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Fractional order theory in thermoelastic solid with three-phase lag heat transfer

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Abstract In this work, the field equations of the linear theory of thermoelasticity have been constructed in the context of a new consideration of Fourier law of heat conduction with time-fractional order and three-phase lag. A uniqueness and reciprocity theorems are proved. One-dimensional application for a half-space of elastic material in the presence of heat sources has been solved using Laplace transform and state space techniques Ezzat (Canad J Phys Rev 86:1241–1250, 2008). According to the numerical results and its graphs, conclusion about the new theory has been established.

Keywords Thermoelasticity · Modified Fourier law · Three-phase lag · Modified Riemann–Liouville fractional derivative · New fractional Taylor's series · State space approach · Fractional calculus

Nomenclature

λ, μ	Lame's constants
ρ	density
C_E	specific heat at constant strain
t	time
T	absolute temperature
Θ	$T - T_0$
υ	thermal displacement
T_o	reference temperature chosen so that $\left\lvert \frac{\theta}{T_o} \right\rvert << 1$
Q	strength of the heat source
F_i	mass force
q_i	components of heat flux
σ_{ij}	components of stress tensor

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e_{ij}	components of strain tensor
u_i	components of displacement vector
k	thermal conductivity
δ_{ij}	Kronecker delta function
α_{T}	coefficient of linear thermal expansion
$\tau_o, \tau_q, \tau_\upsilon, \tau_T$	relaxation times
α	fractional order of the differentiation $0 < \alpha \leq 1$
γ	$= (3\lambda + 2\mu)\alpha_T$
ε	$= \gamma/\rho c_E$, thermal coupling parameter
c_o^2	$= \frac{(\lambda+2\mu)}{\rho}$
H(.)	Heaviside unit step function
δ(.)	Dirac delta function

1 Introduction

The classical uncoupled theory of thermoelasticity predicts two phenomena not compatible with physical observations. First, the equation of heat conduction of this theory does not contain any elastic terms; second, the heat equation is of a parabolic type, predicting infinite speeds of propagation for heat waves.

Biot [2] introduced the theory of coupled thermoelasticity to overcome the first shortcoming. The governing equations for this theory are coupled, eliminating the first paradox of the classical theory. However, both theories share the second shortcoming because the heat equation for the coupled theory is of a mixed parabolic/hyperbolic type.

Lord and Shulman [3] introduced the theory of generalized thermoelasticity with one relaxation time for the special case of an isotropic body. This theory was extended by Dhaliwal and Sherief [4] to include the anisotropic case. In this theory, a modified law of heat conduction including both the heat flux and its time derivative replaces the conventional Fourier's law. The heat equation associated with this theory is hyperbolic and hence eliminates the paradox of infinite speeds of propagation inherent in both the uncoupled and the coupled theories of thermoelasticity. Uniqueness of solution for this theory was proved under different conditions by Ignaczak [5,6], Sherief [7], Chandrasekharaiah [8], Ezzat and El-Karamany [9,10], El-Karamany [11], Ezzat and Awad [12,13] and Ezzat et al. [14]. Ezzat at el. [15] studied the dependence of the modulus of elasticity on the reference temperature in generalized thermoelasticity.

Green and Lindsay [16] developed the theory of generalized thermoelasticity with two relaxation times, which is based on a violate Fourier's law of heat conduction when the body under consideration has a center of symmetry. In this theory, both the equations of motion and of heat conduction are hyperbolic but the equation of motion is modified and differs from that of coupled thermoelasticity theory. The fundamental aspects of this theory are described in Ezzat [17]. Ezzat and El-Karamany [18,19] proved the uniqueness theorem for generalized thermoviscoelasticity with two relaxation times. Othman at el. [20] studied two-dimensional problems of thermoviscoelasticity with two relaxation times.

Other hyperbolic thermoelasticity theory was proposed by Tzou [21]. This theory includes those concerned with material with memory, those which involve the so-called semi-empirical temperature as a constitutive variable and the one takes account of the so-called dual-phase lag effects. In this theory, the Fourier law is replaced by an approximation to a modification of the Fourier law with two different translations for the heat flux and the temperature gradient.

Right now, there are five different generalizations of the coupled theory of thermoelasticity, and the details can be found in Hetnarski and Ignaczak [22]. All five theories are based on assumptions of one kind or another. Also, all these theories model the problem of heat conductions in solids as a purely wave propagation phenomenon.

Recently, Roy Choudhuri [23] was proposed the three-phase lag heat conduction model in which the Fourier law is replaced by an approximation of the equation

$$q_i(p, t + \tau_q) = -\left[k\nabla\Theta(p, t + \tau_T) + k^*\nabla\upsilon(p, t + \tau_\upsilon)\right], \tag{1}$$

where υ is the thermal displacement [24,25] that satisfies $\dot{\upsilon} = T$, $k^*(> 0)$ is a material characteristic in the Green–Naghdi theories and the quantities τ_q, τ_T and τ_υ in this three-phase lag model are, respectively, the phase lag of the heat flux, the phase lag of the temperature gradient and the phase lag of the thermal displacement in which the inequality $0 \leq \tau_\upsilon < \tau_T < \tau_q$ is satisfied, where $P(\overline{x})$ is a point of the body.

Fractional calculus has been used successfully to modify many existing models of physical processes. One can state that the whole theory of fractional derivatives and integrals was established in the 2nd half of the nineteenth century. The first application of fractional derivatives was given by Abel who applied fractional calculus in the solution of an integral equation that arises in the formulation of the tautochrone problem. The generalization of the concept of derivative and integral to a non-integer order has been subjected to several approaches, and some various alternative definitions of fractional derivatives appeared [26–29]. In the last few years, fractional calculus was applied successfully in various areas to modify many existing models of physical processes, e.g., chemistry, biology, modeling and identification, electronics, wave propagation and viscoelasticity [30–33]. One can refer to Podlubny [29] for a survey of applications of fractional calculus.

The theory of fractional calculus has been used successfully to model polymer materials. A quasi-static uncoupled theory of thermoelasticity based on fractional heat conduction equation was put forward by Povstenko [34]. The theory of thermal stresses based on the heat conduction equation with the Caputo time-fractional derivative is used by Povstenko [35] to investigate thermal stresses in an infinite body with a circular cylindrical hole. Sherief et al. [36] introduced new model of thermoelasticity using fractional calculus, proved a uniqueness theorem and derived a reciprocity relation and a variational principle. Youssef [37] introduced another new model of fractional heat conduction equation, proved a uniqueness theorem and presented one-dimensional application. Ezzat [38,39] established a new model of fractional heat conduction equation using the new Taylor series expansion of time-fractional order which developed by Jumarie [40]. El-Karamany and Ezzat [41] introduced two general models of the two-temperature fractional heat conduction law for a non-homogeneous anisotropic elastic solid. For fractional thermoelasticity not involving two temperatures, El-Karamany and Ezzat [42] established the uniqueness, reciprocal theorems and convolution variational principle. The reciprocity relation in case of quiescent initial state is found to be independent of the order of differintegration [41] and [42]. Fractional order theory of a perfect conducting thermoelastic medium not involving two temperatures was investigated by Ezzat and El-Karamany [43]. Povstenko [44] introduced fractional Cattaneo-type equations, in which time non-local generalizations of the Fourier law leading to the fractional telegraph equations were analyzed and the corresponding theories of thermoelasticity were proposed.

The current manuscript is an attempt to derive a new model of thermoelasticity with three-phase lag heat conduction using the methodology of fractional calculus theory. The state space formulation for one-dimensional problem is introduced. The resulting formulation is applied to a problem of a half-space with heat sources distribution. Laplace transform techniques are used to get the solution in a closed form. The inversion of the Laplace transforms is carried out using a numerical approach proposed by Honig and Hirdes [45].

2 The mathematical model

We consider the theory developed by taking a new Taylor series expansion of time-fractional order α, developed in Jumarie [40] on both sides of Eq. (1) and retaining terms up to α-order terms in τ_T and τ_υ and terms up to 2α-order terms in τ_q. One obtained the three-phase lag heat conduction model with time-fractional derivatives in the form

$$\left(1+\frac{\tau_q^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)q=-\left[\tau_\upsilon^*\nabla\Theta+k\frac{\tau_T^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\nabla\Theta+k^*\nabla\upsilon\right]\quad 0<\alpha\leq 1, \tag{2}$$

where

$$\tau_\upsilon^*=k+\frac{k^*\tau_\upsilon^\alpha}{\alpha!}\frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}.$$

In the context of the thermoelasticity theory, the energy equation for a homogenous isotropic thermoelastic solid is given as

$$-\operatorname{div}q+\rho Q=\rho C_E\dot{\Theta}+\gamma T_0\dot{e}_{ii}. \tag{3}$$

Taking divergence and then the time derivative of Eq. (2), we obtain

$$\left(1+\frac{\tau_q^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\operatorname{div}\dot{q}=-\left[\tau_\upsilon^*\nabla^2\dot{\Theta}+k\frac{\tau_T^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\nabla^2\dot{\Theta}+k^*\nabla^2\dot{\upsilon}\right] \tag{4}$$

Further on differentiation of the energy Eq. (3) with respect to time and then elimination of $-\text{div}\dot{q}$ leads to the modified fractional heat transport equation

$$\left(1+\frac{\tau_q^{\alpha}}{\alpha!}\frac{\partial^{\alpha}}{\partial t^{\alpha}}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)(\rho C_E\ddot{\Theta}+\gamma T_0\ddot{e}-\rho\dot{Q})=\tau_{\upsilon}^{*}\nabla^2\dot{\Theta}+k\frac{\tau_T^{\alpha}}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\nabla^2\Theta+k^{*}\nabla^2\Theta. \quad (5)$$

2.1 Special cases

Taking into consideration

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}f(x,t)=\begin{cases} f(x,t)-f(x,0) & \alpha\to 0\\ I^{\alpha-1}\frac{\partial f(x,t)}{\partial t} & 0<\alpha<1\\ \frac{\partial f(x,t)}{\partial t} & \alpha=1\end{cases}$$

where I^{α} is the Riemann–Liouville fractional integral which was introduced as a natural generalization of the well-known n-fold repeated integral $I^n f(t)$ written in a convolution-type form as in Mainardi and Gorenflo [46]:

$$\left.\begin{aligned} I^{\alpha}f(t)&=\int_o^t\frac{(t-\varsigma)^{\alpha-1}}{\Gamma(\alpha)}\,f(\varsigma)\,\mathrm{d}\varsigma\\ I^0 f(t)&=f(t)\end{aligned}\right\}\quad \alpha>0$$

we introduce the following limiting cases:

1. If $\alpha=1$ and the thermal conductivity k is much smaller than k^*, the heat conduction law reduces to $q=-k^*\nabla\upsilon$ and since $\tau_{\upsilon}^{*}=k^*\tau_{\upsilon}$ and ignoring τ_q^2 then Eq. (5) reduces to the following equation

$$\left(1+\tau_q\frac{\partial}{\partial t}\right)\left(\rho C_E\ddot{\Theta}+\gamma T_0\ddot{e}-\rho\dot{Q}\right)=k^*\left(1+\tau_{\upsilon}\frac{\partial}{\partial t}\right)\nabla^2\Theta.$$

   The pervious equation may therefore be considered as an extension of Green–Naghdi theory of type II (thermoelasticity with two-phase lag model in both the heat flux vector and the thermal displacement gradient).
   In this case $\tau_T=0$, $\tau_{\upsilon}=0$ and $\tau_q=0$ reduces to (for much low thermal conductivity) the following equation

$$\rho C_E\ddot{\Theta}+\gamma T_0\ddot{e}=k^*\nabla^2\Theta+\rho\dot{Q},$$

   which is the heat conduction equation of Green–Naghdi theory of type II (generalized thermoelasticity without energy dissipation [24]).
2. When $\alpha=1$, $\tau_T=0$, $\tau_{\upsilon}=0$, and $\tau_q=0$, hence $\tau_{\upsilon}^{*}=k$, Eq. (5) becomes

$$\rho C_{\upsilon}\ddot{\Theta}+\gamma T_0\ddot{e}=k^*\nabla^2\Theta+k\nabla^2\dot{\Theta}+\rho\dot{Q},$$

   which is the heat conduction equation of Green–Naghdi theory of type III [23,47] admitting damped thermoelastic wave solutions.
3. When $\alpha=1$ and $k^*=0$, Eq. (5) reduces to heat conduction equation under dual-phase lag effect [10,21].
4. In the case when $\alpha=1$ and $k^*=0$, $\tau_T=\tau_{\upsilon}=0$, $\tau_{\upsilon}^{*}=k$, $\tau_q=\tau_o>0$ and neglecting τ_q^2, Eq. (5) clearly reduces to Lord–Shulman model [3].
5. When $\alpha=1$, Eq. (5) reduces to heat conduction equation with three-phase lag effect [23].
6. In the case when $0<\alpha\le 1$, $\tau_T=0$, $\tau_{\upsilon}=0$, $\tau_o=\frac{\tau_q^{\alpha}}{\alpha!}$ and neglecting $\tau_q^{2\alpha}$

reduces to the heat transfer with fractional order proposed by Sherief et al. [36] and Ezzat [39].

2.2 The uniqueness theorem

In this section, we shall prove the uniqueness theorem for the system of equations of generalized thermoelasticity with three-phase lag and time-fractional order heat transfer. We consider a linear homogenous isotropic thermoelastic material, for which the governing equations are given by:

(i) The constitutive equation

$$\sigma_{ij} = C_{ijkl} e_{kl} - \gamma_{ij} \Theta. \tag{6}$$

(ii) The strain–displacement relations

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{7}$$

(iii) The equation of motion

$$\sigma_{ji,j} + \rho F_i = \rho \ddot{u}_i. \tag{8}$$

(iv) The modified heat equation

$$\left(1 + \frac{\tau_q^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_q^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\left(\rho C_E \ddot{\Theta} + \gamma T_0 \ddot{e} - \rho \dot{Q}\right) = \left[\tau_\upsilon^* \frac{\partial}{\partial t} + k\frac{\tau_T^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}} + k^*\right]\nabla^2\Theta, \tag{9}$$

where a superposed dot is the time derivative, and a comma followed by index i denotes the derivative with respect to x_i.

Theorem *Assuming that a linear isotropic thermoelastic material occupies a regular region D with boundary surface B in the three-dimensional space, there is only one system of functions: $u_i(x,t)$, $\Theta(x,t)$ of class C^2, and $\sigma_{ij}(x,t)$, $e_{ij}(x,t)$ of class C^1, in the point $P \in (D \cup B)$ having coordinates $x = (x_1, x_2, x_3)$ at $t \geq 0$ which satisfy Eqs. (6) and (7) for $x \in (D \cup B)$, $t \geq 0$ and Eq. (8) and (9) for $x \in D$, $t > 0$, with the boundary conditions:*

$$\Theta = \phi^{(1)}(x_B, t), u_i = G_i^1(x_B, t), \quad x_B \in B, \quad t > 0, \tag{10}$$

and the initial conditions

$$\Theta = \phi^{(2)}(x, 0), u_i = G_i^{(2)}(x, 0), \dot{u}_i = G_i^{(3)}(x, 0)\, x \in D, \quad t = 0. \tag{11}$$

Proof Let $u_i^{(1)}, T^{(1)}, \ldots$ and $u_i^{(2)}, T^{(2)}, \ldots$ be two solution sets of Eqs. (6)–(9) with the same body forces, same material constants, same boundary conditions (10) and same initial conditions (11). Consider the difference functions

$$u_i^* = u_i^{(1)} - u_i^{(2)}, \Theta^* = \Theta^{(1)} - \Theta^{(2)}, e_{ij}^* = e_{ij}^1 - e_{ij}^2, \ldots \tag{12}$$

Eqs. (6)–(9) for the difference functions become

$$\sigma_{ij,j}^* = \rho \ddot{u}_i, \quad \sigma^* = C_{ijkl} e_{kl} - \gamma_{ij}\Theta, \tag{13}$$

$$\left(1 + \frac{\tau_q^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_q^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\left(\rho C_E \ddot{\Theta} + \gamma T_0 \ddot{e} - \rho \dot{Q}\right) = \left[\tau_\upsilon^* \frac{\partial}{\partial t} + k\frac{\tau_T^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}} + k^*\right]\Theta_{,ii}. \tag{14}$$

The difference functions (12) satisfy the homogeneous boundary and initial conditions, thus

$$\begin{aligned} &\Theta^*(x_B, t) = 0, \quad u_i^*(x_B, t) = 0, \quad t > 0, \quad x_B \in B, \\ &\Theta^*(x, 0) = 0, \quad u_i^*(x, 0) = 0, \quad \dot{u}_i^*(x, 0) = 0, \quad x \in B. \end{aligned} \tag{15}$$

The Laplace transform of the Caputo derivative has the following form

$$L\{D_C^\alpha g(t)\} = \{s^\alpha \overline{g}(s)\} - \sum_{K=0}^{n-1} f^k(0^+) s^{\alpha-1-k}, \quad n - 1 < \alpha < n. \tag{16}$$

Applying (16) to the system of equations obtained for the difference functions and omitting the asterisks Eqs. (12)–(15) and bars since the following analysis concerns only the difference functions, we get

$$\left(1+\frac{\tau_q^\alpha}{\alpha!}s^\alpha+\frac{\tau_q^{2\alpha}}{(2\alpha)!}s^{2\alpha}\right)\left(\rho C_E s^2\Theta+\gamma T_0 s^2 e-\rho s Q\right)=\left[\overline{\tau}_\upsilon^s+k\frac{\tau_T^\alpha}{\alpha!}s^{\alpha+1}+k^*\right]\Theta_{,ii}, \quad (17)$$

$$\sigma_{ij,j}=s^2\rho u_i, \quad (18)$$

$$e_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}), \quad (19)$$

$$\Theta(x_B,s)=0,\quad u_i(x_B,s)=0,\quad x_B\in B. \quad (20)$$

Consider the integral

$$\int_D \sigma_{ij}e_{ij}\mathrm{d}V=\int_D \sigma_{ij}u_{i,j}\mathrm{d}V=\int_D (\sigma_{ij}u_i)_{,j}\mathrm{d}V-\int_D \sigma_{ij,j}u_i\mathrm{d}V. \quad (21)$$

Using the divergence theorem and taking into consideration Eq. (20), one obtains

$$\int_D (\sigma_{ij}u_i)_{,j}\mathrm{d}V=\int_B u_i\sigma_{ij}n_i\mathrm{d}A=0, \quad (22)$$

Thus, Eq. (21) takes the form

$$\int_D \sigma_{ij}e_{ij}\mathrm{d}V+\int_D \sigma_{ij,j}u_i\mathrm{d}V=0, \quad (23)$$

From Eq. (13) we obtain

$$\sigma_{ij}e_{ij}=C_{ijkl}e_{ij}e_{kl}-\gamma_{ij}\Theta e_{ij}. \quad (24)$$

Substituting from Eqs. (18) and (24) into (23), we get

$$\int_D (C_{ijkl}e_{ij}e_{kl}-\gamma_{ij}\Theta e_{ij}+\rho s^2u_i^2)\mathrm{d}V=0. \quad (25)$$

Since $\left[\Theta\Theta_{,ii}=\left(\Theta\Theta_{,i}\right)_{,i}-\Theta_{,i}\Theta_{,i}\right]$ and

$$\int_D \left(\Theta\Theta_{,i}\right)_{,i}\mathrm{d}V=\int_B \Theta\Theta_{,i}n_i\mathrm{d}A=0,$$

we have:

$$\int_D \Theta\left(\xi\Theta_{,ii}\right)\mathrm{d}V=-\xi\int_D \Theta_{,i}\Theta_{,i}\mathrm{d}V, \quad (26)$$

where

$$\xi=\left(\overline{\tau}_\upsilon^* s+k\frac{\tau_T^\alpha}{\alpha!}s^{\alpha+1}+k^*\right).$$

Substituting from Eq. (17) into Eq. (26), one obtains

$$\int_D \gamma_{ij}\Theta\varepsilon_{ij}\mathrm{d}V=-\frac{\eta_1}{T_0}\int_D \Theta^2\mathrm{d}V-\frac{\eta_2}{T_0}\int_D \Theta_{,i}\Theta_{,i}\mathrm{d}V, \quad (27)$$

where

$$\eta_1 = \rho\, C_E, \eta_2 = \frac{\xi}{s^2\left(1 + \frac{\tau_q^\alpha}{\alpha!}s^\alpha + \frac{\tau_q^{2\alpha}}{(2\alpha)!}s^{2\alpha}\right)},$$

and Eq. (25) with (27) gives

$$\int_D \left\{ C_{ijkl}e_{ij}e_{kl} + \rho s^2 u_i^2 + \frac{\eta_1}{T_0}\Theta^2 + \frac{\eta_2}{T_0}\Theta_{,i}\Theta_{,i} \right\} \mathrm{d}V = 0, \tag{28}$$

where $C_{ijkl}, C_E, \rho, \gamma, T_0, \tau_T, \tau_q, \tau_\upsilon, K^*, K$ are positive constants, together with the previous equations constitutes a complete system of generalized thermoelasticity for isotropic medium. The integrated function in (28) is the sum of squares; thus, we conclude that

$$u_i = 0, \quad \Theta = 0, \quad e = 0.$$

Then, the Laplace transforms of all the difference functions (9) are zeros and according to Learch's theorem [48]. The inverse Laplace transforms of each is unique, consequently

$$u_i^* = 0, \quad \Theta^* = 0, \quad \sigma_{ij}^* = 0.$$

This completes the proof of uniqueness of solution under the aforementioned assumptions.

2.3 Reciprocity theorem

We shall prove a reciprocity relation for the theory developed in the previous theorem. Reciprocity theorems have become increasingly important lately because of their uses for the numerical solution of boundary value problems by the boundary element method (BEM) [49,50]. This method is rapidly replacing the finite element method in many engineering applications. BEM needs some theoretical preparations, namely a reciprocity theorem and fundamental solutions [17], as a starting point.

Theorem *We consider a homogeneous, bounded, isotropic, perfectly elastic body occupying the region V and bounded by the surface S subject to the action of a body forces F_i, surface tractions P_i, heat sources Q and heating of the surface to the temperature T under homogeneous initial conditions. We write these causes symbolically as $C = \{F_i, P_i, Q, \upsilon\}$.*

The causes C and C' produce in the body the displacements u_i and the temperature increment Θ. We write these results as $R = \{u_i, \Theta\}$. We assume that the stresses σ_{ij} and the strains e_{ij} are continuous together with their first derivatives, whereas the displacement and the temperature are continuous with continuous derivatives up to the second order, for $x \in V \cup S$, $t > 0$. Assume now that there exists another system of causes and effects, namely

$$C' = \left\{F_i', P_i', Q', \upsilon'\right\}, \quad R' = \left\{u_i', \Theta'\right\}.$$

Proof These functions satisfy the equation of motion (8), and they should obey the generalized heat transport equation

$$\left(1 + \frac{\tau_q^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_q^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\left(\rho C_E\ddot{\Theta} + \gamma T_o\ddot{e} - \rho\dot{Q}\right) = \left[\tau_\upsilon^*\frac{\partial}{\partial t} + k\frac{\tau_T^\alpha}{\alpha!}\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}} + k^*\right]\Theta_{,ii}, \tag{29}$$

We also have the constitutive equations

$$\sigma_{ij} = 2\mu e_{ij} + (\lambda e - \gamma\Theta)\,\delta_{ij}. \tag{30}$$

If we perform over Eq. (30) the Laplace integral transform, we get the relation

$$\overline{\sigma}_{ij} = 2\mu\overline{e}_{ij} + \left(\lambda\overline{e} - \gamma\overline{\Theta}\right)\delta_{ij}. \tag{31}$$

For the second system, the equivalent equation is

$$\overline{\sigma}'_{ij} = 2\mu\overline{e}'_{ij} + \left(\lambda\overline{e}' - \gamma\overline{\Theta}'\right)\delta_{ij}, \tag{32}$$

multiplying Eq. (31) by $\overline{e}'_{ij}$ Eq. (32) by $\overline{e}_{ij}$ and integrating the difference over V, we get

$$\int_V (\overline{\sigma}_{ij}\overline{e}'_{ij} - \overline{\sigma}'_{ij}\overline{e}_{ij})\mathrm{d}V = \gamma \int_V (\overline{\Theta}'\overline{e} - \overline{\Theta}\overline{e}')\mathrm{d}V. \tag{33}$$

Now, we have

$$\int_V \overline{\sigma}_{ij}\overline{e}'_{ij}\mathrm{d}V = \int_V \overline{\sigma}_{ij}\overline{u}'_{i,j}\mathrm{d}V = \int_S \overline{\sigma}_{ij}n_j\overline{u}'_i\mathrm{d}S - \int_V \overline{\sigma}_{ij}\overline{u}'_i\mathrm{d}V, \tag{34}$$

Performing the Laplace transform over Eq. (8) and using the homogeneous initial conditions, we obtain

$$\overline{\sigma}_{ij,j} + \rho\overline{F}_i = \rho s^2\overline{u}_i, \quad x \in V \tag{35}$$

Combining Eqs. (30), (34) and (35), we get

$$\int_V \overline{\sigma}_{ij}\overline{e}'_{ij}\mathrm{d}V = \int_S \overline{P}_i\overline{u}'_i\mathrm{d}S + \rho\int_V \overline{F}_i\overline{u}'_i\mathrm{d}V - \rho\int_V \overline{\sigma}_{ij}\overline{u}_i\overline{u}'_i\mathrm{d}V. \tag{36}$$

Substituting from (36) and an analogous integral $\int_V \overline{\sigma}'_{ij}\overline{e}_{ij}\mathrm{d}V$ into (33), we obtain the equation

$$\int_S \left(\overline{P}_i\overline{u}'_i - \overline{P}'_i\overline{u}_i\right)\mathrm{d}S + \rho\int_V (\overline{F}_i\overline{u}'_i - \overline{F}'_i\overline{u}_i)\mathrm{d}V + \gamma\int_V \left(\overline{\Theta}'\overline{e} - \overline{\Theta}\overline{e}'\right)\mathrm{d}V = 0. \tag{37}$$

Equation (37) constitutes the first part of the reciprocity theorem since it contains only causes of a mechanical nature, namely, the mechanical forces and the surface tractions. To derive the second part, we take Laplace transform of both sides of (30) and use the initial condition, and we obtain

$$\left[\overline{\tau}^*_\upsilon s + k\frac{\tau_T^\alpha}{\alpha!}s^{\alpha+1} + k^*\right]\overline{\Theta}_{,ii} = \left(1 + \frac{\tau_q^\alpha}{\alpha!}s^\alpha + \frac{\tau_q^{2\alpha}}{(2\alpha)!}s^{2\alpha}\right)\left(\rho C_\upsilon s^2\overline{\Theta} + \gamma T_0 s^2\overline{e} - \rho s\overline{Q}\right). \tag{38}$$

The analogous equation for (38)

$$\left[\overline{\tau}^*_\upsilon s + k\frac{\tau_T^\alpha}{\alpha!}s^{\alpha+1} + k^*\right]\overline{\Theta}'_{,ii} = \left(1 + \frac{\tau_q^\alpha}{\alpha!}s^\alpha + \frac{\tau_q^{2\alpha}}{(2\alpha)!}s^{2\alpha}\right)\left(\rho C_\upsilon s^2\overline{\Theta}' + \gamma T_0 s^2\overline{e}' - \rho s\overline{Q}'\right). \tag{39}$$

Multiplying Eq. (38) by $\overline{\Theta}'$ and Eq. (39) by subtracting the result and integrating over the region V, we arrive at the identity

$$\left[\overline{\tau}^*_\upsilon s + k\frac{\tau_T^\alpha}{\alpha!}s^{\alpha+1} + k^*\right]\int_V [\overline{\Theta}'\overline{\Theta}_{,ii} - \overline{\Theta}\overline{\Theta}'_{,ii}]\mathrm{d}V = \gamma T_0\left(s^2 + \frac{\tau_q^\alpha}{\alpha!}s^{\alpha+2} + \frac{\tau_q^{2\alpha}}{(2\alpha)!}s^{2\alpha+2}\right)\int_V (\overline{\Theta}'\overline{e} - \overline{\Theta}\overline{e}')\mathrm{d}V$$
$$-\left(s + \frac{\tau_q^\alpha}{\alpha!}s^{\alpha+1} + \frac{\tau_q^{2\alpha}}{(2\alpha)!}s^{2\alpha+1}\right)\int_V [\overline{Q}\overline{\Theta}' - \overline{Q}'\overline{\Theta}]\mathrm{d}V. \tag{40}$$

Integrating by parts we find, after using the transformed boundary condition, that

$$\int_V \overline{\Theta}'\overline{\Theta}_{,ii}\mathrm{d}V = \int_S \overline{\Theta}'\overline{\Theta}_{,i}n_i\mathrm{d}A - \int_V \overline{\Theta}'_{,i}\overline{\Theta}_{,i}\mathrm{d}V = \int_S \overline{\upsilon}'\overline{\Theta}_{,ii}\mathrm{d}A - \int_V \overline{\Theta}'_{,i}\overline{\Theta}_{,i}\mathrm{d}V. \tag{41}$$

In Eq. (41), we have used the notation $\overline{\Theta}_{,n} = \overline{\Theta}_{,i} n_i$. to denote the derivative of $\overline{\Theta}$ in the direction of the normal to the surface S. Substituting from (41) and an analogous expression for $\int_V \overline{\Theta\Theta}'_{,ii}\mathrm{d}V$ into Eq. (40), we obtain

$$\left[\overline{\tau}_{\upsilon}^{*} s + k\frac{\tau_T^{\alpha}}{\alpha!}s^{\alpha+1} + k^{*}\right]\int\limits_S \left[\overline{\upsilon}'\overline{\Theta}_{,n} - \overline{\upsilon}\overline{\Theta}'_{,n}\right]\mathrm{d}S = \gamma T_0\left(s^2 + \frac{\tau_q^{\alpha}}{\alpha!}s^{\alpha+2} + \frac{\tau_q^{2\alpha}}{(2\alpha)!}s^{2\alpha+2}\right)\int\limits_V (\overline{e}\overline{\Theta}' - \overline{e}'\overline{\Theta})\mathrm{d}V$$
$$-\left(s + \frac{\tau_q^{\alpha}}{\alpha!}s^{\alpha+1} + \frac{\tau_q^{2\alpha}}{(2\alpha)!}s^{2\alpha+1}\right)\int\limits_V \left[\overline{Q\Theta}' - \overline{Q}'\overline{\Theta}\right]\mathrm{d}V. \tag{42}$$

Equation (42) is the second part of the reciprocity theorem. It contains thermal causes, namely: the heat sources and heating of the surface S.

Eliminating the integral $\int_V (\overline{e}'\overline{\Theta} - \overline{e}\overline{\Theta}')\mathrm{d}V$ from Eqs. (33) and (42), we arrive at a reciprocity theorem containing both systems of causes C and C' and results R and R'

$$\left[\overline{\tau}_{\upsilon}^{*} s + k\frac{\tau_T^{\alpha}}{\alpha!}s^{\alpha+1} + k^{*}\right]\int\limits_S \left[\overline{\upsilon}'\overline{\Theta}_{,n} - \overline{\upsilon}\overline{\Theta}'_{,n}\right]\mathrm{d}S + \left(s + \frac{\tau_q^{\alpha}}{\alpha!}s^{\alpha+1} + \frac{\tau_q^{2\alpha}}{(2\alpha)!}s^{2\alpha+1}\right)\int\limits_V \left[\overline{Q\Theta}' - \overline{Q}'\overline{\Theta}\right]\mathrm{d}V$$
$$= T_0\left(s^2 + \frac{\tau_q^{\alpha}}{\alpha!}s^{\alpha+2} + \frac{\tau_q^{2\alpha}}{(2\alpha)!}s^{2\alpha+2}\right)\left(\int\limits_S (\overline{P}_i\overline{u}'_i - \overline{P}'_i\overline{u}_i)\mathrm{d}S + \rho\int\limits_V (\overline{F}_i\overline{u}'_i - \overline{F}'_i\overline{u}_i)\mathrm{d}V\right). \tag{43}$$

To invert the Laplace transform in (43), we use the convolution theorem

$$L^{-1}[\overline{f}(s)\overline{g}(s)] = \int\limits_0^t f(\tau)g^{*}\mathrm{d}\tau = \int\limits_0^t g(\tau)f^{*}\mathrm{d}\tau,\ f^{*} = f(t-\tau),\ g^{*} = g(t-\tau) \tag{44}$$

We have thus arrived at a reciprocity theorem containing the system of causes C and C'.

Finally, Eq. (43) finally reduces to

$$K^{*}\int\limits_s \mathrm{d}S\int\limits_0^t \left[\upsilon'\Theta^{*}_{,n} - \upsilon(x,\tau)\Theta^{*}_{,n}\right]\mathrm{d}\tau + K\int\limits_s \mathrm{d}S\int\limits_0^t \left[\upsilon'^{*}\frac{\partial\Theta_{,n}}{\partial\tau} - \upsilon^{*}\frac{\partial\Theta_{,n}}{\partial\tau}\right]\mathrm{d}\tau$$
$$+\frac{k^{*}\tau_{\upsilon}^{\alpha}}{\alpha!}\int\limits_s \mathrm{d}S\int\limits_0^t \left[\upsilon'^{*}\frac{\partial^{\alpha}\Theta_{,n}}{\partial\tau} - \upsilon^{*}\frac{\partial^{\alpha}\Theta'_{,n}}{\partial\tau}\right]\mathrm{d}\tau + \frac{k\tau_T^{\alpha}}{\alpha!}\int\limits_s \mathrm{d}S\int\limits_0^t \left[\upsilon'^{*}\frac{\partial^{1+\alpha}\Theta_{,n}}{\partial\tau} - \upsilon^{*}\frac{\partial^{1+\alpha}\Theta'_{,n}}{\partial\tau}\right]\mathrm{d}\tau$$
$$+\int\limits_V \mathrm{d}V\int\limits_0^t \left[Q^{*}\frac{\partial\Theta'}{\partial\tau} - Q'^{*}\frac{\partial\Theta}{\partial\tau}\right]\mathrm{d}\tau + \frac{\tau_q^{\alpha}}{\alpha!}\int\limits_V \mathrm{d}V\int\limits_0^t \left[Q^{*}\frac{\partial^{1+\alpha}\Theta'}{\partial\tau} - Q'^{*}\frac{\partial^{1+\alpha}\Theta}{\partial\tau}\right]\mathrm{d}\tau$$
$$+\frac{\tau_q^{2\alpha}}{(2\alpha)!}\int\limits_V \mathrm{d}V\int\limits_0^t \left[Q^{*}\frac{\partial^{1+2\alpha}\Theta'}{\partial\tau} - Q'^{*}\frac{\partial^{1+2\alpha}\Theta}{\partial\tau}\right]\mathrm{d}\tau$$
$$= T_0\left(\int\limits_s \mathrm{d}S\int\limits_0^t \left(P_i^{*}\frac{\partial^2 u'_i}{\partial\tau^2} - P_i'^{*}\frac{\partial^2 u_i}{\partial\tau^2}\right)\mathrm{d}\tau + \rho\int\limits_V \mathrm{d}V\int\limits_0^t \left(F_i^{*}\frac{\partial^2 u'_i}{\partial\tau^2} - F_i'^{*}\frac{\partial^2 u_i}{\partial\tau^2}\right)\mathrm{d}\tau\right)$$

$$+\frac{T_0\tau_q^{\alpha}}{\alpha!}\left(\int_s \mathrm{d}S\int_0^t\left(P_i^*\frac{\partial^{\alpha+2}u_i'}{\partial\tau^{\alpha+2}}-P_i'^*\frac{\partial^{\alpha+2}u_i}{\partial\tau^{\alpha+2}}\right)\mathrm{d}\tau+\rho\int_V \mathrm{d}V\int_0^t\left(F_i^*\frac{\partial^{\alpha+2}u_i'}{\partial\tau^{\alpha+2}}-F_i'^*\frac{\partial^{\alpha+2}u_i}{\partial\tau^{\alpha+2}}\right)\mathrm{d}\tau\right)$$
$$+\frac{T_0\tau_q^{2\alpha}}{(2\alpha)!}\left(\int_s \mathrm{d}S\int_0^t\left(P_i^*\frac{\partial^{2\alpha+2}u_i'}{\partial\tau^{2\alpha+2}}-P_i'^*\frac{\partial^{2\alpha+2}u_i}{\partial\tau^{2\alpha+2}}\right)\mathrm{d}\tau+\rho\int_V \mathrm{d}V\int_0^t\left(F_i^*\frac{\partial^{2\alpha+2}u_i'}{\partial\tau^{2\alpha+2}}-F_i'^*\frac{\partial^{2\alpha+2}u_i}{\partial\tau^{2\alpha+2}}\right)\mathrm{d}\tau\right) \tag{45}$$

Equation (45) is the final form of the reciprocity theorem in the physical domain.

3 One-dimensional application

We shall consider a homogeneous isotropic thermoelastic medium at a uniform reference temperature θ_0 occupying the region $-\infty \leq x \leq \infty$, whose state depends only on the space variables x and the time t [51] and that the displacement vector has components $u = (u(x,t), 0, 0)$.

The governing one-dimensional equations for the dynamic coupled generalized fractional thermoelasticity based on the three-phase lag thermoelasticity model considered as [52]

(i) The equation of motion without the body force

$$\frac{\partial\sigma}{\partial x} = \rho\ddot{u} \tag{46}$$

(ii) The fractional heat equation corresponding to thermoelasticity with three-phase lags

$$\left(1+\frac{\tau_q^{\alpha}}{\alpha!}\frac{\partial^{\alpha}}{\partial t^{\alpha}}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\left(\rho C_E\ddot{\Theta}+\gamma T_0\ddot{e}-\rho\dot{Q}\right)=\left[k^*\left(1+\frac{\tau_\upsilon^{\alpha}}{\alpha!}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)+k\frac{\partial}{\partial t}\left(1+\frac{\tau_T^{\alpha}}{\alpha!}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\right]\frac{\partial^2\Theta}{\partial x^2} \tag{47}$$

(iii) The strain–displacement relations

$$e = e_{xx} = \frac{\partial u}{\partial x} \tag{48}$$

(iv) The constitutive equation

$$\sigma = \sigma_{xx} = (\lambda+2\mu)e - \gamma\Theta \tag{49}$$

where in the above equations, a superposed dot denotes the derivative with respect to time.

Let us introduce the following non-dimensional variables:

$$x' = \frac{x}{l},\quad t' = \frac{\upsilon_0}{l}t,\quad \Theta\prime = \frac{\Theta}{\Theta_0},\quad u' = \frac{(\lambda+2\mu)}{l\gamma\Theta_0}u,\quad \sigma' = \frac{\sigma}{\gamma\Theta_0}$$
$$e' = \frac{(\lambda+2\mu)}{\gamma\Theta_0}e,\quad \tau_q' = \frac{\upsilon_0}{l}\tau_q,\quad \tau_T' = \frac{\upsilon_0}{l}\tau_T,\quad \tau_\upsilon' = \frac{\upsilon_0}{l}\tau_\upsilon$$

where l is a standard length, and υ_0 **is a standard speed**.

Using the above values, for the one-dimensional problem, Eqs. (46)–(49) (dropping the primes for convenience) reduce to

$$\frac{1}{C_P^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial\Theta}{\partial x} \tag{50}$$

$$\left(1+\frac{\tau_q^{\alpha}}{\alpha!}\frac{\partial^{\alpha}}{\partial t^{\alpha}}+\frac{\tau_q^{2\alpha}}{(2\alpha)!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\left(\ddot{\Theta}+\varepsilon\frac{\partial^3 u}{\partial t^2\partial x}-Q\right)=\frac{\partial^2}{\partial x^2}\left[C_T^2\left(1+\frac{\tau_\upsilon^{\alpha}}{\alpha!}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)+C_k^2\frac{\partial}{\partial t}\left(1+\frac{\tau_T^{\alpha}}{\alpha!}\frac{\partial^{\alpha}}{\partial t^{\alpha}}\right)\right]\Theta, \tag{51}$$

$$\sigma = \frac{\partial u}{\partial x} - \Theta \tag{52}$$

where

$$C_P = \frac{1}{\nu}\sqrt{\frac{\lambda+\mu}{\rho}}, \quad C_S = \frac{1}{\nu}\sqrt{\frac{\mu}{\rho}}, \quad C_T = \frac{1}{\nu}\sqrt{\frac{k^*}{\rho C_E}},$$
$$C_k = \sqrt{\frac{k}{\rho C_E l \nu}}, \quad Q = \frac{\dot{Q} l}{C_E \theta_0 \nu}, \quad \varepsilon = \frac{\gamma^2 \theta_0}{\rho C_E(\lambda+2\mu)}.$$

From now on, we shall consider a heat source of the form $Q = Q_0\delta(x)H(t)$,

where $\delta(x)$ and $H(t)$ are the Dirac delta function and Heaviside unit step function, respectively, and Q_0 is a constant.

To simplify the algebra, only problems with zero initial conditions are considered.

Appling the Laplace transforms with parameter s (denoted by a bar) defined by the formula (16) on both sides of Eqs. (50)–(52), and writing the resulting equations in matrix form, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{Bmatrix} \overline{\Theta}'(x,s) \\ \overline{u}'(x,s) \\ \overline{\Theta}'(x,s) \\ \overline{u}'(x,s) \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & \varepsilon\alpha \\ 0 & \frac{s^2}{C_p^2} & 1 & 0 \end{bmatrix} \begin{Bmatrix} \overline{\Theta}(x,s) \\ \overline{u}(x,s) \\ \overline{\Theta}'(x,s) \\ \overline{u}'(x,s) \end{Bmatrix} - \frac{Q_o\beta\delta(x)}{s}\begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix} \tag{53}$$

where

$$\alpha = \frac{s^2\left(1+\frac{\tau_q^\alpha}{\alpha!}s^\alpha + \frac{\tau_q^{2\alpha}}{(2\alpha)!}s^{2\alpha}\right)}{C_T^2(1+\frac{\tau_\upsilon^\alpha}{\alpha!}s^\alpha) + C_k^2 s(1+\frac{\tau_T^\alpha}{\alpha!}s^\alpha)}, \quad \text{and} \quad \beta = \frac{\left(1+\frac{\tau_q^\alpha}{\alpha!}s^\alpha + \frac{\tau_q^{2\alpha}}{(2\alpha)!}s^{2\alpha}\right)}{C_T^2(1+\frac{\tau_\upsilon^\alpha}{\alpha!}s^\alpha) + C_k^2 s(1+\frac{\tau_T^\alpha}{\alpha!}s^\alpha)}$$

Equation (53) can be written in constricted form as

$$\overline{\boldsymbol{G}}'(x,s) = \boldsymbol{A}(s)\overline{\boldsymbol{G}}(x,s) + \boldsymbol{B}(x,s), \tag{54}$$

where $\overline{\boldsymbol{G}}(x,s)$ denotes the state vector in the transform domain whose components consist of the transformed temperature and velocity as well as their gradients.

The formal solution of Eq. (54) can be expressed as [53]

$$\overline{\boldsymbol{G}}(x,s) = \exp\left[\boldsymbol{A}(x,s)x\right]\left(\overline{\boldsymbol{G}}(0,s) + \int_0^x \exp\left[-\boldsymbol{A}(s)z\right]\boldsymbol{B}(z,s)\mathrm{d}z\right). \tag{55}$$

Evaluating the integral in Eq. (55) using the integral properties of the Dirac delta function, we obtain

$$\overline{\boldsymbol{G}}(x,s) = L_{ij}(x,s)[\overline{\boldsymbol{G}}(0,s)] + \boldsymbol{\xi}(s), \quad i,j = 1,2,3,4, \tag{56}$$

where

$$\boldsymbol{\xi}(s) = -\frac{\beta Q_o}{4s}\begin{bmatrix} \frac{K_1K_2+\left(\frac{s^2}{C_p^2}\right)}{K_1K_2(K_1+K_2)} \\ 0 \\ 1 \\ \frac{1}{K_1+K_2} \end{bmatrix}.$$

and $\boldsymbol{L}_{ij}(x,s)$ are defined in the Appendix.

Equation (56) expresses the solution of the problem in the Laplace transform domain in terms of the vector $\boldsymbol{\xi}(s)$ representing the applied heat source and the vector $\overline{\boldsymbol{G}}(0,s)$ representing the conditions at the plane source of heat. In order to evaluate the components of this vector, we note first that due to the symmetry of the problem, the temperature is a symmetric of y while the velocity is anti-symmetric. It thus follows that

$$u(0,t)=0 \quad \text{or} \quad \overline{u}(0,s)=0. \tag{57}$$

Gauss's divergence theorem will now be used to obtain the thermal condition at the plane source. We consider a short cylinder of unit base whose axis is perpendicular to the plane source of heat and whose bases lie on opposite sides of it.

Taking limits as the height of the cylinder tends to zero, and noting that there is no heat flux through the lateral surface, upon using the symmetry of the temperature field we get

$$q(0,t)=\frac{Q_o}{2}H(t) \quad \text{or} \quad \overline{q}(0,s)=\frac{Q_o}{2s} \tag{58}$$

Using modification fractional Fourier's law of heat conduction in the non-dimensional form, we obtain the condition

$$\overline{\Theta}'(0,s)=-\frac{\beta Q_o}{2s} \tag{59}$$

Equations (63) and (65) give two components of the vector $\overline{\boldsymbol{G}}(0,s)$. In order to obtain the remaining two components, we substitute $x=0$ on both sides of Eq. (56) obtaining a system of linear equations whose solution gives

$$\overline{\Theta}(0,s)=\frac{\beta Q_o\left[K_1K_2+(s^2/C_p^2)\right]}{2sK_1K_2(K_1+K_2)} \tag{60}$$

$$\overline{u'}(0,s)=\frac{\beta Q_o}{2s(k_1+k_2)} \tag{61}$$

Inserting the values from Eqs. (63), (65), (66) and (67) into the right-hand side of Eq. (62) and performing the necessary matrix operations, we obtain:

$$\overline{\Theta}(x,s)=\frac{\beta Q_o}{2s(K_1^2-K_2^2)}\left[\frac{K_1^2-(s^2/C_p^2)}{K_1}e^{\pm K_1x}-\frac{K_2^2-(s^2/C_p^2)}{K_2}e^{\pm K_2x}\right], \tag{62}$$

$$\overline{u}(x,s)=\frac{\pm\beta Q_o}{2s(K_1^2-K_2^2)}\left[e^{\pm K_1x}-e^{\pm K_2x}\right], \tag{63}$$

Using these equations together with Eq. (52), we obtain

$$\overline{\sigma}(x,s)=\frac{\beta Q_o s}{2C_p^2K_1K_2(K_1^2-K_2^2)}\left[K_2e^{\pm K_1x}-K_1e^{\pm K_2x}\right]. \tag{64}$$

In the above equations, the upper (plus) sign indicates the solution in the region $x<0$, while the lower (minus) sign indicates the region $x\geq 0$, respectively.

4 Numerical results and discussion

In order to invert the Laplace transform in the above equations, we adopt a numerical inversion method based on a Fourier series expansion [45]. The numerical code has been prepared using Fortran 77 programming language. For computational purpose, copper like material has been taken into consideration. The values of the material constants are taken as Roy Choudhuri and Dutta [54]:

$$\varepsilon=1.618,\quad \lambda=7.76(10)^{10}\mathrm{N/m^2},\quad \mu=3.86(10)^{10}\mathrm{N/m^2},\quad \alpha_T=1.78(10)^{-5}\,1/\mathrm{K}, \tag{65}$$

$$\rho=8.954\,\mathrm{kg/m^3},\quad T_0=293\,\mathrm{K},\quad C_E=383.1\,\mathrm{m^2/s^2K} \tag{66}$$

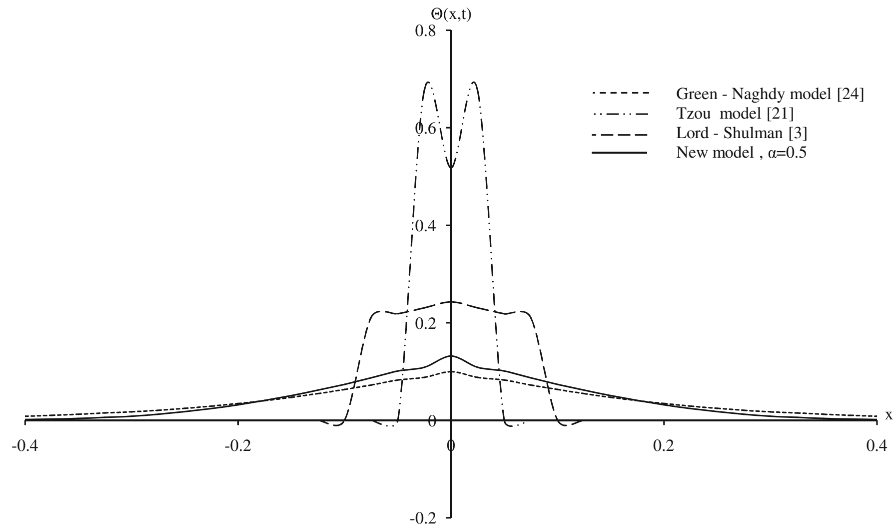

Fig. 1 Dependence of temperature on distance for different values of α

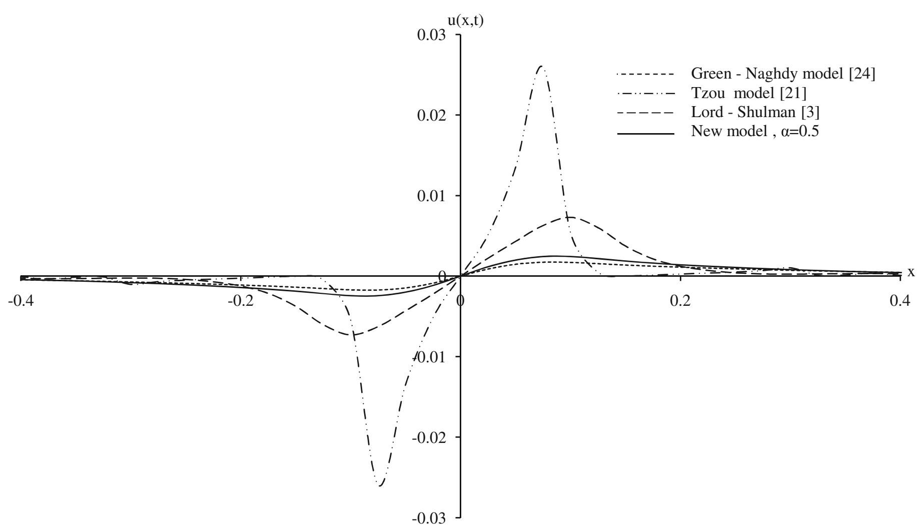

Fig. 2 Dependence of displacement on distance for different values of α

and the hypothetical values of relaxation time parameters are taken as

$$\tau_q = 0.001, \tau_\upsilon = 0.025, \tau_T = 0.015, \quad (67)$$

Also, we have taken $\theta_0 = 1C^0$, $Q^*_\circ = 1$, $\tau = 1$, $C_p = 1$, $C_K = 0.6$, $C_T > 1$; say $C_T = 2$, so the faster wave is the thermal wave.

The computations were carried out for $t = 0.1$. The numerical technique was used to obtain the temperature, the displacement and the stress distributions along x-axis for the infinite space problem in the presence of heat sources and represented graphically in Figs. 1, 2 and 3, respectively. Comparisons are made with the results predicted by all the considered theories. It is noticed that the fractional parameter α has a significant effect on all the fields. The important phenomenon observed in this problem where the medium is of infinite extent is that the solution of any of the considered function for the generalized theory vanishes identically outside a bounded region of space. This demonstrates clearly the difference between the coupled and generalized theories of thermoelasticity.

For a normal conductivity $\alpha = 1.0$, the results coincide with all the previous results of applications that are taken in the context of the generalized thermoelasticity as in Refs. [3,21,54].

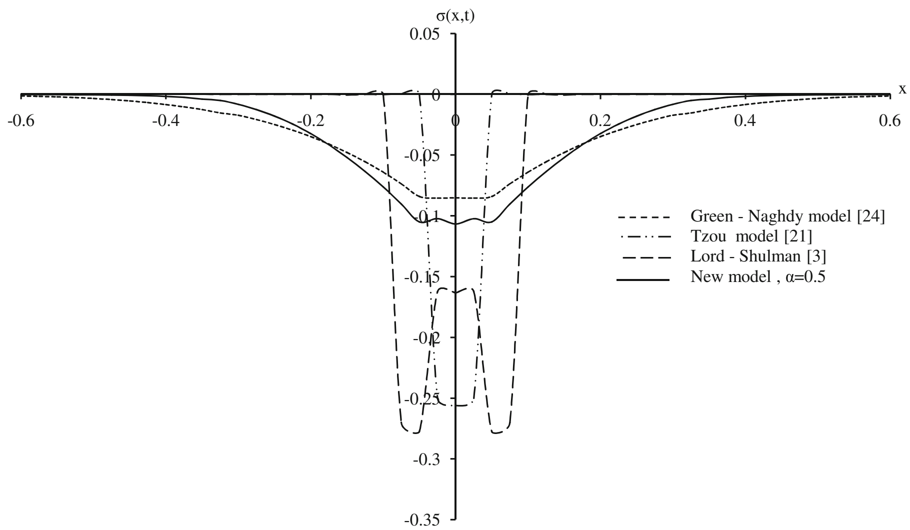

Fig. 3 Dependence of stress on distance for different values of α

For a weak conductivity $0 < \alpha = 0.5 < 1$ [55], we notice that the particles transport the heat to the other particles easily and this makes the decreasing rate of the temperature greater than the other one. In Figs. 2 and 3, the displacement and the stress fields have the same behavior as the temperature. The sharp descent of temperature distribution on space (i.e., the thermal wave front) will not exist.

From the above discussions, the presence of fractional order parameter α has significant effect on the solutions of the temperature, displacement, stress and strain distributions.

5 Conclusion

The main goal of this work is to introduce a new mathematical model for Fourier law of heat conduction with three-phase lag time-fractional order q for isotropic material. According to this theory, we have to construct a new classification to the materials according to its fractional parameter a where this parameter becomes a new indicator of its ability to conduct the heat under the effect of three-phase lag.

Appendix

$$\ell_{11} = \frac{1}{K_1^2 - K_2^2}\left[(K_1^2 - \alpha)\cosh(K_1 x) - (K_2^2 - \alpha)\cosh(K_2 x)\right],$$

$$\ell_{12} = \frac{\alpha a s^2 \varepsilon}{K_1^2 - K_2^2}\left[\frac{\sinh(K_1 x)}{K_1} - \frac{\sinh(K_2 x)}{K_2}\right],$$

$$\ell_{13} = \frac{1}{K_1^2 - K_2^2}\left[\frac{[K_1^2 - (s^2/C_p^2)]}{K_1}\sinh(K_1 x) - \frac{[K_2^2 - (s^2/C_p^2)]}{K_2}\sinh(K_2 x)\right],$$

$$\ell_{14} = \frac{\alpha\varepsilon}{K_1^2 - K_2^2}\left[\cosh(K_1 x) - \cosh(K_2 x)\right], \ell_{21} = \frac{\alpha}{K_1^2 - K_2^2}\left[\frac{\sinh(K_1 x)}{K_1} - \frac{\sinh(K_2 x)}{K_2}\right],$$

$$\ell_{23} = \frac{1}{K_1^2 - K_2^2}\left[\cosh(K_1 x) - \cosh(K_2 x)\right], \ell_{31} = \alpha\ell_{13}, \ell_{32} = \frac{\alpha\varepsilon s^2}{C_p^s}\ell_{23},$$

$$\ell_{24} = \frac{1}{K_1^2 - K_2^2}\left[\frac{(K_1^2 - \alpha)\sinh(K_1 x)}{K_1} - \frac{(K_2^2 - \alpha)\sinh(K_2 x)}{K_2}\right],$$

$$\ell_{33} = \frac{1}{K_1^2 - K_2^2}\left[\left(K_1^2 - \frac{s^2}{C_p^2}\right)\cosh(K_1 x) - (K_2^2 - \frac{s^2}{C_p^2})\cosh(K_2 x)\right],$$

$$\ell_{34} = \frac{\alpha\varepsilon}{K_1^2 - K_2^2}\left[K_1 \sinh(K_1 x) - K_2 \sinh(K_2 x)\right], \quad \ell_{41} = \alpha\ell_{14}, \quad \ell_{42} = \frac{s^2}{C_p^2}\ell_{24},$$

$$\ell_{43} = \alpha\varepsilon\ell_{34}, \ell_{44} = \frac{1}{K_1^2 - K_2^2}\left[(K_1^2 - \alpha)\cosh(K_1 x) - (K_2^2 - \alpha)\cosh(K_2 x)\right],$$

where K_1 and K_2 are the roots of the characteristic equation

$$K^4 - [s^2 + \alpha(1+\varepsilon)]K^2 + \alpha s^2 = 0.$$

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