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An exact closed-form procedure for free vibration analysis of laminated spherical shell panels based on Sanders theory

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Abstract This paper deals with closed-form solutions for in-plane and out-of-plane free vibration of moderately thick laminated transversely isotropic spherical shell panels on the basis of Sanders theory without any usage of approximate methods. The governing equations of motion and the boundary conditions are derived using Hamilton's principle. The highly coupled governing equations are recast to some uncoupled equations by introducing four potential functions. Also, some relations were presented for the unknowns of the original set of equations in terms of the unknowns of the uncoupled equations. According to the proposed analytical approach, both Navier and Lévy-type explicit solutions are developed for moderately thick laminated spherical shell panels. The efficiency and high accuracy of the present approach are investigated by comparing some of the present study with the available results in the literature and the results of 3D finite element method. The effects of various shell parameters like shear modulus ratio of transversely isotropic materials and curvature ratio on the natural frequencies are studied. Clearly, the proposed solutions can accurately predict the in-plane and out-of-plane natural frequencies of moderately thick transversely isotropic spherical shell panels.

Keywords Exact closed-form solution · Laminated spherical shell panel · Sanders theory · Natural frequency

1 Introduction

Laminated shells are widely used for many purposes, especially in design of lightweight structures. This widespread application is due to the superior properties of such structures like high stiffness/strength to weight ratio and excellent thermal properties. The uses of laminated curved panel-type elements are common in many activities of modern lightweight engineering structures. Aircraft and spacecraft structures consist of some spherical and cylindrical shell panel elements. Due to the intensive use of laminated spherical shell panels in lightweight structures, the study of the dynamic behavior of these components has received much attention. Therefore, the free vibration analysis of the laminated spherical shell panels is of much importance.

Many theories were introduced by researchers for the analysis of laminated shells, which many of them were developed for thin shells. Ambartsumyan [1,2] was the first investigator to present an appropriate theory

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for the analysis of laminated thin shells. Dong et al. [3] developed a theory for laminated thin shells that is an extension of the proposed theory of Stavsky [4] for laminated thin shells. Widera and Chung [5] formulated a theory for composite thin cylindrical shells. For a special case that the cylindrical shell is homogeneous and isotropic, the theory reduces to Donnell's shallow shell theory [6]. These theories are based on the Kirchhoff-Love kinematic assumptions [7] and neglect the effect of transverse shear deformation that results in the underestimation of deflections and overestimation of the natural frequencies of the shell structures. The first-order shear deformation theory (FSDT), also known as the Sanders shell theory [8], incorporates the effects of transverse shear deformation and rotary inertia. Hence, this theory eliminates the deficiency of thin shell theories and gives reliable prediction of the natural frequencies of shell structures and is suitable for analyzing moderately thick shells, which the thickness-to-side ratio takes from 0.05 to 0.2.

There are some studies in the literature related to the free vibration analysis of laminated spherical shell panels. Reddy [9] presented exact solutions for free vibration and buckling of simply supported laminated spherical shell panels according to the FSDT. Later, Reddy and Liu [10] developed a higher order shear deformation theory for laminated shells and obtained exact solutions for bending and free vibration of simply supported spherical shell panels. Also, Chaudhuri and Kabir [11] presented an exact solution for free vibration of simply supported laminated doubly curved panel using the four classical shallow shell theories. Singh [12] investigated the free vibration of moderately thick and thick doubly curved open deep sandwich shells by the use of Rayleigh-Ritz method. Liew et al. [13] presented the elasticity solutions for free vibration analysis of spherical shell panels. They employed the p-Ritz method for the solution of the problem. Lee and Reddy [14] presented an exact solution for vibration suppression in laminated composite shells with surface mounted smart material layers based on the linear versions of the Donnell and Sanders shell theories and for simply supported boundary conditions. The nonlinear dynamic response of doubly curved shallow shells resting on Winkler–Pasternak elastic foundation was studied by Civalek [15] for step and sinusoidal loadings. He applied the harmonic differential quadrature (HDQ) and finite differences (FD) methods to solve the governing equations. Later, Civalek [16] utilized the discrete singular convolution method (DSC) to study the free vibration behavior of conical panels. He investigated the effects of boundary conditions, vertex, and subtended angle on the natural frequencies of the shell panel. Hasheminejad and Maleki [17] presented an exact analysis for interaction of a time-harmonic plane-progressive sound field with a laminated transversely isotropic hollow sphere shell with interlaminar bonding imperfections. A nonlinear finite element model for geometrically large amplitude free vibration analysis of laminated spherical shell panels was presented by Panda and Singh [18] using the third-order shear deformation theory. Biglari and Jafari [19] studied the free vibrations of simply supported doubly curved sandwich panels with flexible core based on the first-order shear deformation theory and assumptions of linear distribution of transverse normal stress and uniform shear stresses over the thickness of core. Panda and Singh [20,21] analyzed nonlinear free vibration behavior of thermally post-buckled laminated composite spherical and doubly curved shell panels based on third-order shear deformation theory. They used a direct iterative method in conjunction with nonlinear finite element approach to solve the system of equations. The finite element model of vibrating laminated spherical shell panels with delamination around a central cutout was developed by Lee and Chung [22] based on the third-order shear deformation theory. Tornabene [23,24] applied the generalized differential quadrature (GDQ) method to study the free vibration of laminated composite doubly curved shells and laminated spherical shell panels according to the FSDT. Nguyen-Van et al. [25] presented buckling and free vibration analysis of moderately thick laminated composite plate and shell panel structures of various shapes via a novel smoothed quadrilateral flat element.

Most of studies for laminated shell panels were performed by the use of numerical methods such as finite element, differential quadrature, and Ritz methods, and the exact closed-form solutions were only presented for fully simply supported laminated panels. According to the authors' knowledge, no exact closed-form solution exists for the free vibration of Lévy-type laminated spherical shell panels based on the FSDT. The objective of the present study is to propose an exact closed-form solution for the free vibration of moderately thick laminated spherical shell panels having arbitrary boundary conditions at two opposite edges. To this end, after deriving the governing equations, a powerful decoupling method is applied by introducing four new functions. Also, some relations are presented for the unknowns of the original set of equations in terms of the unknowns of the uncoupled equations. The reformulated equations are exactly solved, and arbitrary boundary conditions are satisfied at two opposite edges. The efficiency and high accuracy of the present approach are investigated by comparing some of the present study with the available results in the literature and the results of 3D finite element method. The effects of various shell parameters like shear modulus ratio of transversely isotropic materials and curvature ratio on the natural frequencies are studied. The present closed-form solutions can accurately predict natural frequencies of moderately thick transversely isotropic spherical shell panels.

2 Governing equations of motion

An orthogonal curvilinear coordinate system (ξ_1, ξ_2, ξ_3) is considered to represent the geometry and deformation of the laminated spherical shell panel. The axes ξ_1 and ξ_2 are located in the mid-plane of the shell panel. Consider a laminated spherical shell panel of length a, width b, uniform thickness h, and mean radius R (Fig. 1). It is assumed that the shell panel has two opposite edges simply supported at the edges $\xi_1 = 0, a$. The other edges have arbitrary boundary conditions.

Based on the first-order shear deformation theory (FSDT), the displacement field is assumed to be

$$\begin{aligned} u_1(\xi_1, \xi_2, \xi_3, t) &= u(\xi_1, \xi_2, t) + \xi_3 \psi_1(\xi_1, \xi_2, t) \\ u_2(\xi_1, \xi_2, \xi_3, t) &= v(\xi_1, \xi_2, t) + \xi_3 \psi_2(\xi_1, \xi_2, t) \\ u_3(\xi_1, \xi_2, \xi_3, t) &= w(\xi_1, \xi_2, t) \end{aligned} \tag{1}$$

where u and v denote the in-plane displacements of the shell middle surface and w is the transverse deflection. Also, ψ_1 and ψ_2 are rotation functions and variable t is the time. Under the assumption of small deformation, the strain-displacement relations are given as

$$\left[\varepsilon_{11}\ \varepsilon_{22}\ \varepsilon_{33}\ 2\varepsilon_{12}\ 2\varepsilon_{13}\ 2\varepsilon_{23}\right]^T = \begin{bmatrix} \chi_{11}^0 & \chi_{22}^0 & 0 & \chi_{12}^0 & \chi_{13}^0 & \chi_{23}^0 \\ \chi_{11}^1 & \chi_{22}^1 & 0 & \chi_{12}^1 & 0 & 0 \end{bmatrix}^T \begin{Bmatrix} 1 \\ \xi_3 \end{Bmatrix} \tag{2}$$

where

$$\begin{aligned} \chi_{11}^0 &= \frac{\partial u}{\partial \xi_1} + \frac{w}{R}, \quad \chi_{11}^1 = \frac{\partial \psi_1}{\partial \xi_1} \\ \chi_{22}^0 &= \frac{\partial v}{\partial \xi_2} + \frac{w}{R}, \quad \chi_{22}^1 = \frac{\partial \psi_2}{\partial \xi_2} \\ \chi_{12}^0 &= \frac{\partial u}{\partial \xi_2} + \frac{\partial v}{\partial \xi_1}, \quad \chi_{12}^1 = \frac{\partial \psi_1}{\partial \xi_2} + \frac{\partial \psi_2}{\partial \xi_1} \\ \chi_{13}^0 &= \psi_1 + \frac{\partial w}{\partial \xi_1} - \frac{u}{R}, \quad \chi_{23}^0 = \psi_2 + \frac{\partial w}{\partial \xi_2} - \frac{v}{R} \end{aligned} \tag{3}$$

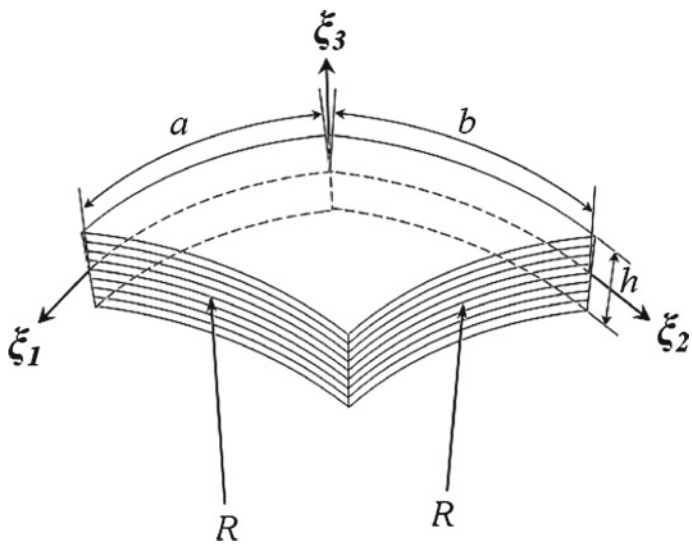

Fig. 1 Geometry and curvilinear coordinate system (ξ_1, ξ_2, ξ_3) of a laminated spherical shell panel

The stress–strain relations for each layer of the laminated transversely isotropic shell are expressed as

$$\begin{aligned}
\sigma_{11}^{(k)} &= \frac{E^{(k)}}{1-\left(\nu^{(k)}\right)^2}\left(\varepsilon_{11}+\nu^{(k)}\varepsilon_{22}\right)\\
\sigma_{22}^{(k)} &= \frac{E^{(k)}}{1-\left(\nu^{(k)}\right)^2}\left(\nu^{(k)}\varepsilon_{11}+\varepsilon_{22}\right)\\
\sigma_{12}^{(k)} &= \frac{E^{(k)}}{1+\nu^{(k)}}\varepsilon_{12}\\
\sigma_{13}^{(k)} &= 2G_3^{(k)}\varepsilon_{13}\\
\sigma_{23}^{(k)} &= 2G_3^{(k)}\varepsilon_{23}
\end{aligned} \tag{4}$$

where $E^{(k)}$, $\nu^{(k)}$ and $G_3^{(k)}$ are modulus of elasticity, Poisson's ratio and the transverse shear modulus of the k^{th} layer, respectively.

The equations of motion of the FSDT in terms of the stress resultants are obtained by using the extended Hamilton's principle as follows

$$\begin{aligned}
&\frac{\partial N_{11}}{\partial \xi_1}+\frac{\partial N_{12}}{\partial \xi_2}+\frac{Q_1}{R}=I_{11}\ddot{u}+I_{22}\ddot{\psi}_1\\
&\frac{\partial N_{22}}{\partial \xi_2}+\frac{\partial N_{12}}{\partial \xi_1}+\frac{Q_2}{R}=I_{11}\ddot{v}+I_{22}\ddot{\psi}_2\\
&\frac{\partial M_{11}}{\partial \xi_1}+\frac{\partial M_{12}}{\partial \xi_2}-Q_1=I_{22}\ddot{u}+I_3\ddot{\psi}_1\\
&\frac{\partial M_{22}}{\partial \xi_2}+\frac{\partial M_{12}}{\partial \xi_1}-Q_2=I_{22}\ddot{v}+I_3\ddot{\psi}_2\\
&\frac{\partial Q_1}{\partial \xi_1}+\frac{\partial Q_2}{\partial \xi_2}-\frac{N_{11}+N_{22}}{R}=I_1\ddot{w}
\end{aligned} \tag{5}$$

In Eq. (5), dot-overscript convention indicates the differentiation with respect to time and the stress resultants and inertias are defined by

$$\begin{aligned}
Q_i &= K_S\int_{-h/2}^{h/2}\sigma_{i3}\,\mathrm{d}\xi_3\\
\left(N_{ij},M_{ij}\right) &= \int_{-h/2}^{h/2}\sigma_{ij}\left(1,\xi_3\right)\mathrm{d}\xi_3\\
I_i &= \int_{-h/2}^{h/2}\rho\xi_3^{i-1}\mathrm{d}\xi_3\\
I_{11} &= I_1+\frac{2}{R}I_2,\quad I_{22}=I_2+\frac{1}{R}I_3
\end{aligned} \tag{6}$$

where K_S is shear correction factor and is usually taken to be 5/6. Using the Hamilton's principle, the boundary conditions are obtained:

$$\text{at } \xi_2=\text{const.}\begin{cases}
\text{either}\quad \delta u=0 \quad\text{or}\quad N_{12}=0\\
\text{either}\quad \delta v=0 \quad\text{or}\quad N_{22}=0\\
\text{either}\quad \delta\psi_1=0 \quad\text{or}\quad M_{12}=0\\
\text{either}\quad \delta\psi_2=0 \quad\text{or}\quad M_{22}=0\\
\text{either}\quad \delta w=0 \quad\text{or}\quad Q_2=0
\end{cases} \tag{7}$$

The inertias and stress resultants can be simplified for laminated shells as follows

$$
\begin{aligned}
I_1 &= \sum_{k=1}^{N} \rho^{(k)} \left(\xi_3^{(k+1)} - \xi_3^{(k)}\right) \\
I_2 &= \frac{1}{2}\sum_{k=1}^{N} \rho^{(k)} \left[\left(\xi_3^{(k+1)}\right)^2 - \left(\xi_3^{(k)}\right)^2\right] \\
I_3 &= \frac{1}{3}\sum_{k=1}^{N} \rho^{(k)} \left[\left(\xi_3^{(k+1)}\right)^3 - \left(\xi_3^{(k)}\right)^3\right]
\end{aligned} \tag{8}
$$

and

$$
\begin{aligned}
N_{11} &= \sum_{k=1}^{N} \frac{E^{(k)}}{1-\left(\nu^{(k)}\right)^2}\left[\left(\left(\frac{\partial u}{\partial \xi_1}+\frac{w}{R}\right)+\nu^{(k)}\left(\frac{\partial v}{\partial \xi_2}+\frac{w}{R}\right)\right)\left(\xi_3^{(k+1)}-\xi_3^{(k)}\right)\right. \\
&\quad \left. +\frac{1}{2}\left(\frac{\partial \psi_1}{\partial \xi_1}+\nu^{(k)}\frac{\partial \psi_2}{\partial \xi_2}\right)\left[\left(\xi_3^{(k+1)}\right)^2-\left(\xi_3^{(k)}\right)^2\right]\right] \\
N_{22} &= \sum_{k=1}^{N} \frac{E^{(k)}}{1-\left(\nu^{(k)}\right)^2}\left[\left(\nu^{(k)}\left(\frac{\partial u}{\partial \xi_1}+\frac{w}{R}\right)+\left(\frac{\partial v}{\partial \xi_2}+\frac{w}{R}\right)\right)\left(\xi_3^{(k+1)}-\xi_3^{(k)}\right)\right. \\
&\quad \left. +\frac{1}{2}\left(\nu^{(k)}\frac{\partial \psi_1}{\partial \xi_1}+\frac{\partial \psi_2}{\partial \xi_2}\right)\left[\left(\xi_3^{(k+1)}\right)^2-\left(\xi_3^{(k)}\right)^2\right]\right] \\
N_{12} &= \frac{1}{2}\sum_{k=1}^{N} \frac{E^{(k)}}{1+\nu^{(k)}}\left(\left(\frac{\partial u}{\partial \xi_2}+\frac{\partial v}{\partial \xi_1}\right)\left(\xi_3^{(k+1)}-\xi_3^{(k)}\right)+\frac{1}{2}\left(\frac{\partial \psi_1}{\partial \xi_2}+\frac{\partial \psi_2}{\partial \xi_1}\right)\left[\left(\xi_3^{(k+1)}\right)^2-\left(\xi_3^{(k)}\right)^2\right]\right.
\end{aligned} \tag{9}
$$

and

$$
\begin{aligned}
M_{11} &= \sum_{k=1}^{N} \frac{E^{(k)}}{1-\left(\nu^{(k)}\right)^2}\left[\frac{1}{2}\left(\left(\frac{\partial u}{\partial \xi_1}+\frac{w}{R}\right)+\nu^{(k)}\left(\frac{\partial v}{\partial \xi_2}+\frac{w}{R}\right)\right)\left[\left(\xi_3^{(k+1)}\right)^2-\left(\xi_3^{(k)}\right)^2\right]\right. \\
&\quad \left. +\frac{1}{3}\left(\frac{\partial \psi_1}{\partial \xi_1}+\nu^{(k)}\frac{\partial \psi_2}{\partial \xi_2}\right)\left[\left(\xi_3^{(k+1)}\right)^3-\left(\xi_3^{(k)}\right)^3\right]\right] \\
M_{22} &= \sum_{k=1}^{N} \frac{E^{(k)}}{1-\left(\nu^{(k)}\right)^2}\left[\frac{1}{2}\left(\nu^{(k)}\left(\frac{\partial u}{\partial \xi_1}+\frac{w}{R}\right)+\left(\frac{\partial v}{\partial \xi_2}+\frac{w}{R}\right)\right)\left[\left(\xi_3^{(k+1)}\right)^2-\left(\xi_3^{(k)}\right)^2\right]\right. \\
&\quad \left. +\frac{1}{3}\left(\nu^{(k)}\frac{\partial \psi_1}{\partial \xi_1}+\frac{\partial \psi_2}{\partial \xi_2}\right)\left[\left(\xi_3^{(k+1)}\right)^3-\left(\xi_3^{(k)}\right)^3\right]\right] \\
M_{12} &= \frac{1}{2}\sum_{k=1}^{N} \frac{E^{(k)}}{1+\nu^{(k)}}\left[\frac{1}{2}\left(\frac{\partial u}{\partial \xi_2}+\frac{\partial v}{\partial \xi_1}\right)\left[\left(\xi_3^{(k+1)}\right)^2-\left(\xi_3^{(k)}\right)^2\right]+\frac{1}{3}\left(\frac{\partial \psi_1}{\partial \xi_2}+\frac{\partial \psi_2}{\partial \xi_1}\right)\left[\left(\xi_3^{(k+1)}\right)^3-\left(\xi_3^{(k)}\right)^3\right]\right]
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
Q_1 &= K_S\sum_{k=1}^{N} G_3^{(k)}\left[\left(\psi_1+\frac{\partial w}{\partial \xi_1}-\frac{u}{R}\right)\left(\xi_3^{(k+1)}-\xi_3^{(k)}\right)\right] \\
Q_2 &= K_S\sum_{k=1}^{N} G_3^{(k)}\left[\left(\psi_2+\frac{\partial w}{\partial \xi_2}-\frac{v}{R}\right)\left(\xi_3^{(k+1)}-\xi_3^{(k)}\right)\right]
\end{aligned} \tag{11}
$$

It is supposed that the laminated shell panel is symmetric with respect to its middle surface. Thus, after substituting Eqs. (8) through (11) into the equations of motion (5) and simplifying the results, the following governing equations are obtained for free vibration analysis of laminated shell panels

$$\overline{\overline{\alpha_1}}u_{,11}+\overline{\overline{\alpha_2}}u_{,22}+\left(\overline{\overline{\alpha_1}}-\overline{\overline{\alpha_2}}\right)v_{,12}+\frac{1}{R}\left[2\left(\overline{\overline{\alpha_1}}-\overline{\overline{\alpha_2}}\right)w_{,1}+K_S\overline{\overline{\alpha_3}}\left(\psi_1+w_{,1}-\frac{1}{R}u\right)\right]$$
$$=\overline{\overline{I_{11}}}\ddot{u}+\overline{\overline{I_{22}}}\ddot{\psi}_1 \tag{12a}$$
$$\overline{\overline{\alpha_1}}v_{,22}+\overline{\overline{\alpha_2}}v_{,11}+\left(\overline{\overline{\alpha_1}}-\overline{\overline{\alpha_2}}\right)u_{,12}+\frac{1}{R}\left[2\left(\overline{\overline{\alpha_1}}-\overline{\overline{\alpha_2}}\right)w_{,2}+K_S\overline{\overline{\alpha_3}}\left(\psi_2+w_{,2}-\frac{1}{R}v\right)\right]$$
$$=\overline{\overline{I_{11}}}\ddot{v}+\overline{\overline{I_{22}}}\ddot{\psi}_2 \tag{12b}$$
$$\overline{\overline{\alpha_4}}\psi_{1,11}+\overline{\overline{\alpha_5}}\psi_{1,22}+\left(\overline{\overline{\alpha_4}}-\overline{\overline{\alpha_5}}\right)\psi_{2,12}-K_S\overline{\overline{\alpha_3}}\left(w_{,1}+\psi_1-\frac{1}{R}u\right)=\overline{\overline{I_{22}}}\ddot{u}+\overline{\overline{I_3}}\ddot{\psi}_1 \tag{12c}$$
$$\overline{\overline{\alpha_4}}\psi_{2,22}+\overline{\overline{\alpha_5}}\psi_{2,11}+\left(\overline{\overline{\alpha_4}}-\overline{\overline{\alpha_5}}\right)\psi_{1,12}-K_S\overline{\overline{\alpha_3}}\left(w_{,2}+\psi_2-\frac{1}{R}v\right)=\overline{\overline{I_{22}}}\ddot{v}+\overline{\overline{I_3}}\ddot{\psi}_2 \tag{12d}$$
$$K_S\overline{\overline{\alpha_3}}\left[\nabla^2 w+\left(\psi_{1,1}+\psi_{2,2}\right)-\frac{1}{R}\left(u_{,1}+v_{,2}\right)\right]-\frac{2}{R}\left(\overline{\overline{\alpha_1}}-\overline{\overline{\alpha_2}}\right)\left[\left(u_{,1}+v_{,2}\right)+\frac{2}{R}w\right]=\overline{\overline{I_1}}\ddot{w} \tag{12e}$$

where $\nabla^2=\partial^2/\partial\xi_1^2+\partial^2/\partial\xi_2^2$ is the two-dimensional Laplacian operator. Also, the coefficients $\overline{\overline{\alpha_i}}$ are given in “Appendix 1”.

3 Reformulation of the governing equations

In order to solve the Eqs. (12) for the free vibration analysis of laminated shell panels, the unknown functions of the displacement field are assumed to vary harmonically with respect to the time variable t as follows

$$\begin{aligned}
u\left(\xi_1,\xi_2,t\right)&=\tilde{u}\left(\xi_1,\xi_2\right)\exp(j\omega t)\\
v\left(\xi_1,\xi_2,t\right)&=\tilde{v}\left(\xi_1,\xi_2\right)\exp(j\omega t)\\
w\left(\xi_1,\xi_2,t\right)&=\tilde{w}\left(\xi_1,\xi_2\right)\exp(j\omega t)\\
\psi_1\left(\xi_1,\xi_2,t\right)&=\tilde{\psi}_1\left(\xi_1,\xi_2\right)\exp(j\omega t)\\
\psi_2\left(\xi_1,\xi_2,t\right)&=\tilde{\psi}_2\left(\xi_1,\xi_2\right)\exp(j\omega t),\qquad j=\sqrt{-1}
\end{aligned} \tag{13}$$

Substituting the above relations into Eqs. (12) yields the following equations

$$\overline{\overline{\alpha_1}}\tilde{u}_{,11}+\overline{\overline{\alpha_2}}\tilde{u}_{,22}+\left(\overline{\overline{\alpha_1}}-\overline{\overline{\alpha_2}}\right)\tilde{v}_{,12}+\frac{1}{R}\left[2\left(\overline{\overline{\alpha_1}}-\overline{\overline{\alpha_2}}\right)\tilde{w}_{,1}+K_S\overline{\overline{\alpha_3}}\left(\tilde{\psi}_1+\tilde{w}_{,1}-\frac{1}{R}\tilde{u}\right)\right]$$
$$+\omega^2\overline{\overline{I_{11}}}\tilde{u}+\omega^2\overline{\overline{I_{22}}}\tilde{\psi}_1=0 \tag{14a}$$
$$\overline{\overline{\alpha_1}}\tilde{v}_{,22}+\overline{\overline{\alpha_2}}\tilde{v}_{,11}+\left(\overline{\overline{\alpha_1}}-\overline{\overline{\alpha_2}}\right)\tilde{u}_{,12}+\frac{1}{R}\left[2\left(\overline{\overline{\alpha_1}}-\overline{\overline{\alpha_2}}\right)\tilde{w}_{,2}+K_S\overline{\overline{\alpha_3}}\left(\tilde{\psi}_2+\tilde{w}_{,2}-\frac{1}{R}\tilde{v}\right)\right]$$
$$+\omega^2\overline{\overline{I_{11}}}\tilde{v}+\omega^2\overline{\overline{I_{22}}}\tilde{\psi}_2=0 \tag{14b}$$
$$\overline{\overline{\alpha_4}}\tilde{\psi}_{1,11}+\overline{\overline{\alpha_5}}\tilde{\psi}_{1,22}+\left(\overline{\overline{\alpha_4}}-\overline{\overline{\alpha_5}}\right)\tilde{\psi}_{2,12}-K_S\overline{\overline{\alpha_3}}\left(\tilde{w}_{,1}+\tilde{\psi}_1-\frac{1}{R}\tilde{u}\right)+\omega^2\overline{\overline{I_{22}}}\tilde{u}+\omega^2\overline{\overline{I_3}}\tilde{\psi}_1=0 \tag{14c}$$
$$\overline{\overline{\alpha_4}}\tilde{\psi}_{2,22}+\overline{\overline{\alpha_5}}\tilde{\psi}_{2,11}+\left(\overline{\overline{\alpha_4}}-\overline{\overline{\alpha_5}}\right)\tilde{\psi}_{1,12}-K_S\overline{\overline{\alpha_3}}\left(\tilde{w}_{,2}+\tilde{\psi}_2-\frac{1}{R}\tilde{v}\right)+\omega^2\overline{\overline{I_{22}}}\tilde{v}+\omega^2\overline{\overline{I_3}}\tilde{\psi}_2=0 \tag{14d}$$
$$K_S\overline{\overline{\alpha_3}}\left(\nabla^2\tilde{w}+\left(\tilde{\psi}_{1,1}+\tilde{\psi}_{2,2}\right)-\frac{1}{R}\left(\tilde{u}_{,1}+\tilde{v}_{,2}\right)\right)-\frac{2}{R}\left(\overline{\overline{\alpha_1}}-\overline{\overline{\alpha_2}}\right)\left[\left(\tilde{u}_{,1}+\tilde{v}_{,2}\right)+\frac{2}{R}\tilde{w}\right]+\omega^2\overline{\overline{I_1}}\tilde{w}=0 \tag{14e}$$

The governing Eqs. (14) are five highly coupled partial differential equations. In order to present exact closed-form solutions for such equations, it is reasonable to reformulate them to the uncoupled form. To this end, we introduce four potential functions as

$$\begin{aligned}\tilde{\eta} &= \tilde{u}_{,2} - \tilde{v}_{,1} \\ \tilde{\mu} &= \tilde{u}_{,1} + \tilde{v}_{,2} \\ \tilde{\varphi} &= \tilde{\psi}_{1,2} - \tilde{\psi}_{2,1} \\ \tilde{\lambda} &= \tilde{\psi}_{1,1} + \tilde{\psi}_{2,2}\end{aligned} \tag{15}$$

Using the potential functions (15), the governing Eqs. (14) can be rewritten in a simpler form as

$$\overline{\overline{l_1}}\tilde{u} - \overline{\overline{l_2}}\tilde{\psi}_1 = \overline{\overline{\alpha_1}}\tilde{\mu}_{,1} + \overline{\overline{\alpha_2}}\tilde{\eta}_{,2} + \overline{\overline{l_3}}\tilde{w}_{,1} \tag{16a}$$

$$\overline{\overline{l_1}}\tilde{v} - \overline{\overline{l_2}}\tilde{\psi}_2 = \overline{\overline{\alpha_1}}\tilde{\mu}_{,2} - \overline{\overline{\alpha_2}}\tilde{\eta}_{,1} + \overline{\overline{l_3}}\tilde{w}_{,2} \tag{16b}$$

$$-\overline{\overline{l_4}}\tilde{u} + \overline{\overline{l_5}}\tilde{\psi}_1 = \overline{\overline{\alpha_4}}\tilde{\lambda}_{,1} + \overline{\overline{\alpha_5}}\tilde{\varphi}_{,2} - K_S\overline{\overline{\alpha_3}}\tilde{w}_{,1} \tag{16c}$$

$$-\overline{\overline{l_4}}\tilde{v} + \overline{\overline{l_5}}\tilde{\psi}_2 = \overline{\overline{\alpha_4}}\tilde{\lambda}_{,2} - \overline{\overline{\alpha_5}}\tilde{\varphi}_{,1} - K_S\overline{\overline{\alpha_3}}\tilde{w}_{,2} \tag{16d}$$

$$\tilde{\lambda} = -\nabla^2\tilde{w} + \overline{\overline{l_6}}\tilde{\mu} + \overline{\overline{l_7}}\tilde{w} \tag{16e}$$

where the coefficients $\overline{\overline{l_i}}$ are defined in "Appendix 2". Differentiating Eqs. (16a) and (16b) with respect to ξ_2 and ξ_1, respectively, and subtracting the results yields

$$\overline{\overline{\alpha_2}}\nabla^2\tilde{\eta} - \overline{\overline{l_1}}\tilde{\eta} + \overline{\overline{l_2}}\tilde{\varphi} = 0 \tag{17}$$

Differentiating Eqs. (16c) and (16d) with respect to ξ_2 and ξ_1, respectively, and subtracting the results yields

$$\overline{\overline{\alpha_5}}\nabla^2\tilde{\varphi} - \overline{\overline{l_5}}\tilde{\varphi} + \overline{\overline{l_4}}\tilde{\eta} = 0 \tag{18}$$

Also, differentiating Eqs. (16a) and (16b) with respect to ξ_1 and ξ_2, respectively, and adding the results yields the following equation

$$\overline{\overline{\alpha_1}}\nabla^2\tilde{\mu} - \overline{\overline{l_1}}\tilde{\mu} + \overline{\overline{l_2}}\tilde{\lambda} + \overline{\overline{l_3}}\nabla^2\tilde{w} = 0 \tag{19}$$

In a similar way, differentiating Eqs. (16c) and (16d) with respect to ξ_1 and ξ_2, respectively, and adding them will result in

$$\overline{\overline{\alpha_4}}\nabla^2\tilde{\lambda} - \overline{\overline{l_5}}\tilde{\lambda} + \overline{\overline{l_4}}\tilde{\mu} - K_S\overline{\overline{\alpha_3}}\nabla^2\tilde{w} = 0 \tag{20}$$

Eliminating $\tilde{\eta}$ from Eqs. (17) and (18) yields an uncoupled partial differential equation in terms of the potential function $\tilde{\varphi}$ as

$$\nabla^4\tilde{\varphi} - \overline{\overline{\Lambda_1}}\nabla^2\tilde{\varphi} + \overline{\overline{\Lambda_2}}\tilde{\varphi} = 0 \tag{21}$$

where $\nabla^4 = \nabla^2\nabla^2 = \left(\partial^2/\partial\xi_1^2 + \partial^2/\partial\xi_2^2\right)\left(\partial^2/\partial\xi_1^2 + \partial^2/\partial\xi_2^2\right)$. Also, the coefficients $\overline{\overline{\Lambda_i}}$ are given in "Appendix 3". Now, by substituting $\tilde{\lambda}$ from Eq. (16e) into Eqs. (19) and (20), the following two equations in terms of $\tilde{\mu}$ and $\tilde{w}$ are obtained

$$\overline{\overline{\alpha_1}}\nabla^2\tilde{\mu} + \left(\overline{\overline{l_2 l_6}} - \overline{\overline{l_1}}\right)\tilde{\mu} + \left(\overline{\overline{l_3}} - \overline{\overline{l_2}}\right)\nabla^2\tilde{w} + \overline{\overline{l_2 l_7}}\tilde{w} = 0 \tag{22a}$$

$$\overline{\overline{\alpha_4 l_6}}\nabla^2\tilde{\mu} + \left(\overline{\overline{l_4}} - \overline{\overline{l_5 l_6}}\right)\tilde{\mu} - \overline{\overline{\alpha_4}}\nabla^4\tilde{w} + \left(\overline{\overline{l_5}} + \overline{\overline{\alpha_4 l_7}} - K_S\overline{\overline{\alpha_3}}\right)\nabla^2\tilde{w} - \overline{\overline{l_5 l_7}}\tilde{w} = 0 \tag{22b}$$

Eliminating $\nabla^2\tilde{\mu}$ from Eqs. (22a) and (22b) and simplifying the results will result in

$$\tilde{\mu} = \overline{\overline{l_8}}\nabla^4\tilde{w} + \overline{\overline{l_9}}\nabla^2\tilde{w} + \overline{\overline{l_{10}}}\tilde{w} \tag{23}$$

Substituting Eq. (23) into either Eq. (22a) or (22b) yields an uncoupled partial differential equation in terms of the transverse deflection $\tilde{w}$ as

$$\nabla^6\tilde{w} + \overline{\overline{\Lambda_3}}\nabla^4\tilde{w} + \overline{\overline{\Lambda_4}}\nabla^2\tilde{w} + \overline{\overline{\Lambda_5}}\tilde{w} = 0 \tag{24}$$

The two uncoupled equations (21) and (24) are exactly equal to the original governing Eqs. (14) and can be solved instead of them. When the uncoupled equations (21) and (24) are used, it is necessary to express the unknown functions of the displacement field (i.e., the functions $\tilde{u}$, $\tilde{v}$, $\tilde{\psi}_1$, and $\tilde{\psi}_2$) in terms of $\tilde{w}$ and $\tilde{\varphi}$. To this end, we substitute $\tilde{\lambda}$, $\tilde{\eta}$, and $\tilde{\mu}$ from Eqs. (16e), (18), and (23) into (16a) through (16d) and obtain

$$\overline{\overline{l_1}}\tilde{u} - \overline{\overline{l_2}}\tilde{\psi}_1 = \overline{\overline{\alpha_1}}\,\overline{\overline{l_8}}\nabla^4\tilde{w}_{,1} + \overline{\overline{\alpha_1}}\,\overline{\overline{l_9}}\nabla^2\tilde{w}_{,1} + \left(\overline{\overline{\alpha_1}}\,\overline{\overline{l_{10}}} + \overline{\overline{l_3}}\right)\tilde{w}_{,1} - \frac{\overline{\overline{\alpha_2}}\,\overline{\overline{\alpha_5}}}{\overline{\overline{l_4}}}\nabla^2\tilde{\varphi}_{,2} + \frac{\overline{\overline{\alpha_2}}\,\overline{\overline{l_5}}}{\overline{\overline{l_4}}}\tilde{\varphi}_{,2} \tag{25a}$$

$$\overline{\overline{l_1}}\tilde{v} - \overline{\overline{l_2}}\tilde{\psi}_2 = \overline{\overline{\alpha_1}}\,\overline{\overline{l_8}}\nabla^4\tilde{w}_{,2} + \overline{\overline{\alpha_1}}\,\overline{\overline{l_9}}\nabla^2\tilde{w}_{,2} + \left(\overline{\overline{\alpha_1}}\,\overline{\overline{l_{10}}} + \overline{\overline{l_3}}\right)\tilde{w}_{,2} + \frac{\overline{\overline{\alpha_2}}\,\overline{\overline{\alpha_5}}}{\overline{\overline{l_4}}}\nabla^2\tilde{\varphi}_{,1} - \frac{\overline{\overline{\alpha_2}}\,\overline{\overline{l_5}}}{\overline{\overline{l_4}}}\tilde{\varphi}_{,1} \tag{25b}$$

$$-\overline{\overline{l_4}}\tilde{u} + \overline{\overline{l_5}}\tilde{\psi}_1 = \overline{\overline{\alpha_4}}\,\overline{\overline{l_6}}\,\overline{\overline{l_8}}\nabla^4\tilde{w}_{,1} + \overline{\overline{\alpha_4}}\left(\overline{\overline{l_6}}\,\overline{\overline{l_9}} - 1\right)\nabla^2\tilde{w}_{,1} + \left(\overline{\overline{\alpha_4}}\,\overline{\overline{l_6}}\,\overline{\overline{l_{10}}} + \overline{\overline{\alpha_4}}\,\overline{\overline{l_7}} - K_S\overline{\overline{\alpha_3}}\right)\tilde{w}_{,1} + \overline{\overline{\alpha_5}}\tilde{\varphi}_{,2} \tag{25c}$$

$$-\overline{\overline{l_4}}\tilde{v} + \overline{\overline{l_5}}\tilde{\psi}_2 = \overline{\overline{\alpha_4}}\,\overline{\overline{l_6}}\,\overline{\overline{l_8}}\nabla^4\tilde{w}_{,2} + \overline{\overline{\alpha_4}}\left(\overline{\overline{l_6}}\,\overline{\overline{l_9}} - 1\right)\nabla^2\tilde{w}_{,2} + \left(\overline{\overline{\alpha_4}}\,\overline{\overline{l_6}}\,\overline{\overline{l_{10}}} + \overline{\overline{\alpha_4}}\,\overline{\overline{l_7}} - K_S\overline{\overline{\alpha_3}}\right)\tilde{w}_{,2} - \overline{\overline{\alpha_5}}\tilde{\varphi}_{,1} \tag{25d}$$

By solving Eqs. (25a) and (25c) simultaneously, the functions $\tilde{u}$ and $\tilde{\psi}_1$ are obtained in terms of $\tilde{w}$ and $\tilde{\varphi}$ as follows

$$\tilde{u} = \overline{\overline{\Lambda_6}}\nabla^4\tilde{w}_{,1} + \overline{\overline{\Lambda_7}}\nabla^2\tilde{w}_{,1} + \overline{\overline{\Lambda_8}}\tilde{w}_{,1} - \overline{\overline{\Lambda_9}}\nabla^2\tilde{\varphi}_{,2} + \overline{\overline{\Lambda_{10}}}\tilde{\varphi}_{,2} \tag{26a}$$

$$\tilde{\psi}_1 = \overline{\overline{\Lambda_{11}}}\nabla^4\tilde{w}_{,1} + \overline{\overline{\Lambda_{12}}}\nabla^2\tilde{w}_{,1} + \overline{\overline{\Lambda_{13}}}\tilde{w}_{,1} - \overline{\overline{\Lambda_{14}}}\nabla^2\tilde{\varphi}_{,2} + \overline{\overline{\Lambda_{15}}}\tilde{\varphi}_{,2} \tag{26b}$$

Also, solving Eqs. (25b) and (25d) simultaneously will result in some relations for $\tilde{v}$ and $\tilde{\psi}_2$ as

$$\tilde{v} = \overline{\overline{\Lambda_6}}\nabla^4\tilde{w}_{,2} + \overline{\overline{\Lambda_7}}\nabla^2\tilde{w}_{,2} + \overline{\overline{\Lambda_8}}\tilde{w}_{,2} + \overline{\overline{\Lambda_9}}\nabla^2\tilde{\varphi}_{,1} - \overline{\overline{\Lambda_{10}}}\tilde{\varphi}_{,1} \tag{27a}$$

$$\tilde{\psi}_2 = \overline{\overline{\Lambda_{11}}}\nabla^4\tilde{w}_{,2} + \overline{\overline{\Lambda_{12}}}\nabla^2\tilde{w}_{,2} + \overline{\overline{\Lambda_{13}}}\tilde{w}_{,2} + \overline{\overline{\Lambda_{14}}}\nabla^2\tilde{\varphi}_{,1} - \overline{\overline{\Lambda_{15}}}\tilde{\varphi}_{,1} \tag{27b}$$

Clearly by substituting Eqs. (26) and (27) into (9), (10), and (11), the stress resultants will be obtained explicitly in terms of $\tilde{w}$ and $\tilde{\varphi}$.

This procedure is used for developing exact closed-form Navier and Lévy-type solutions for the free vibration of laminated transversely isotropic spherical shell panels.

4 Solution of the reformulated equations

4.1 Navier-type solution

Herein, it is assumed that the laminated spherical shell panel is simply supported at all four edges. Thus, the functions $\tilde{\varphi}$ and $\tilde{w}$ are assumed to be

$$\tilde{\varphi} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Phi_{mn}\cos\alpha_m\xi_1\cos\beta_n\xi_2 \tag{28a}$$

$$\tilde{w} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}W_{mn}\sin\alpha_m\xi_1\sin\beta_n\xi_2 \tag{28b}$$

where $\alpha_m = m\pi/a$ and $\beta_n = n\pi/b$. Clearly, Eqs. (28) satisfy the simply supported boundary conditions at the edges $\xi_1 = 0, a$ and $\xi_2 = 0, b$. Introducing Eq. (28a) into (21), an algebraic equation is obtained in terms of the natural frequencies of the laminated shell panels

$$\overline{\overline{\Lambda_1}}(\omega) + \left(\alpha_m^2 + \beta_n^2\right)^{-1}\overline{\overline{\Lambda_2}}(\omega) + \left(\alpha_m^2 + \beta_n^2\right) = 0 \tag{29}$$

The natural frequencies which are obtained from the above equation are related to the in-plane vibration modes. Replacing the maximum transverse deflection function $\tilde{w}$ from Eq. (28b) into the uncoupled equation (24) will result in the following algebraic equation for the out-of-plane natural frequencies of the laminated shell panel

$$\overline{\overline{\Lambda_3}}(\omega) - \left(\alpha_m^2 + \beta_n^2\right)^{-1}\overline{\overline{\Lambda_4}}(\omega) + \left(\alpha_m^2 + \beta_n^2\right)^{-2}\overline{\overline{\Lambda_5}}(\omega) - \left(\alpha_m^2 + \beta_n^2\right) = 0 \tag{30}$$

It is to be noted that by using Eqs. (29) and (30), the natural frequencies of simply supported laminated spherical shell panel can be determined explicitly.

4.2 Lévy-type solution

In this section, Lévy-type solution is developed for the laminated spherical shell panel. Therefore, two opposite edges of the shell panel are assumed to be simply supported and the other sides have arbitrary boundary conditions. The symbolism SCSF, for example, identifies a spherical shell panel with edges $\xi_1 = 0$, $\xi_2 = 0,\ \xi_1 = a,\ \xi_2 = b$ having simply supported, clamped, simply supported, and free boundary condition, respectively. Due to this kind of boundary conditions, the functions $\tilde{\varphi}$ and $\tilde{w}$ are represented as follows

$$\tilde{\varphi} = \sum_{m=1}^{\infty} \Phi_m\left(\xi_2\right) \cos \alpha_m \xi_1 \tag{31a}$$

$$\tilde{w} = \sum_{m=1}^{\infty} W_m\left(\xi_2\right) \sin \alpha_m \xi_1 \tag{31b}$$

Clearly, Eqs. (31a) and (31b) satisfy the simply supported boundary conditions at the edges $\xi_1 = 0, a$. Substituting Eqs. (31a) and (31b) into the uncoupled equations (21) and (24), respectively, two ordinary differential equations are obtained

$$\Phi_m^{(4)}\left(\xi_2\right) - \left(2\alpha_m^2 + \overline{\overline{\Lambda_1}}\right) \Phi_m''\left(\xi_2\right) + \left(\alpha_m^4 + \alpha_m^2 \overline{\overline{\Lambda_1}} + \overline{\overline{\Lambda_2}}\right) \Phi_m\left(\xi_2\right) = 0 \tag{32a}$$

$$W_m^{(6)}\left(\xi_2\right) + \delta_1 W_m^{(4)}\left(\xi_2\right) + \delta_2 W_m''\left(\xi_2\right) + \delta_3 W_m\left(\xi_2\right) = 0 \tag{32b}$$

where

$$\begin{aligned}
\delta_1 &= -\left(3\alpha_m^2 - \overline{\overline{\Lambda_3}}\right)\\
\delta_2 &= 3\alpha_m^4 - 2\alpha_m^2 \overline{\overline{\Lambda_3}} + \overline{\overline{\Lambda_4}}\\
\delta_3 &= -\left(\alpha_m^6 - \alpha_m^4 \overline{\overline{\Lambda_3}} + \alpha_m^2 \overline{\overline{\Lambda_4}} - \overline{\overline{\Lambda_5}}\right)
\end{aligned} \tag{33}$$

The solution of Eq. (32a) can be presented as

$$\Phi_m\left(\xi_2\right) = C_{m1} \cosh \kappa_1 \xi_2 + C_{m2} \sinh \kappa_1 \xi_2 + C_{m3} \cosh \kappa_2 \xi_2 + C_{m4} \sinh \kappa_2 \xi_2 \tag{34}$$

where

$$\left\{\begin{matrix} \kappa_1 \\ \kappa_2 \end{matrix}\right\} = \frac{1}{2}\sqrt{4\alpha_m^2 + 2\overline{\overline{\Lambda_1}} \pm 2\sqrt{\overline{\overline{\Lambda_1}}^2 - 4\overline{\overline{\Lambda_2}}}} \tag{35}$$

Also, it is easy to show that the solution of Eq. (32b) is written as follows

$$W_m\left(\xi_2\right) = \begin{cases} C_{m5} \cosh \kappa_3 \xi_2 + C_{m6} \sinh \kappa_3 \xi_2 + C_{m7} \cosh \kappa_4 \xi_2 + C_{m8} \sinh \kappa_4 \xi_2 \\ C_{m9} \cosh \kappa_5 \xi_2 + C_{m10} \sinh \kappa_5 \xi_2 \qquad \kappa_3^2 > 0 \\ \\ + C_{m5} \cos \tilde{\kappa}_3 \xi_2 + C_{m6} \sin \tilde{\kappa}_3 \xi_2 + C_{m7} \cosh \kappa_4 \xi_2 + C_{m8} \sinh \kappa_4 \xi_2 \\ + C_{m9} \cosh \kappa_5 \xi_2 + C_{m10} \sinh \kappa_5 \xi_2 \qquad \kappa_3^2 < 0 \end{cases} \tag{36}$$

where the coefficients $\kappa_i\,(i = 3, 4, 5)$ are defined as

$$\begin{aligned}
\kappa_3 &= \sqrt{\frac{2}{3}\sqrt{\delta_1^2 - 3\delta_2} \cos\left(\frac{1}{3}\cos^{-1}\left(\frac{9\delta_1\delta_2 - 27\delta_3 - 2\delta_1^3}{2\left(3\delta_2 - \delta_1^2\right)\sqrt{\delta_1^2 - 3\delta_2}}\right)\right) - \frac{1}{3}\delta_1}\\
\kappa_4 &= \sqrt{\frac{2}{3}\sqrt{\delta_1^2 - 3\delta_2} \cos\left(\frac{1}{3}\cos^{-1}\left(\frac{9\delta_1\delta_2 - 27\delta_3 - 2\delta_1^3}{2\left(3\zeta_2 - \zeta_1^2\right)\sqrt{\zeta_1^2 - 3\zeta_2}}\right) + \frac{2\pi}{3}\right) - \frac{1}{3}\delta_1}\\
\kappa_5 &= \sqrt{\frac{2}{3}\sqrt{\delta_1^2 - 3\delta_2} \cos\left(\frac{1}{3}\cos^{-1}\left(\frac{9\delta_1\delta_2 - 27\delta_3 - 2\delta_1^3}{2\left(3\delta_2 - \delta_1^2\right)\sqrt{\delta_1^2 - 3\delta_2}}\right) + \frac{4\pi}{3}\right) - \frac{1}{3}\delta_1}
\end{aligned} \tag{37}$$

and

$$\tilde{\kappa}_3^2 = -\kappa_3^2 \tag{38}$$

Introducing Eqs. (34) and (36) into (31a) and (31b), respectively, the explicit expressions are obtained for the transverse deflection $\tilde{w}$ and potential function $\tilde{\varphi}$ in terms of ten constants C_{m1}through C_{m10}. In order to satisfy the arbitrary boundary conditions at the edges $\xi_2 = 0, b$, five boundary conditions should be chosen from Eq. (7) for each edge. For example, we have:

$$\begin{array}{ll} \text{Free Edge:} & N_{12} = N_{22} = M_{12} = M_{22} = Q_2 = 0 \\ \text{Simply Supported Edge:} & u = N_{22} = \psi_1 = M_{22} = w = 0 \\ \text{Clamped Edge:} & u = v = \psi_1 = \psi_2 = w = 0 \end{array} \tag{39}$$

Substituting the obtained displacement components and stress resultants into the appropriate boundary conditions (39) for the edges $\xi_2 = 0, b$ leads to a set of algebraic equations for analyzing the free vibration of moderately thick laminated spherical shell panel. This set consists of ten equations in terms of the constants C_{m1}through C_{m10}. By setting the determinant of the coefficient matrix of equations equal to zero, the exact characteristic equation is obtained. These equations give the natural frequencies related to both in-plane and out-of-plane vibration modes.

5 Numerical results and discussion

A computer code has been developed according to the foregoing analytical approach to calculate the numerical results for the free vibration of moderately thick laminated spherical shell panels. For generality of the numerical results, the following non-dimensional parameters are defined

$$\begin{aligned} \overline{\omega} &= \omega \frac{a^2}{h_M} \sqrt{\frac{\rho_M}{E_M}} \\ \overline{h} &= h/a \\ \overline{R} &= R/a \\ \overline{G} &= G/G_3 \end{aligned} \tag{40}$$

where $\overline{\omega}$ is the non-dimensional natural frequency, $\overline{h}$ is thickness-to-length ratio, $\overline{R}$ is curvature ratio, and $\overline{G}$ is shear modulus ratio. Also, G and G_3 are the in-plane and out-of-plane shear modulus of the transversely isotropic materials, respectively. It is noticeable that the subscript M in Eq. (40) represents the middle layer of the laminates. The results are provided for both single-layer and three-layer transversely isotropic shell panels. The non-dimensional properties of the three-layer shell panel is assumed as

$$\begin{aligned} &G^{(3)}/G^{(1)} = 1, \quad G^{(2)}/G^{(1)} = 1/2 \\ &\rho^{(3)}/\rho^{(1)} = 1, \quad \rho^{(2)}/\rho^{(1)} = 2/3 \\ &\nu^{(1)} = 0.35, \quad \nu^{(2)} = 0.25, \quad \nu^{(3)} = 0.35 \\ &\overline{G}^{(1)} = 2, \quad \overline{G}^{(2)} = 1.25, \quad \overline{G}^{(3)} = 2 \\ &\overline{h}^{(1)} = \overline{h}^{(2)} = \overline{h}^{(3)} = 1/15 \end{aligned} \tag{41}$$

Also, unless mentioned otherwise, the shear correction factor K_S is taken to be 5/6. In order to investigate the efficiency and accuracy of the present approach, some comparison studies have been provided with the available results in the literature. Also, some comparisons have been made with the results of finite element method using a software package with 3D solid elements. The comparison results have been presented for different classical boundary conditions (i.e., simply supported, clamped, and free edges).

Table 1 shows the first eight out-of-plane natural frequencies of a homogeneous simply supported spherical shell panel together with the corresponding results of the 3D elasticity [26], finite element FSDT [27], and finite element HSDT [27]. The number of half-waves in the ξ_1 and ξ_2 direction are denoted by m and n, respectively. The physical and geometric properties of the spherical shell panel are as: curved lengths $a = b = 1.0118$,

Table 1 Comparison study of the natural frequencies of fully simply supported thin spherical shell panels (SSSS)

Method	Mode (m,n)							
	(1,1)	(2,1)	(1,2)	(2,2)	(3,1)	(1,3)	(3,2)	(2,3)
3D Elasticity [26]	0.52543	0.58420	0.58487	0.67676	0.75219	0.75220	0.87811	0.87804
FSDT [27]	0.50211	0.56247	0.56248	0.65706	0.73915	0.74035	0.86359	0.86360
| Error (%) |	4.44	3.72	3.83	2.91	1.73	1.58	1.65	1.65
HSDT [27]	0.50223	0.56276	0.56277	0.65788	0.73966	0.74081	0.86493	0.86494
| Error (%) |	4.42	3.67	3.78	2.79	1.67	1.51	1.50	1.49
Present	0.52830	0.58853	0.58853	0.68232	0.75818	0.75818	0.88507	0.88507
| Error (%) |	0.55	0.74	0.63	0.82	0.80	0.80	0.79	0.80

Table 2 Non-dimensional natural frequencies $\overline{\omega}$ of moderately thick single-layer spherical shell panels ($\overline{h} = 0.1$, $\overline{R} = 5$, $a/b = 1$)

B.Cs	Method	Mode sequences					
		1	2	3	4	5	6
SSSS	Present	6.0768	13.880	13.880	**19.483**	**19.483**	21.189
	FEM (3D)	6.0812	13.930	13.930	**19.492**	**19.492**	21.326
	| Error (%) |	0.07	0.36	0.36	0.05	0.05	0.64
SCSC	Present	8.3597	14.978	18.006	**19.483**	23.916	26.333
	FEM (3D)	8.4119	15.068	18.186	**19.494**	24.158	26.519
	| Error (%) |	0.62	0.60	0.99	0.06	1.00	0.70
SFSF	Present	2.8727	4.6253	10.408	11.030	12.937	**14.826**
	FEM (3D)	2.8854	4.6485	10.437	11.093	13.033	**14.801**
	| Error (%) |	0.44	0.50	0.28	0.57	0.74	0.17

Table 3 Comparison study of the frequency parameters $\omega a^2\sqrt{\rho h/\overline{\overline{D}}}$ for laminated spherical shell panels

$\overline{R}$	B.Cs		
	SSSS	SCSC	SFSF
5	18.388	22.909	8.736
10	17.653	22.096	8.716
10^2	17.403	21.820	8.709
10^3	17.400	21.818	8.709
Exact FSDT [28]	17.400	21.818	8.709

thickness $h = 0.0191$, radius $R = 1.91$, modulus of elasticity $E = 1$, density $\rho = 1$, and Poisson's ratio $\nu = 0.3$. The percentage error given in Table 1 has been calculated on the basis of the following equation

$$|\text{Error }(\%)| = \left|\frac{(\text{Exact Sanders}) - (\text{3D Elasticity})}{(\text{3D Elasticity})}\right| \times 100 \qquad (42)$$

The results of Table 1 reveal that excellent agreement exists between the results of the present exact solution and those of other methods. It is evident from Table 1 that the results of present study are closer to the results of 3D elasticity [26] than those of FSDT [27] and HSDT [27]. This closeness is apparent even for higher vibration modes.

The first six non-dimensional natural frequencies $\overline{\omega}$ of moderately thick single-layer spherical shell panels are tabulated in Table 2, for various boundary conditions. The curvature and thickness-to-length ratios are assumed to be $\overline{R} = 5$ and $\overline{h} = 0.1$, respectively. The bold-faced values indicate the natural frequencies of the in-plane modes. In order to demonstrate the high accuracy of the obtained results, the spherical shell panels are modeled using a FEM software package with 3D solid elements. It can be observed that the percentage errors given in this table does not exceed 1% even for higher modes. The percentage errors related to the in-plane vibration modes are less that 0.2%.

In Table 3, the non-dimensional natural frequencies $\omega a^2(\rho h/\overline{\overline{D}})^{1/2}$ of three-layer transversely isotropic spherical shell panels obtained by the present approach are compared with those of the exact FSDT of Nosier and Reddy [28] for laminated plates. The comparison results are provided for various boundary conditions. It should be pointed out that $\overline{\overline{D}}$ is the flexural rigidity of laminates and is defined by $\overline{\overline{\alpha_4}}$ (see "Appendix 1"). The assumed physical and geometric properties for the laminated spherical shell panel given in this comparison are as follows

Table 4 Non-dimensional natural frequencies $\overline{\omega}$ of moderately thick laminated spherical shell panels ($\overline{h}_{\text{Total}} = 0.2$, $\overline{R} = 10$, $a/b = 1$)

B.Cs	Method	Mode sequences					
		1	2	3	4	5	6
SSSS	FEM (3D)	17.714	**32.305**	**32.321**	35.851	35.853	**45.727**
	Present	17.840	**32.318**	**32.318**	36.246	36.246	**45.705**
	| Error (%) |	0.71	**0.04**	**0.01**	1.10	1.09	**0.05**
SCSC	FEM (3D)	20.943	**32.315**	36.561	39.644	51.983	**57.535**
	Present	21.164	**32.318**	37.118	39.438	51.736	**57.306**
	| Error (%) |	1.05	**0.01**	1.52	0.52	0.48	**0.39**
SFSF	FEM (3D)	9.3666	14.133	**24.662**	28.754	29.884	**32.337**
	Present	9.4161	14.240	**24.657**	28.908	30.164	**32.318**
	| Error (%) |	0.53	0.76	**0.02**	0.54	0.94	**0.06**

$$
\begin{aligned}
&a = b = 20h^{(2)} \\
&h^{(1)} = h^{(3)} = 0.25,\ h^{(2)} = 0.5 \\
&E^{(1)} = E^{(3)} = 20.83 \times 10^6\text{psi},\ E^{(1)} = 19.20 \times 10^6\text{psi} \\
&G_3^{(1)} = G_3^{(3)} = 3.71 \times 10^6\text{psi},\ G_3^{(2)} = 0.82 \times 10^6\text{psi} \\
&\rho^{(1)} = \rho^{(2)} = \rho^{(3)} = \rho = \text{Const.}
\end{aligned}
\tag{43}
$$

Also, the shear correction factor is taken to be 2/3 (the value is chosen in order to be able to compare with the results of Ref. [28]). It is expected that the laminated spherical shell panel is converted to a laminated plate when the curvature ratio approaches infinity. It can be observed from Table 3 that the non-dimensional natural frequencies of laminated spherical shell panel exactly converge to the results of Ref. [28] when the curvature ratio of the shell panels approaches infinity. The results of this table confirm the high accuracy of the present approach for laminated shell panels.

The first six non-dimensional natural frequencies $\overline{\omega}$ of a moderately thick laminated spherical shell panel ($\overline{h}_{\text{Total}} = 0.2$) with the boundary conditions SSSS, SCSC, and SFSF are listed in Table 4. The non-dimensional properties given in Eq. (41) are considered for the laminated shell panels. The bold-faced values also indicate the natural frequencies of the in-plane vibration modes. In all cases, the three-dimensional FEM results, obtained from a FEM software package, have been presented to demonstrate the high accuracy of the foregoing closed-form solution for the laminated spherical shell panels. It is seen that the maximum value of the percentage error is about 1.52% for the laminated shell panel with clamped edges. This may be due to the fact that the FSDT cannot capture the boundary layer term for clamped edges. Since the thickness-to-length ratio of the shell panels considered in this example is $\overline{h}_{\text{Total}} = 0.2$, the percentage errors are expected to be more than those for thinner ones. It is worth noting that the percentage error of laminated shell theories with respect to the 3D analyses increases when the thickness of shell panels increases.

In order to investigate the effect of shear modulus ratio $\overline{G}$ on the non-dimensional natural frequencies of the transversely isotropic shell panels, attention is focused on Figs. 2, 3, 4 which are depicted for SSSS, SCSC, and SFSF shell panels, respectively. In these figures, the variations of non-dimensional fundamental natural frequency are illustrated for three different values of thickness-to-length ratio ($\overline{h} = 0.01, 0.1, 0.2$). The curvature ratio is taken to be $\overline{R} = 10$, and the aspect ratio is assumed as unity. It can be seen that the effect of shear modulus ratio on the natural frequency of shell panel increases as the shell panel thickness increases. It is evident that the shear modulus ratio has weak effect on the natural frequency when the shell panel is thin. In order to evaluate the effect of shear modulus ratio for different boundary conditions, we consider the case which the transversely isotropic shell panels are thick (i.e., $\overline{h} = 0.2$). In this case, the non-dimensional natural frequency decreases about 22, 14, and 8% for SCSC, SSSS, and SFSF shell panels, respectively, when the shear modulus ratio increases from 1 to 3. Thus, it can be concluded that the effects of shear modulus ratio on the natural frequencies of transversely isotropic shell panels become more significant with an increase in the boundary constraints.

Figure 5 shows the variation of the non-dimensional fundamental natural frequency versus curvature ratio $\overline{R}$ for laminated spherical shell panels with different boundary conditions. The non-dimensional properties given in Eq. (41) are considered for the laminated shell panels. It can be observed that the fundamental natural frequency of SCSC and SSSS laminated shell panels decreases when the curvature ratio increases. This manner reveals that for these boundary conditions, by increasing the curvature, the panel stiffness increases.

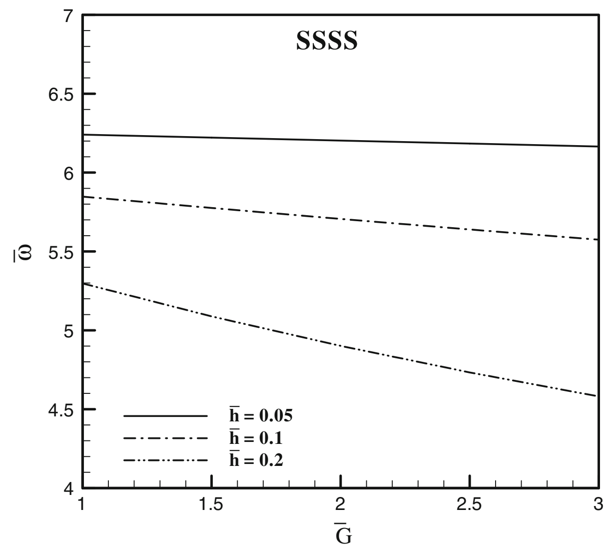

Fig. 2 Variation of the fundamental natural frequency versus shear modulus ratio for SSSS spherical shell panels

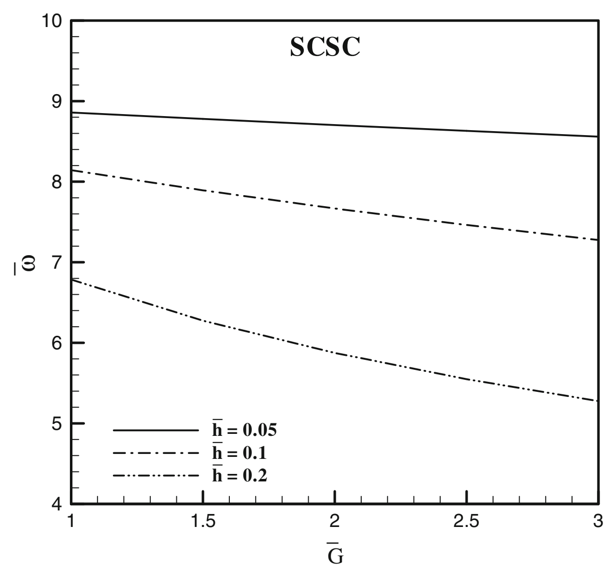

Fig. 3 Variation of the fundamental natural frequency versus shear modulus ratio for SCSC spherical shell panels

This behavior is completely different for SFSF laminated panels. In this case, by increasing the curvature the fundamental natural frequency decreases.

Figure 6 exhibits the influence of aspect ratio (b/a) on the non-dimensional fundamental natural frequency of the laminated spherical shell panels for different boundary conditions. It can be inferred from Fig. 6 that with the increase in the aspect ratio, the non-dimensional natural frequency of SSSS and SCSC laminated shell panels decreases, while the inverse behavior is experienced for SFSF laminated shell panels. Also, it can be observed that the aspect ratio exerts a greater influence on the natural frequencies of SSSS and SCSC laminated shell panels in comparison with the SFSF ones. It is noticeable that a similar behavior has been previously reported by Ref. [29] for thick rectangular plates.

The variations of non-dimensional fundamental natural frequency of the simply supported laminated spherical shell panels versus the aspect ratio are illustrated in Fig. 7 for different values of curvature ratio. It is seen that the effect of curvature ratio on the natural frequencies becomes more significant when the aspect ratio increases.

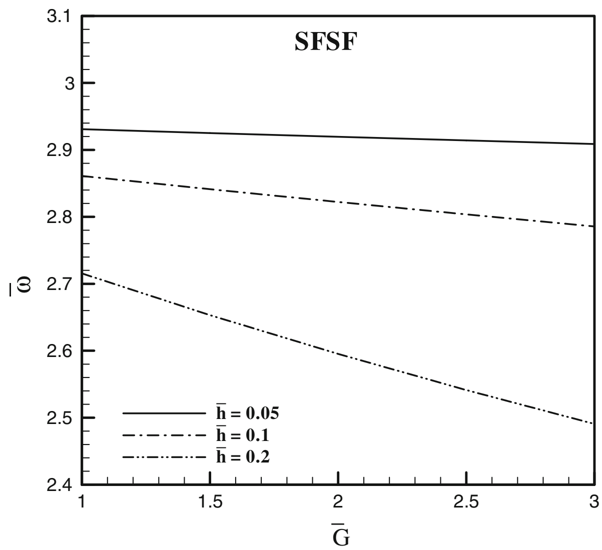

Fig. 4 Variation of the fundamental natural frequency versus shear modulus ratio for SFSF spherical shell panels

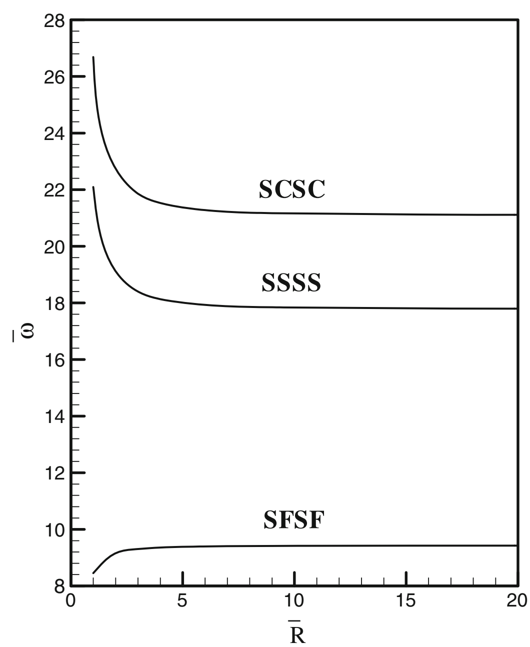

Fig. 5 Variation of the fundamental natural frequency versus curvature ratio for laminated spherical shell panels ($a/b = 1$)

6 Conclusion

This paper dealt with exact closed-form solutions free vibration analysis of moderately thick laminated transversely isotropic spherical shell panels on the basis of Sanders theory. The governing equations of motion and the boundary conditions were derived. The highly coupled governing equations were recast to some uncoupled equations by introducing four potential functions. Also, some relations were presented for the unknowns of the original set of equations in terms of the unknowns of the uncoupled equations. According to the proposed analytical approach, both Navier and Lévy-type explicit solutions were developed for moderately thick laminated spherical shell panels. The efficiency and high accuracy of the present approach were shown by providing some comparison studies with the available results in the literature and the results of 3D finite element method. It was shown that the effect of shear modulus ratio on the natural frequency of transversely isotropic shell

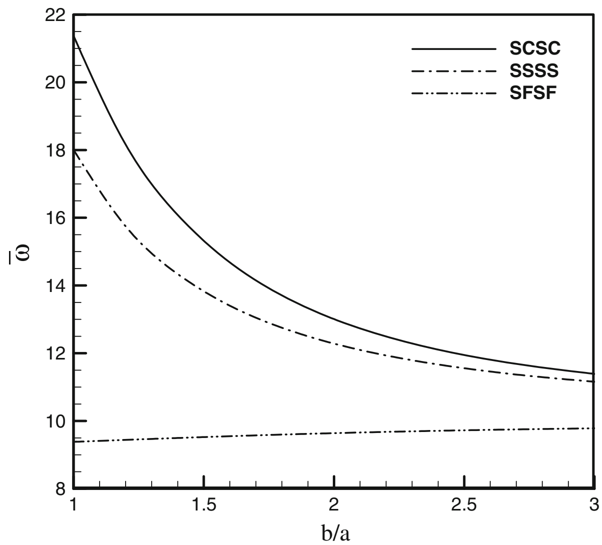

Fig. 6 Variation of the fundamental natural frequency versus aspect ratio for laminated spherical shell panels with various boundary conditions ($\overline{R} = 5$)

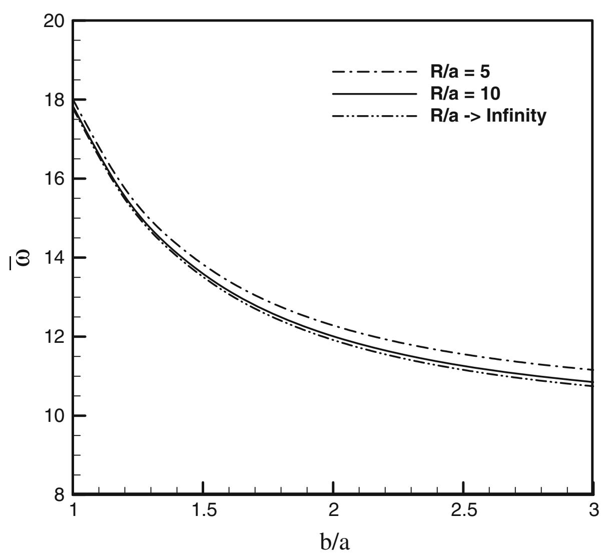

Fig. 7 Variation of the fundamental natural frequency versus aspect ratio for SSSS laminated spherical shell panels

panels increases as the shell panel thickness increases. Also, it was shown that the effects of shear modulus ratio on the natural frequencies of shell panels become more significant with an increase in the boundary constraints. It was observed that the fundamental natural frequency of SCSC and SSSS laminated shell panels decreases when the curvature ratio increases, whereas for the laminated panels with free boundary conditions, the fundamental natural frequency increases when the curvature ratio increases.

Appendix 1

The coefficients $\overline{\overline{\alpha_i}}$ are expressed as

$$\overline{\overline{\alpha_1}} = \sum_{k=1}^{N} \frac{E^{(k)}}{1 - \left(\nu^{(k)}\right)^2} \left(\xi_3^{(k+1)} - \xi_3^{(k)}\right)$$

$$\overline{\overline{\alpha_2}} = \frac{1}{2}\sum_{k=1}^{N} \frac{E^{(k)}}{1+\nu^{(k)}} \left(\xi_3^{(k+1)} - \xi_3^{(k)}\right)$$

$$\overline{\overline{\alpha_3}} = \sum_{k=1}^{N} G_3^{(k)} \left(\xi_3^{(k+1)} - \xi_3^{(k)}\right)$$

$$\overline{\overline{\alpha_4}} = \frac{1}{3}\sum_{k=1}^{N} \frac{E^{(k)}}{1-\left(\nu^{(k)}\right)^2} \left[\left(\xi_3^{(k+1)}\right)^3 - \left(\xi_3^{(k)}\right)^3\right]$$

$$\overline{\overline{\alpha_5}} = \frac{1}{6}\sum_{k=1}^{N} \frac{E^{(k)}}{1+\nu^{(k)}} \left[\left(\xi_3^{(k+1)}\right)^3 - \left(\xi_3^{(k)}\right)^3\right]$$

Appendix 2

The coefficients $\overline{\overline{l_1}}$ through $\overline{\overline{l_{10}}}$ are defined as

$$\overline{\overline{l_1}} = \frac{1}{R^2} K_S \overline{\overline{\alpha_3}} - \omega^2 \overline{\overline{I_{11}}}$$

$$\overline{\overline{l_2}} = \frac{1}{R} K_S \overline{\overline{\alpha_3}} + \omega^2 \overline{\overline{I_{22}}}$$

$$\overline{\overline{l_3}} = \frac{1}{R}\left[2\left(\overline{\overline{\alpha_1}} - \overline{\overline{\alpha_2}}\right) + K_S \overline{\overline{\alpha_3}}\right]$$

$$\overline{\overline{l_4}} = \frac{1}{R} K_S \overline{\overline{\alpha_3}} + \omega^2 \overline{\overline{I_{22} l_5}} = \left(K_S \overline{\overline{\alpha_3}} - \omega^2 \overline{\overline{I_3}}\right)$$

$$\overline{\overline{l_6}} = \frac{1}{R K_S \overline{\overline{\alpha_3}}}\left[K_S \overline{\overline{\alpha_3}} + 2\left(\overline{\overline{\alpha_1}} - \overline{\overline{\alpha_2}}\right)\right]$$

$$\overline{\overline{l_7}} = \frac{1}{K_S \overline{\overline{\alpha_3}}}\left[\frac{4}{R^2}\left(\overline{\overline{\alpha_1}} - \overline{\overline{\alpha_2}}\right) - \omega^2 \overline{\overline{I_1}}\right]$$

$$\overline{\overline{l_8}} = \frac{\overline{\overline{\alpha_1 \alpha_4}}}{\overline{\overline{\alpha_1}}\left(\overline{\overline{l_4}} - \overline{\overline{l_5 l_6}}\right) - \overline{\overline{\alpha_4}}\overline{\overline{l_6}}\left(\overline{\overline{l_2 l_6}} - \overline{\overline{l_1}}\right)}$$

$$\overline{\overline{l_9}} = \frac{\overline{\overline{\alpha_4}}\overline{\overline{l_6}}\left(\overline{\overline{l_3}} - \overline{\overline{l_2}}\right) - \overline{\overline{\alpha_1}}\left(\overline{\overline{l_5}} + \overline{\overline{\alpha_4}}\overline{\overline{l_7}} - K_S \overline{\overline{\alpha_3}}\right)}{\overline{\overline{\alpha_1}}\left(\overline{\overline{l_4}} - \overline{\overline{l_5 l_6}}\right) - \overline{\overline{\alpha_4}}\overline{\overline{l_6}}\left(\overline{\overline{l_2 l_6}} - \overline{\overline{l_1}}\right)}$$

$$\overline{\overline{l_{10}}} = \frac{\overline{\overline{l_7}}\left(\overline{\overline{\alpha_4}}\overline{\overline{l_6 l_2}} + \overline{\overline{\alpha_1}}\overline{\overline{l_5}}\right)}{\overline{\overline{\alpha_1}}\left(\overline{\overline{l_4}} - \overline{\overline{l_5 l_6}}\right) - \overline{\overline{\alpha_4}}\overline{\overline{l_6}}\left(\overline{\overline{l_2 l_6}} - \overline{\overline{l_1}}\right)}$$

Appendix 3

The coefficients $\overline{\overline{\Lambda_1}}$ through $\overline{\overline{\Lambda_{15}}}$ are given by

$$\overline{\overline{\Lambda_1}} = \frac{\overline{\overline{l_5}}}{\overline{\overline{\alpha_5}}} + \frac{\overline{\overline{l_1}}}{\overline{\overline{\alpha_2}}}$$

$$\overline{\overline{\Lambda_2}} = \frac{\overline{\overline{l_1 l_5}} - \overline{\overline{l_2 l_4}}}{\overline{\overline{\alpha_2 \alpha_5}}}$$

$$\overline{\overline{\Lambda_3}} = \frac{\overline{\overline{l_9}}}{\overline{\overline{l_8}}} + \frac{\overline{\overline{l_2 l_6}} - \overline{\overline{l_1}}}{\overline{\overline{\alpha_1}}}$$

$$\overline{\overline{\Lambda_4}} = \frac{1}{\overline{\overline{l_8}}}\left(\overline{\overline{l_{10}}} + \frac{\overline{\overline{l_2 l_6 l_9}} - \overline{\overline{l_1 l_9}} + \overline{\overline{l_3}} - \overline{\overline{l_2}}}{\overline{\overline{\alpha_1}}}\right)$$

$$\overline{\overline{\Lambda_5}} = \frac{\overline{\overline{l_2 l_6 l_{10}}} - \overline{\overline{l_1 l_{10}}} + \overline{\overline{l_2 l_7}}}{\overline{\overline{l_8}}\,\overline{\overline{\alpha_1}}}$$

$$\overline{\overline{\Lambda_6}} = \frac{\overline{\overline{l_8}}\left(\overline{\overline{\alpha_1}}\,\overline{\overline{l_5}} + \overline{\overline{\alpha_4}}\,\overline{\overline{l_2 l_6}}\right)}{\overline{\overline{l_1 l_5}} - \overline{\overline{l_2 l_4}}}$$

$$\overline{\overline{\Lambda_7}} = \frac{\overline{\overline{\alpha_1}}\,\overline{\overline{l_5 l_9}} + \overline{\overline{\alpha_4}}\,\overline{\overline{l_2}}\left(\overline{\overline{l_6 l_9}} - 1\right)}{\overline{\overline{l_1 l_5}} - \overline{\overline{l_2 l_4}}}$$

$$\overline{\overline{\Lambda_8}} = \frac{\overline{\overline{l_5}}\left(\overline{\overline{\alpha_1}}\,\overline{\overline{l_{10}}} + \overline{\overline{l_3}}\right) + \overline{\overline{l_2}}\left(\overline{\overline{\alpha_4}}\,\overline{\overline{l_6 l_{10}}} + \overline{\overline{\alpha_4}}\,\overline{\overline{l_7}} - K_S\overline{\overline{\alpha_3}}\right)}{\overline{\overline{l_1 l_5}} - \overline{\overline{l_2 l_4}}}$$

$$\overline{\overline{\Lambda_9}} = \frac{\overline{\overline{\alpha_2 \alpha_5}}\,\overline{\overline{l_5}}}{\overline{\overline{l_4}}\left(\overline{\overline{l_1 l_5}} - \overline{\overline{l_2 l_4}}\right)}$$

$$\overline{\overline{\Lambda_{10}}} = \frac{\overline{\overline{\alpha_2}}\,\overline{\overline{l_5}}^{2} + \overline{\overline{\alpha_5}}\,\overline{\overline{l_2 l_4}}}{\overline{\overline{l_4}}\left(\overline{\overline{l_1 l_5}} - \overline{\overline{l_2 l_4}}\right)}$$

$$\overline{\overline{\Lambda_{11}}} = \frac{\overline{\overline{l_8}}\left(\overline{\overline{\alpha_1}}\,\overline{\overline{l_4}} + \overline{\overline{\alpha_4}}\,\overline{\overline{l_1 l_6}}\right)}{\overline{\overline{l_1 l_5}} - \overline{\overline{l_2 l_4}}}$$

$$\overline{\overline{\Lambda_{12}}} = \frac{\overline{\overline{\alpha_1}}\,\overline{\overline{l_4 l_9}} + \overline{\overline{\alpha_4}}\,\overline{\overline{l_1}}\left(\overline{\overline{l_6 l_9}} - 1\right)}{\overline{\overline{l_1 l_5}} - \overline{\overline{l_2 l_4}}}$$

$$\overline{\overline{\Lambda_{13}}} = \frac{\overline{\overline{l_4}}\left(\overline{\overline{\alpha_1}}\,\overline{\overline{l_{10}}} + \overline{\overline{l_3}}\right) + \overline{\overline{l_1}}\left(\overline{\overline{\alpha_4}}\,\overline{\overline{l_6 l_{10}}} + \overline{\overline{\alpha_4}}\,\overline{\overline{l_7}} - K_S\overline{\overline{\alpha_3}}\right)}{\overline{\overline{l_1 l_5}} - \overline{\overline{l_2 l_4}}}$$

$$\overline{\overline{\Lambda_{14}}} = \frac{\overline{\overline{\alpha_2 \alpha_5}}}{\overline{\overline{l_1 l_5}} - \overline{\overline{l_2 l_4}}}$$

$$\overline{\overline{\Lambda_{15}}} = \frac{\overline{\overline{\alpha_2}}\,\overline{\overline{l_5}} + \overline{\overline{\alpha_5}}\,\overline{\overline{l_1}}}{\overline{\overline{l_1 l_5}} - \overline{\overline{l_2 l_4}}}$$

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