Arch Appl Mech (2012) 82:1325–1335
DOI 10.1007/s00419-012-0669-6

SPECIAL ISSUE

G. Dassios

Directional dependent Green's function and Kelvin images

Received: 22 August 2011 / Accepted: 14 December 2011 / Published online: 27 June 2012

Abstract The inverse of a linear differential operator is an integral operator with a kernel which is commonly known as the Green's function of the differential operator. Therefore, the knowledge of the Green's function of a linear problem leads directly to an integral representation of its solution. Any Green's function is split into a singular part that carries the localized singularity of the Dirac measure and a regular part that is controlled by the Dirichlet boundary condition. In some relatively simple cases, this regular part can be interpreted as the contribution of imaginary sources which lie in the complement of the fundamental domain. If a problem is associated with the Laplace operator, such as the biharmonic operator or the Papkovitch potentials, which both govern Linear Elastostatics, the construction of such Green's functions are of extremely large importance. All these are well-behaving procedures as long as we live in the highly symmetric geometry represented by the spherical system. But, if we live in a directional-dependent environment, such as the one imposed by the ellipsoidal geometry, the above procedures become extremely complicated, if not impossible. In the present work, the Green's functions and their Kelvin image systems are obtained for the interior and the exterior regions of an ellipsoid. It is amazing, although not unjustified, that besides the point image source, that is needed for the isotropic spherical case, in the case of ellipsoidal domains, the necessary image system involves a full two-dimensional distribution of imaginary sources to account for the anisotropic character of the ellipsoidal domains.

Keywords Green's function · Ellipsoidal harmonics · Kelvin's transformation · Images

1 Introduction

Green, in his monumental essay on Electricity and Magnetism [8], introduced the foundations of what it is known today as Mathematical Physics. One of his achievements there was to formulate mathematically the principle of superposition for linear problems. This formulation provided the ground on which Linear Algebra was built a few decades later. In the case of partial differential equations, where the solution set is an infinite-dimensional functional space, linearity and superposition is incorporated in Green's function, which is nothing more than the mathematical realization of the physical property that some quantities have, according to which the effects of individual sources at some point are added together. Standard references for Green's functions, among others, are [2,7,14,17].

In the same essay, Green also set the rudiments of the theory of images for a boundary value problem, a method that replaces the boundary conditions of a problem with a set of sources that lie in the complementary domain. Nevertheless, it was Kelvin [20–22] who gradually transformed this idea to an intelligent method for solving boundary value problems. Besides its mathematical effectiveness, the method of images provides a

G. Dassios (✉)
Department of Chemical Engineering, University of Patras and FORTH/ICE-HT, 265 04 Patras, Greece
E-mail: gdassios@chemeng.upatras.gr; gdassios@otenet.gr

deep physical understanding, mainly through the symmetries that are present in each particular problem. A first attempt to construct the interior ellipsoidal Green's function was reported in [6] while the corresponding problem for a spherical shell can be found in [19]. The complete solution of the problem, both for the interior and exterior ellipsoidal domains, is presented here.

There are many problems in Mechanics of Solids and Fluids that are formulated in terms of harmonic functions [1,10,12,13,15,18]. Elastostatics and potential flow are two well-known such cases. In fact, for the case of Elastostatics, there are many approaches that utilize solutions of the Laplace equation. One of them is the well-known differential representation of the displacement field known as Papkovitch representation of the displacement field in terms of a vector and a scalar harmonic function [9]. Therefore, any representation of a harmonic function is readily applicable to many different problems in Mechanics and Physics.

This paper is organized as follows. Section 2 provides a very short introduction to the ellipsoidal coordinate system and to the ellipsoidal harmonics at the level of notation. Then, in Sects. 3 and 4, the interior and exterior Green's function for the Laplacian in ellipsoidal geometry, as well as their image systems, are constructed. In order to be able to compare the form of the Green's function in spherical and ellipsoidal coordinates, we provide the form of the corresponding results for the sphere as well. A final Sect. 5 states the obtained results and compares the character of the Green's functions in every particular case.

2 Elements of ellipsoidal harmonics

In an anisotropic environment, the ellipsoid plays the role that the sphere plays in an isotropic one. Since there is a unique isotropic but infinitely many anisotropic behaviors, it follows that there are infinitely many ellipsoidal coordinate systems, one for every anisotropic structure. A spherical system is identified by a center and a unit sphere. An ellipsoidal system is identified by a center, an orientation, and the three semi-axes of the reference ellipsoid, which plays the role of the unit sphere. Given the orientation of the three principal axes, the ellipsoidal system is defined in terms of three semi-axes a_1, a_2, a_3, $0 < a_3 < a_2 < a_1 < \infty$, which define the reference ellipsoid

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1 \tag{1}$$

where a_1, a_2, a_3 specify the semi-focal distances

$$h_1 = \sqrt{a_2^2 - a_3^2}, h_2 = \sqrt{a_1^2 - a_3^2}, h_3 = \sqrt{a_1^2 - a_2^2} \tag{2}$$

related by

$$h_1^2 - h_2^2 + h_3^2 = 0. \tag{3}$$

The ellipsoidal coordinate system (ρ, μ, ν) [11], associated with the fundamental ellipsoid (1), is connected to the Cartesian system (x_1, x_2, x_3) via the formulae

$$x_1 = \frac{\rho\mu\nu}{h_2 h_3}, \quad h_2 < \rho < +\infty \tag{4}$$

$$x_2 = \frac{\sqrt{\rho^2 - h_3^2}\sqrt{\mu^2 - h_3^2}\sqrt{h_3^2 - \nu^2}}{h_1 h_3}, \quad h_3 < \mu < h_2 \tag{5}$$

$$x_3 = \frac{\sqrt{\rho^2 - h_2^2}\sqrt{h_2^2 - \mu^2}\sqrt{h_2^2 - \nu^2}}{h_1 h_2}, \quad 0 < \nu < h_3. \tag{6}$$

The variable ρ, which corresponds to the radial spherical coordinate, defines a family of confocal ellipsoids, and the variables μ and ν, which correspond to the angular spherical coordinates, define two confocal families of hyperboloids of one and two sheets, respectively. The metric coefficients of the ellipsoidal system are given by

$$h_\rho = \|\boldsymbol{r}_\rho\| = \frac{\sqrt{\rho^2 - \mu^2}\sqrt{\rho^2 - \nu^2}}{\sqrt{\rho^2 - h_3^2}\sqrt{\rho^2 - h_2^2}} \tag{7}$$

$$h_\mu = \|\boldsymbol{r}_\mu\| = \frac{\sqrt{\rho^2-\mu^2}\sqrt{\mu^2-\nu^2}}{\sqrt{\mu^2-h_3^2}\sqrt{h_2^2-\mu^2}} \tag{8}$$

$$h_\nu = \|\boldsymbol{r}_\nu\| = \frac{\sqrt{\rho^2-\nu^2}\sqrt{\mu^2-\nu^2}}{\sqrt{h_3^2-\nu^2}\sqrt{h_2^2-\nu^2}} \tag{9}$$

where the lower index denotes partial differentiation with respect to the indicated variable.

Lamé used an ingenious technique to separate variables for the Laplace equation [11]. In fact, he proved that all three separated ordinary differential equations are identical and the only difference among them is the domain where each one is defined. Hence, an interior eigen-solution of the Laplace equation has the form

$$\mathbb{E}_n^m(\rho,\mu,\nu) = E_n^m(\rho)E_n^m(\mu)E_n^m(\nu) \tag{10}$$

where E_n^m is the Lamé functions of the first kind of degree $n = 0, 1, 2, \ldots,$ and of order m=1,2,…,2n+1. Similarly, an exterior eigen-solution has the form

$$\mathbb{F}_n^m(\rho,\mu,\nu) = F_n^m(\rho)E_n^m(\mu)E_n^m(\nu) \tag{11}$$

where F_n^m is the Lamé functions of the second kind, which is given by the elliptic integral

$$\begin{aligned} F_n^m(\rho) &= (2n+1)E_n^m(\rho)\int_\rho^\infty \frac{\mathrm{d}x}{[E_n^m(x)]^2\sqrt{x^2-h_3^2}\sqrt{x^2-h_2^2}} \\ &= (2n+1)E_n^m(\rho)I_n^m(\rho). \end{aligned} \tag{12}$$

The functions $\mathbb{E}_n^m(\rho,\mu,\nu)$ and $\mathbb{F}_n^m(\rho,\mu,\nu)$ are called Lamé products or interior and exterior ellipsoidal harmonics, respectively.

The surface ellipsoidal harmonics

$$S_n^m(\mu,\nu) = E_n^m(\mu)E_n^m(\nu),\ n = 0, 1, 2, \ldots,\ m = 1, 2, \ldots, 2n+1 \tag{13}$$

form a complete set of eigen-functions [11] over the surface S_ρ of any ellipsoid, $\rho =$ constant, and they satisfy the orthogonality relation

$$\oint_{S_\rho} S_n^m(\mu,\nu)S_{n'}^{m'}(\mu,\nu)l_\rho(\mu,\nu)\mathrm{d}s_\rho(\mu,\nu) = \gamma_n^m\delta_{nn'}\delta_{mm'} \tag{14}$$

with respect to the weighting function

$$l_\rho(\mu,\nu) = \frac{1}{\sqrt{\rho^2-\mu^2}\sqrt{\rho^2-\nu^2}} \tag{15}$$

where

$$\mathrm{d}s_\rho(\mu,\nu) = h_\mu h_\nu \mathrm{d}\mu\mathrm{d}\nu \tag{16}$$

defines the differential surface element on the ellipsoid. The constant γ_n^m is the normalization constant of the surface harmonic S_n^m.

Therefore, any smooth function f defined over S_ρ has the expansion

$$f(\mu,\nu) = \sum_{n=0}^\infty \sum_{m=1}^{2n+1} c_n^m S_n^m(\mu,\nu) \tag{17}$$

with the coefficients c_n^m given by

$$c_n^m = \frac{1}{\gamma_n^m}\oint_{S_\rho} f(\mu,\nu)S_n^m(\mu,\nu)l_\rho(\mu,\nu)\mathrm{d}s_\rho(\mu,\nu). \tag{18}$$

The harmonics $S_n^m(\mu,\nu)$ over the surface of an ellipsoid correspond to the harmonics $Y_n^m(\theta,\varphi)$ over the surface of a sphere. For an introduction to the theory of ellipsoidal harmonics we refer to the book: Ellipsoidal Harmonics. Theory and Applications, written by the present author and published by Cambridge University Press.

3 The interior Green's function

For comparison reasons, we first remind the form of the Green's function in spherical coordinates. The interior Green's function for the Laplace operator in spherical geometry is defined as the solution of the boundary value problem

$$\Delta_{\boldsymbol{r}} G_s^i(\boldsymbol{r}, \boldsymbol{r}_0) = \delta(\boldsymbol{r} - \boldsymbol{r}_0), \quad r < a \tag{19}$$

$$G_s^i(\boldsymbol{r}, \boldsymbol{r}_0) = 0, \quad r = a \tag{20}$$

with a being the radius of the sphere and a unit source point $\boldsymbol{r}_0$ lying in the interior of the sphere. As it is well known, the solution of this problem is given by [3]

$$G_s^i(\boldsymbol{r}, \boldsymbol{r}_0) = -\frac{1}{4\pi} \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_0|} + \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{1}{2n+1} \frac{r_0^n r^n}{\alpha^{2n+1}} Y_n^m(\hat{\boldsymbol{r}}) Y_n^m(\hat{\boldsymbol{r}}_0)^* \tag{21}$$

where Y_n^m denotes the normalized complex form of surface spherical harmonics [16]. It is also known [20] that the sum on the right hand side of (21) can be interpreted as the contribution of an image point of strength $-\alpha/r_0$, which is located at the exterior point

$$\boldsymbol{r}_0' = \frac{\alpha^2}{r_0^2} \boldsymbol{r}_0. \tag{22}$$

For the case of the ellipsoid (1), the corresponding Green's function G_e^i has to solve the problem

$$\Delta_{\boldsymbol{r}} G_e^i(\boldsymbol{r}, \boldsymbol{r}_0) = \delta(\boldsymbol{r} - \boldsymbol{r}_0), \quad \rho < a \tag{23}$$

$$G_e^i(\boldsymbol{r}, \boldsymbol{r}_0) = 0, \quad \rho = a \tag{24}$$

where the ellipsoidal representations of the observation point $\boldsymbol{r}$ and the interior source point $\boldsymbol{r}_0$ are given by (ρ, μ, ν) and (ρ_0, μ_0, ν_0), respectively. In view of the expansion [4,5]

$$-\frac{1}{4\pi} \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_0|} = -\sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \frac{1}{2n+1} \frac{1}{\gamma_n^m} \begin{cases} \mathbb{E}_n^m(\rho_0, \mu_0, \nu_0) \mathbb{F}_n^m(\rho, \mu, \nu), \, \rho_0 < \rho \\ \mathbb{E}_n^m(\rho, \mu, \nu) \mathbb{F}_n^m(\rho_0, \mu_0, \nu_0), \, \rho < \rho_0 \end{cases} \tag{25}$$

we can assume the representation

$$\begin{aligned} G_e^i(\boldsymbol{r}, \boldsymbol{r}_0) = & -\sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \frac{1}{2n+1} \frac{1}{\gamma_n^m} \mathbb{E}_n^m(\rho_0, \mu_0, \nu_0) \mathbb{F}_n^m(\rho, \mu, \nu) \\ & + \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \frac{1}{2n+1} \frac{1}{\gamma_n^m} B_n^m \mathbb{E}_n^m(\rho_0, \mu_0, \nu_0) \mathbb{E}_n^m(\rho, \mu, \nu) \end{aligned} \tag{26}$$

which holds for $\rho_0 < \rho < a_1$. Then the boundary condition implies that

$$B_n^m = \frac{F_n^m(a_1)}{E_n^m(a_1)} \tag{27}$$

and the Green's function is written as

$$G_e^i(\boldsymbol{r}, \boldsymbol{r}_0) = -\frac{1}{4\pi} \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_0|} + \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \frac{1}{2n+1} \frac{1}{\gamma_n^m} \frac{F_n^m(a_1)}{E_n^m(a_1)} \mathbb{E}_n^m(\rho_0, \mu_0, \nu_0) \mathbb{E}_n^m(\rho, \mu, \nu). \tag{28}$$

Let's attempt now to construct an image system that will generate the same potential as the one given by the series expansion on the right hand side of Eq. (28). We observe that the lack of symmetry, in the ellipsoidal case, is reflected upon the dependence of the constants B_n^m on both the degree n and the order m, while the corresponding constants, in the case of the sphere, were dependent only on n. This observation makes the identification of an image system for the ellipsoid substantially more difficult than for the sphere.The determination of a monopolic image demands the calculation of four independent numbers, three for its location and one for its strength. Let us then put one such monopolic image at the point $(\rho_0', \mu_0', \nu_0')$ with strength Q, and let us calculate these four unknown quantities by demanding that the four first terms of the potential

$$U_e^i(\boldsymbol{r}) = -Q \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \frac{1}{2n+1} \frac{1}{\gamma_n^m} \mathbb{F}_n^m(\rho_0', \mu_0', \nu_0') \mathbb{E}_n^m(\rho, \mu, \nu) \tag{29}$$

generated by this monopole, coincide with the four first terms of the expansion on the right hand side of (28). That is, we demand that

$$-QI_0^1(\rho_0') = I_0^1(a_1) \tag{30}$$

$$-Q\mathbb{E}_1^1(\rho_0', \mu_0', \nu_0')I_1^1(\rho_0') = \mathbb{E}_1^1(\rho_0, \mu_0, \nu_0)I_1^1(a_1) \tag{31}$$

$$-Q\mathbb{E}_1^2(\rho_0', \mu_0', \nu_0')I_1^2(\rho_0') = \mathbb{E}_1^2(\rho_0, \mu_0, \nu_0)I_1^2(a_1) \tag{32}$$

$$-Q\mathbb{E}_1^3(\rho_0', \mu_0', \nu_0')I_1^3(\rho_0') = \mathbb{E}_1^3(\rho_0, \mu_0, \nu_0)I_1^3(a_1) \tag{33}$$

where I_n^m denote the elliptic integrals defined in Eq. (12).

If we express the strength as

$$Q = -\frac{I_0^1(a_1)}{I_0^1(\rho_0')} \tag{34}$$

and write the internal harmonics of the first degree in terms of Cartesian coordinates, we can rewrite the Eqs. (31)–(33) as

$$x_{01}' \frac{I_1^1(\rho_0')}{I_0^1(\rho_0')} = x_{01} \frac{I_1^1(a_1)}{I_0^1(a_1)} \tag{35}$$

$$x_{02}' \frac{I_1^2(\rho_0')}{I_0^1(\rho_0')} = x_{02} \frac{I_1^2(a_1)}{I_0^1(a_1)} \tag{36}$$

$$x_{03}' \frac{I_1^3(\rho_0')}{I_0^1(\rho_0')} = x_{03} \frac{I_1^3(a_1)}{I_0^1(a_1)}. \tag{37}$$

Since the image $\boldsymbol{r}_0'$ lies on the ellipsoid $\rho = \rho_0'$ we have

$$\frac{x_{01}'^2}{\rho_0'^2} + \frac{x_{02}'^2}{\rho_0'^2 - h_3^2} + \frac{x_{03}'^2}{\rho_0'^2 - h_2^2} = 1 \tag{38}$$

and if we insert the expressions (35)–(37), for the coordinates $x_{01}', x_{02}', x_{03}'$, in (38) we obtain the expression

$$\frac{x_{01}^2}{\rho_0'^2} \frac{I_1^1(a_1)^2}{I_1^1(\rho_0')^2} + \frac{x_{02}^2}{\rho_0'^2 - h_3^2} \frac{I_1^2(a_1)^2}{I_1^2(\rho_0')^2} + \frac{x_{03}^2}{\rho_0'^2 - h_2^2} \frac{I_1^3(a_1)^2}{I_1^3(\rho_0')^2} = \frac{I_0^1(a_1)^2}{I_0^1(\rho_0')^2}. \tag{39}$$

This is a highly nonlinear algebraic equation for the determination of the ellipsoidal variable ρ_0'. In fact, since the source $\boldsymbol{r}_0$ is located in the interior of the reference ellipsoid, the continuous function

$$f(\rho_0') = \frac{x_{01}^2}{\rho_0'^2} \frac{I_1^1(a_1)^2}{I_1^1(\rho_0')^2} + \frac{x_{02}^2}{\rho_0'^2 - h_3^2} \frac{I_1^2(a_1)^2}{I_1^2(\rho_0')^2} + \frac{x_{03}^2}{\rho_0'^2 - h_2^2} \frac{I_1^3(a_1)^2}{I_1^3(\rho_0')^2} - \frac{I_0^1(a_1)^2}{I_0^1(\rho_0')^2} \tag{40}$$

assumes the value

$$f(a_1) = \frac{x_{01}^2}{a_1^2} + \frac{x_{02}^2}{a_2^2} + \frac{x_{03}^2}{a_3^2} - 1 < 0 \tag{41}$$

and in view of the asymptotic forms

$$I_0^1(\rho_0') = O\left(\frac{1}{\rho_0'}\right) \tag{42}$$

$$I_1^m(\rho_0') = O\left(\frac{1}{\rho_0'^3}\right), \quad m = 1, 2, 3 \tag{43}$$

as $\rho_0' \to \infty$, we also have

$$\lim_{\rho_0' \to \infty} f(\rho_0') = \infty. \tag{44}$$

Consequently, there exists a root of Eq. (39) in the interval (a_1, ∞), which specifies the ellipsoid on which the image point lies.

Once the value ρ_0' is known, Eq. (34) determines the strength of the image, and Eqs. (35)–(37) determine its exact location. In fact, if we define the dyadic

$$\widetilde{\boldsymbol{D}}_G^i(\rho_0') = \frac{I_0^1(\rho_0')}{I_0^1(a_1)} \sum_{m=1}^{3} \frac{I_1^m(a_1)}{I_1^m(\rho_0')} \hat{\boldsymbol{x}}_m \otimes \hat{\boldsymbol{x}}_m \tag{45}$$

then the image point is given by

$$\boldsymbol{r}_0' = \widetilde{\boldsymbol{D}}_G^i(\rho_0') \cdot \boldsymbol{r}_0. \tag{46}$$

Formula (46) provides the Cartesian coordinates of $\boldsymbol{r}_0'$ once the ellipsoidal coordinate ρ_0' is known. Nevertheless, in order to calculate the other two ellipsoidal coordinates μ_0' and ν_0', we need to solve the system (31)–(33).

Next, we investigate the image system that will represent the part of the expansion (29) that corresponds to the terms $n \geqslant 2$. This part of the expansion will be represented by the potential generated by a distribution of monopoles with density $d^i(\rho_1', \mu_1', \nu_1')$, over an exterior confocal ellipsoid specified by $\rho = \rho_1'$. We will demonstrate in the sequel that it is possible to choose this surface distribution in such a way, as to provide no contribution to the monopolic ($n = 0$) and the dipolic ($n = 1$) terms of the potential. This is desirable, since the $n = 0$ and $n = 1$ terms have already been matched with the corresponding terms coming from the image point at $\boldsymbol{r}_0'$. Then, in the presence of the monopolic image and the surface distribution, the generated potential reads

$$\begin{aligned} V_e^i(\boldsymbol{r}) = & \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \frac{I_0^1(a_1)}{I_0^1(\rho_0')} \frac{1}{\gamma_n^m} \mathbb{E}_n^m(\rho_0', \mu_0', \nu_0') I_n^m(\rho_0') \mathbb{E}_n^m(\rho, \mu, \nu) \\ & - \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \frac{1}{2n+1} \frac{1}{\gamma_n^m} C_n^m \mathbb{E}_n^m(\rho, \mu, \nu) \end{aligned} \tag{47}$$

where

$$\begin{aligned} C_n^m &= \oint_{S_{\rho_1'}} \mathbb{F}_n^m(\rho_1', \mu, \nu) d^i(\rho_1', \mu, \nu) \mathrm{d}S(\mu, \nu) \\ &= (2n+1) I_n^m(\rho_1') E_n^m(\rho_1') \oint_{S_{\rho_1'}} S_n^m(\mu, \nu) d^i(\rho_1', \mu, \nu) \mathrm{d}S(\mu, \nu) \end{aligned} \tag{48}$$

and $S_{\rho_1'}$ denotes the ellipsoid $\rho = \rho_1'$. It is obvious from the expression (48) that the vanishing of the $n = 0$ term demands that the total charge on $S_{\rho_1'}$ is equal to zero, and that the $n = 1$ terms also vanish if we choose a symmetric distribution with its centroid at the origin.

Hence, by comparison, we arrive at the following values of the integrals

$$\oint_{S_{\rho_1'}} S_n^m(\mu,\nu) d^i(\rho_1',\mu,\nu) \mathrm{d}S(\mu,\nu) = \frac{I_n^m(\rho_0')}{I_n^m(\rho_1')E_n^m(\rho_1')}\left[\frac{I_0^1(a_1)}{I_0^1(\rho_0')}\mathbb{E}_n^m(\rho_0',\mu_0',\nu_0') - \frac{I_n^m(a_1)}{I_n^m(\rho_0')}\mathbb{E}_n^m(\rho_0,\mu_0,\nu_0)\right] \tag{49}$$

for every $n \geqslant 2$ and $m = 1, 2, \ldots, 2n+1$. The function $d^i/l_{\rho_1'}$, where $l_{\rho_1'}$ is the weighting function on the ellipsoid $S_{\rho_1'}$, can be expanded as

$$\frac{d^i(\rho_1',\mu_1',\nu_1')}{l_{\rho_1'}(\mu_1',\nu_1')} = \sum_{n=2}^{\infty}\sum_{m=1}^{2n+1} D_n^m S_n^m(\mu_1',\nu_1') \tag{50}$$

from which we obtain, by orthogonality, the values

$$D_n^m = \frac{1}{\gamma_n^m}\oint_{S_{\rho_1'}} S_n^m(\mu,\nu) d^i(\rho_1',\mu,\nu)\mathrm{d}S(\mu,\nu). \tag{51}$$

But these integrals are known from (49) and therefore, the density on the image ellipsoid is given by

$$d^i(\rho_1',\mu_1',\nu_1') = l_{\rho_1'}(\mu_1',\nu_1')\sum_{n=2}^{\infty}\sum_{m=1}^{2n+1}\frac{1}{\gamma_n^m}\frac{I_n^m(\rho_0')}{I_n^m(\rho_1')E_n^m(\rho_1')}\left[\frac{I_0^1(a_1)}{I_0^1(\rho_0')}\mathbb{E}_n^m(\rho_0',\mu_0',\nu_0') - \frac{I_n^m(a_1)}{I_n^m(\rho_0')}\mathbb{E}_n^m(\rho_0,\mu_0,\nu_0)\right] S_n^m(\mu_1',\nu_1'). \tag{52}$$

Therefore, the image system for the Green's function, in the interior of an ellipsoid, consist of a monopole at the point $\boldsymbol{r}_0'$, given by the solution of (39) and, (45), (46), with strength Q, given in (34), and a surface distribution of monopoles with density d^i, given in (52), over an exterior confocal ellipsoid $\rho_1' > a_1$.

In the case of the sphere, we have

$$I_0^1(x) = \frac{1}{x} \tag{53}$$

$$I_1^m(x) = \frac{1}{3x^3}, \quad m = 1, 2, 3 \tag{54}$$

Eq. (39) is reduced to

$$r_0 r_0' = \alpha^2 \tag{55}$$

while the dyadic is reduced to

$$\widetilde{\boldsymbol{D}}_G^i(r_0') = \frac{r_0'^2}{\alpha^2}\tilde{\boldsymbol{I}}. \tag{56}$$

Then, the mapping (46) reads

$$\boldsymbol{r}_0' = \frac{r_0'^2}{\alpha^2}\boldsymbol{r}_0 = \frac{\alpha^2}{r_0^2}\boldsymbol{r}_0 \tag{57}$$

and recovers the Kelvin image of the source.

It can be shown that, as the ellipsoid degenerates to a sphere

$$\frac{\mathbb{E}_n^m(\rho_0',\mu_0',\nu_0')}{\mathbb{E}_n^m(\rho_0,\mu_0,\nu_0)} \to \left(\frac{\alpha^2}{r_0^2}\right)^n \tag{58}$$

and

$$\frac{I_0^1(a_1)}{I_0^1(\rho_0')}\frac{I_n^m(\rho_0')}{I_n^m(a_1)} \to \left(\frac{\alpha^2}{r_0'^2}\right)^n. \tag{59}$$

Hence, in the spherical limit, the coefficients of the expansion (52) vanish, and therefore, the surface distribution of images disappears. Consequently the spherical case is fully recovered.

Note that, as it was expected, the ellipsoid is endowed with a distinct behavior in every direction, and this is encoded in the form of the dyadic $\widetilde{\boldsymbol{D}}_G^i$. Formula (46) is the ellipsoidal generalization of the Kelvin transformation.

4 The exterior Green's function

In the case of the exterior of a sphere of radius α and a unit source $\boldsymbol{r}_0$ lying outside the sphere, the Green's function satisfies the boundary value problem

$$\Delta_{\boldsymbol{r}} G^e(\boldsymbol{r},\boldsymbol{r}_0) = \delta(\boldsymbol{r}-\boldsymbol{r}_0), \quad r > a \tag{60}$$

$$G^e(\boldsymbol{r},\boldsymbol{r}_0) = 0, \quad r = a \tag{61}$$

$$G^e(\boldsymbol{r},\boldsymbol{r}_0) = O\left(\frac{1}{r}\right), \quad r \to \infty. \tag{62}$$

Following exactly the same steps as in the interior case, with the appropriate expansions in spherical harmonics, we arrive at the solution

$$G_s^e(\boldsymbol{r},\boldsymbol{r}_0) = -\frac{1}{4\pi}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0|} + \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{1}{2n+1}\frac{\alpha^{2n+1}}{r_0^{n+1}r^{n+1}}Y_n^m(\hat{\boldsymbol{r}})Y_n^m(\hat{\boldsymbol{r}}_0)^* \tag{63}$$

which holds for $r > \alpha$.

In order to construct an image system for this case, we consider a monopole, with strength Q, at the interior to the sphere point $\boldsymbol{r}_0'$, which provides the potential

$$U_s^e(\boldsymbol{r}) = -Q\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{1}{2n+1}\frac{r_0'^n}{r^{n+1}}Y_n^m(\hat{\boldsymbol{r}})Y_n^m(\hat{\boldsymbol{r}}_0)^*. \tag{64}$$

Comparing the two expansions on the right hand sides of Eqs. (63) and (64), we obtain

$$\frac{\alpha^{2n+1}}{r_0^{n+1}} = -Q r_0'^n, \quad n \geqslant 0 \tag{65}$$

and if we choose $Q = -r_0'/\alpha$, as in (34), we obtain

$$\left(\frac{\alpha^2}{r_0}\right)^{n+1} = r_0'^{n+1}, \quad n \geqslant 0 \tag{66}$$

which implies that the location of the image is again at the Kelvin image (22) of the source.

Note that, as $r \to \infty$,

$$G_s^e(\boldsymbol{r},\boldsymbol{r}_0) = \frac{1}{4\pi r}\left(\frac{\alpha}{r_0}-1\right) + O\left(\frac{1}{r^2}\right) \tag{67}$$

where we have used the normalized form $Y_0^0(\hat{\boldsymbol{r}}) = (4\pi)^{-1/2}$.

Next, we consider the Green's function for the exterior of the reference ellipsoid, that is for $\rho > a_1$ and a source point $\boldsymbol{r}_0 = (\rho_0, \mu_0, \nu_0)$ with $\rho_0 > a_1$. Working as before with the appropriate interchanges between interior and exterior ellipsoidal harmonics, we produce the representation

$$G_e^e(\boldsymbol{r},\boldsymbol{r}_0) = -\frac{1}{4\pi}\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_0|} + \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\frac{1}{2n+1}\frac{1}{\gamma_n^m}\frac{E_n^m(a_1)}{F_n^m(a_1)}\mathbb{F}_n^m(\rho_0,\mu_0,\nu_0)\mathbb{F}_n^m(\rho,\mu,\nu). \tag{68}$$

In building the image system for the expansion on the right hand side of (68), we start with a monopole in the interior of the ellipsoid and calculate its position $\boldsymbol{r}_0'$ and strength Q from the demand that the terms $n = 0$ and $n = 1$ of the potential it generates, coincide with the corresponding terms of the regular part of the expansion in (68). The monopolic image will generate the potential

$$U_e^e(\boldsymbol{r}) = -Q \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \frac{1}{2n+1} \frac{1}{\gamma_n^m} \mathbb{E}_n^m(\rho_0', \mu_0', \nu_0') \mathbb{F}_n^m(\rho, \mu, \nu). \tag{69}$$

Hence, the strength has to be

$$Q = -\frac{I_0^1(\rho_0)}{I_0^1(a_1)} \tag{70}$$

and from the $n = 1$, we obtain the relations

$$I_0^1(a_1) I_1^m(\rho_0) x_{0m} = I_0^1(\rho_0) I_1^m(a_1) x_{0m}', \quad m = 1, 2, 3. \tag{71}$$

Therefore, if we define the dyadic

$$\widetilde{\boldsymbol{D}}_G^e(\rho_0) = \frac{I_0^1(a_1)}{I_0^1(\rho_0)} \sum_{m=1}^{3} \frac{I_1^m(\rho_0)}{I_1^m(a_1)} \hat{\boldsymbol{x}}_m \otimes \hat{\boldsymbol{x}}_m \tag{72}$$

which now depends on ρ_0, then the position of the image is given by

$$\boldsymbol{r}_0' = \widetilde{\boldsymbol{D}}_G^e(\rho_0) \cdot \boldsymbol{r}_0. \tag{73}$$

We also assume a continuous distribution of monopoles on the interior confocal ellipsoid $\rho_1' < a_1$, with density $d^e(\rho_1', \mu_1', \nu_1')$. As in the interior case, we can pick up the density in such a way so that the $n = 0$ and $n = 1$ terms vanish. Then, in an exterior neighborhood of the boundary, we would have

$$\begin{aligned} V_e^e(\boldsymbol{r}) = &\sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \frac{I_0^1(\rho_0)}{I_0^1(a_1)} \frac{1}{\gamma_n^m} \mathbb{E}_n^m(\rho_0', \mu_0', \nu_0') I_n^m(\rho) \mathbb{E}_n^m(\rho, \mu, \nu) \\ &- \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \frac{1}{2n+1} \frac{1}{\gamma_n^m} G_n^m \mathbb{F}_n^m(\rho, \mu, \nu) \end{aligned} \tag{74}$$

where

$$\begin{aligned} G_n^m &= \oint_{S_{\rho_1'}} \mathbb{E}_n^m(\rho_1', \mu, \nu) d^e(\rho_1', \mu, \nu) \mathrm{d}S(\mu, \nu) \\ &= E_n^m(\rho_1') \oint_{S_{\rho_1'}} S_n^m(\mu, \nu) d^e(\rho_1', \mu, \nu) \mathrm{d}S(\mu, \nu). \end{aligned} \tag{75}$$

The terms for $n \geqslant 2$ should recover the corresponding part of the expansion in (68) and that demands that

$$\oint_{S_{\rho_1'}} S_n^m(\mu_1', \nu_1') d^e(\rho_1', \mu_1', \nu_1') \mathrm{d}S(\mu_1', \nu_1') = \frac{I_0^1(\rho_0)}{I_0^1(a_1)} \frac{\mathbb{E}_n^m(\rho_0', \mu_0', \nu_0')}{E_n^m(\rho_1')} - \frac{I_n^m(\rho_0)}{I_n^m(a_1)} \frac{\mathbb{E}_n^m(\rho_0, \mu_0, \nu_0)}{E_n^m(\rho_1')} \tag{76}$$

for every $n \geqslant 2$and $m = 1, 2, \ldots, 2n + 1$. Working as in the interior case, we can calculate the density of the surface distribution and obtain

$$d^e(\rho_1', \mu_1', \nu_1') = l_{\rho_1'}(\mu_1', \nu_1') \sum_{n=2}^{\infty} \sum_{m=1}^{2n+1} \frac{1}{\gamma_n^m} \left[\frac{I_0^1(\rho_0)}{I_0^1(a_1)} \mathbb{E}_n^m(\rho_0', \mu_0', \nu_0') - \frac{I_n^m(\rho_0)}{I_n^m(a_1)} \mathbb{E}_n^m(\rho_0, \mu_0, \nu_0) \right] \frac{S_n^m(\mu_1', \nu_1')}{E_n^m(\rho_1')}. \tag{77}$$

Therefore, the image system for the Green's function, in the exterior of an ellipsoid, consist of a monopole at the point $\boldsymbol{r}_0'$, given in (72), (73), with strength Q, given in (70), and a surface distribution of monopoles with density d^e, given in (77), over an interior confocal ellipsoid $\rho_1' < a_1$.

5 Conclusions

In the present work, we construct the Green's function for the Laplace operator in the case where the fundamental domain is either the interior or the exterior of an ellipsoid. In addition, we propose Kelvin type image systems that represent the regular parts of the corresponding Green's functions. In the highly symmetric case of a sphere, these systems consist solely of a single point source located at the Kelvin image point of the source. On the other hand, the complete loss of rotational symmetries in the case of the ellipsoid demands an additional two-dimensional distribution of imaginary sources on a confocal ellipsoid outside the fundamental domain.

Two basic remarks are in order here. In the first place by comparing the dyadics $\widetilde{\boldsymbol{D}}^i_G(\rho_0')$, given in (45) and $\widetilde{\boldsymbol{D}}^e_G(\rho_0)$, given in (72), we immediately observe that they are both invertible and that

$$\widetilde{\boldsymbol{D}}^i_G(\rho_0')^{-1} = \widetilde{\boldsymbol{D}}^e_G(\rho_0'). \tag{78}$$

This shows that if the $\boldsymbol{r}_0'$ is the image point for the interior source $\boldsymbol{r}_0$, then $\boldsymbol{r}_0$ is the image point for the exterior source $\boldsymbol{r}_0'$. Indeed, because of (80), we have that

$$\boldsymbol{r}_0' = \widetilde{\boldsymbol{D}}^i_G(\rho_0') \cdot \boldsymbol{r}_0 = \widetilde{\boldsymbol{D}}^i_G(\rho_0') \cdot \widetilde{\boldsymbol{D}}^e_G(\rho_0') \cdot \boldsymbol{r}_0' = \tilde{\boldsymbol{I}} \cdot \boldsymbol{r}_0'. \tag{79}$$

Furthermore, the invertibility of these dyadics imply that for each source point, there exists a *unique* image point, that is the function f, defined in (40) has a unique solution ρ_0' in the interval (a_1,∞). Consequently, we obtain the same relations between source and image points as that in the case of the sphere.

The second remark has to do with the fact that there is an important difference, in the process of calculating the image systems, between the interior an the exterior problems for the ellipsoid. In calculating both the strength and the position of the isolated monopole in the interior case, we need first to solve Eq. (39) to obtain ρ_0', while in the exterior problem, everything is given in terms of ρ_0 which corresponds to the source point, and therefore, it is known. Hence, it is easier to find the image system for the exterior than the interior Green's function. This difference is not easily recognizable in the case of the sphere, because of the trivial way that the two variables r_0 and r_0' are connected. Indeed, in the case of the sphere, formula (46) gives

$$\boldsymbol{r}_0' = \frac{r_0'^2}{\alpha^2}\boldsymbol{r}_0 \tag{80}$$

for the interior problem, and formula (73) gives

$$\boldsymbol{r}_0' = \frac{\alpha^2}{r_0^2}\boldsymbol{r}_0 \tag{81}$$

for the exterior problem, and we can trivially switch from one to the other via the Kelvin relation $r_0 r_0' = \alpha^2$.

Since the Green's function in a given domain is basically equivalent to the solution of the Dirichlet problem for the Laplace equation in this domain, it follows that the knowledge of the Green's function in any ellipsoidal coordinate system can be used to generate the corresponding solutions in many boundary value problems related to harmonic functions in ellipsoidal geometry. Furthermore, by choosing appropriately the orientation and the size of a particular ellipsoid, we can always approximate any convex body by the ellipsoid. Hence, the proposed construction covers a large domain of applications in Mechanics as well as in Physics.

References

1. Aki, K., Richards, P.G.: Quantitative Seismology. University Science Books, Sausalito (2002)
2. Barton, G.: Elements of Green's Functions and Propagation, Potentials, Diffusion and Waves, 1st edn. Oxford University Press, New York (1989)
3. Courant, R., Hilbert, D.: Methods of Mathematical Physics II, 1st English edn. Interscience, New York (1962)
4. Dassios, G.: The magnetic potential for the ellipsoidal MEG problem. J. Comput. Math. **25**, 145–156 (2007)
5. Dassios, G.: Neuronal Currents and EEG—MEG fields. IMA Math. Med. Biol. 2541–2549 (2008). doi:10.1093/imammb/dqn007
6. Dassios, G., Sten, J.C.-E.: The image system and Green's function for the ellipsoid. In: Ammari, H., Kang, H. (eds) Imaging Microstructures. Mathematical and Computational Challenges, vol. 494 of Conteborary Mathematics, pp. 185–195. American Mathematical Society, Providence (2009)

7. Duffy, D.G.: Green's Functions with Applications, 1st edn. Chapman and Hall, Boca Raton (2001)
8. Green, G.: An assay on the application of mathematical analysis to the theories of electricity and magnetism. In: Private Publication, Nottingham. Also in Mathematical Papers of the Late George Green, N.M. Ferrers (Editor). MacMillan and Co, London (1871)
9. Gurtin, M.E.: The Linear Theory of Elasticity, vol. VIa/2 of Handbuch der Physik. Springer, Berlin (1972)
10. Hancock, H.: Low Reynolds Number Hydrodynamics, 2nd edn. Martinus Nijhoff Publishers, Dordrecht (1973)
11. Hobson, E.W.: The Theory of Spherical and Ellipsoidal Harmonics, 1st edn. Cambridge University Press, Cambridge (1931)
12. Kupradge, V.D.: Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity, 1st edn. North-Holland, Amsterdam (1979)
13. Love, A.E.H.: A Treatise on the Mathematical Theory of Elasticity, 4th edn. Dover, New York (1995)
14. Melnikov, Y.A.: Green's Functions in Applied Mechanics, 2nd edn. Computational Mechanics Publications, Southampton (1997)
15. Milne-Thomson, L.M.: Theoretical Hydrodynamics, 5th edn. MacMillan, London
16. Nédélec, J.C.: Acoustic and Electromagnetic Equations. Springer, New York (2001)
17. Roach, G.H.: Green's Functions. Introductory Theory with Applications, 1st edn. Van Nostrand Reinhold, London (1970)
18. Sokolnikoff, I.S.: Mathematical Theory of Elasticity. McGraw-Hill, New York (1946)
19. Sten, J.C.-E., Dassios, G.: Image distribution and surface potential of a dipole in a one-shell conducting sphere. IMA J. Appl. Math. **75**, 720–731 (2010)
20. Thomson, W. (Lord Kelvin): Extrait d'une lettre de M. William Thomson (reported by A. M. Liouville). Journal de Mathématiques Pures et Appliquées **10**, 364–367 (1845)
21. Thomson, W. (Lord Kelvin): Extraits de Deux lettres adressees A. M. Liouville. Journal de Mathématiques Pures et Appliquées, **12**, 256–264 (1847)
22. Thomson, W. (Lord Kelvin): Reprint of Papers on Electrostatics and Magnetism, chapter XV. Determination of the distribution of electricity on a circular segment of plane or spherical conducting surface, under any given influence, pp. 178–191. MacMillan, London (1872)