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Nonlinear vibration of stringer shell by means of extended Hamiltonian approach

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Abstract In this paper, the nonlinear free vibration of a stringer shell is studied. The mathematical model of the string shell, which is the most convenient for frequency analysis, is considered. Due to the geometrical properties of the vibrating shell, strong nonlinearities are evident. Approximate analytical expressions for the nonlinear vibration are provided by introducing the extended version of the Hamiltonian approach. The method suggested in the paper gives the approximate solution for the differential equation with dissipative term for which the Lagrangian exists. The aim of this study is to provide engineers and designers with an easy method for determining the shell nonlinear vibration frequency and nonlinear behavior. The effects of different parameters on the ratio of nonlinear to linear natural frequency of shells are studied. This analytical representation gives excellent approximations to the numerical solutions for the whole range of the oscillation amplitude, reducing the respective error of the angular frequency in comparison with the Hamiltonian approach. This study shows that a first-order approximation of the Hamiltonian approach leads to highly accurate solutions that are valid for a wide range of vibration amplitudes.

Keywords Stringer shell · Nonlinear vibration · Extended Hamiltonian approach

List of Symbols

ϕ	Airy function
w	Normal displacement
E	Young's modulus of shell
E_1	Young's modulus of rib
ν	Poisson's ratio
h	Shell thickness
t	Time
R	Shell radius
ρ_0	Densities of shell
ρ_1	Densities of rib
N	Number of stringer
F	Square stringer cross-section
I	Statical moment of stringer cross-section
A	Dimensionless maximum amplitude of oscillation
ω_{NL}	Nonlinear frequency
ω_L	Linear frequency

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1 Introduction

Nonlinear vibrations of shell-type structures have occurred in many branches of engineering sciences. Most of the studies on the shell-type structures have been done on the buckling of cylindrical shells under axial step loading of infinite or finite duration and impulse loads [1–3]. Liew et al. [4] considered the doubly tapered cylindrical shells and their flexural vibration. Mikulas et al. [5] analyzed the free vibration of the eccentrically stiffened cylindrical shells and flat plates. Finding the modal characteristics of the free and forced vibration of shell-type structures by considering the complicating parameters is an interest area in engineering vibration. In fact, linear analysis is not sufficient to describe the behavior of physical systems. To obtain improved performance of these structures, it is better to consider the nonlinear effects in the design process. The governing equations of the nonlinear vibrations of shells are presented by linear and nonlinear partial differential equations in space and time. Generally, it is very difficult to find an exact solution for nonlinear equations; therefore, many researchers have worked on the analytical approximate methods for nonlinear equations.

Recently, some approximate methods have been proposed to solve nonlinear equations such as homotopy perturbation [6–8], energy balance [9–13], variational approach [14–16], iteration perturbation method [17], max-min approach [18,19], variational iteration method [20], Hamiltonian approach [21,22] and other analytical and numerical methods [23,24].

In this study, we extended and adopted the analytical method called Hamiltonian approach (HA) to solve the nonlinear vibration of a stringer shell. Namely, the generalization to the HA is done to give the approximate analytical solution even for the differential equation which has dissipative terms. The method can be used for all of the problems for which the Lagrange function exists.

The paper has the following sections: After the Introduction, in Sect. 2., the mathematical formulation of the problem of vibration of the stringer shell is given. In Sect. 3., the HA is extended and adopted for solving the differential equation with dissipative terms. The extended Hamiltonian approach (EHA) developed in the paper is applied for solving the governing nonlinear equation (Sect. 4). To show the applicability and accuracy of the proposed approach, some comparisons between analytical and numerical solutions are presented in Sect. 5. The paper ends with Conclusions (Sect. 6).

2 Governing equation of stringer shell

We consider a closed circle cylindrical shell supported in two principal directions. Supporting ribs are the one-dimensional elastic elements, situated uniformly with the same constant distance between them. We assume that the ribs height is small in comparison with the curvature radius. There is no interaction between the two ribs lying in two directions. Therefore, we can define with high-accuracy displacements and vibration frequencies.

Dynamics of a structurally orthotropic stringer shell for large displacements (achieving its thickness order) is analyzed. Applying the semi-inextensional theory, the following governing equations are used [25,26].

$$L_1(w) = \nabla_1^4 w - R\frac{\partial^2 \phi}{\partial x^2} + \rho \frac{\partial^2 w}{\partial t_1^2} - L(w,\phi) = 0 \tag{1}$$

$$\frac{1}{B_1}\frac{\partial^4 \phi}{\partial y^4} + \frac{1}{R}\frac{\partial^2 w}{\partial x^2} = \frac{1}{2R^2} L(w,\phi), \tag{2}$$

where

$$\nabla_1^4 = \frac{1}{R^2}\left(D_1 \frac{\partial^4}{\partial x^4} + 2D_3 \frac{\partial^4}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4}{\partial y^4} \right) \tag{3}$$

$$L(w,\phi) = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \phi}{\partial y^2} - 2\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}, \tag{4}$$

and coefficients of Eqs. (1)–(4) are presented in Appendix A.

We suppose that the ribs are symmetric with respect to the shell middle surface [27–34] and the shell is simply supported.

After some mathematical transformation and using the assumption of the space distribution function, the following second-order differential equation with constant coefficient is obtained [29]:

$$\frac{d^2W}{dt^2}+\alpha W\left(\frac{dW}{dt}\right)^2+\alpha W^2\frac{d^2W}{dt^2}+\beta W+\eta W^3+\lambda W^5=0, \tag{5}$$

where coefficients of Eq. (5) are presented in Appendix B, and W is an unknown time-dependent function. Since the periodic solutions are considered, the following initial conditions can be applied:

$$\text{for}\, t=0 W=A, \frac{dW}{dt}=0 \tag{6}$$

3 Extended Hamiltonian approach (EHA)

The Hamiltonian approach is a method that was proposed by He [21] and recently widely applied (see for example [35]). The Hamiltonian approach is one of the simple and effective approaches for conservative oscillatory systems.

Here, we give the generalization of this approach to the differential equation

$$\ddot{W}+f(W,\dot{W},\ddot{W})=0 \tag{7}$$

with initial conditions:

$$W(0)=A,\quad \dot{W}(0)=0, \tag{8}$$

where f is the implicit function of W and its time derivatives $(\dot{\cdot})=d/dt$ and $(\ddot{\cdot})=d^2/dt^2$. The necessary condition is the existence of the Lagrangian for (5)

$$\Im=\frac{1}{2}\dot{W}^2-F(W,\dot{W}), \tag{9}$$

where the function F has to satisfy the relation

$$\frac{\partial F}{\partial W}-\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{W}}\right)+\frac{\partial F}{\partial W}=f(W,\dot{W},\ddot{W}) \tag{10}$$

Based on (9), the ‘Hamiltonian function’ reads

$$H=\frac{1}{2}\dot{W}^2+F(W,\dot{W})=H_0\equiv const. \tag{11}$$

Using the standard procedure [21], a new function is introduced

$$\bar{H}=\int_0^{T/4}\left(\frac{1}{2}\dot{W}^2+F(W,\dot{W})\right)dt, \tag{12}$$

which is the integral of the function (11) for the period of vibration T, i.e.,

$$H=\frac{\partial\bar{H}}{\partial T}. \tag{13}$$

Oscillatory systems contain two important physical parameters, i.e., the frequency ω and the amplitude of oscillation A. Assuming the approximate solution of (7) in the form of a periodic function $W(\omega t, A)$ and substituting into (12), the obtained function is $\bar{H}(\omega, A)$. Substituting this function into (13) and using the relation (11), we obtain the following:

$$H_0=\frac{\partial\bar{H}}{\partial\left(1/\omega\right)} \tag{14}$$

The difference between H_0 and $\partial \bar{H}/\partial(1/\omega)$ gives the residual function

$$R(A) \equiv H_0 - \frac{\partial \bar{H}}{\partial\,(1/\omega)} \tag{15}$$

Using the condition of the minimum of the residual function $\partial R/\partial A = 0$, i.e.,

$$\frac{\partial}{\partial A}\left(\frac{\partial \bar{H}}{\partial(1/\omega)}\right) = 0, \tag{16}$$

the approximate frequency of vibration ω is obtained.

4 Application of the extended Hamiltonian approach

The 'Hamiltonian' of Eq. (5) is constructed as

$$H = \frac{1}{2}\dot{W}^2 + \frac{1}{2}\alpha W^2\dot{W}^2 + \frac{1}{2}\beta W^2 + \frac{1}{4}\eta W^4 + \frac{1}{6}\lambda W^6. \tag{17}$$

Integrating Eq. (17) with respect to t from 0 to $T/4$, we have the following:

$$\bar{H} = \int_0^{T/4}\left(\frac{1}{2}\dot{W}^2 + \frac{1}{2}\alpha W^2\dot{W}^2 + \frac{1}{2}\beta W^2 + \frac{1}{4}\eta W^4 + \frac{1}{6}\lambda W^6\right)\mathrm{d}t. \tag{18}$$

Let us assume the approximate solution as

$$W(t) = A\cos(\omega t) \tag{19}$$

Substituting Eq. (19) into Eq. (18) and integrating, we obtain the following:

$$\bar{H} = \frac{\omega A^2\pi}{8}\left(1 + \frac{\alpha A^2}{4}\right) + \frac{A^2\pi}{8\omega}\left(\beta + \frac{3}{8}\eta A^2 + \frac{5}{24}\lambda A^4\right). \tag{20}$$

Setting (20) into (16), it follows

$$\frac{\partial}{\partial A}\left(\frac{\partial \bar{H}}{\partial(1/\omega)}\right) \equiv -\frac{A\omega^2}{4}\left(1 + \frac{\alpha A^2}{8}\right) + \frac{A}{4}\left(\beta + \frac{3}{4}\eta A^2 + \frac{5}{8}\lambda A^4\right) = 0 \tag{21}$$

Solving the above equation, an approximate nonlinear frequency as a function of amplitude is obtained as follows:

$$\omega = \frac{1}{2}\sqrt{\frac{8\beta + 6\eta A^2 + 5\lambda A^4}{2 + \alpha A^2}} \tag{22}$$

Hence, the approximate solution yields the following:

$$W(t) = A\cos\left(t\frac{1}{2}\sqrt{\frac{8\beta + 6\eta A^2 + 5\lambda A^4}{2 + \alpha A^2}}\right). \tag{23}$$

Comparing the frequency (22) with the linear one ($\omega_L = \sqrt{\beta}$), the frequency ratio is

$$\frac{\omega}{\omega_L} = \frac{1}{2}\sqrt{\frac{8\beta + 6\eta A^2 + 5\lambda A^4}{(2 + \alpha A^2)\beta}}. \tag{24}$$

5 Results and discussion

To show the accuracy of the EHA results, the numerically obtained solutions, obtained by the Runge-Kutta procedure (RK), are compared with analytical ones. In Fig. 1, the $W - t$ diagrams for two certain groups of parameter values are obtained analytically (EHA) and numerically (RKM). The curves are in a good agreement. Figures 2 and 3 show the effect of various values of the parameters α, η, λ, β on the ratio of nonlinear to linear frequency versus nondimensional amplitude ratio for different cases. The effects of different parameters A, α and A, β and A, η are studied in Figs. 4 and 5 simultaneously. For small amplitudes, the rate of increase in nonlinear fundamental frequency is quite small. The effect of nonlinearity becomes more obvious when the maximum amplitude increases. It is evident that the result of EHA shows agreement with the numerical solution and is a quickly convergent function and valid for a wide range of vibration amplitudes and initial conditions. The accuracy of the results shows that EHA is very suitable for the analysis of the strong nonlinear oscillation problems. The limitation of EHA is that this method is valid for conservative nonlinear problems and nonlinear problems without damping. When we have damping, EHA and also the other analytical methods cannot be applied. EHA provides most minimum residual function resulting in more accurate approximate frequency.

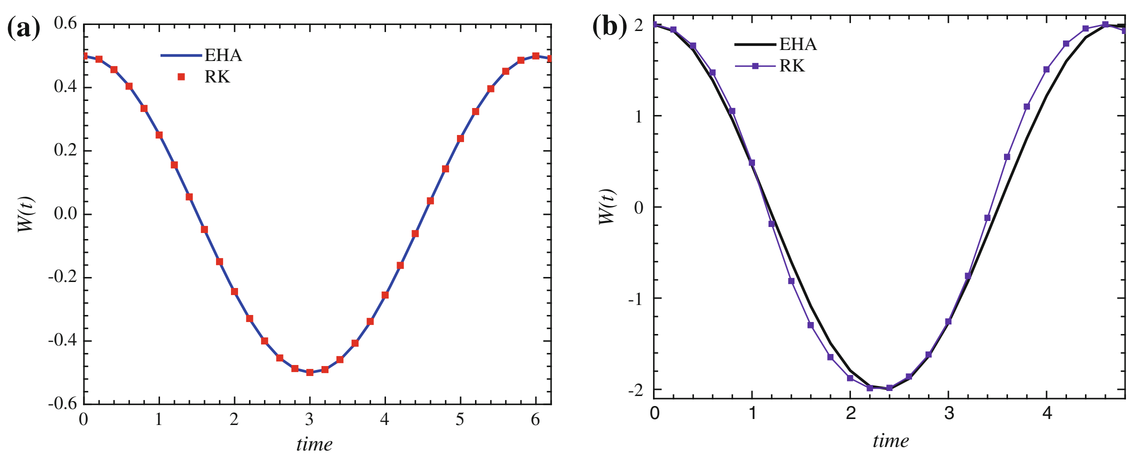

Fig. 1 Comparison of $W(t)$ versus time diagrams: EHA analytical and Runge–Kutta solution **a** for $A = 0.5,\ \alpha = 0.1, \beta = 1,\ \eta = 0.5,\ \lambda = 0.2$ **b** for $A = 4,\ \alpha = 0.5,\ \beta = 2,\ \eta = 0.2,\ \lambda = 0.1$

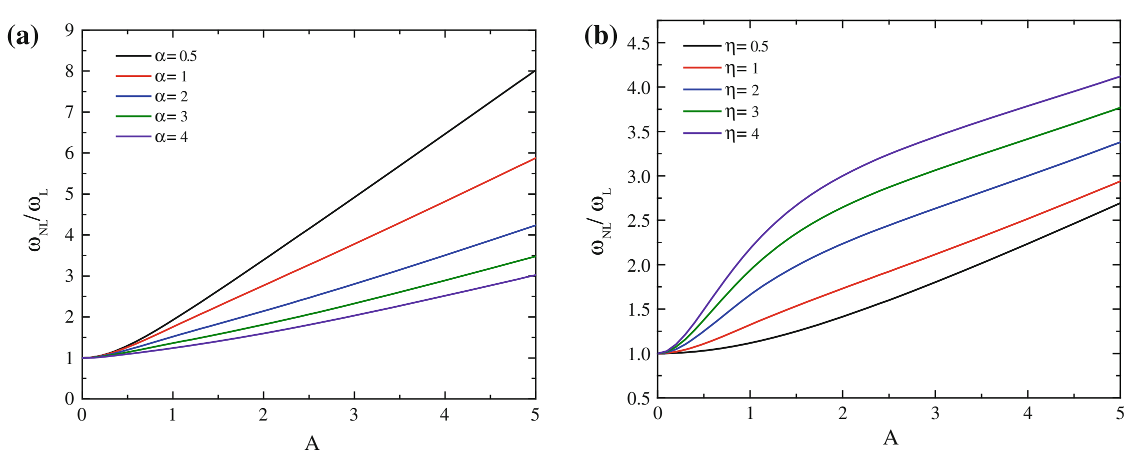

Fig. 2 Comparison of nonlinear to linear frequency corresponding to various parameters of α for **a** $\beta = 0.5,\ \eta = 2,\ \lambda = 0.5$ and η for **b** $\alpha = 1,\ \beta = 0.5,\ \lambda = 0.1$

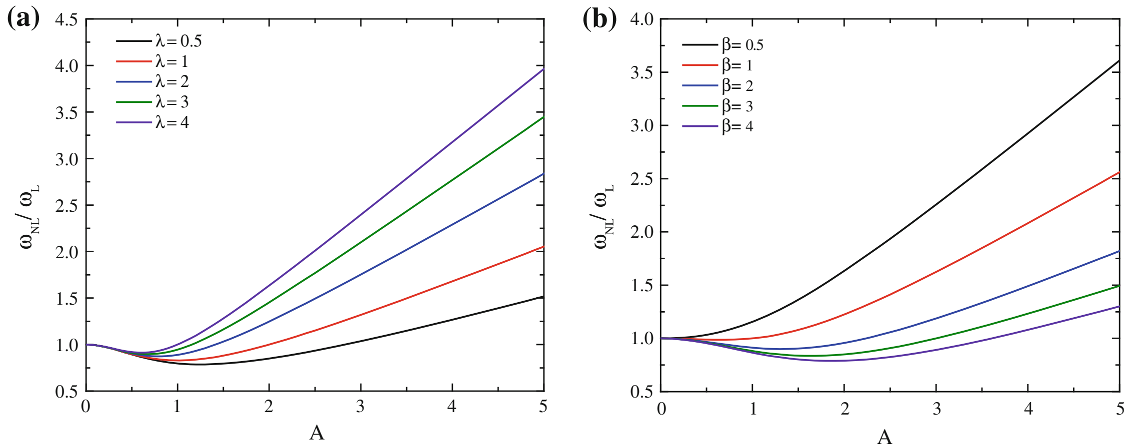

Fig. 3 Comparison of nonlinear to linear frequency corresponding to various parameters of λ for **a** $\alpha = 4,\ \beta = 2,\ \eta = 2$ and β for **b** $\alpha = 1,\ \eta = 0.5,\ \lambda = 0.2$

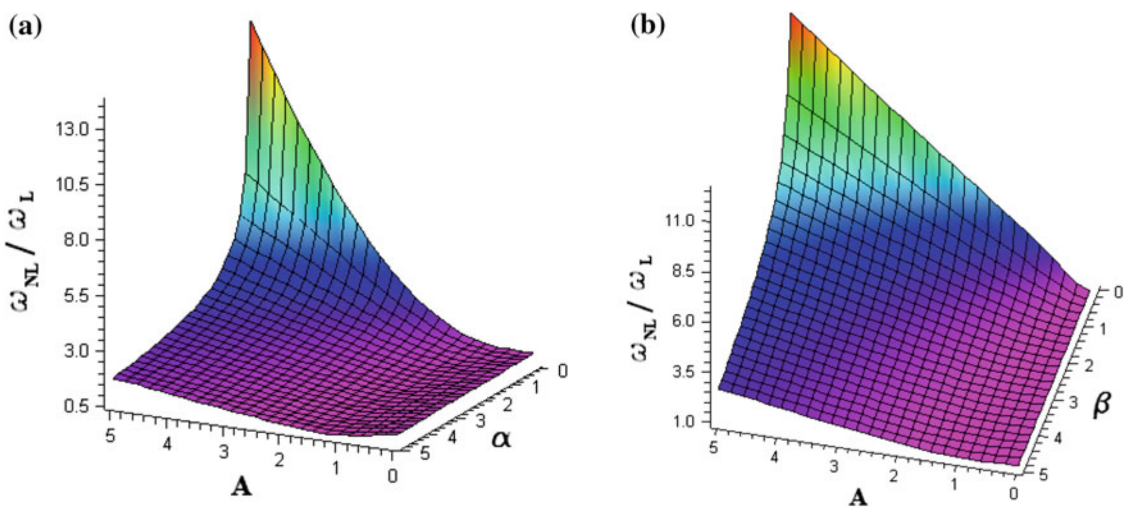

Fig. 4 Sensitivity analysis of nonlinear to linear frequency for **a** $\beta = 4,\ \eta = 1,\ \lambda = 2$ and **b** $\alpha = 2,\ \eta = 4,\ \lambda = 2$

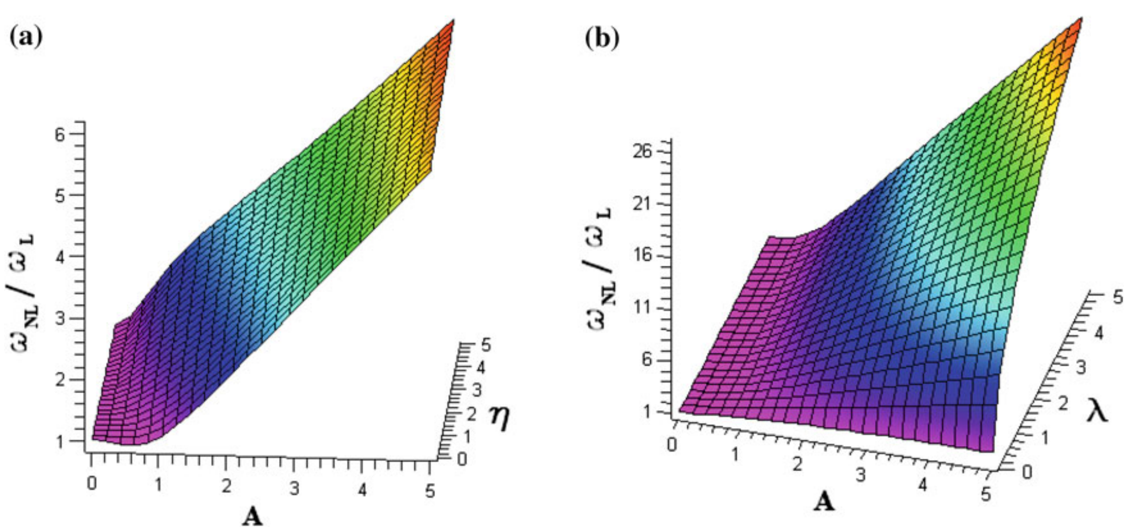

Fig. 5 Sensitivity analysis of nonlinear to linear frequency for **a** $\alpha = 2,\ \beta = 0.5,\ \lambda = 1$ and **b** $\alpha = 1,\ \beta = 0.2,\ \eta = 0.5$

6 Conclusions

It can be concluded:

1. The EHA developed in the paper has been successfully applied to obtain an accurate analytical solution for the nonlinear vibration of a stringer shell.
2. It has illustrated that the results of EHA are in an agreement with those obtained by the numerical method.
3. The influence of the stringer shell parameters on the vibration are as follows: As it is shown, the amplitude of the oscillation has a great effect on the vibrations of the shell. From the relationships of the parameters in Appendix A and B, the radius, thickness and densities of the shell also have great effects on the response of the vibration.
4. EHA does not need any linearization or small perturbation. The method can be a powerful mathematical tool for studying nonlinear oscillators.

Appendix A

$$D_1 = D + \frac{NE_1 I}{(2\pi R)},\ D = \frac{Eh^3}{\left(12\left(1-\nu^2\right)\right)};$$
$$\rho = \rho_0 h + \frac{N\rho_1 F}{(2\pi R)};\ B_1 = \frac{Eh}{\left(1-\nu^2\right)} + \frac{NE_1 F}{(2\pi R)},$$

Appendix B

$$t = \sqrt{\frac{B_1}{\rho}}\frac{t_1}{R}, \quad \beta = \varepsilon_1 + 2\varepsilon_2 p^{-2} + \varepsilon_2 p^{-4} + n^{-4}$$
$$\eta = 0.0625 + 0.5n^2\varepsilon_1 - 0.75, \quad \lambda = 0.25n^4, \quad \alpha = 0.09375n^4$$
$$p = m_1/n, \quad \varepsilon_1 = D_1/\left(B_1 R^2\right), \quad \varepsilon_1 = D_2/\left(B_1 R^2\right), \quad \varepsilon_3 = D_3/\left(B_1 R^2\right)$$

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