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Size-dependent free vibration analysis of composite laminated Timoshenko beam based on new modified couple stress theory

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Abstract A micro-scale free vibration analysis of composite laminated Timoshenko beam (CLTB) model is developed based on the new modified couple stress theory. In this theory, a new anisotropic constitutive relation is defined for modeling the CLTB. This theory uses rotation–displacement as dependent variable and contains only one material length scale parameter. Hamilton's principle is employed to derive the governing equations of motion and boundary conditions. This new model can be reduced to composite laminated Bernoulli–Euler beam model of the couple stress theory. An example analysis of free vibration of the cross-ply simply supported CLTB model is adopted, and an explicit expression of analysis solution is given. Additionally, the numerical results show that the present beam models can capture the scale effects of the natural frequencies of the micro-structure.

Keywords Composite laminated beam · Timoshenko beam · Modified couple stress theory · Material length parameter · Scale effect · Vibration · Natural frequency

1 Introduction

Experimental works made by Fleck et al. [1], Ma and Clarke [2], Stolken and Evans [3], Chong and Lam [4], Lam et al. [5], and McFarland and Colton [6] show that size effects play an important role in micro-scale structures of materials ranging from thin copper wires, silver single crystal, nickel beams, or epoxy polymeric beams. As the conventional continuum theory cannot explain or solve the problems of the scale effects, theories for micro-structures need to be developed.

Theories for micro-structures include couple stress theory and strain gradient theory. A series of researches in the couple stress/strain gradient theories have been made. The couple stress theories (e.g., [7–10]) and the strain gradient theories (e.g., [11–15]) have been established. Recently, unlike the couple stress theories mentioned above, Yang et al. [16] proposed a modified couple stress theory (C^1 theory). In this modified couple stress theory, the stress tensor is symmetric. In the past few years, many researchers have been attracted to this theory. For example, Park and Gao [17] studied the static behaviors of the Bernoulli–Euler micro-scale beam, and Kong et al. [18] developed this model for vibration analysis; Ma et al. [19] developed the Timoshenko micro-scale beam model for static bending and free vibration; Tsiatas [20] studied the static behaviors of the Kirchhoff micro-scale plate model and Yin et al. [21] developed this model for vibration analysis; a nonclassical Mindlin plate model is developed by Ma et al. [22].

Jomehzadehei et al. [23], based on a modified couple stress theory, analyzed the size-dependent vibration of micro-plates. Wang et al. [24], based on the strain gradient elasticity theory, proposed a Kirchhoff micro-plate

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model. Lately, Reddy et al. developed models of functionally graded beams and nonlinear formulations based on the nonlocal/couple stress theory [25,26]. The couple stress theory is used by Reddy [25] to analyze functionally graded beams. The nonlocal nonlinear formulations for beams and plates were developed by Reddy [26] and Reddy et al. [27] based on the modified couple stress theory proposed a nonlinear third-order theory of functionally graded plates.

More recently, a new micro-model for bending analysis of composite laminated beams with the first-order shear deformation has been developed in the first time by Chen et al. [28], based on the new modified couple stress theory. Chen et al. [29] have also established a model of composite laminated Reddy plate based on a new modified couple stress theory. The composite laminated Reddy beam model based on a modified couple stress theory has also been established by Chen et al. [30]. In this model, a new curvature tensor is defined for establishing the constitutive relations of laminated beam for anisotropy materials.

The objective of this paper is to develop the micro-scale composite laminated Timoshenko beam (CLTB) model for dynamic analysis, based on the new modified couple stress theory. The rest of this paper is organized as follows. In Sects. 2, 3, the modified couple stress theory, the displacement field, and the constitutive equations are described. In Sect. 4, using Hamilton's principle, the governing equations of motion and corresponding boundary conditions are obtained. Then, in Sects. 5 and 6, the free vibration problem of a cross-ply simply supported micro beam is solved and the numerical results are analyzed. This paper concludes with a summary in Sect. 7.

2 Modified couple stress theory

The modified couple stress theory proposed by Yang et al. [16] contains only one additional material length scale parameter. In this theory, the constitutive relations can be given as:

$$\begin{cases} \sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}, \\ m_{ij} = 2\mu\ell^2\chi_{ij}, \end{cases} \tag{1}$$

where

$$\begin{cases} \varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \\ \chi_{ij} = \frac{1}{2}\left(\omega_{i,j} + \omega_{j,i}\right), \end{cases} \tag{2}$$

and σ_{ij}, ε_{ij}, m_{ij}, χ_{ij} are the stress tensor, strain tensor, couple stress moment tensor, and the symmetric curvature tensor, respectively. In Eqs. (1) and (2), λ and μ are the elastic coefficients, ℓ is the micro-material's constants, δ_{ij} is the Kronecker delta, $\mathbf{u}$ (u_i) is the displacement vector, and $\boldsymbol{\omega}$ (ω_i) is the rotation vector as

$$\boldsymbol{\omega} = \frac{1}{2}\operatorname{curl}\mathbf{u}. \tag{3}$$

In the modified couple stress theory, both the curvatures ($\chi_{ij} = \chi_{ji}$) and the couple stress moments ($m_{ij} = m_{ji}$) are symmetric; however, it can be used only for isotropic materials.

3 Basic equations of composite laminated beam of new modified couple stress theory

3.1 Displacement field and strain

Using the Cartesian coordinate system (x, y, z) as shown in Fig. 1, where the x-axis is coincident with the centroidal axis of the undeformed beam, the displacement field in a Timoshenko beam can be described by Reddy [31] as follows:

$$\begin{cases} u(x,z,t) = u_0(x,t) - z\theta(x,t), \\ v = 0, \\ w = w(x,t), \end{cases} \tag{4}$$

where θ is the angle of rotation around the y-axis of the cross-section.

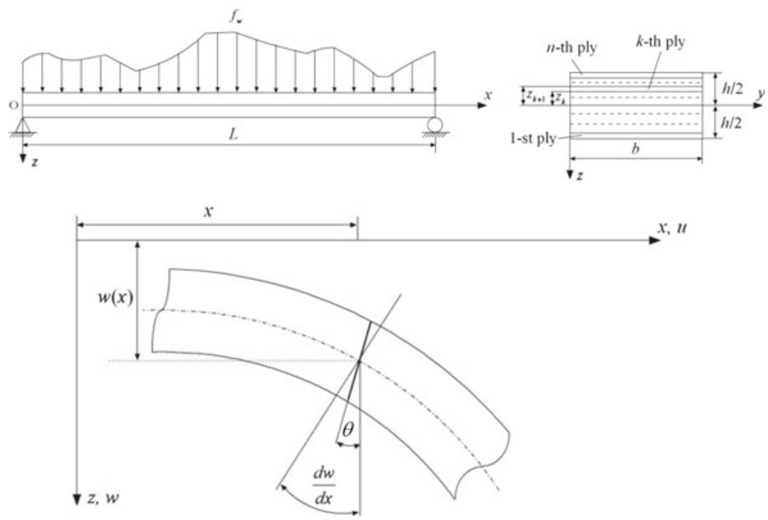

Fig. 1 Schematic diagram of Timoshenko beam

The rotational displacements are

$$\begin{cases} \omega_x = \frac{1}{2}\left(w_{,y} - v_{,z}\right) = 0 \\ \omega_y = \frac{1}{2}\left(u_{,z} - w_{,x}\right) = -\frac{1}{2}\left(\theta + \frac{\partial w}{\partial x}\right) \\ \omega_z = \frac{1}{2}\left(v_{,x} - u_{,y}\right) = 0 \end{cases} \tag{5}$$

where ω_y is the rotational displacement of the centroidal axis which is unrelated to z. Equation (5) shows that the rotational displacement of this model is constant along the thickness direction.

From Eqs. (2), (4), and (5), the strain–displacement and the couple strain–displacement relationships can be described as follows:

$$\begin{Bmatrix} \varepsilon_x \\ \gamma_{xz} \\ \chi_{xy} \\ \chi_{yx} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \frac{\partial \omega_x}{\partial y} + \frac{\partial \omega_y}{\partial x} \\ \frac{\partial \omega_x}{\partial y} + \frac{\partial \omega_y}{\partial x} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_0}{\partial x} - z\frac{\partial \theta}{\partial x} \\ \frac{\partial w}{\partial x} - \theta \\ -\frac{1}{4}\left(\frac{\partial \theta}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) \\ -\frac{1}{4}\left(\frac{\partial \theta}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) \end{Bmatrix} \tag{6}$$

3.2 Constitutive relations for composite laminated beam based on the new modified couple stress theory

In this section, a new modified couple stress theory for anisotropic materials will be established, in which the constitutive relations for anisotropic micro-composite plate/beam are proposed. In this theory, rotation displacements are dependent variables (C^1 theory), and two different length scale parameters, one related to fiber and one related to matrix, are introduced to establish the modified couple stress theory model for composite plate/beam. The following will elaborate on the establishment of the new constitutive equation of anisotropic materials

Adding a rotation balance condition in the equilibrium equations (i.e., the rotational equilibrium rotating about the micro-impurity) is the foundation for the couple stress theory. In the laminated plate or beam, the micro-scale material constants ℓ_b^2 and ℓ_m^2 are related to the fiber and matrix of the same ply, respectively. ℓ_b^2 represents the micro-scale material constant of the fiber rotating in the y–z plane where the fiber cross-section and the matrix interact, and the fiber is viewed as the impurity affecting the rotational equilibrium, as shown in Fig. 2. ℓ_m^2 is the micro-scale material constant within the matrix rotating about the impurity in the x–z plane, as shown in Fig. 3.

In the couple stress theory, the constitutive equation is only contributed to shear stress and is uncoupled with the micro-scale parameters of the shear modules; unlike in the strain gradient theory, where the microscopic parameters of the constitutive equation have a coupling effect.

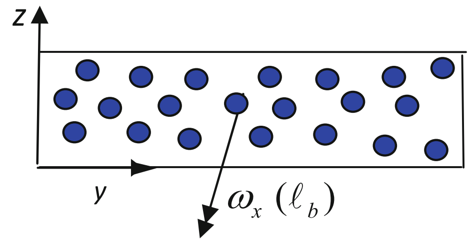

Fig. 2 y–z plane fiber cross-section

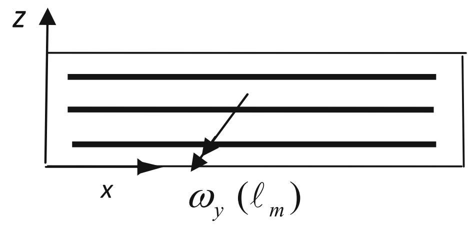

Fig. 3 Fiber within x–z plane

According to the isotropic modified couple stress theory and the classical couple stress theory, a new constitutive equation could be built, and the relationship between couple stress and curvatures is defined as follows:

$$\begin{cases} m_x = \tilde{m}_x = 2C_{44}\ell_b^2 \frac{\partial \omega_x}{\partial x}, \\ m_y = \tilde{m}_y = 2C_{55}\ell_m^2 \frac{\partial \omega_y}{\partial y}, \\ m_{xy} = \frac{1}{2}\left(\tilde{m}_{xy} + \tilde{m}_{yx}\right) = C_{44}\ell_b^2 \frac{\partial \omega_x}{\partial y} + C_{55}\ell_m^2 \frac{\partial \omega_y}{\partial x}, \\ \tilde{m}_{yx} = \frac{1}{2}\left(\tilde{m}_{xy} + \tilde{m}_{yx}\right) = C_{44}\ell_b^2 \frac{\partial \omega_x}{\partial y} + C_{55}\ell_m^2 \frac{\partial \omega_y}{\partial x}. \end{cases} \tag{7}$$

where $C_{44} = G_{12}, C_{55} = G_{22}$ are the shear modules, in y–z and x–z planes, respectively.

Equation (7) can be expressed in matrix form:

$$\begin{Bmatrix} m_x \\ m_y \\ m_{xy} \\ m_{yx} \end{Bmatrix} = \begin{bmatrix} 2C_{44}\ell_{kb}^2 & & & \\ & 2C_{55}\ell_{km}^2 & & \\ & & C_{44}\ell_b^2 & C_{55}\ell_m^2 \\ & & C_{44}\ell_b^2 & C_{55}\ell_m^2 \end{bmatrix} \begin{Bmatrix} \frac{\partial \omega_x}{\partial x} \\ \frac{\partial \omega_y}{\partial y} \\ \frac{\partial \omega_x}{\partial y} \\ \frac{\partial \omega_y}{\partial x} \end{Bmatrix} \tag{8}$$

where the x-axis direction is coincident with the fiber's plying direction. In Eq. (8), the curvatures $\chi_{ij} = \omega_{i,j}$ (i.e., $\chi_{12} = \frac{\partial \omega_{x'}}{\partial y'}$, $\chi_{21} = \frac{\partial \omega_{y'}}{\partial x'}$) are asymmetric; however, couple stress moments are symmetric as seen in $m_{x'y'} = m_{y'x'}$.

For isotropic materials,

$$C_{44} = G_{12} = C_{55} = G_{22} = G, \ \ell_b = \ell_m = \ell \tag{9}$$

From Eqs. (7, 8), we can see

$$m_{xy} = \frac{1}{2}\left(\tilde{m}_{xy} + \tilde{m}_{yx}\right) = G\ell^2 \frac{\partial \omega_x}{\partial y} + G\ell^2 \frac{\partial \omega_y}{\partial x} \tag{10}$$

which is equivalent to Eq. (1), namely, for isotropic materials, the new modified couple stress theory is equivalent to the modified couple stress theory proposed by Yang et al.

For anisotropic modified couple stress theory, the curvatures can be written as

$$\chi = \begin{Bmatrix} \frac{\partial \omega_x}{\partial x} \\ \frac{\partial \omega_y}{\partial y} \\ \frac{\partial \omega_x}{\partial y} \\ \frac{\partial \omega_y}{\partial x} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial^2 w}{\partial x \partial y} \\ -\frac{\partial^2 w}{\partial x \partial y} \\ \frac{\partial^2 w}{\partial y^2} \\ -\frac{\partial^2 w}{\partial x^2} \end{Bmatrix} \tag{11}$$

A laminate beam is made up of a group of single layers bonded to each other. For the composite laminated beam, in the local coordinate (x', y', z) of kth layer, the curvatures $\frac{\partial \omega_{x'}}{\partial y'}$ and $\frac{\partial \omega_{y'}}{\partial x'}$ should be expressed as independent components in terms of the micro-scale material constants ℓ_{kb}^2 and ℓ_{km}^2 related to the fiber and matrix of the same ply, respectively. The constitutive relations in Eq. (1) are for isotropic materials. A new expression of the constitutive relations is defined for the kth ply as follows:

$$\begin{Bmatrix} m_{x'y'} \\ m_{y'x'} \end{Bmatrix} = \begin{bmatrix} C_{44}^k \ell_{kb}^2 & C_{55}^k \ell_{km}^2 \\ C_{44}^k \ell_{kb}^2 & C_{55}^k \ell_{km}^2 \end{bmatrix} \begin{Bmatrix} \frac{\partial \omega_{x'}}{\partial y'} \\ \frac{\partial \omega_{y'}}{\partial x'} \end{Bmatrix} \tag{12}$$

where $C_{44}^k = G_{13}^k, C_{55}^k = G_{23}^k$.

The stress–strain relations for the kth lamina in the local coordinate $\left(x', y', z\right)$ can be written as:

$$\boldsymbol{\sigma}'^k = \boldsymbol{C}^k \boldsymbol{\varepsilon}'^k \tag{13}$$

where

$$\boldsymbol{\sigma}'^k = \left[\sigma_{x'}^k \; \sigma_{y'}^k \quad \tau_{x'z}^k \; \tau_{y'z}^k \quad m_{x'y'}^k \quad m_{y'x'}^k \right]^T \tag{14}$$

$$\boldsymbol{\varepsilon}'^k = \left[\varepsilon_{x'} \; \varepsilon_{y'} \quad \gamma_{x'z} \; \gamma_{y'z} \quad \chi_{x'y'} \quad \chi_{y'x'} \right]^T \tag{15}$$

in which τ denotes shear stress, τ_{xy}^k is neglected, and

$$\boldsymbol{\varepsilon}'^k = \begin{Bmatrix} \varepsilon_{x'} \\ \varepsilon_{y'} \\ \gamma_{x'z'} \\ \gamma_{y'z'} \\ \chi_{x'y'} \\ \chi_{y'x'} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u'}{\partial x'} \\ \frac{\partial v'}{\partial y'} \\ \frac{\partial u'}{\partial z} + \frac{\partial w}{\partial x'} \\ \frac{\partial v'}{\partial z} + \frac{\partial w}{\partial y'} \\ \frac{\partial \omega_{x'}}{\partial y'} \\ \frac{\partial \omega_{y'}}{\partial x'} \end{Bmatrix} \tag{16}$$

$$\boldsymbol{C}^k = \begin{bmatrix} C_{11}^k & C_{12}^k & & & & \\ C_{21}^k & C_{22}^k & & & & \\ & & C_{44}^k & & & \\ & & & C_{55}^k & & \\ & & & & \ell_{kb}^2 C_{44}^k & \ell_{km}^2 C_{55}^k \\ & & & & \ell_{kb}^2 C_{44}^k & \ell_{km}^2 C_{55}^k \end{bmatrix} \tag{17}$$

where x' aligns with the direction of the fiber in the kth layer. Referring to the book by Reddy [32], we have $C_{11}^k = \frac{E_1^k}{(1-(v_{12}^k)^2)}$, $C_{12}^k = \frac{v_{12}^k E_2^k}{(1-v_{12}^k v_{21}^k)}$, $C_{22}^k = \frac{E_2^k}{(1-(v_{22}^k)^2)}$, $C_{44}^k = G_{13}^k$, $C_{55}^k = G_{23}^k$, $v_{21}^k = \frac{E_2^k v_{12}^k}{E_1^k}$. Here, E_1^k and E_2^k are the elastic modules; G_{13}^k and G_{23}^k are shear modules; v_{12}^k, v_{21}^k are the Poisson ratios; ℓ_{kb}^2 and ℓ_{km}^2 are the material micro-structural constants related to the fiber and matrix.

After coordinating the transformation, the stress–strain relations of the kth layer in the global coordinate (x, y, z) can be written as

$$\boldsymbol{\sigma}^k = \boldsymbol{Q}^k \boldsymbol{\varepsilon} \tag{18}$$

where

$$\boldsymbol{\sigma}^k = \left[\sigma_x^k \; \sigma_y^k \quad \tau_{xz}^k \; \tau_{yz}^k \quad m_{xy}^k \quad m_{yx}^k \right]^T \tag{19}$$

$$\varepsilon = \left[\varepsilon_x \; \varepsilon_y \quad \gamma_{xz} \; \gamma_{yz} \quad \chi_{xy} \quad \chi_{yx} \right]^T \tag{20}$$

$$\boldsymbol{Q}^k = \boldsymbol{T}^{kT} \boldsymbol{C}^k \boldsymbol{T}^k \tag{21}$$

The coordinate transformation matrix $\boldsymbol{T}^k$ is expressed as

$$\boldsymbol{T}^k = \begin{bmatrix} m^2 & n^2 & & & & \\ n^2 & m^2 & & & & \\ & & m & n & & \\ & & -n & m & & \\ & & & & m^2 & -n^2 \\ & & & & -n^2 & m^2 \end{bmatrix} \tag{22}$$

where $m = \cos\phi^k$, $\quad n = \sin\phi^k$, and ϕ^k is fiber angle with respect to the x-axis.

As $\varepsilon_y = \gamma_{yz} = 0$ for beam, the 2nd and 3rd columns as well as 2nd and 3rd rows $\boldsymbol{Q}^k$ need not to be considered in the strain energy. With them eliminated, $\boldsymbol{Q}^k$ becomes

$$\boldsymbol{Q}^k = \begin{bmatrix} Q^k_{11} & & & \\ & Q^k_{44} & & \\ & & \ell_k^2 \hat{Q}^k_{44} & \ell_k^2 \hat{Q}^k_{55} \\ & & \ell_k^2 \hat{Q}^k_{44} & \ell_k^2 \hat{Q}^k_{55} \end{bmatrix} \tag{23}$$

Furthermore, carrying out the matrix multiplication in Eq. (23) for the isotropic as well as the anisotropic beam, we could obtain

$$\begin{cases} Q^k_{11} = C^k_{11} m^4 + C^k_{22} n^4 + 2\left(C^k_{12} + 2C^k_{66}\right) m^2 n^2, \\ Q^k_{44} = C^k_{44} m^2 + C^k_{55} n^2 + 2\left(C^k_{12} + 2C^k_{66}\right) m^2 n^2, \\ \ell_k^2 \hat{Q}^k_{44} = \ell_{kb}^2 C^k_{44} m^4 + \ell_{km}^2 C^k_{55} n^4 + \left(\ell_{kb}^2 C^k_{44} + \ell_{km}^2 C^k_{55}\right) m^2 n^2, \\ \ell_k^2 \hat{Q}^k_{55} = \ell_{kb}^2 C^k_{44} n^4 + \ell_{km}^2 C^k_{55} m^4 + \left(\ell_{kb}^2 C^k_{44} + \ell_{km}^2 C^k_{55}\right) m^2 n^2 \end{cases} \tag{24}$$

For the cross-ply laminates, $\phi^k = 0$ or $\pi/2$ which leads to $mn = 0$, the Eq. (24) becomes

$$\begin{cases} Q^k_{11} = C^k_{11} m^4 + C^k_{22} n^4, \\ Q^k_{44} = C^k_{44} m^2 + C^k_{55} n^2, \\ \ell_k^2 \hat{Q}^k_{44} = \ell_{kb}^2 C^k_{44} m^4 + \ell_{km}^2 C^k_{55} n^4, \\ \ell_k^2 \hat{Q}^k_{55} = \ell_{kb}^2 C^k_{44} n^4 + \ell_{km}^2 C^k_{55} m^4 \end{cases} \tag{25}$$

In practice, $\ell_b \gg \ell_m$ and one assume $\ell_m = 0$. Then, Eq. (25) for cross-ply laminated beams can be further simplified as:

$$\begin{cases} Q^k_{11} = C^k_{11} m^4 + C^k_{22} n^4, \\ Q^k_{44} = C^k_{44} m^2 + C^k_{55} n^2, \\ \ell_k^2 \hat{Q}^k_{44} = \ell_k^2 C^k_{44} m^4, \\ \ell_k^2 \hat{Q}^k_{55} = \ell_k^2 C^k_{44} n^4 \end{cases} \tag{26}$$

where $\ell_k = \ell_{kb}$.

However, for isotropic beams, $\ell_{kb} = \ell_{km} = \ell$, $C^k_{11} = C^k_{22} = E$ and $\hat{Q}^k_{44} = C^k_{44} = G$, $\hat{Q}^k_{55} = C^k_{55} = G$. Thus, Eq. (26) can be simplified to be

$$\begin{cases} Q^k_{11} = E, \\ Q^k_{44} = G, \\ \ell_k^2 \hat{Q}^k_{44} = \ell^2 G, \\ \ell_k^2 \hat{Q}^k_{55} = \ell^2 G, \end{cases} \tag{27}$$

where E is the elastic module and G the shear module. Then, substituting Eq. (27) into Eq. (23), the result is the same as shown in Eq. (1b), which is used for isotropic materials.

4 Governing motion equations of composite laminated Timoshenko beam based on the new modified couple stress theory

4.1 Hamilton's principle for composite laminated beam of modified couple stress theory

The Hamilton's principle can be used to identify the equilibrium conditions and the boundary conditions. For the composite laminated beam in Fig. 1, if unit width (i.e., $b = 1$) is assumed, the principle can be expressed by

$$\int_0^T [\delta K - (\delta U - \delta W)]\mathrm{d}t = 0 \tag{28}$$

where

$$\delta K = \int_0^L \left[\sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} \rho^k \left(\frac{\partial \boldsymbol{u}}{\partial t}\right)^T \delta \frac{\partial \boldsymbol{u}}{\partial t} \mathrm{d}z\right] \mathrm{d}x, \tag{29}$$

$$\delta U = \int_0^L \left[\sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} \left(\sigma^k\right)^T \delta\varepsilon \mathrm{d}z\right] \mathrm{d}x, \tag{30}$$

$$\delta W = \int_{\Omega} \bar{\boldsymbol{f}}^T \delta u \mathrm{d}v + \int_{\partial\Omega} \bar{\boldsymbol{T}}^T \delta u \mathrm{d}s. \tag{31}$$

In Eq. (29), ρ^k is the mass density of the kth ply material, and in Eq. (31), $\bar{\boldsymbol{f}}$ and $\bar{\boldsymbol{T}}$ are the prescribed body force and boundary traction vectors, respectively.

Substituting Eq. (6) into the Eq. (30), by the integration on z and x coordinates in the section of beam, the first variation of the total strain energy in the beam becomes

$$\begin{aligned}
\delta U &= \int_0^L \left[\sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} \left(\sigma_x^k \delta\varepsilon_x + \tau_{xz}^k \delta\gamma_{xz} + 2m_{xy}^k \delta\chi_{xy}\right) \mathrm{d}z\right] \mathrm{d}x \\
&= \int_0^L \left\{-\frac{\partial N}{\partial x}\delta u_0 + \left[-\frac{\partial Q}{\partial x} - \frac{1}{2}\frac{\partial^2 Y}{\partial x^2}\right]\delta w + \left[\frac{\partial M}{\partial x} - Q + \frac{1}{2}\frac{\partial Y}{\partial x}\right]\delta\theta\right\}\mathrm{d}x \\
&\quad + \left\{N\delta u_0 + \left[Q + \frac{1}{2}\frac{\partial Y}{\partial x}\right]\delta w - \left[M + \frac{Y}{2}\right]\delta\theta - \frac{Y}{2}\delta\left(\frac{\partial w}{\partial x}\right)\right\}\Bigg|_{x=0}^{x=L}
\end{aligned} \tag{32}$$

where

$$\{N, M, Q, Y\} = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} \left\{\sigma_x^k, z\sigma_x^k, \tau_{xz}^k, m_{xy}^k\right\} \mathrm{d}z \tag{33}$$

are the stress and the couple stress resultants.

Under the modified couple stress theory, the external virtual work can be expanded as

$$\delta W = \int_0^L (f_u \delta u_0 + f_w \delta w + f_c \delta w)\mathrm{d}x + \left(\bar{N}\delta u_0 + \bar{V}\delta w + \bar{M}\delta\theta\right)\Big|_{x=0}^{x=L}, \tag{34}$$

where $\boldsymbol{f}_u$ and $\boldsymbol{f}_w$ are the x- and z-components of the body force per unit length beam; $\boldsymbol{f}_c$ is the body moment about the z-axis per unit length of the beam; $\bar{N}$, $\bar{V}$, and $\bar{M}$ are the applied axial force, transverse force, and bending moment at the two ends of the beam, respectively. Moreover,

$$\int_0^L f_c \delta\omega \mathrm{d}x = -\frac{1}{2}\int_0^L f_c \delta\left(\theta + w_{,x}\right)\mathrm{d}x = -\frac{1}{2}\left(\int_0^L f_c\delta\theta\mathrm{d}x + f_c\delta w|_{x=0}^{x=L} - \int_0^L \frac{\partial f_c}{\partial x}\delta w\mathrm{d}x\right). \tag{35}$$

The first variation of kinetic energy is

$$\delta K = \delta\int_0^L \frac{1}{2}\rho\left\{\left(\frac{\partial u_1}{\partial t}\right)^2 + \left(\frac{\partial u_2}{\partial t}\right)^2 + \left(\frac{\partial u_3}{\partial t}\right)^2\right\}\mathrm{d}x = \delta\int_0^L \frac{1}{2}\rho\left\{\left[\frac{\partial\left(u_0 - z\theta\right)}{\partial t}\right]^2 + \left(\frac{\partial w}{\partial t}\right)^2\right\}\mathrm{d}x. \tag{36}$$

Substituting Eqs. (32), (34), and (36) into the Eq. (28), we have

$$\begin{aligned}
&\int_0^T\int_0^L\left\{\left(\frac{\partial N}{\partial x} + f_u - m_0\frac{\partial^2 u_0}{\partial t^2}\right)\delta u_0 + \left[\frac{\partial Q}{\partial x} + \frac{1}{2}\frac{\partial^2 Y}{\partial x^2} + \frac{1}{2}\frac{\partial f_c}{\partial x} + f_w - m_0\frac{\partial^2 w}{\partial t^2}\right]\delta w\right.\\
&\left.+\left(-\frac{\partial M}{\partial x} + Q - \frac{1}{2}\frac{\partial Y}{\partial x} - \frac{1}{2}f_c - m_2\frac{\partial^2\theta}{\partial t^2}\right)\delta\theta\right\}\mathrm{d}x\mathrm{d}t\\
&+\int_0^T\left\{\left(\bar{N} - N\right)\delta u_0 + \left[-Q - \frac{1}{2}\frac{\partial Y}{\partial x} - \frac{f_c}{2} + \bar{V}\right]\delta w\right.\\
&\left.+\left[\frac{Y}{2} + \bar{Y}\right]\delta\left(\frac{\partial w}{\partial x}\right) + \left[M + \frac{Y}{2} + \bar{M}\right]\delta\theta\right\}\Bigg|_{x=0}^{x=L}\mathrm{d}t\\
&+\int_0^L\left[m_0\left(\frac{\partial u_0}{\partial t}\delta u_0 + \frac{\partial w}{\partial t}\delta w\right) + m_2\frac{\partial\theta}{\partial t}\delta\theta\right]\Bigg|_{t=0}^{t=T}\mathrm{d}x = 0.
\end{aligned} \tag{37}$$

From Eq. (37), the equations of motion can be obtained as

$$\begin{cases}
\frac{\partial N}{\partial x} + f_u = m_0\frac{\partial^2 u_0}{\partial t^2},\\
k_s\frac{\partial Q}{\partial x} + \frac{1}{2}\frac{\partial^2 Y}{\partial x^2} + \frac{1}{2}\frac{\partial f_c}{\partial x} + f_w = m_0\frac{\partial^2 w}{\partial t^2},\\
-\frac{\partial M}{\partial x} + k_s Q - \frac{1}{2}\frac{\partial Y}{\partial x} - \frac{1}{2}f_c = m_2\frac{\partial^2\theta}{\partial t^2}.
\end{cases} \tag{38}$$

the traction boundary conditions at $x = 0$ and $x = L$ are

$$\begin{cases}
N = \bar{N},\\
Q + \frac{1}{2}\frac{\partial Y}{\partial x} + \frac{1}{2}f_c = \bar{V},\\
-\frac{Y}{2} = \bar{Y},\\
-M - \frac{Y}{2} = \bar{M},
\end{cases} \tag{39}$$

and the displacement boundary conditions are

$$\begin{cases}
u_0 = \bar{u}_0,\\
w = \bar{w},\\
\frac{\partial w}{\partial x} = \frac{\partial\bar{w}}{\partial x},\\
\theta = \bar{\theta}.
\end{cases} \tag{40}$$

4.2 Equations of motion in terms of displacements for the composite laminated Timoshenko beam of modified couple stress theory

Substituting geometric Eq. (6) and stress–strain relations in Eq. (13) into Eq. (33), we have

$$\begin{cases} N = \bar{Q}_{11}\frac{\partial u_0}{\partial x} - \bar{J}_{11}\frac{\partial \theta}{\partial x}, \\ M = \bar{J}_{11}\frac{\partial u_0}{\partial x} - \bar{I}_{11}\frac{\partial \theta}{\partial x}, \\ Q \ = \bar{Q}_{44}\left(\frac{\partial w}{\partial x} - \theta\right), \\ Q_2 = \bar{I}_{44}\left(\frac{\partial w}{\partial x} - \theta\right), \\ Y = -\frac{1}{2}\ell_a^2\bar{\bar{Q}}_{44}\left(\frac{\partial \theta}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right), \end{cases} \tag{41}$$

where

$$\begin{cases} \left\{\bar{Q}_{jj},\ \bar{I}_{jj}\right\} = \sum_{k=1}^{n}\left\{z_{k+1} - z_k,\ \frac{z_{k+1}^3 - z_k^3}{3}\right\} Q_{jj}^k \quad (j = 1, 4), \\ \bar{J}_{11} = \sum_{k=1}^{n}\frac{z_{k+1}^2 - z_k^2}{2} Q_{11}^k, \\ \bar{\bar{Q}}_{44} = \sum_{k=1}^{n}(z_{k+1} - z_k)\,\hat{Q}_{44}^k, \\ \ell_a^2 = \ell_k^2\hat{Q}_{44}^k\sum_{k=1}^{n}\frac{\overline{z_{k+1}z_k}}{\bar{\bar{Q}}_{44}}. \end{cases} \tag{42}$$

The equations of motion in terms of displacements as u_0, w and rotation as θ of the composite laminated Timoshenko beam of couple stress theory can be obtained as

$$\begin{cases} \bar{Q}_{11}\frac{\partial^2 u_0}{\partial x^2} - \bar{J}_{11}\frac{\partial^2 \theta}{\partial x^2} + f_u = m_0\frac{\partial^2 u_0}{\partial t^2}, \\ k_s\bar{Q}_{44}\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \theta}{\partial x}\right) - \frac{\ell_a^2\bar{\bar{Q}}_{44}}{4}\left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^3 \theta}{\partial x^3}\right) + \frac{1}{2}\frac{\partial f_c}{\partial x} + f_w = m_0\frac{\partial^2 w}{\partial t^2}, \\ -\bar{J}_{11}\frac{\partial^2 u_0}{\partial x^2} + \bar{I}_{11}\frac{\partial^2 \theta}{\partial x^2} + k_s\bar{Q}_{44}\left(\frac{\partial w}{\partial x} - \theta\right) + \frac{\ell_a^2\bar{\bar{Q}}_{44}}{4}\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^2 \theta}{\partial x^2}\right) - \frac{1}{2}f_c = m_2\frac{\partial^2 \theta}{\partial t^2}. \end{cases} \tag{43}$$

For the same materials of each layer, we can take $\ell = \ell_k$ $(k = 1, 2, 3)$ and $\ell_a^2 = \ell^2$.

If we substitute $\theta = \frac{\partial w}{\partial x}$ and $\frac{\partial^2 \theta}{\partial t^2} = 0$ into Eq. (43), the equations of motion in terms of displacement of the composite laminated Bernoulli–Euler beam (CLBB) model of couple stress theory can be obtained as follows:

$$\begin{cases} \bar{Q}_{11}\frac{\partial^2 u_0}{\partial x^2} - \bar{J}_{11}\frac{\partial^3 w}{\partial x^3} + f_u = m_0\frac{\partial^2 u_0}{\partial t^2}, \\ \bar{J}_{11}\frac{\partial^3 u_0}{\partial x^3} - \left(\bar{I}_{11} + \ell^2\bar{\bar{Q}}_{44}\right)\frac{\partial^4 w}{\partial x^4} + f_w = m_0\frac{\partial^2 w}{\partial t^2}. \end{cases} \tag{44}$$

Introducing the relations of $u_0 = 0$, $\bar{\bar{Q}}_{44} = GA$, and $\bar{I}_{11} = EI$ [28] into Eq. (44), the equation of motion in terms of displacement of isotropic Bernoulli–Euler beam model of the couple stress theory may be written as follows:

$$(EI + \ell^2 GA)\frac{\partial^4 w}{\partial x^4} + m_0\frac{\partial^2 w}{\partial t^2} + f_w = 0 \tag{45}$$

which is identical to the result as in the reference of Kong et al. [18]. In this case, we can see the new modified couple stress theory can be used for isotropic materials.

The new equations (43), (44), and (45) can help us explain the size effects. Moreover, by letting $\ell = 0$, the new models can be reduced to the corresponding classical beam models.

5 Free vibration

To illustrate the size effects of the new model, a simply supported beam model is analyzed. The corresponding boundary conditions are

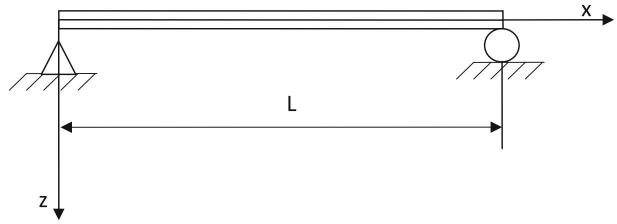

Fig. 4 Simply supported beam

$$\begin{cases} u_0\,|_{x=0} = u_0\,|_{x=L} = 0, \\ w\,|_{x=0} = w\,|_{x=L} = 0, \\ \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=0} = \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=L} = 0, \\ \left.\frac{\partial \theta}{\partial x}\right|_{x=0} = \left.\frac{\partial \theta}{\partial x}\right|_{x=L} = 0. \end{cases} \tag{46}$$

5.1 Natural frequency of Timoshenko beam model

A simply supported beam as shown in Fig. 4 is used for the natural vibration analysis with all external forces vanished (i.e., $f_u = 0$, $f_w = 0$, and $f_c = 0$; and $\bar{N} = 0$, $\bar{V} = 0$, and $\bar{M} = 0$). In order to simplify processes for analysis of the scale effects with respect to the deflection w, we can assume $f_u = 0$, $N = 0$ and $u = u_0(x, t) \equiv 0$ for any $x \in [0, L]$. The trial functions are assumed [31] as follows:

$$\begin{cases} w(x,t) = \sum_{n=0}^{\infty} W_n^V \sin\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}, \\ \theta(x,t) = \sum_{n=0}^{\infty} \Theta_n^V \cos\left(\frac{n\pi x}{L}\right) e^{i\omega_n t}, \end{cases} \tag{47}$$

where ω_n is the vibration frequency, W_n^V and Θ_n^V are the Fourier coefficients, and i is the usual imaginary number satisfying $i^2 = -1$.

If we substitute Eq. (47) into Eq. (43), we have

$$(\boldsymbol{K} - \lambda \boldsymbol{M}) \begin{Bmatrix} W_n^V \\ \Theta_n^V \end{Bmatrix} = 0, \tag{48}$$

where $\lambda = \omega_n^2$, for any fixed values of n. The elements of coefficient matrix $\boldsymbol{K}$ and $\boldsymbol{M}$, which refer to stiffness matrix and mass matrix, respectively, are given as

$$\boldsymbol{K} = \begin{bmatrix} \frac{4n^4\pi^4 L^2 k_s \bar{Q}_{44} + \ell^2 n^4 \pi^4 \bar{\bar{Q}}_{44}}{4L^4} & \frac{\ell^2 n^3 \pi^3 L \bar{\bar{Q}}_{44} - 4n\pi L^3 k_s \bar{Q}_{44}}{4L^4} \\ \frac{\ell^2 n^3 \pi^3 \bar{\bar{Q}}_{44} - 4n\pi L^2 k_s \bar{Q}_{44}}{4L^3} & \frac{4n^2\pi^2 L \bar{I}_{11} + 4k_s L^3 \bar{Q}_{44} + \ell^2 n^2 \pi^2 L \bar{\bar{Q}}_{44}}{4L^3} \end{bmatrix}, \tag{49}$$

$$\boldsymbol{M} = \begin{bmatrix} m_0 & \\ & m_2 \end{bmatrix}, \tag{50}$$

where

$$\begin{cases} m_0 = \rho b h, \\ m_2 = \rho b h^3/12. \end{cases} \tag{51}$$

For nontrivial solutions of W_n^V $(\neq 0)$ and Θ_n^V $(\neq 0)$, it is required that the determinant of the coefficient matrix of Eq. (48) vanishes, which leads to

$$C_2\lambda^2 + C_1\lambda + C_0 = 0, \tag{52}$$

where

$$\begin{cases} C_2 = -m_0 m_2, \\ C_1 = \frac{1}{4L^6}\left(4k_s n^2\pi^2 L^4 \bar{Q}_{44} m_2 + \ell^2 n^4 \pi^4 L^2 \bar{\bar{Q}}_{44} m_2 + 4n^2\pi^2 L^4 \bar{I}_{11} m_0 + 4k_s L^6 \bar{Q}_{44} m_0 + \ell^2 n^2 \pi^2 L^4 \bar{\bar{Q}}_{44} m_0\right), \\ C_0 = -\frac{1}{4L^6}\left(4n^4\pi^4 k_s L^2 \bar{I}_{11} \bar{Q}_{44} + 4\ell^2 n^4 \pi^4 k_s L^2 \bar{Q}_{44} \bar{\bar{Q}}_{44} + \ell^2 n^6 \pi^6 \bar{I}_{11} \bar{\bar{Q}}_{44}\right). \end{cases} \tag{53}$$

The λ can be obtained from Eq. (52)

$$\lambda = \frac{-C_1 + \sqrt{C_1^2 - 2C_2 C_0}}{2C_2}. \tag{54}$$

The positive solution of ω_n determined from Eq. (54) is the nth order natural frequency of the simply supported beam.

For the isotropic Timoshenko beam, the elastic constants become as follows: $\bar{Q}_{44} = GA$, $\bar{\bar{Q}}_{44} = GA$, $\bar{I}_{11} = EI$, and $I = \frac{bh^3}{12}$. To substitute these constants into Eq. (52), the equation becomes as

$$\begin{cases} C_2 = -m_0 m_2, \\ C_1 = \frac{1}{4L^6}\left(\left(4n^2\pi^2 k_s L^4 GA + \ell^2 n^4 \pi^4 L^2 GA\right) m_2 + \left(4n^2\pi^2 L^4 EI + 4k_s L^6 GA + \ell^2 n^2 \pi^2 L^4 GA\right) m_0\right), \\ C_0 = -\frac{1}{4L^6}\left(4n^4\pi^4 k_s L^2 EIGA + 4\ell^2 n^4 \pi^4 k_s L^2 (GA)^2 + \ell^2 n^6 \pi^6 EIGA\right). \end{cases} \tag{55}$$

They are identical to the results as shown in the reference of Ma et al. [19]. This also illustrates the validity of the new modified couple stress theory used for isotropic materials.

The natural frequency of the simply supported classical CLTB model can be obtained from Eqs. (52) and (53), when $l = 0$, and Eq. (53) is written as

$$\begin{cases} C_2 = -m_0 m_2, \\ C_1 = \frac{1}{L^2}\left(k_s n^2 \pi^2 \bar{Q}_{44} m_2 + \left(n^2\pi^2 \bar{I}_{11} + k_s L^2 \bar{Q}_{44}\right) m_0\right), \\ C_0 = -\frac{1}{L^4} n^4 \pi^4 k_s \bar{I}_{11} \bar{Q}_{44}. \end{cases} \tag{56}$$

5.2 Natural frequency of Bernoulli–Euler beam model

Similarly, when we substitute Eq. (47) into Eq. (44), we can have the natural frequency of CLBB model of the modified couple stress theory, which is as follows:

$$\omega_n = \sqrt{\frac{\bar{I}_{11} + \ell^2 \bar{\bar{Q}}_{44}}{m_0 L^4}} (n\pi)^2. \tag{57}$$

By letting $\ell = 0$, the new model then reduces to the classical CLBB model

$$\omega_n = \sqrt{\frac{\bar{I}_{11}}{m_0 L^4}} (n\pi)^2. \tag{58}$$

For the isotropic Bernoulli–Euler beam, the elastic constants become $\bar{\bar{Q}}_{44} = GA$ and $\bar{I}_{11} = EI$. Substituting these constants into Eq. (57), the natural frequency of isotropic Bernoulli–Euler beam model can be obtained as

$$\omega_n = \sqrt{\frac{EI + \ell^2 GA}{m_0 L^4}} (n\pi)^2. \tag{59}$$

This is identical to the result as shown in the reference Kong et al. [18].

As illustrated above, the natural frequencies of the simply supported CLTB model of the modified couple stress theory can be reduced to the classical CLTB model and the isotropic Timoshenko beam model of the modified couple stress theory. Similarly, the natural frequency of the simply supported CLBB model of the modified couple stress theory can be reduced to the classical CLBB model and the isotropic Bernoulli–Euler beam model of the modified couple stress theory.

Table 1 Natural frequency of the CLTB model

ω_n (MHz)	Classical	Present						
	$\ell = 0.0$	$\ell = 0.1$	$\ell = 0.5$	$\ell = 1.0$	$\ell = 1.5$	$\ell = 3$	$\ell = 6$	$\ell = 9$
ω_1	5.00999	5.00999	5.00999	5.00999	5.02991	5.05964	5.14782	5.33854
ω_2	17.6324	17.6324	17.6380	17.6465	17.6664	17.7651	18.1466	18.7617
ω_3	33.9470	33.9470	33.9544	33.9765	34.0147	34.2184	35.0186	36.2960
ω_4	51.6120	51.6130	51.6256	51.6643	51.7291	52.0807	53.4528	55.6327
ω_5	69.6326	69.6333	69.6534	69.7151	69.8176	70.3676	72.5114	75.8933

Table 2 Natural frequency of the CLBB model

ω_n (MHz)	Classical	Present						
	$\ell = 0.00$	$\ell = 0.1$	$\ell = 0.5$	$\ell = 1.0$	$\ell = 1.5$	$\ell = 3$	$\ell = 6$	$\ell = 9$
ω_1	5.28539	5.28544	5.28645	5.28959	5.29483	5.32304	5.43439	5.61509
ω_2	21.1416	21.1417	21.1458	21.1583	21.1794	21.2922	21.7376	22.4603
ω_3	47.5686	47.5690	47.5780	47.6063	47.6534	47.9073	48.9095	50.5358
ω_4	84.5663	84.5671	84.5832	84.6334	84.7172	85.1685	86.9503	89.8414
ω_5	132.135	132.136	132.162	132.240	132.371	133.076	135.860	140.377

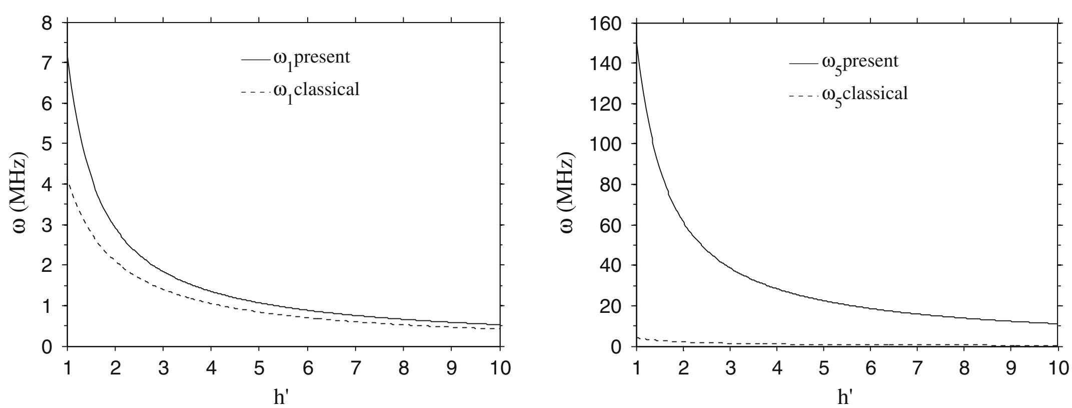

Fig. 5 Natural frequency of Timoshenko beam model

6 Numerical results and discussion

To illustrate the size effects of the composite laminated micro-beams, both the CLTB model and the CLBB model, different numerical results are compared in the present study.

Consider the three-layer ([90°/0°/90°]) micro-beams with the size of width $b = 25 \times 10^{-6}$ m, length $L = 200 \times 10^{-6}$ m, thickness $h = 25 \times 10^{-6}$ m, and the material constants [33]: $E_2 = 6.9 \times 10^9$ Pa, $E_1 = 25E_2$, $G_{12} = G_{13} = 0.5E_2$, $G_{23} = 0.2E_2$, $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$, in which subscripts 1 and 2 represent the direction of fiber and matrix, respectively. In addition, the density of the beam is taken to be $\rho = 1,578$ kg/m^3. In order to explore the effect of length scale parameter on free vibration behavior of micro-beams, different values of material length scale parameter ℓ are taken, and the corresponding natural frequencies are shown in Tables 1 and 2. It is obvious that as material length scale parameter ℓ increases, natural frequency increases accordingly, especially for high-order frequency.

To further illustrate the size effects, we change the sizes of the beam into width $b = 2h$, length $L = 20h$, and the material length scale parameter $\ell = 4 \times 10^{-6}$ m. The first and fifth natural frequencies of both present and classical Timoshenko beam models vary with h' (i.e., $h' = h/\ell = h/(4 \times 10^{-6}\,\text{m})$) are given in Fig. 5, which show that the natural frequencies of the present Timoshenko beam model in the couple stress theory are more than the classical beam model. For various values of h/ℓ, the first and fifth natural frequencies of both present and classical Bernoulli–Euler beam models with h' are given in Fig. 6, which show that the natural frequencies of the Bernoulli–Euler beam model in the couple stress theory are more than the classical beam model.

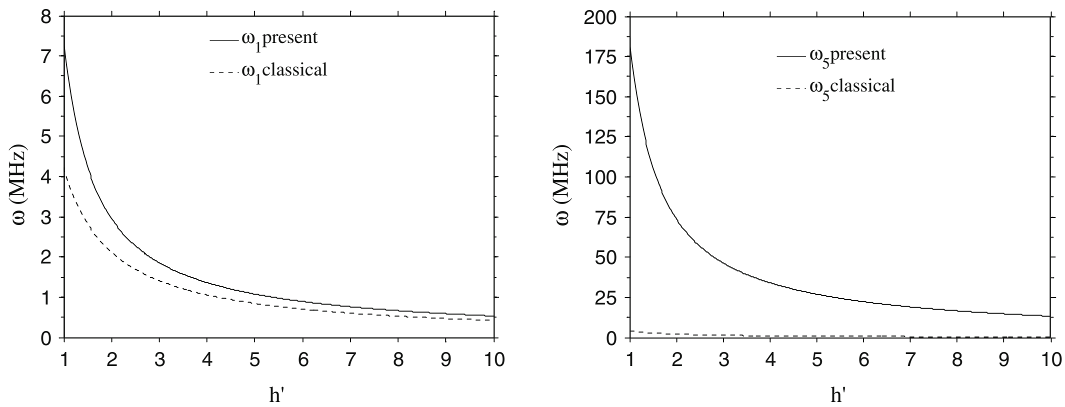

Fig. 6 Natural frequency of Bernoulli–Euler beam model

7 Conclusions

A new modified couple stress theory is developed for establishing the model of composite laminated Timoshenko beam. This theory contains the rotation and displacement as dependent variables and the only one material length scale parameter for micro-structures. In this theory, a new constitutive relation for laminated beam with anisotropic materials is defined. The present beam model can be viewed as a simplified couple stress theory in engineering mechanics. An example as analysis of the free vibration of the cross-ply simply supported CLTB model is adopted.

Numerical results show that the present beam model can capture the scale effects of micro-structures. The numerical results given by the CLTB model and the CLBB model show that the natural frequency of the nonclassical beam model is always higher than that of the classical beam model.

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