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TECHNICAL NOTE

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Vibration of a tapered cantilever of constant thickness and linearly tapered width

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Abstract The vibration of a cantilever beam with constant thickness and linearly tapered sides is solved using a novel accurate, efficient initial value numerical method. The effects of tip mass, base fixity, and taper on the natural frequencies are determined. This geometrically anisotropic beam vibrates in a mixture of modes in two perpendicular directions.

Keywords Beam · Vibration · Taper · Cantilever

1 Introduction and formulation

The vibration of a cantilever with tip mass is important in a wide variety of mechanical applications. For strength reasons usually the cantilever is tapered, being wider at the base and narrower at the tip. The vibrations of a class of linearly tapered beams were first studied by Kirchkoff [1] who expressed the solutions in terms of what is now known as Bessel functions.

Ignoring rotational inertia and shear deformation, the Euler–Bernoulli small-deflection beam equation can be shown to be (e.g. Magrab [2])

$$\frac{\partial^2}{\partial x'^2}\left(EI(x')\frac{\partial^2 y'}{\partial x'^2}\right)+\rho(x')\frac{\partial^2 y'}{\partial t'^2}=0 \tag{1}$$

Here x' is the axial distance from the base of the cantilever, y' is the transverse deflection, *EI* is the flexural rigidity,ρ is the mass per length, and $t'$is the time. Let

$$EI(x')=EI_0 l(x'),\ \rho(x')=\rho_0 r(x') \tag{2}$$

where EI_0 is the maximum of *EI* and ρ_0 is the maximum of ρ, both occurring at the base. Consider a harmonic vibration with frequency ω'

$$y'=w'(x')e^{i\omega' t'} \tag{3}$$

Normalize all lengths by the beam length L, the time by $L^2\sqrt{\rho_0/EI_0}$ and drop primes. Equation 1 becomes

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[l(x)\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right]-\omega^2 r(x)w=0 \tag{4}$$

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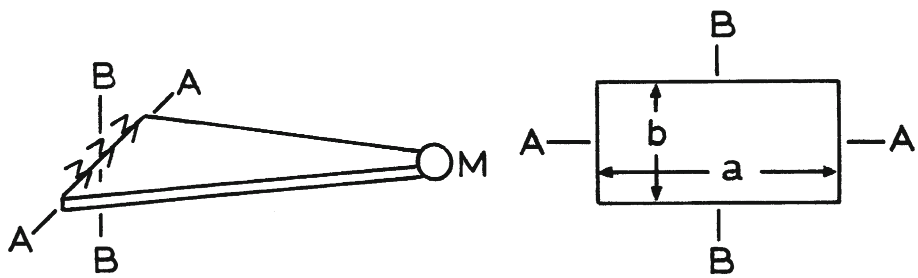

Fig. 1 **a** The cantilever beam of constant thickness and tapered sides. **b** Cross-section at the base

Here

$$\omega = \omega' L^2 \sqrt{\rho_0 / E I_0} \tag{5}$$

is the non-dimensional frequency. Let

$$z = 1 - cx, \quad 0 \le c \le 1 \tag{6}$$

where c is the degree of taper. Assume power law variations of rigidity and density

$$l(z) = z^m, \quad r(z) = z^n \tag{7}$$

Equation (4) becomes

$$c^4 \frac{d^2}{dz^2} \left[z^m \frac{d^2 w}{dz^2} \right] - \omega^2 z^n w = 0 \tag{8}$$

Kirchkoff considered the cases $m = 4$, $n = 2$ which represents a conical beam, and $m = 3$, $n = 1$ which is a beam with constant thickness and linear width. In general, if $m = n + 2$ Eq. (8) can be factored into

$$\left[z^{-n} \frac{d}{dz} \left(z^{n+1} \frac{d}{dz} \right) + \frac{\omega}{c^2} \right] \left[z^{-n} \frac{d}{dz} \left(z^{n+1} \frac{d}{dz} \right) - \frac{\omega}{c^2} \right] w = 0 \tag{9}$$

Each one of the brackets in Eq. (9) is a Bessel operator. When n is an integer, the exact solution is

$$w = z^{-n/2} [C_1 J_n(u) + C_2 Y_n(u) + C_3 I_n(u) + C_4 K_n(u)], \;\; u = 2\sqrt{\omega z}/c \tag{10}$$

Here J an Y are Bessel functions and I and K are modified Bessel functions. Many papers have extended Kirchkoff's work, notably Sanger [3].

Consider the vibration of a cantilever beam with constant thickness and linearly varying width. Figure 1a shows the beam with a tip mass and a rotational spring-hinged base. Note the beam can vibrate in two different directions. If vibration is about the axis A–A, which is perpendicular to the direction of thickness, then $m = 1$, $n = 1$. If vibration is about the axis B–B, which is in the direction of thickness, then $m = 3$, $n = 1$.

For the $m = 3$, $n = 1$ case, using the exact solution of Kirchkoff, Mabie and Rogers [4] computed the frequencies for a cantilever beam with tip mass. Sankaran et al. [5] and Lee [6] added a spring-hinged base, and the problem is considered solved.

However, the $m = 1$, $n = 1$ case does not have simple Bessel-type solutions, and numerical methods are needed. Wang [7] expressed the solution in terms of hypergeometric functions, but the frequencies need to be evaluated by infinite series expansions. Downs [8] used a complicated dynamic discretization method which subdivides the beam into segments. Naguleswaran [9] used Frobenius series expansions but have convergence problems. None of the previous reports considered tip mass or base flexibility.

The purpose of the present paper is to present the complete frequencies for the cantilever beam, which has constant thickness and linearly varying width and with a tip mass and a spring- hinged base. We shall use a novel initial value method which will be accurate and efficient.

Table 1 First three frequencies for a beam with constant thickness and linearly tapered width ($m = n = 1$) with clamped base and no tip mass

c	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.999
ω_1	3.5160	3.6310	3.7629	3.9160 {3.9160}	4.0970 (4.5857) [4.0970]	4.3152 [4.3152]	4.5853 [4.5853] {4.5853}	4.9317 [4.9316]	5.3976 (5.3969) [5.3976] {5.3976}	6.0704 [6.0704] {6.0704}	7.1422 {7.1565}
ω_2	22.035	22.254	22.502	22.786 {22.786}	24.021 (24.021) [23.119]	23.519 [23.519]	24.021 [24.021] {24.021}	24.687 [24.687]	25.656 (25.656) [25.656] {25.656}	27.299 [27.299] {27.299}	30.970 {31.041}
ω_3	61.697	61.910	62.153	62.436 {62.436}	62.776 [62.776]	63.199 [63.199]	63.751 [63.751] {63.752}	64.527 [64.527]	65.747 [65.747] {65.747}	68.115 [68.115] {68.115}	75.653 {75.487}

Values in parentheses are from Wang [7], square brackets from Naguleswaran [9] and flower brackets from Downs [8]

2 The initial value method

The boundary conditions for a spring-hinged base at $x' = 0$ with rotational spring constant k is

$$w = 0,\ EI\frac{\partial^2 y'}{\partial x'^2} = k\frac{\partial y'}{\partial x'} \tag{11}$$

or in normalized form at $z = 1$

$$w = 0,\ -c\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = \gamma\frac{\mathrm{d}w}{\mathrm{d}z} \tag{12}$$

where $\gamma = kL/EI_0$ is the normalized spring constant. At the tip at $x' = L$ the moment is zero and the shear balances the tip mass M

$$EI\frac{\partial^2 y'}{\partial x'^2} = 0, \quad \frac{\partial}{\partial x'}\left(EI\frac{\partial^2 y'}{\partial x'^2}\right) = M\frac{\partial^2 y'}{\partial t'^2} \tag{13}$$

or in normalized form at $z = 1 - c$

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} = 0, \quad -c^3(1-c)^m\frac{\mathrm{d}^3 w}{\mathrm{d}z^3} + \nu\omega^2 w = 0 \tag{14}$$

Here $\nu = M/\rho_0 L$ is a mass ratio. Equation (8) is to be solved with the boundary conditions Eqs. (12), (14).

The boundary value problem is inconvenient since the four boundary conditions are evenly divided at both ends of the beam. We shall adapted an initial value method originally suggested Barasch and Chen [10] for rotating plates but seldom advocated. Let

$$w = C_1 w_1(z) + C_2 w_2(z) \tag{15}$$

where w_1 and w_2 each satisfies four initial conditions including Eq. (12). Furthermore, w_1 and w_2 have independent initial conditions, i.e.,

$$w_1(1) = 0,\ \frac{\mathrm{d}w_1}{\mathrm{d}z}(1) = 1,\ \frac{\mathrm{d}^2 w_1}{\mathrm{d}z^2}(1) = -\gamma/c,\ \frac{\mathrm{d}^3 w_1}{\mathrm{d}z^3}(1) = 0 \tag{16}$$

$$w_2(1) = 0,\ \frac{\mathrm{d}w_2}{\mathrm{d}z}(1) = 0,\ \frac{\mathrm{d}^2 w_2}{\mathrm{d}z^2}(1) = 0,\ \frac{\mathrm{d}^3 w_2}{\mathrm{d}z^3}(1) = 1 \tag{17}$$

Then Eq. (8) is integrated by the Runge–Kutta method backwards from $z = 1$ to $z = 1 - c$ for both w_1 and w_2. Eq. (14) gives, for non-trivial solutions, the condition

$$\begin{vmatrix} \frac{\mathrm{d}^2 w_1}{\mathrm{d}z^2}\big|_{1-c} & \frac{\mathrm{d}^2 w_2}{\mathrm{d}z^2}\big|_{1-c} \\ -c^3(1-c)^m\frac{\mathrm{d}^3 w_1}{\mathrm{d}z^3}\big|_{1-c} & -c^3(1-c)^m\frac{\mathrm{d}^3 w_2}{\mathrm{d}x^3}\big|_{1-c} \\ +\nu\omega^2 w_1\big|_{1-c} & +\nu\omega^2 w_2\big|_{1-c} \end{vmatrix} = 0 \tag{18}$$

Table 2 Frequencies for the $m = 3$, $n = 1$ case

$\nu \backslash c$	0.1	0.3	0.5	0.7	0.9
(a) $\gamma = \infty$ *or clamped base*					
0	3.5587	3.6668	3.8238	4.0817	4.6307
	21.338	19.881	18.317	16.625	14.931
	58.980	53.322	47.265	40.588	32.833
0.1	2.9591	2.9304	2.8737	2.7494	2.3864
	18.533	16.766	14.276	12.432	9.3434
	52.660	46.696	40.299	33.229	24.688
1	1.5121	1.4084	1.2804	1.1112	0.8435
	15.587	14.195	12.681	10.957	8.7053
	48.441	43.342	37.887	31.817	24.254
10	0.5212	0.4770	0.4258	0.3626	0.2700
	14.931	13.700	12.340	10.760	8.6377
	47.722	42.834	37.564	31.652	24.209
(b) $\gamma = 10$					
0	3.0280	3.1718	3.3649	3.6557	4.2131
	18.785	17.592	16.318	14.952	13.642
	53.120	48.135	42.813	36.968	30.237
0.1	2.5474	2.5719	2.5771	2.5292	2.2733
	16.358	14.873	13.188	11.177	8.4699
	47.433	42.135	36.451	30.165	22.559
1	1.3282	1.2616	1.1713	1.0408	0.8131
	13.660	12.486	11.208	9.7465	7.8234
	43.446	38.938	34.126	28.782	22.123
10	0.4607	0.4296	0.3912	0.3406	0.2606
	13.035	12.010	10.876	9.5518	7.7547
	42.751	38.444	33.810	28.619	22.078
(c) $\gamma = 1$					
0	1.6111	1.7400	1.9109	2.1573	2.5741
	15.700	14.569	13.395	12.198	11.211
	48.574	43.778	38.708	33.222	27.101
0.1	1.3870	1.4566	1.5345	1.6149	1.6463
	13.564	12.168	10.607	8.7692	6.2999
	43.162	38.054	32.611	26.644	19.533
1	0.7578	0.7545	0.7432	0.7149	0.6335
	10.973	9.8234	8.5861	7.1999	5.4475
	39.199	34.847	30.245	25.203	19.056
10	0.2674	0.2613	0.2523	0.2374	0.2050
	10.327	9.3168	8.2176	6.9678	5.3526
	38.492	34.340	29.918	25.030	19.006
(d) $\gamma = 0.1$					
0	0.5626	0.6137	0.6823	0.7811	0.9430
	14.949	13.793	12.597	11.395	10.456
	47.744	42.954	37.899	32.444	26.402
0.1	0.4883	0.5202	0.5592	0.6080	0.6689
	12.852	11.429	9.8385	7.9629	5.4107
	42.360	37.255	31.818	25.862	18.777
1	0.2720	0.2765	0.2808	0.2843	0.2835
	10.333	9.0257	7.7117	6.2137	4.2593
	38.383	34.026	29.424	24.392	18.279
10	0.09678	0.09669	0.09641	0.09571	0.09335
	9.5576	8.4839	7.3012	5.9328	4.1128
	37.669	33.513	29.092	24.215	18.227

The frequencies are obtained by bisection to satisfy Eq. (18). The errors of both Runge–Kutta and bisection can be prescribed to any accuracy.

We shall compare our numerical method with existing reports for the special case of a clamped base and zero tip mass ($m = n = 1$) which has no exact solution. The $c = 1$ case is approximated by $c = 0.999$ in our numerical computation. We see that all results agree for $0.5 \leq c \leq 0.9$. However, for $0.1 \leq c \leq 0.4$, the values from the "exact" hypergeometric series and the Frobenius series fail. The method of dynamic discretization seems to be accurate but tedious to implement. For $c = 1$, Eq. (8) is singular at $x = 1$, where all methods encounter some difficulty. Our values for $c = 0.999$ are deemed correct (Table 1).

Table 3 Frequencies for the $m = 1$, $n = 1$ case

$\nu \backslash c$	0.1	0.3	0.5	0.7	0.9
(a) $\gamma = \infty$ *or clamped base*					
0	3.6310	3.9160	4.3152	4.9317	6.0704
	22.254	22.786	23.519	24.687	27.299
	61.910	62.436	63.199	64.527	68.115
0.1	3.0245	3.1534	3.3084	3.4980	3.7298
	19.374	19.393	19.370	19.263	18.938
	55.374	55.028	54.584	54.001	53.130
1	1.5505	1.5337	1.5110	1.4793	1.4308
	16.304	16.431	16.586	16.765	16.914
	50.924	50.997	51.101	51.245	51.363
10	0.5350	0.5212	0.5054	0.4869	0.4631
	15.614	15.841	16.102	16.395	16.665
	50.158	50.367	50.611	50.901	51.170
(b) $\gamma = 10$					
0	3.0715	3.3261	3.6780	4.2111	5.1633
	19.548	20.016	20.663	21.700	24.019
	55.694	56.140	56.807	58.004	61.304
0.1	2.5882	2.7164	2.8711	3.0920	3.3010
	17.063	17.077	17.050	16.941	16.626
	49.822	49.462	49.005	48.407	47.516
1	1.3539	1.3488	1.3392	1.3228	1.2932
	14.258	14.349	14.457	14.573	14.642
	45.622	45.636	45.672	45.738	45.762
10	0.4702	0.4611	0.4502	0.4372	0.4199
	13.602	13.785	13.990	14.211	14.393
	44.882	45.025	45.196	45.400	45.568
(c) $\gamma = 1$					
0	1.6173	1.7630	1.9600	2.2480	2.7300
	16.383	16.721	17.218	18.076	20.119
	51.019	51.355	51.907	52.985	56.134
0.1	1.3933	1.4803	1.5870	1.7219	1.8995
	14.193	14.117	13.991	13.771	13.335
	45.425	44.975	44.420	43.715	42.696
1	0.7624	0.7717	0.7804	0.7879	0.7926
	11.505	11.461	11.405	11.317	11.126
	41.255	41.151	41.060	40.986	40.854
10	0.2692	0.2679	0.2661	0.2638	0.2603
	10.828	10.869	10.903	10.912	10.827
	40.504	40.528	40.570	40.635	40.649
(d) $\gamma = 0.1$					
0	0.5628	0.6147	0.6844	0.7851	0.9496
	15.628	15.929	16.387	17.204	19.203
	50.176	50.491	51.021	52.079	55.211
0.1	0.4886	0.5212	0.5615	0.6130	0.6818
	13.477	13.367	13.201	12.936	12.449
	44.612	44.139	43.560	42.828	41.778
1	0.2722	0.2773	0.2827	0.2883	0.2940
	10.765	10.667	10.545	10.372	10.070
	40.428	40.297	40.177	40.072	39.906
10	0.0969	0.09701	0.0971	0.0972	0.0972
	10.061	10.046	10.011	9.9323	9.7326
	39.671	39.668	39.682	39.716	39.697

3 Results

Since the cantilever beam can vibrate about two different axes (A–A and B–B), one needs to consider both sets of frequencies. We first generate the exact frequencies for $m = 3$, $n = 1$ from Eqs. (10), (12) and (14). The results from our numerical method Eq. (18) completely agree with those of the exact method, further confirming the accuracy.

For completeness, the results for $m = 3$, $n = 1$ are tabulated in Table 2, which are more comprehensive than previous reports. Table 2 is also used for the calculations of the complete lowest frequencies of the beam.

The corresponding results for the $m = 1$, $n = 1$ case, computed by the initial value method, are given in Table 3.

Table 4 Lowest frequencies for $c = 0.5$, $\gamma = \nu = 1$

$a/b = 0.1$	$a/b = 1$	$a/b = 10$
0.07432*	0.7432*	0.7804
0.7804	0.7804	7.4318*
0.8586*	8.5861*	11.405
3.0245*	11.405	41.060
6.6228*	30.245*	85.861*

Asterisk shows vibration is about the B–B axis, otherwise it is about the A–A axis

4 Discussions

The tapered cantilever beam with tip mass and spring-hinged base is fundamental in mechanical vibrations. Our accurate frequency tables would be useful for the design of such structures.

From our Tables, the following can be concluded. For the $m = 3$, $n = 1$ vibration about the axis B–B, we find frequencies decrease with increased tip mass ν and decreased base fixity γ. The increase in taper c decreases frequency, except perhaps the fundamental frequency. For the $m = 1$, $n = 1$ vibration about the axis A–A, the effects of ν and γ are similar to the $m = 3$, $n = 1$ case. However, the effect of taper is quite complex and is found to increase or decrease the frequency.

In practice, a cantilever beam can oscillate in both A–A or B–B directions, which have different EI_0, but the frequencies can only be compared with the same normalization. Consider the base cross-section shown in Fig 1b, where the width and thickness are a and b, respectively. For vibration about A–A and B–B, the rigidities are proportional to

$$EI_A \sim ab^3,\ EI_B \sim a^3 b \tag{19}$$

Let $EI_0 = EI_A$, then the frequencies in Table 3 are unchanged. Using the same EI_A to normalize the frequencies in Table 2, we find the actual frequencies should be multiplied by the aspect ratio a/b. As an example, lets take the beam with $c = 0.5$, $\gamma = \nu = 1$. The lowest five frequencies are listed in Table 4. Notice only for base aspect ratio one, the values can be directly compared.

Notice vibrations in either direction can be excited. This property is peculiar to geometrically anisotropic beams but seldom acknowledged by previous researchers. Such a new perspective is illustrated in Table 4.

We advocate a novel, seldom noticed, efficient, accurate initial value method. As shown in Sect. 2, the method is more advantageous than all existing methods. Aside from linear taper studied in this paper, the method can also be applied to other non-uniform tapers and functionally graded beams.

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