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Elasticity solutions for functionally graded annular plates subject to biharmonic loads

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Abstract Based on England's expansion formula for displacements, the elastic field in a transversely isotropic functionally graded annular plate subjected to biharmonic transverse forces on its top surface is investigated using the complex variables method. The material parameters are assumed to vary along the thickness direction in an arbitrary fashion. The problem is converted to determine the expressions of four analytic functions $\alpha(\zeta)$, $\beta(\zeta)$, $\phi(\zeta)$ and $\psi(\zeta)$ under certain boundary conditions. A series of simple and practical biharmonic loads are presented. The four analytic functions are constructed carefully in a biconnected annular region corresponding to the presented loads, which guarantee the single-valuedness of the mid-plane displacements of the plate. The unknown constants contained in the analytic functions can be determined from the boundary conditions that are similar to those in the plane elasticity as well as those in the classical plate theory. Numerical examples show that the material gradient index and boundary conditions have a significant influence on the elastic field.

Keywords Functionally graded materials · Annular plates · Transversely isotropic · Biharmonic load · Elasticity solutions

1 Introduction

Functionally graded materials (FGMs) are a new type of inhomogeneous materials, which can be used to meet different requirements for material service performance at different locations in structures due to the exhibiting gradient change of macroscopic properties in space. Therefore, a large number of research activities have been directed to the study of elastic responses of FGM plates under various conditions.

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There are several methods that have been proposed to analyze the bending of FGM plates, among which analytical and numerical methods based on certain simplified theories are frequently used. For example, Reddy et al. [1] examined the axisymmetric bending of functionally graded circular and annular plates by developing exact relationships between the solutions of the classical plate theory (CPT) and the first-order shear deformation plate theory (FSDT). Two refined displacement models, RSDT1 and RSDT2, were developed by Tounsi et al. [2] for a bending analysis of functionally graded sandwich plates. Exact analytical solutions directly based on the elasticity theory can only be derived for a relatively few problems, but they can serve as benchmarks for accessing the validity of various approximate plate theories or numerical methods. Cheng and Batra [3] used an asymptotic expansion method to analyze the isotropic FGM elliptic plate with clamped edges based on the three-dimensional (3D) elasticity theory. A 3D elasticity solution for an isotropic FGM rectangular plate with simply supported edges subject to transverse loading was developed by Kashtalyan [4]. Wang et al. [5] investigated the axisymmetric bending of transversely isotropic FGM circular plates subject to arbitrarily transverse loads. More works on FGM plate theories and their applications may be found in the review paper of Birman and Byrd [6].

It is noted that Mian and Spencer [7] developed an ingenious method to obtain a class of 3D solutions for isotropic FGM plates with traction-free surfaces, in which the material properties are assumed to vary arbitrarily with the thickness coordinate. Yang et al. [8] extended the above method to a transversely isotropic FGM annular plate with uniform loads applied on the top and bottom surfaces. Using the complex variables method, England [9] made a noticeable generalization of Mian and Spencer's method [7] by including the effect of top-surface pressure, which satisfies the biharmonic equation or higher-order ones. Recently, Yang et al. [10] extended England's method to the case of functionally graded plates with materials characterizing transverse isotropy; they obtained the elasticity solutions of an FGM rectangular plate with opposite edges simply supported and subject to a special family of biharmonic polynomial loads (totally 12 different types).

To the authors' knowledge, no analytical solution based on the 3D elasticity theory is found, which can be used to predict the asymmetric behavior of functionally graded annular plates. In the present study, a series of biharmonic loads are given using the Fourier expansion technology. Corresponding to different loads, four analytic functions $\alpha(\zeta)$, $\beta(\zeta)$, $\phi(\zeta)$ and $\psi(\zeta)$ that meet the single-valuedness of the mid-plane displacements are constructed by the complex variables method, which is a generalization of England's method. Finally, 3D elasticity solutions are obtained for a transversely isotropic FGM annular plate subject to biharmonic loads under different boundary conditions.

2 Basic formulations

In the Cartesian coordinate system (x, y, z), the equations of equilibrium in the absence of body forces can be written as

$$\sigma_{ij,j} = 0, \tag{1}$$

where the comma denotes differentiation with respect to the indicated variable.

The stress–displacement relations for transversely isotropic materials are expressed as [11]:

$$\begin{aligned} &\sigma_x = c_{11}u_{,x} + c_{12}v_{,y} + c_{13}w_{,z}, \quad \sigma_y = c_{12}u_{,x} + c_{11}v_{,y} + c_{13}w_{,z}, \quad \sigma_{zx} = c_{44}\left(w_{,x} + u_{,z}\right), \\ &\sigma_z = c_{13}u_{,x} + c_{13}v_{,y} + c_{33}w_{,z}, \quad \sigma_{yz} = c_{44}\left(v_{,z} + w_{,y}\right), \quad \sigma_{xy} = c_{66}\left(u_{,y} + v_{,x}\right), \end{aligned} \tag{2}$$

where u, v and w are the displacement components, and c_{ij} with $2c_{66} = c_{11} - c_{12}$ are the elastic constants. For FGMs, they are functions of z, i.e., $c_{ij} = c_{ij}(z)$. If $c_{11} = c_{33}$, $c_{12} = c_{13}$, and $c_{44} = c_{66}$, the material becomes isotropic. The xy plane is an isotropic plane, coinciding with the mid-plane of the plate. The positive z-axis is upward and perpendicular to the mid-plane.

According to England [9], we seek the following solution of (1) and (2):

$$\begin{aligned} u\,(x, y, z) &= \bar{u} + R_1\Delta_{,x} + R_0\bar{w}_{,x} + R_2\nabla^2\bar{w}_{,x} + R_3\nabla^4\bar{w}_{,x} + R_4\nabla^6\bar{w}_{,x}, \\ v\,(x, y, z) &= \bar{v} + R_1\Delta_{,y} + R_0\bar{w}_{,y} + R_2\nabla^2\bar{w}_{,y} + R_3\nabla^4\bar{w}_{,y} + R_4\nabla^6\bar{w}_{,y}, \\ w\,(x, y, z) &= \bar{w} + T_1\Delta + T_2\nabla^2\bar{w} + T_3\nabla^4\bar{w} + T_4\nabla^6\bar{w}, \end{aligned} \tag{3}$$

where $R_0, \ldots R_4, T_1, \ldots T_4$ are functions of z, $\bar{u} = \bar{u}(x, y)$, $\bar{v} = \bar{v}(x, y)$, and $\bar{w} = \bar{w}(x, y)$ are the mid-plane displacements, and

$$\Delta = \bar{u}_{,x} + \bar{v}_{,y}, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \nabla^4 = \nabla^2\nabla^2, \quad \nabla^6 = \nabla^2\nabla^4. \tag{4}$$

Suppose that the plate is free from shear tractions at the upper and lower surfaces, i.e., $\sigma_{zx} = \sigma_{zy} = 0$ at $z = \pm h/2$. Moreover, we have $\sigma_z = 0$ at $z = -h/2$, and $\sigma_z = -p(x, y)$ at $z = h/2$, where p is a biharmonic load. By substituting Eq. (3) into Eq. (2), then into Eq. (1) and making use of the stress boundary conditions on the upper and lower surfaces of the plate, the expressions of functions $R_0, T_1, T_2, R_1, R_2, T_3, R_3, T_4, R_4$ can be determined (see the Appendix A of [10]) and eventually the following general solution of the mid-plane displacements has been obtained [10]:

$$\bar{w} = \bar{\zeta}\beta(\zeta) + \zeta\overline{\beta(\zeta)} + \alpha(\zeta) + \overline{\alpha(\zeta)} + W\left(\zeta, \bar{\zeta}\right), \tag{5}$$

$$\begin{aligned} D = {} & \frac{\kappa_1 + 1}{\kappa_1 - 1}\phi(\zeta) - \zeta\overline{\phi'(\zeta)} - \overline{\psi(\zeta)} - 2\frac{\kappa_2}{\kappa_1}\left[\beta(\zeta) + \zeta\overline{\beta'(\zeta)}\right] \\ & - \frac{2}{\kappa_1}\left(\kappa_2 + \kappa_3\nabla^2 + \kappa_4\nabla^4\right)\frac{\partial W}{\partial \bar{\zeta}}, \end{aligned} \tag{6}$$

where a prime denotes the derivative with respect to ζ, $D = \bar{u} + i\bar{v}$, $\zeta = x + iy$, $\alpha(\zeta)$, $\beta(\zeta)$, $\phi(\zeta)$ and $\psi(\zeta)$ are four analytic functions of the complex variable ζ, κ_1, κ_2, κ_3 and κ_4 are constants, and

$$\begin{aligned} W\left(\zeta, \bar{\zeta}\right) = {} & \bar{\zeta}^3 Q(\zeta) + \zeta^3\overline{Q(\zeta)} + 3\bar{\zeta}^2 P(\zeta) + 3\zeta^2\overline{P(\zeta)} \\ & - 12S_{21}\left[\bar{\zeta}^2 Q'(\zeta) + \zeta^2\overline{Q'(\zeta)}\right]. \end{aligned} \tag{7}$$

where S_{21} is a constant, and $Q(\zeta)$ and $P(\zeta)$ are analytic functions, which are relevant to the following biharmonic load $p(x, y)$:

$$p\,(x, y) = -\left[\bar{\zeta}\,Q''(\zeta) + \zeta\,\overline{Q''(\zeta)} + P''(\zeta) + \overline{P''(\zeta)}\right] 96S_1\,(h/2)\,, \tag{8}$$

where $S_1(h/2)$ is also a constant.

Substituting Eqs. (5) and (6) into Eqs. (3) and (2) gives rise to the displacements and stress components, respectively, all expressed in terms of the four analytic functions $\alpha(\zeta)$, $\beta(\zeta)$, $\phi(\zeta)$ and $\psi(\zeta)$. The following expressions of the resultant forces and moments are obtained by integrating the stress components:

$$\begin{aligned} N_x + N_y &= a_1\left[\phi'(\zeta) + \overline{\phi'(\zeta)}\right] + 4a_2\left[\beta'(\zeta) + \overline{\beta'(\zeta)}\right] + a_2\nabla^2 W - a_3\nabla^4 W - a_4\nabla^6 W, \\ N_y - N_x + 2iN_{xy} &= a_1\left[\bar{\zeta}\phi''(\zeta) + \psi'(\zeta)\right] - a_5\phi'''(\zeta) + 4a_2\bar{\zeta}\beta''(\zeta) + 2a_6\alpha''(\zeta) - a_7\beta'''(\zeta) \\ & \quad + 4\frac{\partial^2}{\partial \zeta^2}\left(a_2 W - a_3\nabla^2 W - a_4\nabla^4 W - a_8\nabla^6 W\right). \end{aligned} \tag{9}$$

$$\begin{aligned} M_x + M_y &= -b_1\left[\phi'(\zeta) + \overline{\phi'(\zeta)}\right] + 4b_2\left[\beta'(\zeta) + \overline{\beta'(\zeta)}\right] + b_2\nabla^2 W + b_3\nabla^4 W + b_4\nabla^6 W, \\ M_y - M_x + 2iM_{xy} &= a_6\left[\bar{\zeta}\phi''(\zeta) + \psi'(\zeta)\right] - b_5\phi'''(\zeta) + b_6\bar{\zeta}\beta''(\zeta) + b_7\alpha''(\zeta) \\ & \quad - b_8\beta'''(\zeta) + \frac{\partial^2}{\partial \zeta^2}\left(b_6 W - b_9\nabla^2 W - b_0\nabla^4 W - a_9\nabla^6 W\right). \end{aligned} \tag{10}$$

$$Q_{xz} - iQ_{yz} = \frac{4}{\kappa_1 - 1}Q_{z1}\phi''(\zeta) + 8Q_{z2}\beta''(\zeta) + 2\frac{\partial}{\partial \zeta}\left(Q_{z2}\nabla^2 W + Q_{z3}\nabla^4 W + Q_{z4}\nabla^6 W\right), \tag{11}$$

where $a_k (k = 1, \ldots, 9)$, $b_j (j = 0, \ldots, 9)$, Q_{z1}, Q_{z2}, Q_{z3} and Q_{z4} are real constants.

3 FGM annular plates subject to biharmonic loads

Consider a transversely isotropic FGM annular plate subject to a transverse biharmonic load with inner radius r_0, outer radius r_1 and thickness h. In the cylindrical coordinate system $(r,\ \theta,\ z)$, the $r-\theta$ plane coincides with the mid-plane of the plate, and the z-axis is vertical to the $r-\theta$ plane. Denote u_r, u_θ and w as the displacement components in the r-, θ- and z-directions, respectively; and $\sigma_r, \sigma_\theta, \sigma_z, \sigma_{r\theta}, \sigma_{rz}$and $\sigma_{\theta z}$ as the stress components.

The following transform relations for the physical quantities between the Cartesian coordinates and cylindrical coordinates hold:

$$\begin{aligned}\bar{u}_r + i\bar{u}_\theta &= De^{-i\theta}, \quad N_r + N_\theta = N_x + N_y, \quad M_r + M_\theta = M_x + M_y,\\ N_\theta - N_r + 2iN_{r\theta} &= \left(N_y - N_x + 2iN_{xy}\right)e^{2i\theta},\\ M_\theta - M_r + 2iM_{r\theta} &= \left(M_y - M_x + 2iM_{xy}\right)e^{2i\theta},\\ Q_{rz} - iQ_{\theta z} &= (Q_{xz} - iQ_{yz})e^{i\theta}.\end{aligned} \tag{12}$$

The biharmonic load $p(r,\theta)$ can be expanded into Fourier series in the circumferential direction as follows:

$$p(r,\theta) = \sum_{-\infty}^{\infty} T_k(r)e^{ik\theta} = \sum_{k=0}^{\infty} p_k(r,\theta),\, T_k(r) = \frac{1}{2\pi}\int_0^{2\pi} p(r,\theta)e^{-ik\theta}d\theta, \tag{13}$$

where $p_0 = T_0(r)$, and $p_k(r,\theta) = T_k(r)e^{ik\theta} + T_{-k}(r)e^{-ik\theta} = T_k(r)e^{ik\theta} + \overline{T_k(r)}e^{-ik\theta}$.

Substituting Eq. (13) into the biharmonic equation yields the following:

$$\Lambda_k^2 T_k(r) = 0,\, \Lambda_k = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{k^2}{r^2}, \quad k = 0,\ \pm1,\ \pm2, \ldots \tag{14}$$

Let $T_k(r) = a_k r^\lambda$. Substituting it into Eq. (14) leads to the following characteristic equation:

$$\left[(\lambda-2)^2 - k^2\right]\left(\lambda^2 - k^2\right) = 0. \tag{15}$$

Therefore, the general solution of $T_k(r)$ has the following form:

$$T_k(r) = a_{k1}r^k + a_{k2}r^{-k} + a_{k3}r^{k+2} + a_{k4}r^{2-k}. \tag{16}$$

Notice that there are multiple roots in Eq. (15) if $k = 0$ or $k = \pm1$, for which

$$T_0(r) = a_{01} + a_{02}\ln\frac{r}{a} + a_{03}r^2 + a_{04}r^2\ln\frac{r}{a}, \tag{17}$$

$$T_1(r) = a_{11}r + a_{12}r\ln\frac{r}{a} + a_{13}r^{-1} + a_{14}r^3, \quad T_{-1}(r) = \overline{T_1(r)}, \tag{18}$$

where a is a known real constant, which can be taken to be $a = r_1$ for instance, and a_{0j} and $a_{kj}(j = 1, 2, 3, 4;\ \ k = \pm1, \pm2, \ldots)$ are real constants and complex constants, respectively. Therefore, a series of biharmonic loads $p_k(r,\theta)$ can be obtained, which give the complete set of periodic solutions in the θ- direction of the general solution of the biharmonic equation (see Timoshenko and Goodier [12]). Then, the response corresponding to $p(r,\theta)$ can be converted to that of $p_k(r,\theta)$ by using Eq. (13) and the principle of superposition.

(1) $k = 0$

This corresponds to the axisymmetric load case, for which we have

$$\begin{aligned}p_0 = T_0(r) &= a_{01} + a_{02}\ln\frac{r}{a} + a_{03}r^2 + a_{04}r^2\ln\frac{r}{a}\\ &= a_{01} + a_{03}\zeta\bar{\zeta} + \frac{1}{2}\left(a_{02} + a_{04}\zeta\bar{\zeta}\right)\left(\ln\frac{\zeta}{a} + \ln\frac{\bar{\zeta}}{a}\right).\end{aligned} \tag{19}$$

Comparing Eq. (19) with (8) leads to the following:

$$
\begin{aligned}
-96S_1\,(h/2)\,P''(\zeta) &= \frac{1}{2}a_{01} + \frac{1}{2}a_{02}\ln\frac{\zeta}{a},\\
-96S_1\,(h/2)\,Q''(\zeta) &= \frac{1}{2}a_{03}\zeta + \frac{1}{2}a_{04}\zeta\ln\frac{\zeta}{a}.
\end{aligned}
\tag{20}
$$

Integrating Eq. (20) gives

$$
P(\zeta) = P_{01}\zeta^2 + P_{02}\zeta^2\ln\frac{\zeta}{a}, \qquad Q(\zeta) = Q_{01}\zeta^3 + Q_{02}\zeta^3\ln\frac{\zeta}{a}, \tag{21}
$$

where

$$
\begin{aligned}
P_{01} &= \frac{3a_{02} - 2a_{01}}{768S_1\,(h/2)}, \qquad P_{02} = -\frac{a_{02}}{384S_1\,(h/2)},\\
Q_{01} &= \frac{5a_{04} - 6a_{03}}{6912S_1\,(h/2)}, \qquad Q_{02} = -\frac{a_{04}}{1152S_1\,(h/2)}.
\end{aligned}
\tag{22}
$$

Substituting Eq. (21) into Eq. (7) leads to the following:

$$
W_0 = W_{01}\zeta^2\bar{\zeta}^2 + W_{02}\zeta^3\bar{\zeta}^3 + \left(W_{03}\zeta^2\bar{\zeta}^2 + W_{04}\zeta^3\bar{\zeta}^3\right)\left(\ln\frac{\zeta}{a} + \ln\frac{\bar{\zeta}}{a}\right), \tag{23}
$$

where W_0 is a single-valued function, and

$$
\begin{aligned}
W_{01} &= 6\left[P_{01} - 4S_{21}\left(3Q_{01} + Q_{02}\right)\right], \qquad W_{02} = 2Q_{01},\\
W_{03} &= 3\left(P_{02} - 12S_{21}Q_{02}\right), \qquad W_{04} = Q_{02}.
\end{aligned}
\tag{24}
$$

Let

$$
\alpha(\zeta) = \alpha_0 + \gamma_0\ln\frac{\zeta}{a}, \quad \beta(\zeta) = \beta_1\zeta + \gamma_1\zeta\ln\frac{\zeta}{a}, \quad \phi(\zeta) = \phi_1\zeta, \quad \psi(\zeta) = \psi_{-1}\zeta^{-1}, \tag{25}
$$

where $\alpha_0, \gamma_0, \beta_1, \gamma_1, \phi_1$ and ψ_{-1} are real constants to be determined.

By substituting Eqs. (25) and (23) into Eqs. (5) and (6) and making use of Eq. (12), we obtain

$$
\begin{aligned}
\bar{w} &= 2\alpha_0 + 2\beta_1\zeta\bar{\zeta} + W_{01}\zeta^2\bar{\zeta}^2 + W_{02}\zeta^3\bar{\zeta}^3\\
&\quad + \left(\gamma_0 + \gamma_1\zeta\bar{\zeta} + W_{03}\zeta^2\bar{\zeta}^2 + W_{04}\zeta^3\bar{\zeta}^3\right)\left(\ln\frac{\zeta}{a} + \ln\frac{\bar{\zeta}}{a}\right).
\end{aligned}
\tag{26}
$$

$$
\bar{u}_r + i\bar{u}_\theta = De^{-i\theta} = D_0(r)r, \tag{27}
$$

in which the expression of the function $D_0(r)$ is given in Appendix A. $\bar{w}$ and $\bar{u}_r$ are real functions of r, which can be obtained from Eqs. (26) and (27), respectively, and $\bar{u}_\theta \equiv 0$. There are 6 real constants in the expressions of $\bar{w}$ and $\bar{u}_r$, which can be determined from the cylindrical boundary conditions of the annular plate at $r = r_0$ and $r = r_1$ (see Yang et al. [8]).

(2) $\boldsymbol{k = 1}$

$$
p_1(r,\theta) = T_1(r)e^{i\theta} + T_{-1}(r)e^{-i\theta} = T_1(r)e^{i\theta} + \overline{T_1(r)}e^{-i\theta}. \tag{28}
$$

Substituting Eq. (18) into Eq. (28) leads to the following:

$$
\begin{aligned}
p_1(r,\theta) &= a_{11}\zeta + \frac{1}{2}a_{12}\zeta\ln\frac{\zeta}{a} + \bar{a}_{13}\zeta^{-1} + \bar{a}_{11}\bar{\zeta} + \frac{1}{2}\bar{a}_{12}\bar{\zeta}\ln\frac{\bar{\zeta}}{a} + a_{13}\bar{\zeta}^{-1}\\
&\quad + \bar{\zeta}\left(\frac{1}{2}\bar{a}_{12}\ln\frac{\zeta}{a} + a_{14}\zeta^2\right) + \zeta\left(\frac{1}{2}a_{12}\ln\frac{\bar{\zeta}}{a} + \bar{a}_{14}\bar{\zeta}^2\right).
\end{aligned}
\tag{29}
$$

Comparing Eq. (29) with (8) leads to the following:

$$-96S_1\,(h/2)\,P''(\zeta) = a_{11}\zeta + \frac{1}{2}a_{12}\zeta\ln\frac{\zeta}{a} + \bar{a}_{13}\zeta^{-1},$$
$$-96S_1\,(h/2)\,Q''(\zeta) = \frac{1}{2}\bar{a}_{12}\ln\frac{\zeta}{a} + a_{14}\zeta^2. \tag{30}$$

Integrating Eq. (30) gives

$$P(\zeta) = P_{11}\zeta^3 + P_{12}\zeta^3\ln\frac{\zeta}{a} + P_{13}\zeta\left(\ln\frac{\zeta}{a} - 1\right),$$
$$Q(\zeta) = Q_{11}\zeta^2\left(\ln\frac{\zeta}{a} - \frac{3}{2}\right) + Q_{12}\zeta^4, \tag{31}$$

where

$$P_{11} = \frac{5a_{12} - 12a_{11}}{6912S_1\,(h/2)},\quad P_{12} = \bar{Q}_{11}/3,\quad P_{13} = -\frac{\bar{a}_{13}}{96S_1\,(h/2)},$$
$$Q_{11} = -\frac{\bar{a}_{12}}{384S_1\,(h/2)},\quad Q_{12} = -\frac{a_{14}}{1152S_1\,(h/2)}. \tag{32}$$

Substituting Eq. (31) into Eq. (7) leads to the following:

$$W_1 = A_1\left(\zeta,\bar{\zeta}\right)\zeta + \overline{A_1\left(\zeta,\bar{\zeta}\right)}\bar{\zeta}, \tag{33}$$

where W_1 is a multi-valued function, and

$$A_1\left(\zeta,\bar{\zeta}\right) = \bar{W}_{13}\zeta\bar{\zeta} + \bar{W}_{12}\zeta^2\bar{\zeta}^2 + Q_{12}\zeta^3\bar{\zeta}^3 + \bar{W}_{11}\zeta\bar{\zeta}\ln\frac{\bar{\zeta}}{a} + \bar{Q}_{11}\zeta^2\bar{\zeta}^2\left(\ln\frac{\zeta}{a} + \ln\frac{\bar{\zeta}}{a}\right),$$
$$W_{11} = 3\left(P_{13} - 8S_{21}Q_{11}\right),\quad W_{12} = 3\left(\bar{P}_{11} - Q_{11}/2 - 16S_{21}\bar{Q}_{12}\right),\quad W_{13} = 3\left(8S_{21}Q_{11} - P_{13}\right) \tag{34}$$

Let

$$\alpha(\zeta) = \alpha_1\zeta + \alpha_{-1}\zeta^{-1} + \bar{B}_0\zeta\ln\frac{\zeta}{a},\quad \beta(\zeta) = \beta_2\zeta^2 + B_0\ln\frac{\zeta}{a} + \bar{W}_{11}\zeta^2\ln\frac{\zeta}{a},$$
$$\phi(\zeta) = \phi_2\zeta^2 + C_0\ln\frac{\zeta}{a},$$
$$\psi(\zeta) = \psi_0 + \psi_{-2}\zeta^{-2} + \left(2\frac{\kappa_2}{\kappa_1}\bar{B}_0 - \frac{\kappa_1+1}{\kappa_1-1}\bar{C}_0\right)\ln\frac{\zeta}{a} + 16\frac{k_3}{k_1}\bar{W}_{11}\ln\frac{\zeta}{a}, \tag{35}$$

where $\alpha_{-1}, \alpha_1, \beta_2, \phi_2, \psi_{-2}, \psi_0, B_0$ and C_0 are complex constants. The terms related to $\bar{W}_{11}$ are particularly introduced as the supplementary terms in order to ensure the single-valuedness of $\bar{w}$ and $D = \bar{u} + i\bar{v}$.

By substituting Eqs. (33) and (35) into Eqs. (5) and (6) and making use of Eq. (12), we obtain

$$\bar{w} = B_1(r)\zeta + \overline{B_1(r)}\bar{\zeta}, \tag{36}$$
$$\bar{u}_r + i\bar{u}_\theta = De^{-i\theta} = C_1(r)r^2e^{i\theta} + D_1(r)e^{-i\theta}, \tag{37}$$

where the expressions of the functions $B_1(r)$, $C_1(r)$ and $D_1(r)$ are given in Appendix A. It is found from Eqs. (36) and (37) that the mid-plane displacements are single-valued.

(3) $k = 2$

$$p_2(r,\theta) = T_2(r)e^{2i\theta} + \overline{T_2(r)}e^{-2i\theta} = a_{21}\zeta^2 + \bar{a}_{21}\bar{\zeta}^2 + a_{22}\bar{\zeta}^{-2} + \bar{a}_{22}\zeta^{-2} + a_{23}\bar{\zeta}\zeta^3$$
$$+\bar{a}_{23}\zeta\bar{\zeta}^3 + a_{24}\zeta\bar{\zeta}^{-1} + \bar{a}_{24}\bar{\zeta}\zeta^{-1}. \tag{38}$$

Comparing Eq. (38) with (8) leads to the following:

$$-96S_1\,(h/2)\,P''(\zeta) = a_{21}\zeta^2 + \bar{a}_{22}\zeta^{-2}, \quad -96S_1\,(h/2)\,Q''(\zeta) = a_{23}\zeta^3 + \bar{a}_{24}\zeta^{-1}. \tag{39}$$

Integrating Eq. (39) gives

$$P(\zeta) = P_{21}\zeta^4 + P_{22}\ln\frac{\zeta}{a}, \quad Q(\zeta) = Q_{21}\zeta^5 + Q_{22}\zeta\left(\ln\frac{\zeta}{a} - 1\right), \tag{40}$$

where

$$\begin{aligned} P_{21} &= -\frac{a_{21}}{1152S_1\,(h/2)}, \quad P_{22} = \frac{\bar{a}_{22}}{96S_1\,(h/2)}, \\ Q_{21} &= -\frac{a_{23}}{1920S_1\,(h/2)}, \quad Q_{22} = -\frac{\bar{a}_{24}}{96S_1\,(h/2)}. \end{aligned} \tag{41}$$

Substituting Eq. (40) into Eq. (7) leads to the following:

$$W_2 = A_2\left(\zeta,\bar{\zeta}\right)\zeta^2 + \overline{A_2\left(\zeta,\bar{\zeta}\right)}\bar{\zeta}^2, \tag{42}$$

where W_2 is a multi-valued function, and

$$\begin{aligned} &A_2\left(\zeta,\bar{\zeta}\right) = \bar{Q}_{22}\zeta\bar{\zeta}\ln\frac{\bar{\zeta}}{a} + \bar{W}_{22}\ln\frac{\bar{\zeta}}{a} + Q_{21}\zeta^3\bar{\zeta}^3 - \bar{Q}_{22}\zeta\bar{\zeta} + W_{21}\zeta^2\bar{\zeta}^2, \\ &W_{21} = 3\left(P_{21} - 20S_{21}Q_{21}\right), \qquad W_{22} = 3\left(P_{22} - 4S_{21}Q_{22}\right). \end{aligned} \tag{43}$$

Let

$$\begin{aligned} &\alpha(\zeta) = \alpha_2\zeta^2 + \alpha_{-2}\zeta^{-2} + \bar{W}_{22}\zeta^2\ln\frac{\zeta}{a}, \\ &\beta(\zeta) = \beta_3\zeta^3 + \beta_{-1}\zeta^{-1} + \bar{Q}_{22}\zeta^3\ln\frac{\zeta}{a}, \qquad \phi(\zeta) = \phi_3\zeta^3 + \phi_{-1}\zeta^{-1}, \\ &\psi(\zeta) = \psi_1\zeta + \psi_{-3}\zeta^{-3} + 4\left(\frac{\kappa_2}{\kappa_1}\bar{W}_{22} + 12\frac{\kappa_3}{\kappa_1}\bar{Q}_{22}\right)\zeta\ln\frac{\zeta}{a}, \end{aligned} \tag{44}$$

where $\alpha_{-2}, \alpha_2, \beta_{-1}, \beta_3, \phi_{-1}, \phi_3, \psi_{-3}$ and ψ_1 are complex constants. The terms related to $\bar{W}_{22}$ and $\bar{Q}_{22}$ are again particularly introduced as the supplementary terms in order to ensure the single-valuedness of $\bar{w}$ and $D = \bar{u} + i\bar{v}$.

By substituting Eqs. (42) and (44) into Eqs. (5) and (6) and making use of Eq. (12), we obtain

$$\bar{w} = B_2(r)\zeta^2 + \overline{B_2(r)}\bar{\zeta}^2, \tag{45}$$

$$\bar{u}_r + i\bar{u}_\theta = De^{-i\theta} = C_2(r)r^3e^{2i\theta} + D_2(r)r^{-1}e^{-2i\theta}, \tag{46}$$

where the expressions of the functions $B_2(r)$, $C_2(r)$ and $D_2(r)$ are given in Appendix A. It is found from Eqs. (45) and (46) that the mid-plane displacements are single-valued.

(4) $k = 3$

$$\begin{aligned} p_3(r,\theta) &= T_3(r)e^{3i\theta} + \overline{T_3(r)}e^{-3i\theta} \\ &= a_{31}\zeta^3 + a_{32}\bar{\zeta}^{-3} + a_{33}\bar{\zeta}\zeta^4 + a_{34}\bar{\zeta}^{-2}\zeta + \bar{a}_{31}\bar{\zeta}^3 + \bar{a}_{32}\zeta^{-3} + \bar{a}_{33}\zeta\bar{\zeta}^4 + \bar{a}_{34}\zeta^{-2}\bar{\zeta}. \end{aligned} \tag{47}$$

Comparing Eq. (47) with (8) leads to the following:

$$-96S_1\,(h/2)\,P''(\zeta) = a_{31}\zeta^3 + \bar{a}_{32}\zeta^{-3}, \qquad -96S_1\,(h/2)\,Q''(\zeta) = a_{33}\zeta^4 + \bar{a}_{34}\zeta^{-2}. \tag{48}$$

Integrating Eq. (48) gives

$$P(\zeta) = P_{31}\zeta^5 + P_{32}\zeta^{-1}, \quad Q(\zeta) = Q_{31}\zeta^6 + Q_{32}\ln\frac{\zeta}{a}, \tag{49}$$

where

$$P_{31} = -\frac{a_{31}}{1920 S_1\,(h/2)}, \quad P_{32} = -\frac{\bar{a}_{32}}{192 S_1\,(h/2)},$$
$$Q_{31} = -\frac{a_{33}}{2880 S_1\,(h/2)}, \quad Q_{32} = \frac{\bar{a}_{34}}{96 S_1\,(h/2)}. \tag{50}$$

Substituting Eq. (49) into Eq. (7) leads to the following:

$$W_3 = A_3\left(\zeta, \bar{\zeta}\right)\zeta^3 + \overline{A_3\left(\zeta, \bar{\zeta}\right)}\bar{\zeta}^3, \tag{51}$$

where W_3 is a multi-valued function, and

$$A_3\left(\zeta, \bar{\zeta}\right) = Q_{31}\zeta^3\bar{\zeta}^3 + W_{31}\zeta^2\bar{\zeta}^2 + \bar{W}_{32}\left(\zeta\bar{\zeta}\right)^{-1} + \bar{Q}_{32}\ln\frac{\bar{\zeta}}{a},$$
$$W_{31} = 3\left(P_{31} - 24 S_{21} Q_{31}\right), \quad W_{32} = 3\left(P_{32} - 4 S_{21} Q_{32}\right). \tag{52}$$

Let

$$\alpha(\zeta) = \alpha_3\zeta^3 + \alpha_{-3}\zeta^{-3} + \bar{Q}_{32}\zeta^3\ln\frac{\zeta}{a}, \quad \beta(\zeta) = \beta_4\zeta^4 + \beta_{-2}\zeta^{-2},$$
$$\phi(\zeta) = \phi_4\zeta^4 + \phi_{-2}\zeta^{-2}, \quad \psi(\zeta) = \psi_2\zeta^2 + \psi_{-4}\zeta^{-4} + 6\frac{\kappa_2}{\kappa_1}\bar{Q}_{32}\zeta^2\ln\frac{\zeta}{a}, \tag{53}$$

where α_{-3}, α_3, β_{-2}, β_4, ϕ_{-2}, ϕ_4, ψ_{-4} and ψ_2 are complex constants. The terms related to $\bar{Q}_{32}$ are also particularly introduced as the supplementary terms in order to ensure the single-valuedness of $\bar{w}$ and $D = \bar{u} + i\bar{v}$.

By substituting Eqs. (51) and (53) into Eqs. (5) and (6) and making use of Eq. (12), we obtain

$$\bar{w} = B_3(r)\zeta^3 + \overline{B_3(r)}\bar{\zeta}^3, \tag{54}$$
$$\bar{u}_r + i\bar{u}_\theta = De^{-i\theta} = C_3(r)r^4 e^{3i\theta} + D_3(r)r^{-2}e^{-3i\theta}, \tag{55}$$

where the expressions of the functions $B_3(r)$, $C_3(r)$ and $D_3(r)$ are given in Appendix A. It is found from Eqs. (54) and (55) that the mid-plane displacements are single-valued.

(5) $\boldsymbol{k} = 4, 5, \ldots$

$$p_k(r,\theta) = T_k(r)e^{ik\theta} + \overline{T_k(r)}e^{-ik\theta}$$
$$= a_{k1}\zeta^k + \bar{a}_{k1}\bar{\zeta}^k + a_{k2}\bar{\zeta}^{-k} + \bar{a}_{k2}\zeta^{-k} + a_{k3}\bar{\zeta}\zeta^{k+1} + \bar{a}_{k3}\zeta\bar{\zeta}^{k+1} + a_{k4}\zeta\bar{\zeta}^{1-k} + \bar{a}_{k4}\bar{\zeta}\zeta^{1-k}. \tag{56}$$

Comparing Eq. (56) with (8) leads to the following:

$$-96 S_1\,(h/2)\,P''(\zeta) = a_{k1}\zeta^k + \bar{a}_{k2}\zeta^{-k},$$
$$-96 S_1\,(h/2)\,Q''(\zeta) = a_{k3}\zeta^{k+1} + \bar{a}_{k4}\zeta^{1-k}. \tag{57}$$

Integrating Eq. (57) gives

$$P(\zeta) = P_{k1}\zeta^{k+2} + P_{k2}\zeta^{-k+2}, \quad Q(\zeta) = Q_{k1}\zeta^{k+3} + Q_{k2}\zeta^{-k+3}, \tag{58}$$

where

$$P_{k1} = -\frac{a_{k1}}{96\,(k+2)\,(k+1)\,S_1\,(h/2)}, \quad P_{k2} = -\frac{\bar{a}_{k2}}{96\,(2-k)\,(1-k)\,S_1\,(h/2)},$$
$$Q_{k1} = -\frac{a_{k3}}{96\,(k+3)\,(k+2)\,S_1\,(h/2)}, \quad Q_{k2} = -\frac{\bar{a}_{k4}}{96\,(3-k)\,(2-k)\,S_1\,(h/2)}. \tag{59}$$

Substituting Eq. (58) into Eq. (7) leads to the following:

$$W_k = A_k(r)\zeta^k + \overline{A_k(r)}\bar{\zeta}^k, \tag{60}$$

where W_k is a single-valued function, and

$$
\begin{aligned}
A_k(r) &= Q_{k1}\zeta^3\bar{\zeta}^3 + W_{k1}\zeta^2\bar{\zeta}^2 + \left(\bar{Q}_{k2}\zeta^3\bar{\zeta}^3 + \bar{W}_{k2}\zeta^2\bar{\zeta}^2\right)\left(\zeta\bar{\zeta}\right)^{-k}, \\
W_{k1} &= 3\left[P_{k1} - 4S_{21}(k+3)\,Q_{k1}\right], \quad W_{k2} = 3\left[P_{k2} - 4S_{21}(3-k)\,Q_{k2}\right].
\end{aligned} \tag{61}
$$

Let

$$
\begin{aligned}
&\alpha(\zeta) = \alpha_k\zeta^k + \alpha_{-k}\zeta^{-k}, \quad \beta(\zeta) = \beta_{k+1}\zeta^{k+1} + \beta_{-k+1}\zeta^{-k+1}, \\
&\phi(\zeta) = \phi_{k+1}\zeta^{k+1} + \phi_{-k+1}\zeta^{-k+1}, \quad \psi(\zeta) = \psi_{k-1}\zeta^{k-1} + \psi_{-k-1}\zeta^{-k-1},
\end{aligned} \tag{62}
$$

where $\alpha_{-k}, \alpha_k, \beta_{-k+1}, \beta_{k+1}, \phi_{-k+1}, \phi_{k+1}, \psi_{-k-1}$ and ψ_{k-1} are complex constants.

By substituting Eqs. (60) and (62) into Eqs. (5) and (6) and making use of Eq. (12), we obtain

$$
\bar{w} = B_k(r)\zeta^k + \overline{B_k(r)}\bar{\zeta}^k, \tag{63}
$$

$$
\bar{u}_r + i\bar{u}_\theta = De^{-i\theta} = C_k(r)r^{1+k}e^{ik\theta} + D_k(r)r^{1-k}e^{-ik\theta}, \tag{64}
$$

where the expressions of the functions $B_k(r)$, $C_k(r)$ and $D_k(r)$ are given in Appendix A.

4 Determination of real or complex constants

The real or complex constants contained in the elastic field corresponding to each kind of load can be determined from the cylindrical boundary conditions of the annular plate at $r = r_i (i = 0, 1)$. The procedure of fixing these constants is shown below for $k = 1$, just as an example.

4.1 The expressions of resultant forces in cylindrical coordinates

By substituting Eqs. (33) and (35) into Eqs. (9–11) and making use of Eq. (12), we obtain the expressions of the resultant forces and moments for $k = 1$ as follows:

$$
N_r + N_\theta = N_x + N_y = N_1(r)re^{i\theta} + \overline{N_1(r)}\,re^{-i\theta}. \tag{65}
$$

$$
N_\theta - N_r + 2iN_{r\theta} = \left(N_y - N_x + 2iN_{xy}\right)e^{2i\theta} = N_{-1}(r)re^{i\theta} + N_{-3}(r)r^3e^{-i\theta}. \tag{66}
$$

$$
M_r + M_\theta = M_x + M_y = M_1(r)re^{i\theta} + \overline{M_1(r)}\,re^{-i\theta}. \tag{67}
$$

$$
M_\theta - M_r + 2iM_{r\theta} = \left(M_y - M_x + 2iM_{xy}\right)e^{2i\theta} = M_{-1}(r)re^{i\theta} + M_{-3}(r)r^3e^{-i\theta}. \tag{68}
$$

$$
Q_{rz} - iQ_{\theta z} = \left(Q_{xz} - iQ_{yz}\right)e^{i\theta} = Q_0(r)e^{i\theta} + Q_{-2}(r)r^2e^{-i\theta}, \tag{69}
$$

in which the expressions of $N_1(r)$, $N_{-1}(r)$, $N_{-3}(r)$, $M_1(r)$, $M_{-1}(r)$, $M_{-3}(r)$, $Q_0(r)$ and $Q_{-2}(r)$ are given in Appendix B.

4.2 Boundary conditions

For the FGM annular plate, there are three types of cylindrical boundary conditions at $r = r_i (i = 0, 1)$; these include simply-supported (S), clamped (C) and free (F), which are expressed respectively as:

$$
\mathbf{S}: \quad \bar{w} = 0, \quad \bar{u}_\theta = 0, \quad N_r = 0, \quad M_r = 0. \tag{70}
$$

$$
\mathbf{C}: \quad \bar{w} = 0, \quad \bar{u}_\theta = 0, \quad \bar{u}_r = 0, \quad \bar{w}_{,r} = 0. \tag{71}
$$

$$
\mathbf{F}: \quad N_r = 0, \ M_r = 0, \quad N_{r\theta} = 0, \quad Q_{rz} + \frac{1}{r}\frac{\partial M_{r\theta}}{\partial\theta} = 0. \tag{72}
$$

We find from Eq. (36) that the following equations are satisfied if $\bar{w}(r_i, \ \theta) = 0$ and $\bar{w}_{,r}(r_i, \theta) = 0$

$$
B_1(r_i) = 0, \quad \left.\frac{\mathrm{d}B_1(r)}{\mathrm{d}r}r\right|_{r=r_i} + B_1(r_i) = 0. \tag{73}
$$

From Eq. (37), the following equations hold if $\bar{u}_r\,(r_i,\ \theta) = \bar{u}_\theta\,(r_i,\ \theta) = 0$,

$$C_1(r_i) = 0, \qquad D_1(r_i) = 0. \tag{74}$$

It follows from the condition of $\bar{u}_\theta\,(r_i,\ \theta) = 0$ that

$$C_1(r_i)r_i^2 - \overline{D_1(r_i)} = 0. \tag{75}$$

The following equations are satisfied if $N_r\,(r_i,\ \theta) = 0$ and $N_{r\theta}\,(r_i,\ \theta) = 0$

$$2N_1(r_i) - N_{-1}(r_i) - r_i^2\overline{N_{-3}(r_i)} = 0, \qquad N_{-1}(r_i) - r_i^2\overline{N_{-3}(r_i)} = 0, \tag{76}$$

which are obtained from Eqs. (65) and (66), respectively.

We find from Eqs. (67) and (68) that the following equation holds if $M_r\,(r_i,\ \theta) = 0$

$$2M_1(r_i) - M_{-1}(r_i) - r_i^2\overline{M_{-3}(r_i)} = 0. \tag{77}$$

The following equation is equivalent to the condition of $Q_{rz}(r_i,\ \theta) + \partial M_{r\theta}(r,\ \theta)/(r\partial\theta)|_{r=r_i} = 0$, which can be found from Eqs. (68) and (69)

$$2Q_0(r_i) + M_{-1}(r_i) + \left[2\overline{Q_{-2}(r_i)} - \overline{M_{-3}(r_i)}\right] r_i^2 = 0. \tag{78}$$

There are 8 different kinds of combination of the cylindrical boundary conditions as shown in Eqs. (70–72), namely SS, CC, SC, CS, CF, FC, SF and FS; here, the first letter denotes the conditions at the inner edge and the second signifies those at the outer edge. Any of the eight types of cylindrical boundary conditions of the annular plate are sufficient to determine the unknown constants $\alpha_{-1}, \alpha_1, \beta_2, \phi_{-2}, \psi_{-2}, \psi_0, B_0$ and C_0, and hence, the displacements and stress components at any position in the plate can be obtained completely for $k = 1$.

5 Numerical results and discussion

The following dimensionless quantities are introduced to show the numerical results:

$\hat{W} = wD_0/qr_1^4$, $\tilde{W} = w/h$, $\bar{\sigma}_{ij} = \sigma_{ij}/q$, $\beta = h/r_2$, $\bar{r} = (r_0 + r_1)/2$, $r_2 = r_1 - r_0$, $D_0 = Eh^3/12(1-\nu^2)$, $q = 1 \times 10^6\text{N/m}^2$.

Example 1: Simply supported homogeneous isotropic annular plate subject to uniform load q

In order to validate the present method, the numerical results are compared in Table 1 with the CPT solution [13] for a simply supported homogeneous isotropic annular plate subject to a uniform load on the top surface, with $h = 0.002$m, $r_1 = 0.1$m and $\nu = 0.3$. In this case, the elasticity solution for axisymmetric bending can be obtained by letting $a_{01} = q$ and $a_{02} = a_{03} = a_{04} = 0$ in Eq. (19).

It can be found from Table 1 that the present elasticity solution agrees well with the classical plate theory prediction.

Example 2: Transversely isotropic FGM annular plate subject to load $qr^3\cos\theta$

The elasticity solutions can be obtained for a transversely isotropic FGM annular plate subject to the load $qr^3\cos\theta$ by letting $a_{14} = q/2$ and $a_{11} = a_{12} = a_{13} = 0$ in Eq. (29). Take $r_0 = 0.25$m, $r_1 = 1$m and the FG model in the following form [1]:

$$C_{ij} = C_{ij}^{0(A)}\,(0.5 - z/h)^{\lambda} + C_{ij}^{0(T)}\left[1 - (0.5 - z/h)^{\lambda}\right]\ (i,\ j = 1, 2, 3, 4, 5, 6)\,,$$

Table 1 Dimensionless deflection $\hat{W}(\bar{r}, 0)$

	r_0/r_1			
	0.1	0.3	0.5	0.7
CPT	0.0060	0.0029	0.0008	0.0001
Present	0.00607	0.00286	0.00079	0.00011

Table 2 Elastic constants of Al_2O_3 and Titanium (Unit: GPa)

Materials	c_{11}^0	c_{12}^0	c_{13}^0	c_{33}^0	c_{55}^0
Al_2O_3	460.2	174.7	127.4	509.5	126.9
Titanium	162.4	92	69	180.7	46.7

Table 3 Dimensionless deflection and stresses ($\beta = 0.2$)

	$\tilde{W}(\bar{r}, \pi/4, 0)(\times 10^{-5})$			$\bar{\sigma}_r(\bar{r}, \pi/4, h/2)$			$\bar{\sigma}_{rz}(\bar{r}, \pi/4, 0)$		
	$\lambda = 0$	$\lambda = 2$	$\lambda = 4$	$\lambda = 0$	$\lambda = 2$	$\lambda = 4$	$\lambda = 0$	$\lambda = 2$	$\lambda = 4$
SS	−4.3915	−8.2590	−9.1450	3.2167	−2.3867	−2.5806	−0.2286	−0.2108	−0.2038
CC	−0.7956	−1.5654	−1.6760	−0.7900	−0.6781	−0.6967	−0.0363	−0.0454	−0.0370
SC	−1.1457	−2.2518	−2.4176	−0.9784	−0.8187	−0.8445	0.0480	0.0378	0.0436
CS	−2.6780	−4.9759	−5.4943	−2.1476	−1.5713	−1.6773	−0.3878	−0.3660	−0.3592
FC	−1.6958	−3.3862	−3.6153	−0.8693	−0.7298	−0.7419	0.1151	0.1063	0.1078
CF	−40.8639	−78.3916	−85.2512	7.9226	6.4045	6.9280	−2.0978	−2.0384	−1.9928
FS	−7.5979	−14.3237	−15.7451	−3.7417	−2.8200	−3.0363	−0.0983	−0.0911	−0.0884
SF	−180.587	−377.580	−422.052	4.2316	4.0488	4.2658	−2.0500	−1.9940	−1.9489

where $C_{ij}^{0(A)}$ are those of Al_2O_3 at $z = -h/2$, and $C_{ij}^{0(T)}$ are those of Titanium at $z = h/2$, both given in Table 2. The parameter λ is the gradient index, which reflects the degree of material inhomogeneity. Obviously, $\lambda = 0$ corresponds to the homogeneous material.

Table 3 gives the dimensionless deflection $\tilde{W}$, radial normal stress $\bar{\sigma}_r$ and shear stress $\bar{\sigma}_{rz}$ of the FGM annular plate for all 8 kinds of combination of the boundary conditions and three values of λ. The thickness-to-span ratio is fixed at $\beta = 0.2$. The following observations can be obtained from the results:

1. The deflection increases with λ, regardless of the boundary conditions. This is simply because the whole rigidity of the FGM plate decreases with λ. The deflections of the SF and CC annular plates are, respectively, the largest and smallest among the plates with different boundary conditions.
2. With the increase of λ, the absolute value of the normal stress decreases firstly and then increases gradually. The normal stress at $z = h/2$ is tensile for the CF and SF boundary conditions, and compressive for the other kinds of boundary conditions.
3. With the increase of λ, the absolute value of the shear stress decreases for the SS, CS, CF, FS and SF annular plates, but first decreases and then gradually increases for the SC and FC plates. For the CC annular plate, the absolute value of the shear stress first increases and then gradually decreases. In addition, the shear stress for the SC and FC plates shows a reverse sign compared with the other 6 plates.
4. The outer boundary conditions have a stronger influence on the deflection and stresses than the inner boundary conditions, as can be seen from the comparison between the SC plate and the CS plate, the FC plate and the CF plate or that between the FS plate and the SF plate.

6 Conclusions

The bending of transversely isotropic functionally graded annular plates subject to biharmonic transverse loads is investigated based on a generalization of the England's method. General solutions of the basic equations of elasticity are presented and the elasticity solutions corresponding to a series of biharmonic loads $p_k(r, \theta)$ in the cylindrical coordinate system are determined from the boundary conditions similar to that in plane elasticity and the classical plate theory. Numerical examples show that the material gradient index and boundary conditions have a significant influence on the elastic field, which can provide guidance for optimizing the design in engineering applications.

Annular plates are biconnected so that the four analytic functions $\alpha(\zeta)$, $\beta(\zeta)$, $\phi(\zeta)$ and $\psi(\zeta)$ in the general solutions and the particular solutions W_k corresponding to $p_k(r, \theta)$ may be multi-valued. This situation is quite different from the simply connected region (e.g., a circular plate). Therefore, in addition to match the particular solutions W_k and introduce enough arbitrary constants to satisfy the boundary conditions, the single-valuedness of the mid-plane displacements of the plate must be guaranteed when constructing the expressions of the four analytic functions $\alpha(\zeta)$, $\beta(\zeta)$, $\phi(\zeta)$ and $\psi(\zeta)$. There are two kinds of multi-valued functions in the expressions of these four analytic functions. The first one is coordinated completely among the four analytic functions and ensures the single-valuedness of the mid-plane displacements, while the other is called the supplementary solutions, from which the multiple values of the mid-plane displacements can be counterbalanced with that from the particular solutions W_k, thus making the final mid-plane displacements single-valued. As a result, the supplementary solutions should be introduced if W_k is a multi-valued function (for $k = 1, 2, 3$). The supplementary solutions are known functions that do not contain any arbitrary constant. In this paper, Eqs. (25), (35), (44), (53) and (62) are the sets of expressions of the four analytic functions constructed in accordance with the above considerations.

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Appendix A: Expressions of functions related to the mid-plane displacements

$$\begin{aligned} D_0(r) = & \frac{2}{\kappa_1 - 1}\phi_1 - 2\frac{\kappa_2}{\kappa_1}(2\beta_1 + \gamma_1) - 32\frac{\kappa_3}{\kappa_1}(W_{01} + 2W_{03}) - 384\frac{\kappa_4}{\kappa_1}(3W_{02} + 8W_{04}) \\ & - \frac{2}{\kappa_1}[\kappa_2(2W_{01} + W_{03}) + 12\kappa_3(6W_{02} + 7W_{04})]\zeta\bar{\zeta} \\ & - 2\frac{\kappa_2}{\kappa_1}(3W_{02} + W_{04})\zeta^2\bar{\zeta}^2 - \frac{2}{\kappa_1}\Big[\kappa_2\gamma_1 + 16(\kappa_3 W_{03} + 36\kappa_4 W_{04}) \\ & + 2(\kappa_2 W_{03} + 36\kappa_3 W_{04})\zeta\bar{\zeta} + 3\kappa_2 W_{04}\zeta^2\bar{\zeta}^2\Big]\left(\ln\frac{\zeta}{a} + \ln\frac{\bar{\zeta}}{a}\right) \\ & - \left(\psi_{-1} + 128\frac{\kappa_4}{\kappa_1}W_{03}\right)(\zeta\bar{\zeta})^{-1}. \end{aligned} \tag{A.1}$$

$$\begin{aligned} B_1(r) = & \alpha_1 + (\beta_2 + \bar{W}_{13})\zeta\bar{\zeta} + \bar{\alpha}_{-1}(\zeta\bar{\zeta})^{-1} + \bar{W}_{12}\zeta^2\bar{\zeta}^2 + Q_{12}\zeta^3\bar{\zeta}^3 \\ & + (\bar{B}_0 + \bar{W}_{11}\zeta\bar{\zeta} + \bar{Q}_{11}\zeta^2\bar{\zeta}^2)\left(\ln\frac{\zeta}{a} + \ln\frac{\bar{\zeta}}{a}\right). \end{aligned} \tag{A.2}$$

$$\begin{aligned} C_1(r) = & \frac{k_1 + 1}{k_1 - 1}\phi_2 - 2\frac{k_2}{k_1}(\beta_2 + \bar{W}_{11} + \bar{W}_{13}) - 8\frac{k_3}{k_1}(11\bar{Q}_{11} + 6\bar{W}_{12}) - 2304\frac{k_4}{k_1}Q_{12} \\ & - \frac{2}{k_1}\left[k_2(\bar{Q}_{11} + 2\bar{W}_{12}) - 96k_3 Q_{12}\right]\zeta\bar{\zeta} - 6\frac{k_2}{k_1}Q_{12}(\zeta\bar{\zeta})^2 \\ & - \frac{2}{k_1}\left[k_2(\bar{W}_{11} + 2\bar{Q}_{11}\zeta\bar{\zeta}) + 24k_3\bar{Q}_{11}\right]\left(\ln\frac{\zeta}{a} + \ln\frac{\bar{\zeta}}{a}\right) \\ & - \left[\frac{16}{k_1}(k_3\bar{W}_{11} + 24k_4\bar{Q}_{11}) + \left(\bar{C}_0 + 2\frac{\kappa_2}{\kappa_1}\bar{B}_0\right)\right](\zeta\bar{\zeta})^{-1} + \left(64\frac{k_4}{k_1}\bar{W}_{11} - \bar{\psi}_{-2}\right)(\zeta\bar{\zeta})^{-2}. \end{aligned} \tag{A.3}$$

$$\begin{aligned} D_1(r) = & -\bar{\psi}_0 - 16\frac{k_3}{k_1}(W_{11} + W_{13}) - 128\frac{k_4}{k_1}(10Q_{11} + 3W_{12}) \\ & - 2\left[\bar{\phi}_2 + \frac{k_2}{k_1}(2\bar{\beta}_2 + 2W_{13} + W_{11}) + 16\frac{k_3}{k_1}(4Q_{11} + 3W_{12}) + 2304\frac{k_4}{k_1}\bar{Q}_{12}\right]\zeta\bar{\zeta} \\ & - \frac{2}{k_1}\left[k_2(Q_{11} + 3W_{12}) + 144k_3\bar{Q}_{12}\right](\zeta\bar{\zeta})^2 - 8\frac{k_2}{k_1}\bar{Q}_{12}(\zeta\bar{\zeta})^3 + \left(\frac{\kappa_1 + 1}{\kappa_1 - 1}C_0\right. \\ & \left. - 2\frac{k_2}{k_1}B_0\right)\left(\ln\frac{\zeta}{a} + \ln\frac{\bar{\zeta}}{a}\right) - \frac{2}{k_1}\left[8k_3 W_{11} + k_2(2W_{11} + 3Q_{11}\zeta\bar{\zeta})\zeta\bar{\zeta}\right. \\ & \left. + 48k_3 Q_{11}\zeta\bar{\zeta} + 192k_4 Q_{11}\right]\left(\ln\frac{\zeta}{a} + \ln\frac{\bar{\zeta}}{a}\right). \end{aligned} \tag{A.4}$$

$$B_2(r) = \alpha_2 + \bar{\alpha}_{-2}\left(\zeta\bar{\zeta}\right)^{-2} + \bar{\beta}_{-1}\left(\zeta\bar{\zeta}\right)^{-1} + Q_{21}\zeta^3\bar{\zeta}^3 + \left(\beta_3 - \bar{Q}_{22}\right)\zeta\bar{\zeta} + W_{21}\zeta^2\bar{\zeta}^2 + \left(\bar{Q}_{22}\zeta\bar{\zeta} + \bar{W}_{22}\right)\left(\ln\frac{\zeta}{a} + \ln\frac{\bar{\zeta}}{a}\right). \tag{A.5}$$

$$C_2(r) = \frac{\kappa_1+1}{\kappa_1-1}\phi_3 - \frac{2}{\kappa_1}\left(\kappa_2\beta_3 + 32\kappa_3 W_{21} + 1920\kappa_4 Q_{21}\right) - \frac{4}{\kappa_1}\left(\kappa_2 W_{21} + 60\kappa_3 Q_{21}\right)\zeta\bar{\zeta} - 6\frac{\kappa_2}{\kappa_1}Q_{21}\zeta^2\bar{\zeta}^2 - 2\frac{\kappa_2}{\kappa_1}\bar{Q}_{22}\left(\ln\frac{\zeta}{a} + \ln\frac{\bar{\zeta}}{a}\right) - \frac{2}{\kappa_1}\left(\kappa_2\bar{W}_{22} + 12\kappa_3\bar{Q}_{22}\right)\left(\zeta\bar{\zeta}\right)^{-1} + \left(\bar{\phi}_{-1} + 2\frac{\kappa_2}{\kappa_1}\bar{\beta}_{-1} + 16\frac{\kappa_3}{\kappa_1}\bar{W}_{22} + 192\frac{\kappa_4}{\kappa_1}\bar{Q}_{22}\right)\left(\zeta\bar{\zeta}\right)^{-2} - \left(128\frac{\kappa_4}{\kappa_1}\bar{W}_{22} + \bar{\psi}_{-3}\right)\left(\zeta\bar{\zeta}\right)^{-3}. \tag{A.6}$$

$$D_2(r) = \frac{\kappa_1+1}{\kappa_1-1}\phi_{-1} - \frac{2}{\kappa_1}\left(\kappa_2\beta_{-1} + 8\kappa_3 W_{22} + 96\kappa_4 Q_{22}\right) - \left(1536\frac{\kappa_4}{\kappa_1}\bar{W}_{21} + \bar{\psi}_1\right)\zeta\bar{\zeta} - \left[2\frac{\kappa_2}{\kappa_1}\left(3\bar{\beta}_3 - 2Q_{22}\right) + 192\frac{\kappa_3}{\kappa_1}\bar{W}_{21} + 11520\frac{\kappa_4}{\kappa_1}\bar{Q}_{21} + 3\bar{\phi}_3\right]\left(\zeta\bar{\zeta}\right)^2 - \frac{8}{\kappa_1}\left(\kappa_2\bar{W}_{21} + 60\kappa_3\bar{Q}_{21}\right)\left(\zeta\bar{\zeta}\right)^3 - 10\frac{\kappa_2}{\kappa_1}\bar{Q}_{21}\left(\zeta\bar{\zeta}\right)^4 - \frac{2}{\kappa_1}\left(3\kappa_2 Q_{22}\zeta\bar{\zeta} + 2\kappa_2 W_{22} + 24\kappa_3 Q_{22}\right)\zeta\bar{\zeta}\left(\ln\frac{\zeta}{a} + \ln\frac{\bar{\zeta}}{a}\right). \tag{A.7}$$

$$B_3(r) = \alpha_3 + \bar{\alpha}_{-3}\left(\zeta\bar{\zeta}\right)^{-3} + \beta_4\zeta\bar{\zeta} + \bar{\beta}_{-2}\left(\zeta\bar{\zeta}\right)^{-2} + Q_{31}\left(\zeta\bar{\zeta}\right)^3 + W_{31}\left(\zeta\bar{\zeta}\right)^2 + \bar{W}_{32}\left(\zeta\bar{\zeta}\right)^{-1} + \bar{Q}_{32}\left(\ln\frac{\zeta}{a} + \ln\frac{\bar{\zeta}}{a}\right). \tag{A.8}$$

$$C_3(r) = \frac{\kappa_1+1}{\kappa_1-1}\phi_4 - 2\frac{\kappa_2}{\kappa_1}\beta_4 - 80\frac{\kappa_3}{\kappa_1}W_{31} - 5760\frac{\kappa_4}{\kappa_1}Q_{31} - \frac{4}{\kappa_1}\left(\kappa_2 W_{31} + 72\kappa_3 Q_{31}\right)\zeta\bar{\zeta} - 6\frac{\kappa_2}{\kappa_1}Q_{31}\left(\zeta\bar{\zeta}\right)^2 - 2\frac{\kappa_2}{\kappa_1}\bar{Q}_{32}\left(\zeta\bar{\zeta}\right)^{-1} + \frac{2}{\kappa_1}\left(\kappa_2\bar{W}_{32} + 12\kappa_3\bar{Q}_{32}\right)\left(\zeta\bar{\zeta}\right)^{-2} + \left(384\frac{\kappa_4}{\kappa_1}\bar{W}_{32} - \bar{\psi}_{-4}\right)\left(\zeta\bar{\zeta}\right)^{-4} + \left(2\bar{\phi}_{-2} + 4\frac{\kappa_2}{\kappa_1}\bar{\beta}_{-2} - 32\frac{\kappa_3}{\kappa_1}\bar{W}_{32} - 384\frac{\kappa_4}{\kappa_1}\bar{Q}_{32}\right)\left(\zeta\bar{\zeta}\right)^{-3}. \tag{A.9}$$

$$D_3(r) = \frac{\kappa_1+1}{\kappa_1-1}\phi_{-2} - 2\frac{\kappa_2}{\kappa_1}\beta_{-2} + 16\frac{\kappa_3}{\kappa_1}W_{32} + 192\frac{\kappa_4}{\kappa_1}Q_{32} - \frac{4}{\kappa_1}\left(\kappa_2 W_{32} + 12\kappa_3 Q_{32}\right)\zeta\bar{\zeta} - \left(\bar{\psi}_2 + 3840\frac{\kappa_4}{\kappa_1}\bar{W}_{31}\right)\left(\zeta\bar{\zeta}\right)^2 - \left(4\bar{\phi}_4 + 8\frac{\kappa_2}{\kappa_1}\bar{\beta}_4 + 320\frac{\kappa_3}{\kappa_1}\bar{W}_{31} + 23040\frac{\kappa_4}{\kappa_1}\bar{Q}_{31}\right)\left(\zeta\bar{\zeta}\right)^3 - 10\left(\frac{\kappa_2}{\kappa_1}\bar{W}_{31} + 72\frac{\kappa_3}{\kappa_1}\bar{Q}_{31}\right)\left(\zeta\bar{\zeta}\right)^4 - 12\frac{\kappa_2}{\kappa_1}\bar{Q}_{31}\left(\zeta\bar{\zeta}\right)^5 - 6\frac{\kappa_2}{\kappa_1}Q_{32}\left(\zeta\bar{\zeta}\right)^2\left(\ln\frac{\zeta}{a} + \ln\frac{\bar{\zeta}}{a}\right). \tag{A.10}$$

$$B_k(r) = \alpha_k + \bar{\alpha}_{-k}\left(\zeta\bar{\zeta}\right)^{-k} + \beta_{k+1}\zeta\bar{\zeta} + \bar{\beta}_{-k+1}\left(\zeta\bar{\zeta}\right)^{-k+1} + A_k(r) \tag{A.11}$$

$$C_k(r) = \frac{\kappa_1+1}{\kappa_1-1}\phi_{k+1} - 2\frac{\kappa_2}{\kappa_1}\beta_{k+1} - 16\frac{\kappa_3}{\kappa_1}(k+2)W_{k1} - 192\frac{\kappa_4}{\kappa_1}(k+3)(k+2)Q_{k1} - \frac{4}{\kappa_1}\left[\kappa_2 W_{k1} + 12\kappa_3(k+3)Q_{k1}\right]\zeta\bar{\zeta} - 6\frac{\kappa_2}{\kappa_1}Q_{k1}\left(\zeta\bar{\zeta}\right)^2 - \left[\bar{\psi}_{-k-1} - \right.$$

$$
\begin{aligned}
&- 64\frac{\kappa_4}{\kappa_1}(2-k)(1-k)\,k\bar{W}_{k2}\Big]\left(\zeta\bar{\zeta}\right)^{-k-1} - (1-k)\Big[\bar{\phi}_{-k+1} + 2\frac{\kappa_2}{\kappa_1}\bar{\beta}_{-k+1}\\
&+ 16\frac{\kappa_3}{\kappa_1}(2-k)\,\bar{W}_{k2} + 192\frac{\kappa_4}{\kappa_1}(3-k)(2-k)\,\bar{Q}_{k2}\Big]\left(\zeta\bar{\zeta}\right)^{-k}\\
&-\frac{2}{\kappa_1}(2-k)\left[\kappa_2\bar{W}_{k2} + 12\kappa_3(3-k)\,\bar{Q}_{k2}\right]\left(\zeta\bar{\zeta}\right)^{1-k} - 2\frac{\kappa_2}{\kappa_1}(3-k)\,\bar{Q}_{k2}\left(\zeta\bar{\zeta}\right)^{2-k}.
\end{aligned}
\tag{A.12}
$$

$$
\begin{aligned}
D_k(r) = &\frac{\kappa_1+1}{\kappa_1-1}\phi_{-k+1} - 2\frac{\kappa_2}{\kappa_1}\beta_{-k+1} - 16\frac{\kappa_3}{\kappa_1}(2-k)\,W_{k2} - 192\frac{\kappa_4}{\kappa_1}(3-k)(2-k)\,Q_{k2}\\
&-\frac{4}{\kappa_1}\left[\kappa_2 W_{k2} + 12\kappa_3(3-k)\,Q_{k2}\right]\zeta\bar{\zeta} - 6\frac{\kappa_2}{\kappa_1}Q_{k2}\left(\zeta\bar{\zeta}\right)^2 - \Big[\bar{\psi}_{k-1}\\
&+ 64\frac{\kappa_4}{\kappa_1}(k+2)(k+1)\,k\bar{W}_{k1}\Big]\left(\zeta\bar{\zeta}\right)^{k-1} - (k+1)\Big[\bar{\phi}_{k+1} + 2\frac{\kappa_2}{\kappa_1}\bar{\beta}_{k+1}\\
&+ 16\frac{\kappa_3}{\kappa_1}(k+2)\,\bar{W}_{k1} + 192\frac{\kappa_4}{\kappa_1}(k+3)(k+2)\,\bar{Q}_{k1}\Big]\left(\zeta\bar{\zeta}\right)^{k}\\
&-\frac{2}{\kappa_1}(k+2)\left[\kappa_2\bar{W}_{k1} + 12\kappa_3(k+3)\,\bar{Q}_{k1}\right]\left(\zeta\bar{\zeta}\right)^{k+1} - 2\frac{\kappa_2}{\kappa_1}(k+3)\,\bar{Q}_{k1}\left(\zeta\bar{\zeta}\right)^{k+2}.
\end{aligned}
\tag{A.13}
$$

Appendix B. Expressions of functions related to the resultant forces for *k*=1

$$
\begin{aligned}
N_1(r) = &\,2a_1\phi_2 + 4a_2\left(2\beta_2 + 3\overline{W}_{11} + 2\overline{W}_{13}\right) - 64a_3\left(3\overline{W}_{12} + 7\overline{Q}_{11}\right) - 9216a_4 Q_{12}\\
&+4\left[a_2\left(6\overline{W}_{12} + 5\overline{Q}_{11}\right) - 288a_3 Q_{12}\right]\zeta\overline{\zeta} + 48a_2 Q_{12}\left(\zeta\overline{\zeta}\right)^2\\
&-\left[32\left(a_3\overline{W}_{11} + 24a_4\overline{Q}_{11}\right) - a_1\bar{C}_0 - 4a_2\bar{B}_0\right]\left(\zeta\overline{\zeta}\right)^{-1}\\
&+8\left(a_2\overline{W}_{11} + 3a_2\overline{Q}_{11}\zeta\overline{\zeta} - 24a_3\overline{Q}_{11}\right)\left(\ln\frac{\zeta}{a} + \ln\frac{\overline{\zeta}}{a}\right).
\end{aligned}
\tag{B.1}
$$

$$
\begin{aligned}
N_{-1}(r) = &\,2\left[a_1\phi_2 + 2a_2\left(2\beta_2 + 3\overline{W}_{11} + 2\overline{W}_{13}\right) - 32a_3\left(3\overline{W}_{12} + 7\overline{Q}_{11}\right) - 4608a_4 Q_{12}\right]\\
&+4\left[a_2\left(6\overline{W}_{12} + 5\overline{Q}_{11}\right) - 288a_3 Q_{12}\right]\zeta\overline{\zeta} + 48a_2 Q_{12}\left(\zeta\overline{\zeta}\right)^2\\
&+2\left[\left(8a_1\frac{\kappa_3}{\kappa_1} - a_7\right)\overline{W}_{11} - 384a_4\overline{Q}_{11} + 2a_2\bar{B}_0 - \frac{\kappa_1+1}{2(\kappa_1-1)}a_1\bar{C}_0\right]\left(\zeta\overline{\zeta}\right)^{-1}\\
&+8\left(a_2\overline{W}_{11} + 3a_2\overline{Q}_{11}\zeta\overline{\zeta} - 24a_3\overline{Q}_{11}\right)\left(\ln\frac{\zeta}{a} + \ln\frac{\overline{\zeta}}{a}\right).
\end{aligned}
\tag{B.2}
$$

$$
\begin{aligned}
N_{-3}(r) = &\,4\left[a_2\left(2W_{12} + 3Q_{11}\right) - 96a_3\overline{Q}_{12}\right] + 24a_2\overline{Q}_{12}\zeta\overline{\zeta}\\
&+4\left(a_2 W_{11} - 24a_3 Q_{11}\right)\left(\zeta\overline{\zeta}\right)^{-1} + \left[32\left(a_3 W_{11} + 24a_4 Q_{11}\right) - a_1 C_0 - 4a_2 B_0\right]\left(\zeta\overline{\zeta}\right)^{-2}\\
&+2\left(2a_6\alpha_{-1} - a_1\psi_{-2} - 128a_4 W_{11} - 3072a_8 Q_{11} - a_5 C_0 - a_7 B_0\right)\left(\zeta\overline{\zeta}\right)^{-3}\\
&+8a_2 Q_{11}\left(\ln\frac{\zeta}{a} + \ln\frac{\overline{\zeta}}{a}\right).
\end{aligned}
\tag{B.3}
$$

$$
\begin{aligned}
M_1(r) = &\,-2b_1\phi_2 + 4b_2\left(2\beta_2 + 3\overline{W}_{11} + 2\overline{W}_{13}\right) + 64b_3\left(3\overline{W}_{12} + 7\overline{Q}_{11}\right) + 9216b_4 Q_{12}\\
&+4\left[b_2\left(6\overline{W}_{12} + 5\overline{Q}_{11}\right) + 288b_3 Q_{12}\right]\zeta\overline{\zeta} + 48b_2 Q_{12}\left(\zeta\overline{\zeta}\right)^2\\
&+\left[32\left(b_3\overline{W}_{11} + 24b_4\overline{Q}_{11}\right) + 4b_2\bar{B}_0 - b_1\bar{C}_0\right]\left(\zeta\overline{\zeta}\right)^{-1}\\
&+8\left(b_2\overline{W}_{11} + 3b_2\overline{Q}_{11}\zeta\overline{\zeta} + 24b_3\overline{Q}_{11}\right)\left(\ln\frac{\zeta}{a} + \ln\frac{\overline{\zeta}}{a}\right).
\end{aligned}
\tag{B.4}
$$

$$
\begin{aligned}
M_{-1}(r) = {} & 2a_6\phi_2 + b_6\left(2\beta_2 + 3\overline{W}_{11} + 2\overline{W}_{13}\right) - 16b_9\left(3\overline{W}_{12} + 7\overline{Q}_{11}\right) - 2304b_0 Q_{12} \\
& + \left[b_6\left(6\overline{W}_{12} + 5\overline{Q}_{11}\right) - 288b_9 Q_{12}\right]\zeta\overline{\zeta} + 12b_6 Q_{12}\left(\zeta\overline{\zeta}\right)^2 \\
& + \left(16a_6\frac{\kappa_3}{\kappa_1}\overline{W}_{11} - 2b_8\overline{W}_{11} - 192b_0\overline{Q}_{11} + b_6\bar{B}_0 - \frac{\kappa_1+1}{\kappa_1-1}a_6\bar{C}_0\right)\left(\zeta\overline{\zeta}\right)^{-1} \\
& + 2\left(b_6\overline{W}_{11} + 3b_6\overline{Q}_{11}\zeta\overline{\zeta} - 24b_9\overline{Q}_{11}\right)\left(\ln\frac{\zeta}{a} + \ln\frac{\overline{\zeta}}{a}\right). \qquad \text{(B.5)}
\end{aligned}
$$

$$
\begin{aligned}
M_{-3}(r) = {} & b_6\left(2W_{12} + 3Q_{11}\right) - 96b_9\overline{Q}_{12} + 6b_6\overline{Q}_{12}\zeta\overline{\zeta} + \left(b_6 W_{11} - 24b_9 Q_{11}\right)\left(\zeta\overline{\zeta}\right)^{-1} \\
& + \left[8\left(b_9 W_{11} + 24b_0 Q_{11}\right) - a_6 C_0 - b_6 B_0\right]\left(\zeta\overline{\zeta}\right)^{-2} + 2b_6 Q_{11}\left(\ln\frac{\zeta}{a} + \ln\frac{\overline{\zeta}}{a}\right) \\
& + 2\left(b_7\alpha_{-1} - a_6\psi_{-2} - 32b_0 W_{11} - 768a_9 Q_{11} - b_5 C_0 - b_8 B_0\right)\left(\zeta\overline{\zeta}\right)^{-3}. \qquad \text{(B.6)}
\end{aligned}
$$

$$
\begin{aligned}
Q_0(r) = {} & 8\left[\frac{1}{\kappa_1-1}Q_{z1}\phi_2 + Q_{z2}\left(2\beta_2 + 5\overline{W}_{11} + 2\overline{W}_{13}\right) + 16Q_{z3}\left(3\overline{W}_{12} + 10\overline{Q}_{11}\right)\right. \\
& \left. + 2304Q_{z4}Q_{12}\right] + 32\left[Q_{z2}\left(3\overline{W}_{12} + 4\overline{Q}_{11}\right) + 144Q_{z3}Q_{12}\right]\zeta\overline{\zeta} \\
& + 288Q_{z2}Q_{12}\left(\zeta\overline{\zeta}\right)^2 + 16\left[Q_{z2}\left(\overline{W}_{11} + 6\overline{Q}_{11}\zeta\overline{\zeta}\right) + 24Q_{z3}\overline{Q}_{11}\right]\left(\ln\frac{\zeta}{a} + \ln\frac{\overline{\zeta}}{a}\right). \qquad \text{(B.7)}
\end{aligned}
$$

$$
\begin{aligned}
Q_{-2}(r) = {} & 8Q_{z2}\left(6W_{12} + 11Q_{11}\right) + 2304Q_{z3}\overline{Q}_{12} + 192Q_{z2}\overline{Q}_{12}\zeta\overline{\zeta} \\
& + 16\left(Q_{z2}W_{11} + 24Q_{z3}Q_{11}\right)\left(\zeta\overline{\zeta}\right)^{-1} - \left[64\left(Q_{z3}W_{11} + 24Q_{z4}Q_{11}\right)\right. \\
& \left. + \frac{4}{\kappa_1-1}Q_{z1}C_0 + 8Q_{z2}B_0\right]\left(\zeta\overline{\zeta}\right)^{-2} + 48Q_{z2}Q_{11}\left(\ln\frac{\zeta}{a} + \ln\frac{\overline{\zeta}}{a}\right). \qquad \text{(B.8)}
\end{aligned}
$$

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