Arch Appl Mech (2013) 83:765–781
DOI 10.1007/s00419-012-0716-3

ORIGINAL

Mostafa Talebitooti

Three-dimensional free vibration analysis of rotating laminated conical shells: layerwise differential quadrature (LW-DQ) method

Received: 3 May 2012 / Accepted: 1 November 2012 / Published online: 21 November 2012

Abstract This paper focuses on the free vibration analysis of thick, rotating laminated composite conical shells with different boundary conditions based on the three-dimensional theory, using the layerwise differential quadrature method (LW-DQM). The equations of motion are derived applying the Hamilton's principle. In order to accurately account for the thickness effects, the layerwise theory is used to discretize the equations of motion and the related boundary conditions through the thickness of the shells. Then, the equations of motion as well as the boundary condition equations are transformed into a set of algebraic equation applying the DQM in the meridional direction. This study demonstrates the applicability, accuracy, stability and the fast rate of convergence of the present method, for free vibration analyses of rotating thick laminated conical shells. The presented results are compared with those of other shell theories obtained using conventional methods and a special case where the angle of the conical shell approaches zero, that is, a cylindrical shell and excellent agreements are achieved.

Keywords Rotating conical shell · Layerwise theory · Natural frequency · Differential quadrature method

1 Introduction

Rotating laminated conical shells are increasingly being used in many engineering applications such as the drive shafts of gas turbines, high-speed centrifugal separators, motors and rotor systems because of their strength to weight ratio. Hence, it is of a great importance to understand the vibration behavior of rotating laminated conical shells for the design of aforementioned structures. Most studies were restricted to the vibration analysis of non-rotating and rotating conical shells based on classical laminated shell theory (CLT). They include the works by Sivadas [1] and Lam and Hua [2] on the rotating conical shells with simply supported boundary condition as well as a discussion on influence of boundary condition on rotating conical shell by Lam and Hua [3] and Ng et al. [4]. For non-rotating conical shell, many papers are involved to discuss the influence of boundary conditions on the frequency characteristics, particularly the works done by Sivadas and Ganesan [5], Thambiratnam and Zhuge [6] and Tong [7,8]. This theory cannot be used for thick shells and even thin shells when the number of circumferential waves increases as a result of neglecting shear deformation and rotary inertia effects in CLT. The refinement of thin-shell theories has resulted in a number of the so-called first- and higher-order shear deformation theories to include the effects of transverse shear deformation [9–13]. These theories behave much more accurately than the thin-shell theories for the analysis of slightly thick shells but are still inadequate for the analysis of thick shells. In order to analyze the thick conical shells precisely, the three-dimensional theory should be used to account all the transverse stress and strain components. The literature shows that the dynamic analysis of rotating conical shells based on 3-D theory is complex and the common

M. Talebitooti (✉)
Center of Excellence in Experimental Solid Mechanics and Dynamics, School of Mechanical Engineering,
Iran University of Science and Technology, Narmak, 16846-13114 Tehran, Iran
E-mail: mtalebi@iust.ac.ir

method used for classical shell theories cannot be able to solve these problems [14–17]. In these researches, numerical methods such as the finite element, the Ritz and the series solutions are the most methods used for vibration and buckling analysis of conical shells.

In recent years, the new numerical method so-called hybrid layerwise differential quadrature method (LW-DQM) is employed to discretize the governing equations of structures [18–21]. In contrast to the equivalent single-layer theories (ESLT), the layerwise theories develop the separate displacement field expansions through the each subdivision. The displacement components are continuous through the laminate thickness but their derivatives with respect to the thickness coordinate may be discontinuous at various points through the thickness, thereby allowing for possibility of continuous transverse stress at interfaces separating contradictory materials [11,22].

The differential quadrature method (DQM) is an efficient numerical technique, which was originated by Bellman [23] to solve linear and nonlinear differential equations. The basic idea of the DQ method is that the partial derivative of a function with respect to a variable at a given discrete point can be approximated as a weighted linear sum of the function values at all discrete points in the domain of that variable. The mathematical fundamentals and recent developments of the DQ method as well as its major applications in engineering are discussed by Shu [24]. It is worthwhile to note that the interest of researches [4,25–31] in this procedure is increasing due to its great simplicity and versatility.

The literature clearly shows that there is no investigation on vibration of rotating laminated conical shells with arbitrary boundary conditions and three-dimensional theory of elasticity, in spite of the fact that the quantity of critical speed and vibration of conical shells is important to their applications. Hence, in this study, based on the 3-D theory, a hybrid LW-DQ method for free vibration analysis of rotating laminated conical shell with arbitrary boundary conditions is developed. The accuracy, convergence and versatility of the algorithm are proved via different examples for both thin and thick shells. Also, the effects of geometrical parameters, laminations scheme and different boundary conditions are demonstrated.

2 Problem formulation

2.1 Geometrical configuration

The rotating conical shell, as shown in Fig. 1, is considered to be thick, laminated and composed of an arbitrary number of layers. In this figure, α is the cone angle, L is the length, h is the thickness, a and b are the mean radii at two ends, and Ω is the constant angular velocity of conical shell about its symmetrical and horizontal axis. The reference surface of the conical shell is taken to be at its middle surface where an orthogonal co-ordinate system (x, θ, z) is fixed, and $r = r(x, z)$ is a radius at any co-ordinate point (x, θ, z). The displacement of the shell in the x, θ and z directions are denoted by u, v and w, respectively.

2.2 Displacement field in layerwise theory

The displacement field for closed conical shells in the circumferential direction can be expanded as follows:

$$\begin{aligned} u(x,\theta,z,t) &= \sum_{n=1}^{\infty} U_n(x,z)\cos(n\theta+\omega t) \\ v(x,\theta,z,t) &= \sum_{n=1}^{\infty} V_n(x,z)\sin(n\theta+\omega t) \\ w(x,\theta,z,t) &= \sum_{n=1}^{\infty} W_n(x,z)\cos(n\theta+\omega t) \end{aligned} \tag{1}$$

where n represents the circumferential wave number.

In order to build a high degree of transverse discretization generality into the model, the layerwise laminate theory of Reddy [11] is extended to be used here. The layerwise concept of Reddy is very general in that the number of subdivisions through the thickness can be greater than, equal to, or less than the number of material layers, and the pattern of layerwise interpolation through thickness is very easy to build into the finite

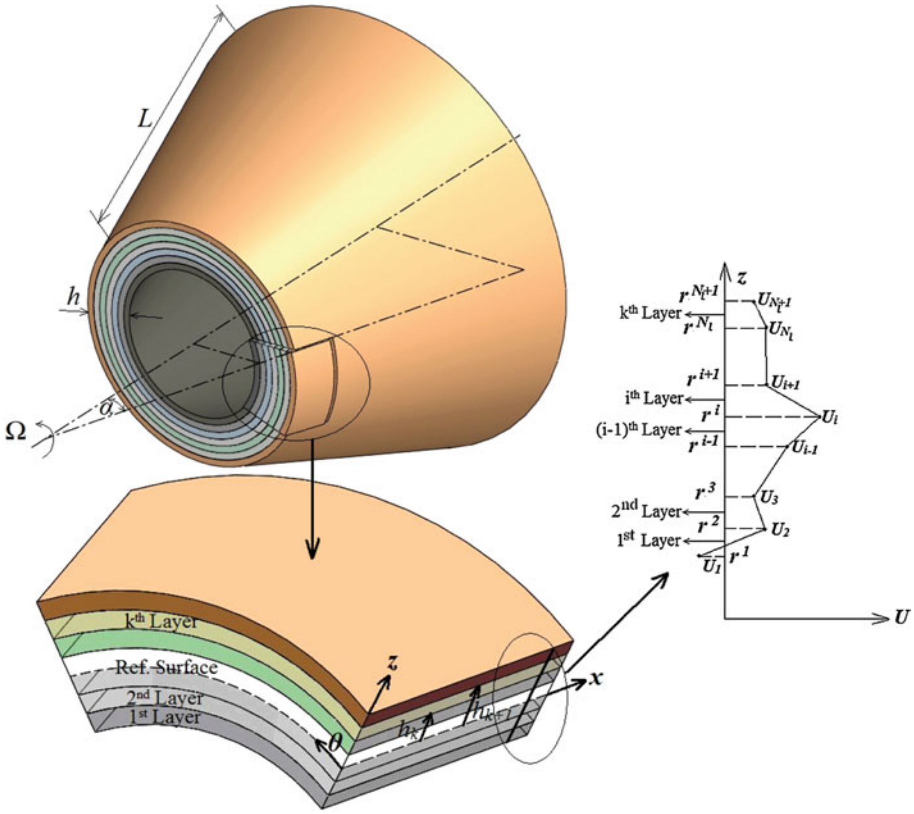

Fig. 1 Configuration of the conical shell and displacement representation used in the layerwise theory

element method. In this theory, the variation of the displacements U_n, V_n and W_n through the thickness can be represented to a certain desired level of accuracy by simply increasing the order of the transverse interpolation polynomials

$$U_n(x,z)=\sum_{i=1}^{N_z}U_{in}(x)\,\psi_i(z)$$
$$V_n(x,z)=\sum_{i=1}^{N_z}V_{in}(x)\,\psi_i(z)$$
$$W_n(x,z)=\sum_{i=1}^{N_z}W_{in}(x)\,\psi_i(z) \qquad (2)$$

where $U_{in}(x)$, $V_{in}(x)$ and $W_{in}(x)$ denote the unknown displacement components at the node ith in the x, θ and z-direction, respectively; N_z represents the number of nodes in the z-direction of the shell, which depends on the number of considered mathematical layers N_l through the thickness and nodes per layer N_{pl} in the thickness direction as $N_z=N_l(N_{pl}-1)+1$. Also, $\psi_i(z)$ is a piecewise continuous Lagrange interpolation function in terms of the thickness coordinate z in the ith mathematical layer and is assumed to take the following form:

$$\psi_i(z)=\begin{cases}0 & z\le z_{i-1}\\ \frac{z-z_{i-1}}{h_{i-1}} & z_{i-1}\le z\le z_i\\ \frac{z_{i+1}-z}{h_i} & z_i\le z\le z_{i+1}\\ 0 & z\ge z_{i+1}\end{cases} \qquad (3)$$

where h_i is the thickness of the ith layer and z_i denotes the global thickness coordinate of the ith layer. Due to this independent interpolation through thickness, this layerwise theory reduces the 3-D problem to a 2-D one.

Table 1 Mechanical properties of the material

Modulus of elasticity (GPa)	Poisson's ratio	Modulus of rigidity (GPa)	Density (kg/m^3)
$E_{11} = 125$	$\upsilon_{12} = 0.4$	$G_{12} = G_{23} = 5.9$	$\rho = 1,643$
$E_{22} = E_{33} = 10$	$\upsilon_{13} = \upsilon_{23} = 0.2$	$G_{13} = 3$	

Table 2 Convergence behavior of the fundamental frequency parameter $f = \omega b\sqrt{\rho h/A_{11}}$ of laminated conical shells with clamped boundary condition ($m = 1$, $L = 3$ m, $a = 1$ m, $\alpha = 15°$, [0°/90°], OM = 10 rev/s, Cs-Cl)

h/a	NGP	$N_l = 2$		$N_l = 4$		$N_l = 6$		$N_l = 8$	
		f_b[a]	f_f[b]	f_b	f_f	f_b	f_f	f_b	f_f
0.02	6	0.12439	0.11135	0.12420	0.11116	0.12416	0.11112	0.12414	0.11110
$n = 5$	8	0.12026	0.10722	0.12007	0.10703	0.12002	0.10698	0.12001	0.10697
	10	0.11765	0.10461	0.11745	0.10441	0.11741	0.10436	0.11739	0.10435
	12	0.11629	0.10325	0.11609	0.10305	0.11605	0.10301	0.11603	0.10299
	14	0.11549	0.10244	0.11529	0.10225	0.11524	0.10220	0.11523	0.10218
	16	0.11501	0.10197	0.11481	0.10177	0.11477	0.10173	0.11475	0.10171
	18	0.11480	0.10176	0.11460	0.10156	0.11456	0.10151	0.11454	0.10150
0.1	6	0.25231	0.23158	0.25184	0.23111	0.25166	0.23093	0.25159	0.23086
$n = 3$	8	0.24748	0.22674	0.24701	0.22627	0.24681	0.22607	0.24673	0.22600
	10	0.24571	0.22495	0.24522	0.22448	0.24502	0.22427	0.24494	0.22419
	12	0.24525	0.22449	0.24476	0.22401	0.24454	0.22379	0.24445	0.22370
	14	0.24512	0.22436	0.24462	0.22387	0.24439	0.22363	0.24429	0.22354
	16	0.24504	0.22428	0.24454	0.22378	0.24430	0.22354	0.24420	0.22345
	18	0.24499	0.22423	0.24447	0.22372	0.24423	0.22348	0.24414	0.22338

[a] f_b = Frequency of backward wave
[b] f_f = Frequency of forward wave

Table 3 Convergence behavior of the fundamental frequency parameter $f = \omega b\sqrt{\rho h/A_{11}}$ of laminated conical shells with clamped boundary condition ($m = 1$, $L = 3$m, $a = 1$m, $\alpha = 30°$, [0°/90°], OM = 10 rev/s, Cs-Cl)

h/a	NGP	$N_l = 2$		$N_l = 4$		$N_l = 6$		$N_l = 8$	
		f_b	f_f	f_b	f_f	f_b	f_f	f_b	f_f
0.02	6	0.09929	0.08795	0.09869	0.08735	0.09855	0.08721	0.09850	0.08716
$n = 7$	8	0.09233	0.08106	0.09174	0.08047	0.09160	0.08033	0.09155	0.08028
	10	0.08811	0.07684	0.08749	0.07622	0.08734	0.07607	0.08729	0.07602
	12	0.08591	0.07465	0.08528	0.07402	0.08514	0.07387	0.08508	0.07382
	14	0.08485	0.07358	0.08422	0.07295	0.08407	0.07280	0.08401	0.07275
	16	0.08457	0.07330	0.08393	0.07266	0.08378	0.07251	0.08372	0.07246
	18	0.08468	0.07342	0.08405	0.07278	0.08389	0.07263	0.08384	0.07257
0.1	6	0.30696	0.28330	0.30655	0.28290	0.30640	0.28276	0.30635	0.28270
$n = 3$	8	0.30041	0.27674	0.29999	0.27633	0.29983	0.27617	0.29977	0.27611
	10	0.29831	0.27461	0.29788	0.27418	0.29770	0.27401	0.29763	0.27394
	12	0.29796	0.27425	0.29751	0.27381	0.29732	0.27362	0.29724	0.27354
	14	0.29782	0.27411	0.29737	0.27366	0.29716	0.27345	0.29707	0.27337
	16	0.29771	0.27400	0.29725	0.27355	0.29703	0.27333	0.29694	0.27324
	18	0.29764	0.27392	0.29717	0.27346	0.29694	0.27324	0.29686	0.27315

2.3 Energy equations of the shell

The strain energy of the laminated composite conical shell is expressed as

$$U_{\varepsilon}^{\text{sh}} = \frac{1}{2}\iiint\limits_{\text{vol}} \varepsilon^T [\bar{C}] \varepsilon r \,\mathrm{d}z \mathrm{d}\theta \mathrm{d}x \tag{4}$$

where $r = r(x, z) = a + x\sin\alpha + z\cos\alpha$ and the strain vector ε based on the three-dimensional theory of elasticity can be written as follows:

$$\varepsilon^T = \left\{ \varepsilon_x \ \varepsilon_\theta \ \varepsilon_z \ \gamma_{\theta z} \ \gamma_{xz} \ \gamma_{x\theta} \right\} \tag{5}$$

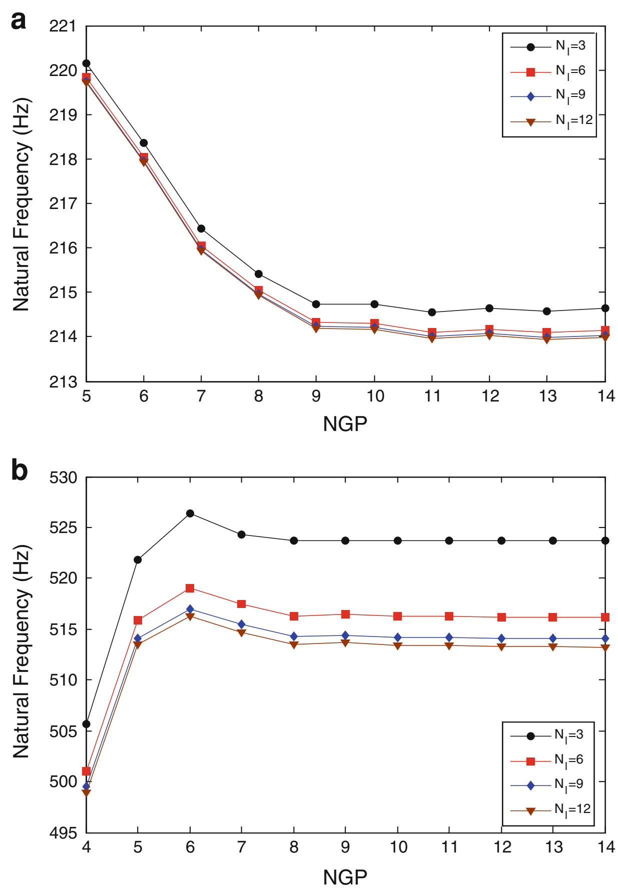

Fig. 2 Convergence behavior of the natural frequency of laminated conical shell with clamped boundary condition ($L = 2$ m, a $= 0.5$ m, $n = 3$, $\alpha = 15°$, [0°/90°/0°], Cs-Cl). **a** $h = 50$ mm, **b** $h = 250$ mm

The linearized strain-displacement relations are

$$\varepsilon = D \cdot \chi \tag{6}$$

where the differential operator D and the vector of displacement χ are defined as follows:

$$D = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ \frac{S_\alpha}{r} & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{C_\alpha}{r} \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} - \frac{C_\alpha}{r} & \frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{\partial}{\partial x} & \frac{-S_\alpha}{r} \end{bmatrix}; \quad \begin{aligned} \chi^T &= \{u, v, w\} \\ S_\alpha &= \sin\alpha \\ C_\alpha &= \cos\alpha \end{aligned} \tag{7}$$

and $[\bar{C}] = [T]^{-1}[C][T]$ is the transformed stiffness matrix for a laminated shell, in which $[C]$ is the principle material stiffness matrix and $[T]$ is the transformation matrix [11,12].

It should be noted that the work is carried out on the shell due to the centrifugal force generated by rotation. The work done on the shell can be written as

$$U_h^{\mathrm{sh}} = \frac{1}{2}\int_0^L\int_0^{2\pi}\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{N_\theta^{\mathrm{sh}}}{r^2}\left\{\left[\left(vC_\alpha - \frac{\partial w}{\partial \theta}\right)\right]^2 + \left[\left(\frac{\partial v}{\partial \theta} + wC_\alpha\right)\right]^2\right\} r\,\mathrm{d}z\,\mathrm{d}\theta\,\mathrm{d}x \tag{8}$$

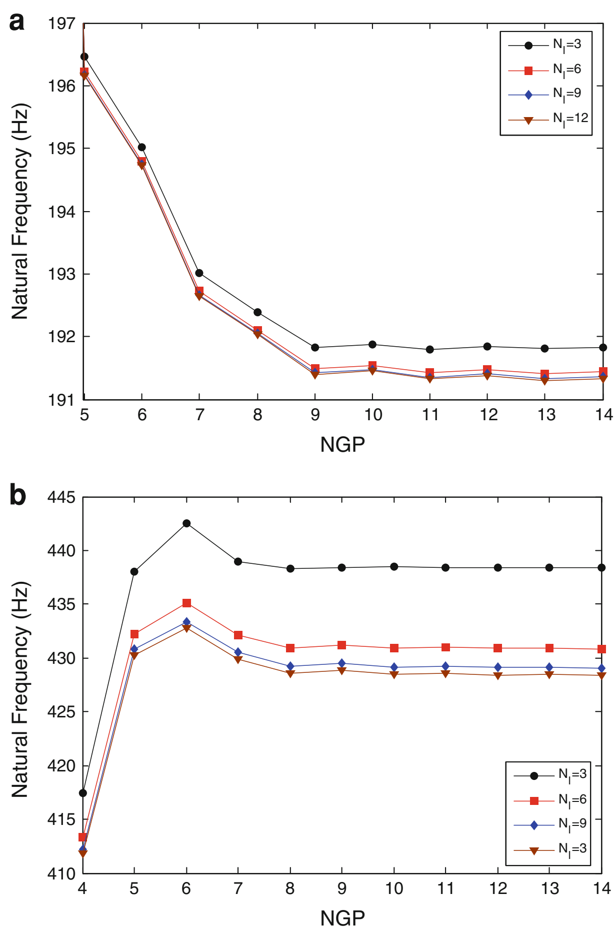

Fig. 3 Convergence behavior of the natural frequency of laminated conical shell with clamped boundary condition ($L = 2\,\mathrm{m}$, $a = 0.5\,\mathrm{m}$, $n = 3$, $\alpha = 30°$, $[0°/90°/0°]$, Cs-Cl). **a** $h = 50\,\mathrm{mm}$, **b** $h = 250\,\mathrm{mm}$

where N_θ^{sh} is the initial hoop stress due to centrifugal force and is defined as

$$N_\theta^{\mathrm{sh}} = \rho(z)\, r^2 \Omega^2 \tag{9}$$

The kinetic energy of the rotating conical shell is expressed as

$$T = \frac{1}{2} \int_0^L \int_0^{2\pi} \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(z)\, \vec{V}.\vec{V} r \mathrm{d}z \mathrm{d}\theta \mathrm{d}x \tag{10}$$

where $\vec{V}$ is the velocity vector at any point of the shell and given by

$$\vec{V} = \dot{\vec{r}} + \left(\Omega C_\alpha \vec{i} - \Omega S_\alpha \vec{k}\right) \times \vec{r} \tag{11}$$

Here, $(\cdot)$ presents differentiation with respect to time.

In Eq. (11), the displacement vector is written as

$$\vec{r} = u\vec{i} + v\vec{j} + w\vec{k} \tag{12}$$

where $\vec{i}$, $\vec{j}$ and $\vec{k}$, respectively, denote the unit vectors in the x, θ and z directions in the non-rotating frame.

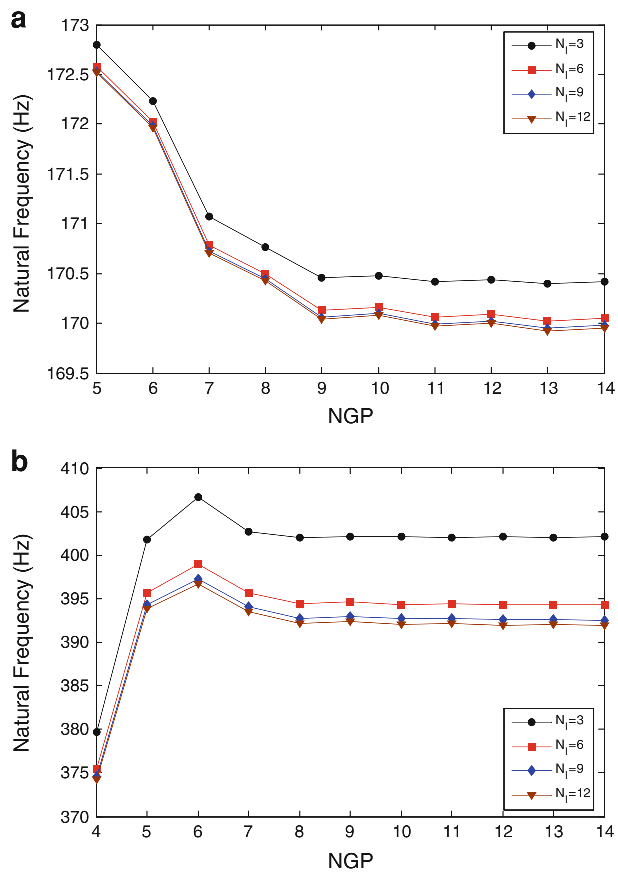

Fig. 4 Convergence behavior of the natural frequency of laminated conical shell with clamped boundary condition ($L = 2\,\mathrm{m}$, $a = 0.5\,\mathrm{m}$, $n = 3$, $\alpha = 45^o$, $[0°/90°/0°]$, Cs-Cl). **a** $h = 50\,\mathrm{mm}$, **b** $h = 250\,\mathrm{mm}$

Table 4 Comparison of frequency parameter $f = \omega R\sqrt{\rho\left(1-\upsilon^2\right)/E}$ for a isotropic laminated cylindrical shell with simply supported boundary condition ($m = 1$, $L/R = 1$, $\upsilon = 0.3$, Ss-Sl)

H/R	H/L											
	0.1				0.2				0.4			
	$n = 1$		$n = 2$		$n = 1$		$n = 2$		$n = 1$		$n = 2$	
	Loy and Lam [33]	Present	Loy and Lam [33]	Present	Loy and Lam [33]	Present	Loy and Lam [33]	Present	Loy and Lam [33]	Present	Loy and Lam [33]	Present
0.2	0.5859	0.5849	0.4208	0.4223	0.9947	0.9921	0.9212	0.9253	2.0664	2.0415	2.1326	2.1432
0.4	0.2646	0.2629	0.3465	0.3417	0.6188	0.5916	0.5806	0.5769	1.2351	1.2216	1.2637	1.1832
0.6	0.1479	0.1475	0.4252	0.4051	0.4032	0.3965	0.5273	0.5114	0.9063	0.8929	0.95010	0.9328

Substituting Eqs. (11) and (12) into Eq. (10), the kinetic energy expression of the shell can be expanded in the form

$$
\begin{aligned}
T^{\mathrm{sh}} = \frac{1}{2}\int_0^L\int_0^{2\pi}\int_{-\frac{h}{2}}^{\frac{h}{2}} & \rho(z)\left[(\dot{u})^2 + (\dot{v})^2 + (\dot{w})^2\right. \\
& +\Omega^2\left(v^2 + 2uwC_\alpha S_\alpha + u^2S_\alpha^2 + w^2C_\alpha^2\right) \\
& \left.+2\Omega S_\alpha\left(\dot{u}v - \dot{v}u\right) + 2\Omega C_\alpha\left(\dot{w}v - \dot{v}w\right)\right] r\,\mathrm{d}z\mathrm{d}\theta\mathrm{d}x \qquad (13)
\end{aligned}
$$

where $S_\alpha^2 = \sin^2\alpha$ and $C_\alpha^2 = \cos^2\alpha$.

Table 5 Comparison of fundamental frequency parameter $f = \left(\omega L^2/100\,h\right)\sqrt{\rho/E_{22}}$ for cross-ply laminated cylindrical shell under various boundary conditions ($h/R = 0.2$, $L/R = 2$, $m = 1$)

	S-S	S-C	C-C	C-F
[0°, 90°]				
CST [34]	0.1630	0.1841	0.2120	0.0938
FSDT [34]	0.1552	0.1697	0.1876	0.0914
HSDT [34]	0.1566	0.1726	0.1928	0.0921
LW-DQ (present)	0.1371	0.1531	0.1718	0.0909
[0°, 90°, 0°]				
CST [34]	0.2073	0.2662	0.3338	0.1099
FSDT [34]	0.1779	0.1945	0.2129	0.0988
HSDT [34]	0.1777	0.1972	0.2191	0.0995
LW-DQ (present)	0.1729	0.1903	0.2108	0.0992
$[0°, 90°]_5$				
CST [34]	0.1958	0.2304	0.2752	0.1077
FSDT [34]	0.1899	0.2012	0.2137	0.1004
HSDT [34]	0.1900	0.2027	0.2166	0.1010
LW-DQ (present)	0.1802	0.1977	0.2104	0.1379

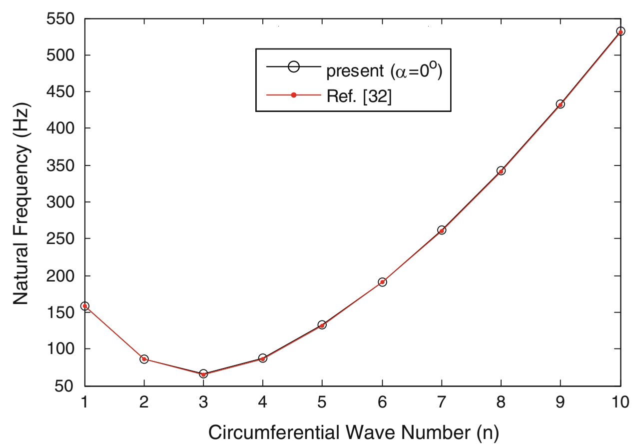

Fig. 5 Variation of the natural frequencies for the thick composite cylindrical shell with the circumferential wave number. Comparison between results of present study and those reported in Ref. [32] ($L = 4$ m, $R = 1$ m, $NGP = 10$, $N_l = 12$, $h = 15$ mm, [0°/90°/0°], Ss-Sl)

2.4 Governing equations of motion

The governing differential equations of motion can be derived by using Hamilton's principle

$$\int_{t_0}^{t_1} \delta\left(\Pi\right) \mathrm{d}t = \int_{t_0}^{t_1} \left(\delta T^{\mathrm{sh}} - \delta U_{\varepsilon}^{\mathrm{sh}} - \delta U_{h}^{\mathrm{sh}}\right) \mathrm{d}t = 0 \tag{14}$$

Substituting Eqs. (4), (8) and (13) into Eq. (14), the equations of motion and the related boundary conditions for free vibration analysis of arbitrary rotating laminated conical shell can be derived as follows:

$$\begin{aligned}
\delta U_{in} : \pi \Bigg\{ & \left(F_{55}^{ij} + n^2 C_{66}^{ij} + S_{\alpha}^2 C_{22}^{ij} - B_{11}^{ij} \frac{\partial^2}{\partial x^2} + G^{ij} \frac{\partial^2}{\partial t^2} + G^{ij} \Omega^2 S_{\alpha}^2 \right) U_{jn} \\
& + \left[-\left(n A_{12}^{ij} + n A_{66}^{ij} \right) \frac{\partial}{\partial x} + n S_{\alpha} \left(C_{22}^{ij} + C_{66}^{ij} \right) + 2 G^{ij} \Omega S_{\alpha} \frac{\partial}{\partial t} \right] V_{jn} \\
& + \left[-\left(C_{\alpha} A_{12}^{ij} + E_{13}^{ji} - E_{55}^{ij} \right) \frac{\partial}{\partial x} + \left(S_{\alpha} C_{\alpha} C_{22}^{ij} + S_{\alpha} D_{23}^{ji} + G^{ij} \Omega^2 S_{\alpha} C_{\alpha} \right) \right] W_{jn} \Bigg\} = 0
\end{aligned} \tag{15}$$

$$\delta V_{in} : \pi \left\{ \left[n \left(A_{12}^{ij} + A_{66}^{ij} \right) \frac{\partial}{\partial x} + n S_{\alpha} \left(C_{22}^{ij} + C_{66}^{ij} \right) - 2 G^{ij} \Omega S_{\alpha} \frac{\partial}{\partial t} \right] U_{jn} \right.$$

Table 6 Comparison of frequency parameter $f = \omega b\sqrt{\left(1-\upsilon^2\right)\rho/E}$ for a non-rotating isotropic conical shell with Cs-Cl boundary condition ($m = 1$, $\upsilon = 0.3$, $h/b = 0.01$, $L\sin\alpha/b = 0.5$)

n	$\alpha = 45°$			$\alpha = 60°$		
	Irie [35]	Lam and Hua [3]	Present	Irie [35]	Lam and Hua [3]	Present
1	0.8120	0.8452	0.8171	0.6316	0.6449	0.6352
2	0.6696	0.6803	0.6724	0.5523	0.5568	0.5549
3	0.5430	0.5553	0.5453	0.4785	0.4818	0.4807
4	0.4570	0.4778	0.4594	0.4298	0.4361	0.4318
5	0.4095	0.4395	0.4121	0.4093	0.4202	0.4113

Table 7 Comparison of frequency parameter $f = \omega b\sqrt{\left(1-\upsilon^2\right)\rho/E}$ for a non-rotating isotropic conical shell with Ss-Cl boundary condition ($m = 1$, $\upsilon = 0.3$, $h/b = 0.01$)

Cone angle	n	$L\sin\alpha/b = 0.5$			$L\sin\alpha/b = 0.75$		
		Irie et al. [35]	Ng et al. [4]	Present	Irie et al. [35]	Ng et al. [4]	Present
$\alpha = 30°$	1	0.8760	0.8808	0.8794	0.7451	0.7452	0.7546
	2	0.6278	0.6241	0.6290	0.4510	0.4530	0.4651
	3	0.4574	0.4537	0.4585	0.2990	0.2998	0.3143
	4	0.3551	0.3506	0.3565	0.2545	0.2520	0.2657
	5	0.3073	0.3017	0.3086	0.2661	0.2620	0.2720
$\alpha = 45°$	5	0.3834	0.3804	0.3831	0.3150	0.3147	0.3198
	6	0.3747	0.3691	0.3740	0.3437	0.3475	0.3464
	7	0.3997	0.3924	0.3988	0.3883	0.3898	0.3902
	8	0.4489	0.4411	0.4479	0.4454	0.4458	0.4478
	9	0.5142	0.5063	0.5134	0.5133	0.5195	0.5168

$$
\begin{aligned}
&+\left[n^2C_{22}^{ij}+F_{44}^{ij}-\left(D_{44}^{ij}+D_{44}^{ji}\right)C_\alpha+C_{44}^{ij}C_\alpha^2-B_{66}^{ij}\frac{\partial^2}{\partial x^2}\right.\\
&\left.+C_{66}^{ij}S_\alpha^2+G^{ij}\Omega^2\left(1-C_\alpha^2-n^2\right)+G^{ij}\frac{\partial^2}{\partial t^2}\right]V_{jn}\\
&\left.+\left[n\left(D_{23}^{ji}+\left(C_{44}^{ij}+C_{22}^{ij}\right)C_\alpha-D_{44}^{ij}\right)-2G^{ij}\Omega C_\alpha\left(\frac{\partial}{\partial t}+\Omega n\right)\right]W_{jn}\right\}=0
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
\delta W_{in}:\pi&\left\{\left[\left(C_\alpha A_{12}^{ij}+E_{13}^{ij}-E_{55}^{ij}\right)\frac{\partial}{\partial x}+S_\alpha C_\alpha C_{22}^{ij}+S_\alpha D_{23}^{ij}+G^{ij}\Omega^2S_\alpha C_\alpha\right]U_{jn}\right.\\
&+\left[n\left(D_{23}^{ij}+\left(C_{44}^{ij}+C_{22}^{ij}\right)C_\alpha-D_{44}^{ji}\right)+2G^{ij}\Omega C_\alpha\left(\frac{\partial}{\partial t}-\Omega n\right)\right]V_{jn}\\
&+\left[C_{22}^{ij}C_\alpha^2+\left(D_{23}^{ji}+D_{23}^{ij}\right)C_\alpha+F_{33}^{ij}\right.\\
&\left.\left.+n^2C_{44}^{ij}-B_{55}^{ij}\frac{\partial^2}{\partial x^2}+G^{ij}\frac{\partial^2}{\partial t^2}-G^{ij}\Omega^2n^2\right]W_{jn}\right\}=0
\end{aligned}
\tag{17}
$$

Geometrical and natural boundary conditions at both ends are derived as:

Either

$$U_{in}=0$$

or

$$N_x^{in}=\pi\left[\left(A_{12}^{ij}C_\alpha+E_{13}^{ji}\right)W_{jn}+nA_{12}^{ij}V_{jn}+\left(B_{11}^{ij}\frac{\partial}{\partial x}+A_{12}^{ij}S_\alpha\right)U_{jn}\right]=0 \tag{18}$$

Either

$$V_{in}=0$$

or

$$M_{x\theta}^{in}=\pi\left[-nA_{66}^{ij}U_{jn}+B_{66}^{ij}\frac{\partial V_{jn}}{\partial x}-S_\alpha A_{66}^{ij}W_{jn}\right]=0 \tag{19}$$

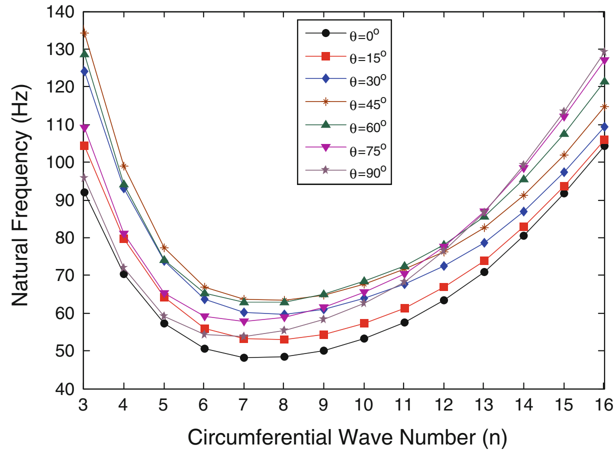

Fig. 6 Variation of natural frequency of backward waves with circumferential mode number for rotating conical shells with different stacking sequence layered ($L = 3\,\text{m}$, $a = 0.5\,\text{m}$, $h = 10\,\text{mm}$, $\alpha = 30°$, $\Omega = 5(\text{rev/s})$, Ss-Cl)

Either

$$W_{in} = 0$$

or

$$M_{xz}^{in} = \pi \left[E_{55}^{ji} U_{jn} + B_{55}^{ij} \frac{\partial W_{jn}}{\partial x} \right] = 0 \tag{20}$$

where

$$A_{kl}^{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} C_{kl} \psi_i \psi_j \mathrm{d}z, \quad B_{kl}^{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} C_{kl} \psi_i \psi_j r \mathrm{d}z$$

$$C_{kl}^{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} C_{kl} \frac{\psi_i \psi_j}{r} \mathrm{d}z, \quad D_{kl}^{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} C_{kl} \psi_i' \psi_j \mathrm{d}z$$

$$E_{kl}^{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} C_{kl} \psi_i' \psi_j r \mathrm{d}z, \quad F_{kl}^{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} C_{kl} \psi_i' \psi_j' r \mathrm{d}z$$

$$G^{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \psi_i \psi_j r \mathrm{d}z \tag{21}$$

2.5 Differential quadrature discretization

The DQ method is a numerical method with flexibility given for the arrangement of the grid points in the domain of interest. It also possesses higher-order interpolation characteristics. In order to use the DQ method to discretize the governing equations in the meridional direction x, each nodal surface is discretized into a set of N_x grid points in this direction [4]. A brief review of the DQ method is presented in "Appendix A". Using the DQ-discretization rules at each domain and boundary condition grid point, it can be deduced that

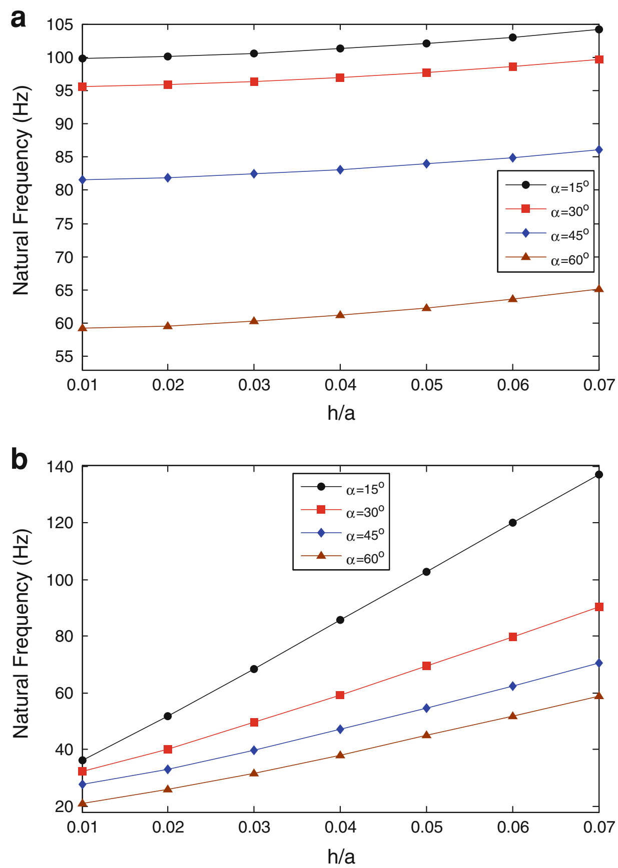

Fig. 7 Variation of natural frequency with geometric ratio h/a for cross-ply laminated conical shell with different cone angle in mode **a** $n = 2$, **b** $n = 6$ ($L = 4$ m, $a = 1$ m, $m = 1$, $\Omega = 0$, Cs-Sl, $[0^\circ/90^\circ/0^\circ]$)

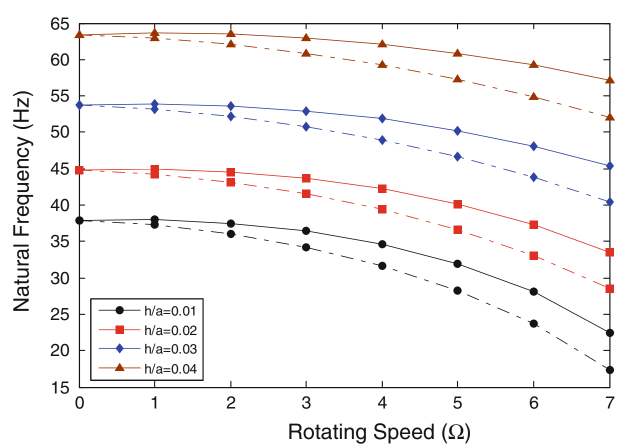

Fig. 8 Variation of natural frequency of cross-ply laminated conical shell with rotating speed Ω for different geometric ratio h/a with Cs-Fl boundary condition ($L = 5$ m, $a = 1$ m, $m = 1$, $n = 5$, $\alpha = 15^\circ$, $[0^\circ/90^\circ/0^\circ]$)

$$\delta U_{inm} : \pi \left\{ \left(F_{55}^{ij} + n^2 C_{66}^{ij} + S_\alpha^2 C_{22}^{ij} + G^{ij} \frac{\partial^2}{\partial t^2} + G^{ij} \Omega^2 S_\alpha^2 \right) U_{jnm} - B_{11}^{ij} \sum_{l=1}^{N_x} \tilde{C}_{ml}^2 U_{jnl} \right.$$
$$- \left(n A_{12}^{ij} + n A_{66}^{ij} \right) \sum_{l=1}^{N_x} \tilde{C}_{ml}^1 V_{jnl} + \left[n S_\alpha \left(C_{22}^{ij} + C_{66}^{ij} \right) + 2 G^{ij} \Omega S_\alpha \frac{\partial}{\partial t} \right] V_{jnm}$$
$$- \left(C_\alpha A_{12}^{ij} + E_{13}^{ji} - E_{55}^{ij} \right) \sum_{l=1}^{N_x} \tilde{C}_{ml}^1 W_{jnl}$$
$$\left. + \left(S_\alpha C_\alpha C_{22}^{ij} + S_\alpha D_{23}^{ji} + G^{ij} \Omega^2 S_\alpha C_\alpha \right) W_{jnm} \right\} = 0 \tag{22}$$

$$\delta V_{inm} : \pi \left\{ n \left(A_{12}^{ij} + A_{66}^{ij} \right) \sum_{l=1}^{N_x} \tilde{C}_{ml}^1 U_{jnl} + \left[n S_\alpha \left(C_{22}^{ij} + C_{66}^{ij} \right) - 2 G^{ij} \Omega S_\alpha \frac{\partial}{\partial t} \right] U_{jnm} \right.$$
$$+ \left[n^2 C_{22}^{ij} + F_{44}^{ij} - \left(D_{44}^{ij} + D_{44}^{ji} \right) C_\alpha + C_{44}^{ij} C_\alpha^2 + C_{66}^{ij} S_\alpha^2 \right.$$
$$\left. + G^{ij} \frac{\partial^2}{\partial t^2} + G^{ij} \Omega^2 \left(1 - C_\alpha^2 - n^2 \right) \right] V_{jnm} - B_{66}^{ij} \sum_{l=1}^{N_x} \tilde{C}_{ml}^2 V_{jnl}$$
$$\left. + \left[n \left(D_{23}^{ji} + \left(C_{44}^{ij} + C_{22}^{ij} \right) C_\alpha - D_{44}^{ij} \right) - 2 G^{ij} \Omega C_\alpha \left(\frac{\partial}{\partial t} + \Omega n \right) \right] W_{jnm} \right\} = 0 \tag{23}$$

$$\delta W_{inm} : \pi \left\{ \left(\cos\alpha A_{12}^{ij} + E_{13}^{ij} - E_{55}^{ij} \right) \sum_{l=1}^{N_x} \tilde{C}_{ml}^1 U_{jnl} \right.$$
$$+ \left[\left(S_\alpha C_\alpha C_{22}^{ij} + S_\alpha D_{23}^{ij} \right) + G^{ij} \Omega^2 S_\alpha C_\alpha \right] U_{jnm}$$
$$+ \left[n \left(D_{23}^{ij} + \left(C_{44}^{ij} + C_{22}^{ij} \right) C_\alpha - D_{44}^{ji} \right) + 2 G^{ij} \Omega C_\alpha \left(\frac{\partial}{\partial t} - \Omega n \right) \right] V_{jnm}$$
$$+ \left[\left(C_{22}^{ij} C_\alpha^2 + \left(D_{23}^{ji} + D_{23}^{ij} \right) C_\alpha + F_{33}^{ij} + n^2 C_{44}^{ij} \right) + G^{ij} \left(\frac{\partial^2}{\partial t^2} - \Omega^2 n^2 \right) \right] W_{jnm}$$
$$\left. - B_{55}^{ij} \sum_{l=1}^{N_x} \tilde{C}_{ml}^2 W_{jnl} \right\} = 0 \tag{24}$$

Either

$$U_{inm} = 0$$

or

$$N_x^{in} = \pi \left[\left(A_{12}^{ij} C_\alpha + E_{13}^{ji} \right) W_{jn} + n A_{12}^{ij} V_{jn} + A_{12}^{ij} S_\alpha U_{jn} + B_{11}^{ij} \sum_{l=1}^{N_x} \tilde{C}_{ml}^1 U_{jnl} \right] = 0 \tag{25}$$

Either

$$V_{inm} = 0$$

or

$$M_{x\theta}^{in} = \pi \left[-n A_{66}^{ij} U_{jn} + B_{66}^{ij} \sum_{l=1}^{N_x} \tilde{C}_{ml}^1 V_{jnl} - S_\alpha A_{66}^{ij} W_{jn} \right] = 0 \tag{26}$$

Either

$$W_{inm} = 0$$

or

$$M_{xz}^{in} = \pi \left[E_{55}^{ji} U_{jn} + B_{55}^{ij} \sum_{l=1}^{N_x} \tilde{C}_{ml}^{1} W_{jnl} \right] = 0 \tag{27}$$

In present paper, one of the following boundary conditions or some combinations of them is considered as follows:

(a) Clamped

$$U_{in} = V_{in} = W_{in} = 0 \tag{28}$$

(b) Simply supported

$$V_{in} = W_{in} = N_x^{in} = 0 \tag{29}$$

(c) Free

$$N_x^{in} = M_{x\theta}^{in} = M_{xz}^{in} = 0 \tag{30}$$

Thus, the whole system of differential equation has been discretized and the global assembling leads to the following set of linear algebraic equations:

$$\left\{ [M]\, \omega^2 \right\} \{d\} + \{[G]\, \omega\} \{d\} + [K_{dd}] \{d\} + [K_{db}] \{b\} = 0 \tag{31}$$

In the above equations, vectors $\{d\}$ and $\{b\}$ with dimensions $3N_z(N_x - 2)$ and $6N_z$ denote the unknowns at the sampling points in the interior domain and those on the boundary. The dimensions of $[M]$, $[K_{dd}]$ and $[G]$ are $N^* \times N^*(N^* = 3N_z(N_x - 2))$ and dimension of $[K_{db}]$ is $N^* \times 6N_z$.

In a similar manner, the discretized form of the boundary conditions become

$$[K_{bd}] \{d\} + [K_{bb}] \{b\} = 0 \tag{32}$$

The dimension of $[K_{bd}]$ is $6N_z \times N^*$ and dimension of $[K_{bb}]$ is $6N_z \times 6N_z$.

Using Eq. (32) to eliminate the boundary degrees of freedom $\{b\}$ from Eq. (31), it can be concluded as

$$\left\{ [M]\, \omega^2 \right\} \{d\} + \{[G]\, \omega\} \{d\} + \left\{ [K_{dd}] - [K_{db}] [K_{bb}]^{-1} [K_{bd}] \right\} \{d\} = 0 \tag{33}$$

Equation (33) is a non-standard eigenvalue equation. For a given frequency, it can be transformed equivalently into a standard form of eigenvalue equation as follows:

$$\left(\underbrace{\begin{bmatrix} 0 & I \\ -K & -G \end{bmatrix}}_{A^*} - \underbrace{\begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}}_{B^*} \omega \right) \begin{Bmatrix} d \\ \omega d \end{Bmatrix} = 0 \tag{34}$$

where I is a $N^* \times N^*$ identity matrix.

Using a conventional eigenvalue approach, the standard eigenvalue Eq. (34) can be solved, and $2N^*$ eigenvalues are obtained. From these eigenvalues, the two eigenvalues for which the absolute of real values are the smallest are chosen. One of these eigenvalues is negative and corresponds to backward wave, and the other one is positive and corresponds to forward wave. In the case of a stationary conical shell, these two eigenvalues are identical and the vibration of the conical shell is a standing wave motion.

3 Numerical results and discussion

At this section, the backward and forward waves are presented as a solid line and a dashed line, respectively; the unit of rotating speed Ω is rps (rps, revolution per second). In addition, five boundary conditions are considered here for the rotating conical shell. These boundary conditions are the fully clamped (Cs-Cl), fully simply supported (Ss-Sl), simply supported at small edge-clamped at large edge (Ss-Cl), clamped at small edge-simply supported at large edge (Cs-Sl), and clamped at small edge-free at large edge (Cs-Fl). Moreover, unless otherwise stated, all layers are assumed to be of the same thickness and material and material properties of the shell used in the present study are listed in Table 1.

Several examples are demonstrated to show the verification and efficiency of the results along with the convergence of the present analysis. The results obtained via the LW-DQ method are compared with those of other shell theories for conical shells. Additionally, the present results are particularly compared with those of Malekzadeh et al. [32] for laminated cylindrical shell, by taking $\alpha = 0$ into the present formulations. Then, the effects of boundary conditions, rotating speed on the natural frequency of conical shells are investigated.

3.1 Convergence checking

In Tables 2 and 3, the convergence behaviors of the non-dimensional fundamental frequency parameter $f = \omega b\sqrt{\rho h/A_{11}}$ of forward and backward waves for rotating cross-ply [0°/90°] laminated conical shell constrained with Cs-Cl boundary condition with cone angles 15° and 30° are presented, respectively. The convergence behaviors versus the number of grid points (NGP) along the meridional direction and the number of mathematical layers through the thickness of the shell (N_l) are examined. The results are performed for two different ratios of thickness to mean small radius of the shell, h/a. In these case studies, the fundamental frequencies are occurred in circumferential mode numbers 5 and 3 for h/a ratios 0.02 and 0.1, respectively. The fast rate of convergence of the method is quite evident.

Also, the convergence behaviors of the method for conical shells with two different values of thicknesses for three different cone angles 15°, 30° and 45° are presented in Figs. 2, 3 and 4, respectively. As it is well anticipated from these figures, with increasing the thickness, the convergence behaviors have been postponed. It should be noted that based on the presented results, it was found that the trend of convergence for conical shell with different cone angles is approximately the same.

3.2 Validation

In order to validate the results, firstly, the formulation is customized for an isotropic cylindrical shell with fully simply supported boundary condition. The results of the first two frequency parameters $f = \omega R\sqrt{\rho\left(1-\upsilon^2\right)/E}$ of thick cylindrical shells, listed in Table 4, are compared with those of LW-exact solutions by Loy and Lam [33] for different H/L and H/R.

The further comparisons are demonstrated with the work done by Khdeir et al. [34] which considers different theories of HSDT, FSDT, and CST for the non-rotating shells. The fundamental frequency parameter $f = \left(\omega L^2/100h\right)\sqrt{\rho/E_{22}}$ of symmetric and antisymmetric cross-ply laminated cylindrical shells under various boundary conditions are listed in Table 5. The trend of the results listed in this table indicates the accuracy of the present work as it is based on 3D theory of elasticity.

The third comparison is done with the work done by Malekzadeh et al. [32] which considers the LW-DQ method for a non-rotating laminated cylindrical shell with fully simply supported boundary condition. As depicted in Fig. 5, comparing the results shows the excellent accuracy of the present work.

Lastly, with taking $\Omega = 0$ into the formulations, the present results have been compared with those of Refs. [3,4,35] for a non-rotating thin conical shell with Ss-Sl and also Ss-Cl boundary conditions as listed in Tables 6 and 7, respectively.

3.3 Parameter studies

The previous sections demonstrate the fast convergence behavior and reliability of the present method. Here, the effects of ply angle, thickness and rotating speed on the natural frequency of the laminated conical shells with different boundary conditions are considered.

The effect of ply angle of symmetric angle-ply laminated conical shells for different values of circumferential wave numbers are shown in Fig. 6. Seven patterns are defined to designate stacking sequence of plies. Pattern [0°/45°/0°] results in high natural frequencies particularly at lower circumferential modes. However, monitoring the other patterns gives a new idea about the middle ply and that when the angle-ply approaches from 0 to 90 provides the high natural frequencies at high circumferential modes. This is due to the fact that they contain the layers which made the structure stiffer in circumference at high circumferential wave numbers where the wavelength decreases.

The influence of geometric ratio h/a on natural frequency of the conical shells with different cone angles, α, are illustrated in Fig. 7. As depicted, increasing the cone angle leads to a decrease in natural frequency of the shell due to more outspread mass and less curvature stiffness. It can also be concluded that the increasing the geometric ratio h/a may enhance the natural frequency of the shell, particularly those occurred in high circumferential mode numbers as a shortening the waves.

Figure 8 shows the influence of the thickness of shell on natural frequency of conical shell with different rotating speed. From the figure, it is seen that with increasing rotating speed, the difference of natural frequency between backward and forward waves increases monotonically for different thicknesses. It is also observed that in high rotating speed, the frequencies of the shells generally decrease with increasing rotating speed for both forward and backward waves. This is due to the fact that the inertia terms generated by rotation of the shell become more significant rather than the stiffening terms influenced by initial hoop tension.

4 Conclusion

In this paper, the three-dimensional free vibration analysis of thick rotating laminated conical shells is developed using a mixed LW-DQM. Hamilton's principle with the layerwise theory is used to discretize the through-thickness form of the equation of motion and the related boundary conditions of the conical shell. Then, the equations of motion as well as the boundary condition equations are transformed into a set of algebraic equation applying the DQM in the meridional direction. It is found that the this approach present a fast rate of convergence and yield accurate results when compared with those of available in the literature and these documented results constitute a useful reference source. It is concluded that the mixed LW-DQM can be used as a robust, effective and accurate numerical method for analysis of thick rotating laminated conical shells.

Appendix A: the basic idea of the DQ method

The DQM is based on a simple mathematical concept that any sufficiently smooth function in a domain can be expressed approximately as an $(N-1)$th order polynomial in the overall domain. In other words, at a discrete mesh point in a domain, the derivative of a sufficiently smooth function with respect to a coordinate direction can be approximated by taking a weighted linear sum of the functional values at all the discrete mesh points in the coordinate direction. Thus, the partial derivatives of a function $f(x)$ as an example, at a point (x_i), are expressed as Shu [24]

$$\left.\frac{\partial^s f(x)}{\partial x^s}\right|_{x=x_i} = \sum_{j=1}^{N} \tilde{C}_{ij}^{s} f\left(x_j\right), \quad i = 1, 2, \ldots, N. \tag{A. 1}$$

where N is the NGP and f can be taken u, v and w in the present study and $\tilde{C}_{ij}^{s}$ are the respective weighting coefficient related to the sth order derivative and is obtained as follows:

If $s=1$, namely for the first-order derivative,

$$\tilde{C}_{ij}^{1} = \frac{M^{(1)}(x_i)}{\left(x_i - x_j\right) M^{(1)}\left(x_j\right)} \quad \text{for} \quad i \neq j \quad \text{and} \quad i, j = 1, 2, \ldots N \tag{A. 2}$$

and

$$\tilde{C}_{ii}^{1} = -\sum_{j=1 (j \neq i)}^{N} \tilde{C}_{ij}^{1} \quad \text{for} \quad i = 1, 2, \ldots, N \tag{A. 3}$$

where $M^{(1)}(x)$ is the first-order derivative of $M(x)$ and they can be defined as

$$M(x) = \prod_{j=1}^{N} (x - x_j) M^{(1)}(x_k) = \prod_{j=1(j \neq k)}^{N} (x_k - x_j) \tag{A. 4}$$

If $s > 1$, namely for the second- and higher-order derivatives, the weighting coefficient are obtained, using the following simple recurrence relationship:

$$\tilde{C}_{ij}^{s} = r\left(\tilde{C}_{ij}^{1} \cdot \tilde{C}_{ii}^{s-1} - \frac{\tilde{C}_{ij}^{s-1}}{x_i - x_j}\right) \quad \text{for } i \neq j \text{ and } \quad i, j = 1, 2, \ldots, N; \quad s = 2, 3, \ldots, N-1 \tag{A. 5}$$

and

$$\tilde{C}_{ii}^{s} = -\sum_{j=1(j \neq i)}^{N} \tilde{C}_{ij}^{s} \quad \text{for} \quad i = 1, 2, \ldots, N \tag{A. 6}$$

Because the coordinate distribution and the number of discrete grid points (NGP) can be chosen arbitrarily in the implementation of the DQM, the following distributions of the grid points in the meridional x direction will be used in the present formulation.

$$x_i = \frac{L}{2}\left(1 - \cos\left(\frac{i-1}{N-1}\pi\right)\right), \quad i = 1, 2, \ldots, N \tag{A. 7}$$

References

1. Sivadas, K.R.: Vibration analysis of pre-stressed rotating thick circular conical shell. J. Sound Vib. **186**, 99–109 (1995)
2. Lam, K.Y., Hua, L.: Vibration analysis of a rotating truncated circular conical shell. Int. J. Solids Struct. **34**(17), 2183–2197 (1997)
3. Lam, K.Y., Hua, L.: Influence of boundary conditions on the frequency characteristic of a rotating truncated circular conical shell. J. Sound Vib. **223**(2), 171–195 (1999)
4. Ng, T.Y., Hua, L., Lam, K.Y.: Generalized differential quadrature for free vibration of rotating composite laminated conical shell with various boundary conditions. Int. J. Mech. Sci. **45**, 567–587 (2003)
5. Sivadas, K.R., Ganesan, N.: Vibration analysis of laminated conical shells with variable thickness. J. Sound Vib. **148**, 477–491 (1991)
6. Thambiratnam, D.P., Zhuge, Y.: Axisymmetric free vibration analysis of conical shells. Eng. Struct. **15**, 83–89 (1993)
7. Tong, L.Y.: Free vibration of orthotropic conical shells. Int. J. Eng. Sci. **31**, 719–733 (1993)
8. Tong, L.Y.: Free vibration of composite laminated conical shells. Int. J. Mech. Sci. **35**, 47–61 (1993)
9. Toorani, M.H., Lakis, A.A.: General equations of anisotropic plates and shells including transverse shear deformations, rotary inertia and initial curvature effects. J. Sound Vib. **237**, 561–615 (2000)
10. Auricchio, F., Sacco, E.: Refined first-order shear deformation theory models for composite laminates. J. Appl. Mech. **70**, 381–390 (2003)
11. Reddy, J.N.: Mechanics of laminated composite plates and shells: theory and analysis. 2nd edn. CRC press, New York (2004)
12. Qatu, M.S.: Vibration of laminated shells and plates. 1st edn. Elsevier Ltd, Amsterdam (2004)
13. Tornabene, F.: Free vibration analysis of functionally graded conical, cylindrical shell and annular plate structures with a four-parameter power-law distribution. Comput. Meth. Appl. Mech. Eng. **198**, 2911–2935 (2009)
14. Smith, T.A.: Numerical analysis of rotationally symmetric shells by the modal superposition method. J. Sound Vib. **233**(3), 515–543 (2000)
15. Buchanan, G.R., Wong, F.T.I.: Frequencies and mode shapes for thick truncated hollow cones. Int. J. Mech. Sci. **43**, 2815–2832 (2001)
16. Gupta, N.K., Sheriff, N.M., Velmurugan, R.: A study on buckling of thin conical frusta under axial loads. Thin-Walled Struct. **44**, 986–996 (2006)
17. Li, F.M., Kishimoto, K., Huang, W.H.: The calculations of natural frequencies and forced vibration responses of conical shell using the Rayleigh–Ritz method. Mech. Res. Commun. **36**, 595–602 (2009)
18. Zhang, Z., Feng, C., Liew, K.M.: Three-dimensional vibration analysis of multilayered piezoelectric composite plates. Int. J. Eng. Sci. **44**, 397–408 (2006)
19. Malekzadeh, P., Setoodeh, A.R., Barmshouri, E.: A hybrid layerwise and differential quadrature method for in-plane free vibration of laminated thick circular arches. J. Sound Vib. **315**, 212–225 (2008)
20. Heydarpour, Y., Malekzadeh, P., Golbahar Haghighi, M.R., Vaghefi, M.: Thermoelastic analysis of rotating laminated functionally graded cylindrical shells using layerwise differential quadrature method. Acta. Mech. **223**(1), 81–93 (2012)
21. Setoodeh, A.R., Tahani, M., Selahi, E.: Hybrid layerwise-differential quadrature transient dynamic analysis of functionally graded axisymmetric cylindrical shells subjected to dynamic pressure. Compos. Struct. **93**, 2663–2670 (2011)
22. Reddy, J.N.: A generalization of two-dimensional theories of laminated composite plates. Commun. Appl. Numer. Methods **3**, 173–180 (1987)

23. Bellman, R.E.: Methods of Nonlinear Analysis. Academic Press, New York (1973)
24. Shu, C.: Differential Quadrature and Its Application in Engineering. Springer, Berlin (2000)
25. Wang, Y., Liu, R., Wang, X.: On free vibration analysis of nonlinear piezoelectric circular shallow spherical shells by the differential quadrature element method. J. Sound Vib. **245**, 179–185 (2001)
26. Liew, K.M., Ng, T.Y., Zhang, J.Z.: Differential quadrature-layerwise modeling technique for three dimensional analysis of cross-ply laminated plates of various edge supports. Comput. Meth. Appl. Mech. Eng. **191**, 3811–3832 (2002)
27. Karami, G., Malekzadeh, P.: A new differential quadrature methodology for beam analysis and the associated differential quadrature element method. Comput. Meth. Appl. Mech. Eng. **191**, 3509–3526 (2002)
28. Huang, Y.Q., Li, Q.S.: Bending and buckling analysis of antisymmetric laminates using the moving least square differential quadrature method. Comput. Meth. Appl. Mech. Eng. **193**, 3471–3492 (2004)
29. Wang, X., Wang, Y.: Free vibration analyses of thin sector plates by the new version of differential quadrature method. Comput. Meth. Appl. Mech. Eng. **193**, 3957–3971 (2004)
30. Wang, X.: Nonlinear stability analysis of thin doubly curved orthotropic shallow shells by the differential quadrature method. Comput. Meth. Appl. Mech. Eng. **196**, 2242–2251 (2007)
31. Haftchenari, H., Darvizeh, M., Darvizeh, A., Ansari, R., Sharama, C.B.: Dynamic analysis of composite cylindrical shells using differential quadrature method (DQM). Compos. Struct. **78**, 292–298 (2007)
32. Malekzadeh, P., Farid, M., Zahedinejad, P.: A three-dimensional layerwise-differential quadrature free vibration analysis of laminated cylindrical shells. Int. J. Press. Vessel. Pip. **85**, 450–458 (2008)
33. Loy, C.T., Lam, K.Y.: Vibration of thick cylindrical shells on the basis of three dimensional theory of elasticity. J. Sound Vib. **226**(4), 719–737 (1999)
34. Khdeir, A.A., Reddy, J.N., Frederick, D.A.: Study of bending, vibration and buckling of cross-ply circular cylindrical shell theories. Int. J. Eng. Sci. **27**, 1337–1351 (1989)
35. Irie, T., Yamada, G., Tanaka, K.: Natural frequencies of truncated conical shells. J. Sound Vib. **92**, 447–453 (1984)