = 2I_3$, $c = 0$ — *for Kovalevskaya's case*), and from the *Euler's case* when all the applied torques are zero ($a = b = c = 0$), or from other particular cases [5].

The key features of exact solution are the next: the center of mass of rigid body is assumed to be located at meridional plane along the main principal axis of inertia of rigid body ($a = 0$), besides, the principal moments of inertia of rigid body are assumed to satisfy to a simple algebraic equality (3.1) under conditions $I_2 > I_1 > I_3$, $c \neq 0$.

Also, there is a proper restriction at choosing of constants of invariants in (1.3) for such a solution: $(C_0)^2 = 2I_1 \cdot C_1$ (*it means a restriction at choosing of initial conditions: one of the initial meanings of* $\Omega_1(0)$, $\Omega_2(0)$ or $\Omega_3(0)$ *should be chosen according to the given meaning of* $(C_0)^2$).

Such a solution is also proved to satisfy to all the Euler–Poinsot equations, including well-known invariants of motion for the case of rotation over the fixed point. Besides, it also satisfies to the additional Euler's invariant

of motion (*the scalar square of the vector of angular momentum is equal to the proper constant*). So, such a solution is a generalization of Euler's case ($a = b = c = 0$) to the case ($a = 0$, $b$, $c \neq 0$).
Besides, we also consider the case ($a = 0$, $b$, $c = 0$).

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