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Controlled motion of a two-module vibration-driven system induced by internal acceleration-controlled masses

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Abstract The rectilinear motion of a vibration-driven mechanical system composed of two identical modules connected by an elastic element is considered in this paper. Each module consists of a main body and an internal mass that can move inside the main body. Anisotropic linear resistance is assumed to act between each module and the resistant medium. The motion of the system is excited by two acceleration-controlled masses inside the respective main bodies. The primary resonance situation that the excitation frequency is close to the natural frequency of the system is considered, and the steady-state motion of the system as a whole is mainly investigated. Both the internal excitation force and the external resistance force contain non-smooth factors and are assumed to be small quantities of the same order when compared with the maximum value of the force developed in the elastic element during the motion. With this assumption, method of averaging can be employed and an approximate value of the average steady-state velocity of the entire system is derived through a set of algebraic equations. The analytical results show that the magnitude of the average steady-state velocity can be controlled by varying the time shift between the excitations in the modules. The optimal value of the time shift that corresponds to the maximal average steady-state velocity exists and is unchanging with the external coefficients of resistance. For a system with specific parameters, numerical simulations are carried out to verify the correctness of the analytical results. The optimal value of the time shift is numerically obtained, and the optimal situation is studied to show the advantages of the control.

Keywords Vibration-driven system · Internal mass · Anisotropic linear resistance · Steady-state motion · Method of averaging · Optimal control

1 Introduction

In recent years, vibration-driven systems have attracted great attention from researchers due to their extensive potential applications in medical treatment, engineering diagnosis, seabed exploration, and disaster rescue, etc. They can move in various environments without propelling components (such as wheels, legs, oars, jets, screws, and other outward devices). The propulsion of the system is provided due to the vibrations of internal masses and the interaction of the system with the resistive environment. This principle can explain some

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motions of limbless animals, e.g., snakes or worms. Such systems have a number of advantages over systems based on conventional principles of locomotion. They are simple in design and their bodies can be fabricated into very small size, and thus can be probably utilized as the dynamic models of certain micro-robots and bionic robots.

A large numbers of publication [1–14] work at the systems with movable internal masses. When a designated excitation force is applied to the internal mass, the reaction force exerts on the main body and changes its velocity. The system can thus move forward under control on account of the anisotropic resistance force acting between the main body and the plane. Mechanisms based on this principle do not require complicated gear case and can be made hermetic and smooth, i.e., without external moving parts, which enable them to be used in capsule-type micro-robots. This kind of robots has a large applicability in restricted place and vulnerable media, for example, robots for inspection in narrow industrial pipelines and self-propelled endoscopes in human vessels [1]. Such system can also be driven to a prescribed position with high degree of accuracy ($\sim 10^{-8}$m), which is promisingly utilized in high-precision units in scanning electron microscopes, as well as micro- and nano-technological equipments [2,3].

A systematic study on the optimal control of the rectilinear motion of a system with an internal mass on a rough horizontal plane was carried out by F.L. Chernousko [4–7,10]. In [4,5], Coulomb's dry friction was assumed to act between the main body and the plane. The internal mass was allowed to move within a fixed limit along a line parallel to the line of motion of the main body. It is assumed that, at the beginning and ending instants of each period, the velocities of the main body equal to zero and the internal mass locates at the extreme left position inside the main body, with zero velocity too. In order to realize a steady-state motion (velocity-periodic motion) of the system, periodic control modes were constructed for the relative motion of the internal mass, including velocity-controlled mode and acceleration-controlled mode, also called "two-phase motion" and "three-phase motion", respectively. In velocity-controlled mode, each period includes two intervals of constant velocity, and the velocity of the internal mass relative to the main body is a piecewise-constant function. When the internal mass arrives at the extreme left-hand and right-hand position, its velocity changes instantaneously both in magnitude and direction, i.e., an impact occurs. The parameters to be varied in this case are the magnitudes of the relative velocities of the internal mass. In acceleration-controlled mode, the relative acceleration is a piecewise-constant function that three intervals of constant acceleration are contained in a period. The magnitudes of the relative acceleration can be varied within a pre-set upper limit. Close attention was paid to the average velocity of the steady-state motion of the system as a whole, which reflects the efficiency and is one of the key characters of the system. For both modes, optimal parameters were decided to realize the maximal average velocity of the steady-state motion. In [6,7], for velocity-controlled mode, the resistance force between the main body and the environment was extended to piecewise-linear resistance and quadratic-law resistance. Without the condition that the velocities of the main body and the internal mass vanish simultaneously, optimal parameters of the velocity-controlled mode were found that corresponds to the maximal average velocity of the steady-state motion. In [8], for acceleration-controlled mode and piecewise-linear resistance, the relationships of the control parameters were found to realize a steady-state motion through both analytical and numerical ways. Optimal and feasible parameters of the acceleration-controlled mode, at which the maximal average velocity of the steady-state motion was reached, were determined. In [9], with no constraints imposed on the structure of the control law and with Coulomb's dry friction acting between the system and the plane, an optimal control was constructed for a two-mass system moving along a rough plane to maximize the average velocity over a period. Unlike [4,5], only periodicity condition was asked to be satisfied by the internal mass. The constraint on the amplitude of the internal motion, as well as the condition that the velocity of the internal mass equals to zero at the instants when the velocity of the main body vanishes, was not imposed on. Some experimental progress, including a pendulum-driven cart, a vibro-robot in a tube, and a capsubot, was made by Li, Furuta, and Chernousko [1,10,11].

Bolotnik, Zimmermann, and Sobolev et al. [12–14] studied the optimal control of the rectilinear motion of a rigid body on a rough horizontal plane by means of two internal masses. One of the masses moves horizontally parallel to the line of the motion of the main body, whereas the other mass moves in the vertical direction, which makes it possible to influence the normal reaction of the supporting surface, and hence, the dry friction force. In [12], both masses perform harmonic oscillations relative to the carrying body. The vertical and horizontal oscillations of the internal masses have the same frequency but are shifted in phase. It was shown that the direction and magnitude of the average velocity of the steady-state motion (velocity-periodic motion) can be controlled through varying the phase shift between the internal horizontal and vertical oscillations, as well as the their frequencies. In the case when small force of dry and linear viscous friction acted between the carrying body and environment, an algebraic equation for calculating the average velocity of the steady-state motion was obtained. The optimal value of the phase shift, at which the magnitude of the average velocity reaches maximum, was found. By means of unbalanced exciters, an experimental investigation based on the above principle was carried out by Sobolev and Sorokin [13]. The optimal control of a rigid body moving along a rough horizontal plane due to motions of two internal masses was designed in [14]. Instead of harmonic oscillations, generic periodic motions were constructed for the internal masses. With the constraints that the period was fixed and the magnitudes of the accelerations of the internal masses relative to the main body did not exceed prescribed limits, optimal motions were obtained to ensure steady-state motion of the system with maximal average velocity.

The rectilinear motions of chain of bodies connected to one another by means of springs and dashpots were studied by a number of authors [15–19]. The asymmetry of the resistance force, which is required for a directional motion, is common in engineering and can be provided through covering the contact surface of the system with scale-like plates. The system is driven by varying forces acting between the bodies or internal periodical excitations. In [15], a forward rectilinear motion of a system of two rigid bodies along a horizontal plane was considered. Dry friction is assumed to act between the bodies and the plane, and the motion is controlled by forces of interaction between the bodies. Without any restriction on the control forces, optimal parameters of the system and a control law were found, corresponding to the maximum mean velocity of the system as a whole. In [16], the dynamics of a system of two bodies joined by a nonlinear elastic element were considered. The motion is excited by harmonic forces acting between the bodies. An approximate steady motion with a constant "on the average" velocity is obtained through method of averaging. The motion of n mass points in a common straight line was studied in [17]. All the masses except the middle one are equipped with scales contacting the ground. The middle mass is subjected to a harmonic external force. In [18], a one-dimensional motion of two-mass points in a resistive medium was considered. The resistance force is described by small non-symmetric viscous (piecewise-linear) friction, whose magnitude depends on the direction of motion. The mass points are interconnected with a kinematic constraint or with an elastic spring. Method of averaging was adopted to obtain the expression of the average velocity of the system's steady-state motion. Work [19] devoted to a system composed of two identical modules connected by a linear spring. Each module consists of a main body and an unbalanced vibration rotor, which represents the system studied in [12]. The steady-state motion was mainly considered and a nearly resonant excitation mode was investigated. Based on method of averaging, possibilities for control of the magnitude and direction of the speed of the system were established when friction was assumed to be small.

As one can see in [19], a two-module vibration-driven system has some advantages over one-module system. Without changing the amplitude of the excitation, the system's average steady-state velocity can be easily controlled through varying the phase shift, i.e., the controllability is improved. It is also a more practical dynamic model for some kinds of bionic robots, such as worm-like robots. In all these studies on system with chain of bodies, harmonic motions or harmonic forces are applied on the joint or inside the bodies, which only involves the analysis on smooth excitation (control). On the other hand, velocity-controlled mode and acceleration-controlled mode raised by Chernousko are two novel excitation modes, which do not require common rotation-type driver and can be possibly achieved through the locomotion of a magnetic device in magnetic field. Systems with such control modes are very promisingly to be further minimized, and become the dynamic model for some micro-robots. However, since the non-smooth characters of these two control modes, little work has been carried out on the motion of two-module vibration-driven systems induced by internal velocity- and acceleration-controlled masses.

The purpose of this study is to examine the rectilinear motion of a two-module vibration-driven system in a resistant medium induced by internal acceleration-controlled masses. Each module of the system under consideration can be used as a single-module vibration-driven system, whose steady-state motion was studied in [8]. It was shown that several rigorous conditions on the internal motion were required to be satisfied to achieve a steady-state (velocity-periodic) motion of the system. Changing the magnitudes of the parameters of the acceleration-controlled mode, one was able to optimize the mean velocity of the system. In this paper, the two modules of the vibration-driven system are connected through a linear elastic element. Anisotropic linear resistance is assumed to act between the modules and the environment. Major attention is given to the primary resonance situation that the excitation frequency is close to the frequency of natural elastic oscillation. We will investigate analytically the steady-state motion of the system in the case when the force of friction is small when compared with the elastic force developed from the elastic element. Though both the internal excitations and external resistance forces contain factors of non-smooth, method of averaging will be employed in this investigation. An approximate value of the average steady-state velocity of the entire system can be obtained through a set of algebraic equations. We will show that, unlike a single-module system, the average steady-state

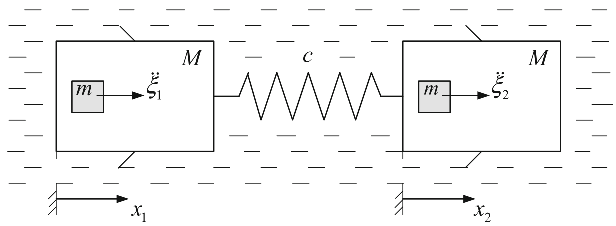

Fig. 1 Two-module system with internal acceleration-controlled masses

velocity can be easily controlled by varying the time shift between the two internal excitations. The optimal value of the time shift corresponding to the maximal magnitude of the average steady-state velocity will be numerically obtained. Numerical integration of the exact full equations of motion demonstrates an acceptable agreement with the results obtained on the basis of the averaged equations.

2 Dynamical model of the vibration-driven system

2.1 Description of the dynamical system

Consider a system consisting of two identical modules connected by an elastic member (e.g., a spring). Each module is composed of a main body and a movable internal mass. Both bodies can move along a same straight line in a resistant medium. The internal masses also move along a horizontal line parallel to the line of motion of the main bodies (see Fig. 1). Acceleration-controlled modes, i.e., three-phase motions, are applied to both the internal masses. In order to avoid collision of the internal mass and the main body, the relative displacement of the internal mass is fixed within an interval, which can be determined by the dimension of the cavity in the main body or by setting up clapboards inside the main body. We assume that both the two internal masses vibrate with the same frequency and amplitude but are shifted in phase. Anisotropic linear resistance is assumed to act between the main bodies and the medium.

In what follows, for sake of brevity, the main bodies and the internal masses will be referred as body 1, body 2 and mass 1, mass 2, respectively. Introduce the following notation: M is the mass of each of the bodies 1 and 2; m is the mass of each of the internal masses 1 and 2; c is the stiffness coefficient of the elastic member; L is the length of the fixed interval for the relative displacement of the internal masses; $F_1, -F_1$ and $F_2, -F_2$ are the interaction forces between the main body and the internal mass in modules 1 and 2, respectively; R_1 and R_2 are the external resistance forces acting on bodies 1 and 2, respectively. Let x_1 and x_2 denote the absolute coordinates measuring the displacements of bodies 1 and 2 relative to the environment; ξ_1 and ξ_2 the coordinates measuring the displacements of mass 1 relative to body 1 and mass 2 relative to body 2. Then, the corresponding velocities and accelerations of body 1, body 2 and mass 1, mass 2 can be expressed into the first-order derivatives and second-order derivatives of x_1, x_2 and ξ_1, ξ_2.

2.2 Equations of motion

The motion of the system is governed by the following differential equations

$$\begin{aligned} m(\ddot{\xi}_1 + \ddot{x}_1) &= -F_1, \\ M\ddot{x}_1 &= F_1 + c(x_2 - x_1) - R_1, \\ m(\ddot{\xi}_2 + \ddot{x}_2) &= -F_2, \\ M\ddot{x}_2 &= F_2 - c(x_2 - x_1) - R_2. \end{aligned} \tag{1}$$

Eliminating the interaction forces F_1 and F_2 of (1), we obtain

$$\begin{aligned} (M+m)\ddot{x}_1 &= -m\ddot{\xi}_1 - c(x_1 - x_2) - R_1, \\ (M+m)\ddot{x}_2 &= -m\ddot{\xi}_2 + c(x_1 - x_2) - R_2. \end{aligned} \tag{2}$$

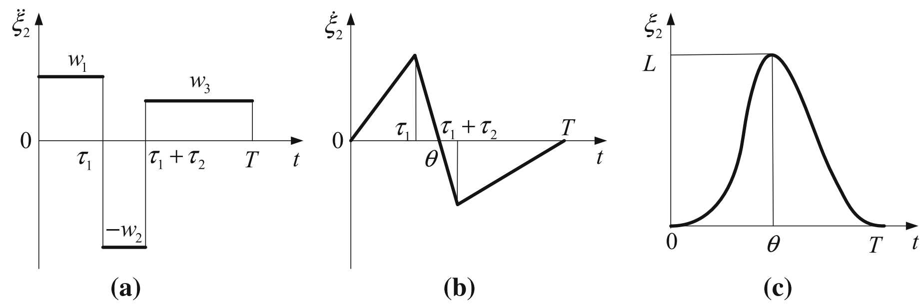

Fig. 2 Acceleration-controlled mode of mass 2: **a** relative acceleration; **b** relative velocity; **c** relative displacement

In these equations, R_i $(i = 1, 2)$ is the resistance force acting on body i. The anisotropic linear resistance is defined as

$$R_i = -(M + m) f(\dot{x}_i)\dot{x}_i, \quad i = 1, 2, \tag{3}$$

in which,

$$f(\dot{x}_i) = \begin{cases} f_+, & \text{if } \dot{x}_i \geq 0, \\ f_-, & \text{if } \dot{x}_i < 0, \end{cases} \quad i = 1, 2. \tag{4}$$

f_+ and f_- are coefficients of resistance, which are all non-negative constants. We assume $f_+ \neq f_-$, which imply the anisotropism of the interface or the asymmetry of the shape of bodies 1 and 2. Without loss of generality, we assume

$$f_+ < f_-, \tag{5}$$

which means the resistance for the forward motion is less than for the backward motion.

2.3 Internal acceleration-controlled mode

In the present study, we confine our consideration to internal acceleration-controlled mode, which was named as internal three-phase motion in [4–7]. Both masses 1 and 2 vibrate with the same frequency and with the same amplitude, while are shifted in phase. Without loss of generality, it is assumed that the motion of mass 1 falls behind that of mass 2, i.e.,

$$\ddot{\xi}_1(t) = \ddot{\xi}_2(t - \tau_0), \tag{6}$$

where $\tau_0 \geq 0$ is the time shift between the internal motions of masses 1 and 2. The situation that $\tau_0 < 0$ corresponds to a form of motion that mass 1 goes ahead of mass 2. However, in view of the periodicity of the internal excitation, a negative value of τ_0 is equivalent to a positive value $T - \tau_0$, which corresponds to the form of motion that mass 1 falls behind mass 2 again. Hence, in what follows, it suffices to consider the value of τ_0 in the range from 0 to T.

For mass 2, the period interval $[0, T]$ consists of three segments in which $\ddot{\xi}_2$ is constant. Each duration of the segments is denoted by τ_j $(j = 1, 2, 3)$ and the corresponding relative acceleration is denoted by w_j $(j = 1, 2, 3)$, respectively. Thus, one has

$$\left.\begin{aligned} &\ddot{\xi}_2(t) = w_1, && \text{when } t \in (0, \tau_1), \\ &\ddot{\xi}_2(t) = -w_2, && \text{when } t \in (\tau_1, \tau_1 + \tau_2), \\ &\ddot{\xi}_2(t) = w_3, && \text{when } t \in (\tau_1 + \tau_2, T),\ T = \tau_1 + \tau_2 + \tau_3, \end{aligned}\right\} \tag{7}$$

where $w_j (j = 1, 2, 3)$ are positive constants (See Fig. 2a).

A fixed interval for the relative displacement of mass 2 relative to body 2 is expressed as

$$0 \leq \xi_2(t) \leq L. \tag{8}$$

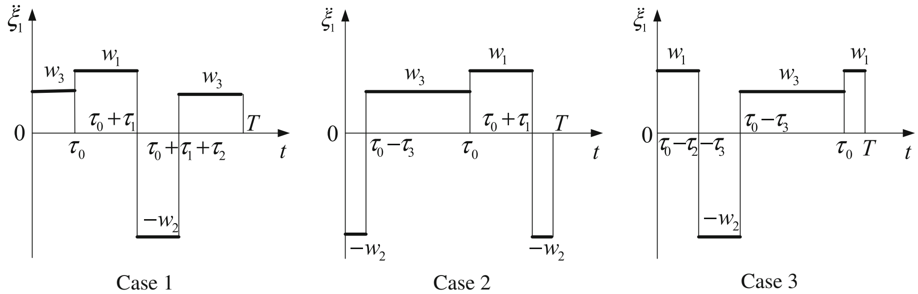

Fig. 3 Three possible cases for acceleration-controlled mode of mass 1

Since the acceleration-controlled mass 2 performs a periodic motion with period T, the functions $\xi_2(t)$, $\dot{\xi}_2(t)$, and $\ddot{\xi}_2(t)$ are all periodic functions with the same period. Without loss of generality, the origin of the time is chosen at the instant when $\xi_2 = 0$, we put

$$\xi_2(0) = \xi_2(T) = 0. \tag{9}$$

As the relative displacement of mass 2 takes its minimal values at the beginning and ending moments of a period, it follows that

$$\dot{\xi}_2(0) = \dot{\xi}_2(T) = 0. \tag{10}$$

Besides, it is required that the maximal relative displacement of mass 2 is attained at at a certain instant of time $\theta \in (0, T)$, i.e.,

$$\xi_2(\theta) = L,\ \dot{\xi}_2(\theta) = 0. \tag{11}$$

Integrating $\ddot{\xi}_2(t)$ with the acceleration (7), and taking the conditions (9)∼(11) that should be satisfied by mass 2 into consideration, one obtain the relative velocity and relative displacement of mass 2, which are illustrated in Fig. 2 b,c. From conditions (9)∼(11), all the other control parameters of mass 2 for acceleration-controlled mode, including τ_j $(j = 1, 2, 3)$ and T can be expressed in terms of w_j $(j = 1, 2, 3)$ and L, namely

$$\begin{aligned}
\tau_1 &= \left[\frac{2w_2L}{w_1(w_1+w_2)}\right]^{\frac{1}{2}},\\
\tau_2 &= \left(\frac{2L}{w_2}\right)^{\frac{1}{2}}\left[\left(\frac{w_1}{w_1+w_2}\right)^{\frac{1}{2}}+\left(\frac{w_3}{w_2+w_3}\right)^{\frac{1}{2}}\right],\\
\tau_3 &= \left[\frac{2w_2L}{w_3(w_2+w_3)}\right]^{\frac{1}{2}},\\
T &= \tau_1+\tau_2+\tau_3 = \left(\frac{2L}{w_2}\right)^{\frac{1}{2}}\left[\left(\frac{w_1+w_2}{w_1}\right)^{\frac{1}{2}}+\left(\frac{w_2+w_3}{w_3}\right)^{\frac{1}{2}}\right].
\end{aligned} \tag{12}$$

The acceleration-controlled mass 1, whose motion is defined by (6), performs a periodic motion too. Noting that a negative value of time may be taken when $t < \tau_0$, $\ddot{\xi}_2(t-\tau_0) = 0$ is additionally defined for $t < \tau_0$ in the first period. Definition (6) remains valid for the left part of the first period and all the other periods. This setting is acceptable for the reason that the change on the first period alone has little influence on the steady-state motion of the system in the whole time histories. Besides, through such additional definition, conditions (9)∼(11) can all be satisfied for mass 1. Then, all the control parameters of mass 2 can be employed by mass 1.

In light of the three segments of the relative acceleration in a period, three possible cases are derived for the motion of mass 1, corresponding to different segments the time shift τ_0 locates in. For case 1, $\tau_0 \in (0, \tau_3)$; for case 2, $\tau_0 \in (\tau_3, \tau_2+\tau_3)$; and for case 3, $\tau_0 \in (\tau_2+\tau_3, T)$. The acceleration–time diagrams of mass 1 in a period for the three possible cases are shown in Fig. 3, marked with case 1, case 2, and case 3. The expressions of relative acceleration of mass 1 for the three cases have the forms:

Case 1:

$$\left.\begin{aligned} \ddot{\xi}_1(t) &= w_3, && \text{when } t \in (0, \tau_0), \\ \ddot{\xi}_1(t) &= w_1, && \text{when } t \in (\tau_0, \tau_0 + \tau_1), \\ \ddot{\xi}_1(t) &= -w_2, && \text{when } t \in (\tau_0 + \tau_1, \tau_0 + \tau_1 + \tau_2), \\ \ddot{\xi}_1(t) &= w_3, && \text{when } t \in (\tau_0 + \tau_1 + \tau_2, T); \end{aligned}\right\} \tag{13}$$

Case 2:

$$\left.\begin{aligned} \ddot{\xi}_1(t) &= -w_2, && \text{when } t \in (0, \tau_0 - \tau_3), \\ \ddot{\xi}_1(t) &= w_3, && \text{when } t \in (\tau_0 - \tau_3, \tau_0), \\ \ddot{\xi}_1(t) &= w_1, && \text{when } t \in (\tau_0, \tau_0 + \tau_1), \\ \ddot{\xi}_1(t) &= -w_2, && \text{when } t \in (\tau_0 + \tau_1, T); \end{aligned}\right\} \tag{14}$$

Case 3:

$$\left.\begin{aligned} \ddot{\xi}_1(t) &= w_1, && \text{when } t \in (0, \tau_0 - \tau_2 - \tau_3), \\ \ddot{\xi}_1(t) &= -w_2, && \text{when } t \in (\tau_0 - \tau_2 - \tau_3, \tau_0 - \tau_3), \\ \ddot{\xi}_1(t) &= w_3, && \text{when } t \in (\tau_0 - \tau_3, \tau_0), \\ \ddot{\xi}_1(t) &= w_1, && \text{when } t \in (\tau_0, T). \end{aligned}\right\} \tag{15}$$

Practically, the relative accelerations of mass 1 and mass 2 should satisfy the constraints

$$0 < w_j < W, \quad j = 1, 2, 3, \tag{16}$$

where W is the highest permissible acceleration of the relative motion and is an assigned positive constant.

2.4 Non-dimensionalization

Introduce the dimensionless variables in according to the following formulas (the asterisk $*$ is a symbol of dimensionless variables):

$$\begin{aligned} & x_i^* = \frac{x}{S}, \quad \xi_i^* = \frac{\xi_i}{S}, \quad t^* = t\sqrt{\frac{c}{M+m}}, \quad \varepsilon = f_-\sqrt{\frac{M+m}{c}}, \\ & \alpha = \frac{m}{M+m}\sqrt{\frac{c}{M+m}}, \quad \beta = \frac{\alpha}{f_-}, \quad f^*(\dot{x}_i^*) = \frac{f(\dot{x}_i)}{f_-} \quad (i = 1, 2). \end{aligned} \tag{17}$$

In the above transformations, S is a unit of length used for non-dimensionalization and can be set arbitrarily. For the sake of simplicity, S is chosen as

$$S = \frac{W(M+m)}{c}.$$

Then, the dimensionless relative acceleration $\ddot{\xi}_i^*$ equal to $\frac{\ddot{\xi}_i}{W}$, $(i = 1, 2)$ and are on the order of $O(10^0)$.

Proceed to the dimensionless variables in (2) and then omit the asterisks, one obtain the dimensionless governing equation

$$\begin{aligned} \ddot{x}_1(t) + (x_1 - x_2) &= -\varepsilon\beta\ddot{\xi}_1(t) - \varepsilon f(\dot{x}_1)\dot{x}_1, \\ \ddot{x}_2(t) - (x_1 - x_2) &= -\varepsilon\beta\ddot{\xi}_2(t) - \varepsilon f(\dot{x}_2)\dot{x}_2. \end{aligned} \tag{18}$$

Here,

$$f(\dot{x}_i) = \begin{cases} k, & \text{if } \dot{x}_i \ge 0, \\ 1, & \text{if } \dot{x}_i < 0, \end{cases} \quad i = 1, 2, \tag{19}$$

where k is the dimensionless coefficient of resistance for forward motion, $k = \frac{f_+}{f_-}$ and $0 \le k \le 1$.

3 The steady-state motion of the system

3.1 Method of averaging

In what follows, we assume that ε is a small parameters, while the quantities β and $|x_2 - x_1|$ are on the following orders

$$\varepsilon \ll 1, \quad \beta \sim O(10^0), \quad |x_2 - x_1| \sim O(10^0). \tag{20}$$

Since ε can be represented as $\varepsilon = f_-\sqrt{\frac{M+m}{c}} = \frac{f_- S\sqrt{c(M+m)}}{cS}$, the smallness of the parameter ε shows that the value of the resistance force is small compared with the amplitude of the elastic force.

By subtracting the first equation of (18) from its second equation, the equation of the relative motion of the system's modules can be obtained as

$$\ddot{\eta} + 2\eta = \varepsilon\left[-\beta(\ddot{\xi}_2 - \ddot{\xi}_1) - [f(\dot{x}_2)\dot{x}_2 - f(\dot{x}_1)\dot{x}_1]\right], \tag{21}$$

where $\eta = x_2 - x_1$.

Assuming that no resistance force and excitation force acts on the bodies, the relative motion oscillates with the natural frequency, i.e., the resonant frequency. This free vibration is governed by equation (21) with zero right-hand side ($\varepsilon = 0$). The general solution of the free vibration has the form

$$\eta = A\cos(\sqrt{2}t - \theta_0), \tag{22}$$

where A and θ_0 are arbitrary constants represent the amplitude and the initial phase of the oscillation, respectively. The natural frequency of the free vibration is $\omega_0 = \sqrt{2}$.

To analyze the response of the system in primary resonance, we assume that the difference of the excitation frequency $\omega = \frac{2\pi}{T}$ from the resonant frequency ω_0 has an order of magnitude of ε, i.e.,

$$\omega^2 = \omega_0^2(1 + \varepsilon\sigma), \tag{23}$$

where σ is the off-resonance detuning. Here, σ is a const and $\sigma \sim O(10^0)$. Then, equation (21) can be transformed into

$$\ddot{\eta} + \omega^2\eta = \varepsilon\left[-\beta(\ddot{\xi}_2 - \ddot{\xi}_1) - [f(\dot{x}_2)\dot{x}_2 - f(\dot{x}_1)\dot{x}_1] + \sigma\omega_0^2\eta\right]. \tag{24}$$

To enable the method of averaging, the following changes of variables are introduced to reduce the system (18) into a standard form

$$\begin{aligned} &x_1 = X - a\cos\varphi, \quad x_2 = X + a\cos\varphi, \\ &\dot{X} = V, \quad \dot{x}_1 = V + a\omega\sin\varphi, \quad \dot{x}_2 = V - a\omega\sin\varphi, \\ &\varphi = \omega t - \theta, \end{aligned} \tag{25}$$

where $X = \frac{x_1+x_2}{2}$ is the absolute coordinate of the center of mass of the two rigid bodies, and V is the absolute velocity. a, φ and θ are the functions of t. Adding the two equations of (18) together, and combining equation(24), system (18) can be rewritten as

$$\begin{aligned} \dot{V} &= -\varepsilon\left[\beta(\ddot{\xi}_1 + \ddot{\xi}_2) + [f(\dot{x}_1)\dot{x}_1 - f(\dot{x}_2)\dot{x}_2]\right], \\ \ddot{\eta} + \omega^2\eta &= \varepsilon\left[-\beta(\ddot{\xi}_2 - \ddot{\xi}_1) - [f(\dot{x}_2)\dot{x}_2 - f(\dot{x}_1)\dot{x}_1] + \sigma\omega_0^2\eta\right]. \end{aligned} \tag{26}$$

From relations (25), we have

$$\eta(t) = x_2(t) - x_1(t) = 2a\cos\varphi, \quad \dot{\eta}(t) = \dot{x}_2 - \dot{x}_1 = -2a\omega\sin\varphi. \tag{27}$$

Differentiating the first equation of (27) with respect to t, one can eliminate the second equation of (27) and obtain

$$\dot{a}\cos\varphi + a\dot{\theta}\sin\varphi = 0. \tag{28}$$

Differentiating the second equation of (27) and substituting the results into equation (24) yields

$$-\dot{a}\sin\varphi + a\dot{\theta}\cos\varphi = \frac{\varepsilon}{2\omega}\left[-\beta(\ddot{\xi}_2 - \ddot{\xi}_1) - [f(\dot{x}_2)\dot{x}_2 - f(\dot{x}_1)\dot{x}_1] + \sigma\omega_0^2\eta\right]. \tag{29}$$

Equations (28), (29), as well as the first equation of (26) give the standard form of equations (18) governing V, a and θ:

$$\begin{aligned}
\dot{V} &= -\frac{\varepsilon}{2}\left[\beta(\ddot{\xi}_1+\ddot{\xi}_2)+(f(V-a\omega\sin\varphi)(V-a\omega\sin\varphi)+f(V+a\omega\sin\varphi)(V+a\omega\sin\varphi))\right],\\
\dot{a} &= -\frac{\varepsilon}{2\omega}\left[-\beta(\ddot{\xi}_2-\ddot{\xi}_1)-(f(V-a\omega\sin\varphi)(V-a\omega\sin\varphi)\right.\\
&\quad \left.-f(V+a\omega\sin\varphi)(V+a\omega\sin\varphi))+2a\sigma\omega_0^2\cos\varphi\right]\sin\varphi,\\
\dot{\theta} &= \frac{\varepsilon}{2a\omega}\left[-\beta(\ddot{\xi}_2-\ddot{\xi}_1)-(f(V-a\omega\sin\varphi)(V-a\omega\sin\varphi)\right.\\
&\quad \left.-f(V+a\omega\sin\varphi)(V+a\omega\sin\varphi))+2a\sigma\omega_0^2\cos\varphi\right]\cos\varphi.
\end{aligned} \tag{30}$$

In equations (30), the quantity φ is the fast variable, while V, a and θ are slow variables. By averaging the right-hand sides of these equations with respect to the fast variable φ from 0 to 2π, we obtain the averaged system in the form

$$\begin{aligned}
\dot{V} &= -\frac{\varepsilon}{2\pi\cdot 2}P,\\
\dot{a} &= -\frac{\varepsilon}{2\pi\cdot 2\omega}Q,\\
\dot{\theta} &= \frac{\varepsilon}{2\pi\cdot 2a\omega}R,
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
P &= \int_0^{2\pi}\left[\beta(\ddot{\xi}_1+\ddot{\xi}_2)+(f(V-a\omega\sin\varphi)(V-a\omega\sin\varphi)\right.\\
&\quad \left.+f(V+a\omega\sin\varphi)(V+a\omega\sin\varphi))\right]\mathrm{d}\varphi,\\
Q &= \int_0^{2\pi}\left[-\beta(\ddot{\xi}_2-\ddot{\xi}_1)-(f(V-a\omega\sin\varphi)(V-a\omega\sin\varphi)\right.\\
&\quad \left.-f(V+a\omega\sin\varphi)(V+a\omega\sin\varphi))+2a\sigma\omega_0^2\cos\varphi\right]\sin\varphi\,\mathrm{d}\varphi,\\
R &= \int_0^{2\pi}\left[-\beta(\ddot{\xi}_2-\ddot{\xi}_1)-(f(V-a\omega\sin\varphi)(V-a\omega\sin\varphi)\right.\\
&\quad \left.-f(V+a\omega\sin\varphi)(V+a\omega\sin\varphi))+2a\sigma\omega_0^2\cos\varphi\right]\cos\varphi\,\mathrm{d}\varphi.
\end{aligned} \tag{32}$$

In the following, this average system will be mainly studied.

On account of the fact that both the excitation and resistance contain the non-smooth characters, the integration will be performed piecewise. The integration range is divided at the instants when the magnitudes of relative accelerations of the internal masses change, as well as the instants when the signs of the absolute velocities of the main bodies change. The results of integration (32) for the three different cases listed in section 2.3 are expressed as P_m, Q_m and R_m $(m = 1, 2, 3)$, respectively:

$$\begin{aligned}
P_m &= \begin{cases}
4a\,\omega\,u\,\pi, & \text{if } u < -1,\\
2a\omega\Big[(1+k)\pi u \\
\qquad -2(1-k)\left(\sqrt{1-u^2}+u\arcsin u\right)\Big], & \text{if } |u|\le 1,\\
4k\,a\,\omega\,u\,\pi, & \text{if } u > 1.
\end{cases}\\
Q_m &= \begin{cases}
A_m + 2a\,\omega\,\pi, & \text{if } u < -1,\\
A_m - a\,\omega\Big[-(1+k)\pi \\
\qquad +2(1-k)\left(u\sqrt{1-u^2}+\arcsin u\right)\Big], & \text{if } |u|\le 1,\\
A_m + 2k\,a\,\omega\,\pi, & \text{if } u > 1.
\end{cases}\\
R_m &= B_m + 2a\,\sigma\,\omega_0^2\,\pi,
\end{aligned} \tag{33}$$

where

$$u = \frac{V}{a\omega}. \tag{34}$$

For the three different motion cases, A_m and B_m $(m = 1, 2, 3)$ are functions with parameters ω_j $(j = 1, 2, 3)$, τ_0 and θ:

$$\begin{aligned} A_m &= \int_0^{2\pi} -\beta(\ddot{\xi}_2 - \ddot{\xi}_1) \sin\varphi \mathrm{d}\varphi, \\ B_m &= \int_0^{2\pi} -\beta(\ddot{\xi}_2 - \ddot{\xi}_1) \cos\varphi \mathrm{d}\varphi. \end{aligned} \tag{35}$$

One should be alerted again that these two integrations are performed piecewise. The integration variable φ for $\ddot{\xi}_i(t)$ is obtained through a transformation $\varphi = \omega t - \theta$. We take A_1 as an example to illustrate the procedures of integrating,

$$\begin{aligned} A_1 &= \int_0^{2\pi} -\beta(\ddot{\xi}_2 - \ddot{\xi}_1) \sin\varphi \mathrm{d}\varphi \\ &= -\beta \left[\int_0^{2\pi} \ddot{\xi}_2 \sin\varphi \mathrm{d}\varphi - \int_0^{2\pi} \ddot{\xi}_1 \sin\varphi \mathrm{d}\varphi \right] \\ &= -\beta \left[\int_{-\theta}^{\omega\tau_1-\theta} w_1 \sin\varphi \mathrm{d}\varphi + \int_{\omega\tau_1-\theta}^{\omega(\tau_1+\tau_2)-\theta} (-w_2) \sin\varphi \mathrm{d}\varphi + \int_{\omega(\tau_1+\tau_2)-\theta}^{\omega T-\theta} w_3 \sin\varphi \mathrm{d}\varphi \right. \\ &\quad - \int_{-\theta}^{\omega\tau_0-\theta} w_3 \sin\varphi \mathrm{d}\varphi - \int_{\omega\tau_0-\theta}^{\omega(\tau_0+\tau_1)-\theta} w_1 \sin\varphi \mathrm{d}\varphi - \int_{\omega(\tau_0+\tau_1)-\theta}^{\omega(\tau_0+\tau_1+\tau_2)-\theta} (-w_2) \sin\varphi \mathrm{d}\varphi \\ &\quad \left. - \int_{\omega(\tau_0+\tau_1+\tau_2)-\theta}^{\omega T-\theta} w_3 \sin\varphi \mathrm{d}\varphi \right]. \end{aligned}$$

3.2 The steady-state motion

Since any non-stationary process approaches a steady state due to the resistance force, in what follows, the steady-state motion of the vibration-driven system as a whole will be of our interest. The variable V, i.e., the velocity of the steady-state motion of the system's center of mass, will be used to characterize the velocity of the system as a whole. When the motion of the system enter the steady state, the velocity of the steady-state motion is constant "on average" with periodic oscillation imposed on. We define the system carry out a steady-state motion if $V = \text{const}$, $a = \text{const}$, and $\theta = \text{const}$, i.e., $\dot{V} = 0$, $\dot{a} = 0$, and $\dot{\theta} = 0$. Thus, the analysis of the steady-state motion is reduced to studying the solution of the algebraic equations:

$$P_m = 0, \quad Q_m = 0, \quad R_m = 0, \ m = 1, 2, 3. \tag{36}$$

From the results of integration (33) and the definition of steady-state motion (36), we know that the system can conduct a steady-state motion only if $|u| \le 1$, then it follows that

$$(1 + k)\pi\, u - 2(1 - k) \left(\sqrt{1 - u^2} + u \arcsin u \right) = 0, \tag{37}$$

$$A_m - a\omega \left[-(1 + k)\pi + 2(1 - k) \left(u\sqrt{1 - u^2} + \arcsin u \right) \right] = 0, \tag{38}$$

$$B_m + 2a\sigma\, \omega_0^2 \pi = 0, \ m = 1, 2, 3. \tag{39}$$

According to equation (34), only the steady-state values of u and a are needed to calculate V. From equation (37), the steady-state value of u can be obtained once the parameters k is fixed. Letting $H(k,\ u) = (1+k)\pi\ u - 2(1-k)\left(\sqrt{1-u^2} + u \arcsin u\right)$, and calculating the derivative H'_u of the function $H(k,\ u)$ with respect to u for $|u| < 1$, we obtain

$$H'_u(k,\ u) = (1+k)\pi - 2(1-k)\arcsin u > 0. \tag{40}$$

Since $H(k,\ 0) = -2(1-k) < 0$ and $H(k,\ 1) = 2k\pi > 0$, the equation (37) must have a unique root $u = u_s$ for a fixed k according to the monotonicity obtained in (40), and the root locates in the interval (0, 1). To investigate the dependence of the steady-state value u_s on k, we differentiate the function $H(k,\ u)$ with respect to k, with $|u| < 1$

$$H'_k(k,\ u) = 2\sqrt{1-u^2} + u\,(\pi + 2\arcsin u) > 0. \tag{41}$$

On the basis of the relation

$$\frac{\mathrm{d}u_s}{\mathrm{d}k} = -\frac{H'_k}{H'_u}, \tag{42}$$

and from equations (40) and (41), it can be concluded that $\frac{\mathrm{d}u_s}{\mathrm{d}k} < 0$. Thus, the steady-state value u_s decreases as k increases.

By eliminating the variable a from (38) and (39), we obtain a new equations for θ

$$A_m + \frac{B_m\,\omega}{2\sigma\,\omega_0^2\,\pi}\left[-(1+k)\pi + 2(1-k)\left(u\sqrt{1-u^2} + \arcsin u\right)\right] = 0,\ m = 1, 2, 3. \tag{43}$$

Since equation (43) contains three unknown parameters, i.e., $\theta,\ \tau_0$ and k (u is a one-variable function of k), the parameter θ can be expressed with the form $\theta = I(\tau_0,\ k)$. When a pair of values of τ_0 and k is given, the steady-state value of θ, i.e., θ_s, can be obtained. Substituting the value of θ_s back to equation (39), the steady-state value of a can be expressed through

$$a_s = -\frac{B_m}{2\sigma\,\omega_0^2\,\pi}. \tag{44}$$

Finally, using equations (34), (37) and (44), the average steady-state velocity of the system can be given by

$$V_s = u_s a_s \omega. \tag{45}$$

Notice that the change on the value of σ leads to changes on the values of the three internal accelerations, hence, the changes on the integrations A_m and B_m ($m = 1, 2, 3$) in equation (43). Then, if the sign of σ is changed, by solving equations (43) and (44), different steady-state values of θ and a will be obtained. It is possible that the new steady-state value θ_s corresponds to a negative steady-state value a_s. However, the negative values of a_s do not mean that the steady-state velocity will change its direction.

In fact, we notice that if $(a_s,\ \theta_s)$ is the solution of equation (38) and (39) (or equation (43) and (44)) for the set of parameters $(\tau_0,\ \Delta,\ k)$, then $(-a_s,\ \theta \pm \pi)$ is also the solution. Since the relation $a\cos(\omega t + \theta) = -a\cos(\omega t + \theta \pm \pi)$ is satisfied, the two solutions are equivalent to each other. Therefore, when σ changes its sign, even we obtain a negative value of a_s, a positive value $(-a_s)$ can always be found if we change θ_s into $\theta_s + \pi$ or $\theta_s - \pi$, which does not affect the steady-state solution of the system. Consequently, being different with [19], the steady-state velocity will not change its direction if we change the sign of the off-resonance detuning σ

We then investigate the derivative $\frac{\mathrm{d}a}{\mathrm{d}\tau_0}$:

$$\frac{\mathrm{d}a}{\mathrm{d}\tau_0} = -\frac{1}{2\sigma\,\omega_0^2\,\pi}\frac{\mathrm{d}B_m}{\mathrm{d}\tau_0} = -\frac{1}{2\sigma\,\omega_0^2\,\pi}\left(\frac{\partial B_m}{\partial\theta}\frac{\mathrm{d}\theta}{\mathrm{d}\tau_0} + \frac{\partial B_m}{\partial\tau_0}\right), \tag{46}$$

in which, $\frac{\partial B_m}{\partial\theta} = -A_m$. The steady-state value a_s will reach the maximum only if the optimal value of τ_0 satisfies the relation $\frac{\mathrm{d}a}{\mathrm{d}\tau_0} = 0$. Besides, it is worthy noting that A_m, $\frac{\mathrm{d}\theta}{\mathrm{d}\tau_0}$ and $\frac{\partial B_m}{\partial\tau_0}$ make no reference to the parameter k. Thus, the equation $\frac{\mathrm{d}a}{\mathrm{d}\tau_0} = 0$ is k independent, and the optimal value of τ_0 is not related to the parameter k.

Based on the above analysis, three important conclusions on the dynamics of the system can be drawn for the case of small resistance and weak excitation.

1. The steady-state value u_s has inverse relation to the parameter k. For a prescribed environment where the system works, i.e., a fixed value of k, the steady-state value u_s can be determined through equation (37). And from equation (43), θ_s depends on τ_0, hence, in views of equation (44), a_s depends on τ_0. Therefore, the magnitude of the average steady-state velocity V_s can be obtained and depends on the time shift τ_0.

2. The magnitudes of the internal accelerations change with the excitation frequency, i.e., the value of off-resonance detuning σ. However, the qualitative characteristics of the steady-state motion of the system do not depend on the sign of the off-resonance detuning σ. Changing the detuning σ in sign does not lead to the reverse in direction of motion of the system.

3. For a fixed value of k, without changing the amplitude and the frequency of the internal excitation, one is able to control the magnitude of the average steady-state velocity of the system by changing the time shift between the internal excitations. The optimal value of τ_0, which corresponds to the maximal magnitude of the average steady-state velocity, has no relation with the parameter k. Once an optimal value of τ_0 is obtained for a system with arbitrary value of k, this is the optimal value for all $k \in [0, 1]$.

4 Numerical analysis

In this section, the analytical results will be verified by several numerical examples. The optimal situation that corresponds to the maximal average steady-state velocity will be discussed.

4.1 The steady-state motion

The following values of parameters are firstly taken to simulate the system:

$$\begin{aligned}
&M = 1.2\,\mathrm{kg}, && m = 0.72\,\mathrm{kg}, && c = 100\,\mathrm{N/m},\\
&L = 0.15\,\mathrm{m}, && f_+ = 0.2\,\mathrm{s}^{-1}, && f_- = 0.4\,\mathrm{s}^{-1},\\
&w_1 = 6\,\mathrm{m/s}^2, && w_2 = 5\,\mathrm{m/s}^2, && w_3 = 4\,\mathrm{m/s}^2,\\
&\tau_1 = 0.15\,\mathrm{s}, && \tau_2 = 0.34\,\mathrm{s}, && \tau_3 = 0.20\,\mathrm{s},\\
&W = 10\,\mathrm{m/s}^2.
\end{aligned} \tag{47}$$

Such parameters locate within a feasible region, and hence, are meaningful in guiding design for practical mechanical systems. With these parameters, the conditions (9)∼(11) are all satisfied for the internal motions of masses 1 and 2. The natural frequency of the system is

$$\omega_0 = \sqrt{\frac{2c}{M+m}} = 10.21\,\mathrm{s}^{-1}, \tag{48}$$

and in this example, the excitation frequency is

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{\tau_1 + \tau_2 + \tau_3} = 8.98\,\mathrm{s}^{-1}. \tag{49}$$

Then, the length scale S and all the dimensionless parameters ε, k, and β are given

$$S = 0.19\,\mathrm{m}, \quad \varepsilon = 0.055 \quad k = 0.5, \quad \beta = 6.77. \tag{50}$$

The parameters ε and β satisfy the first two constraints of (20), and the last one constraint will be verified numerically later. The dimensionless natural frequency and excitation frequency are $\sqrt{2}$ and 1.24, respectively. According to equation (23), the off-resonance detuning is

$$\sigma = \frac{1}{\varepsilon}\left(\frac{\omega^2}{\omega_0^2} - 1\right) = -4.06. \tag{51}$$

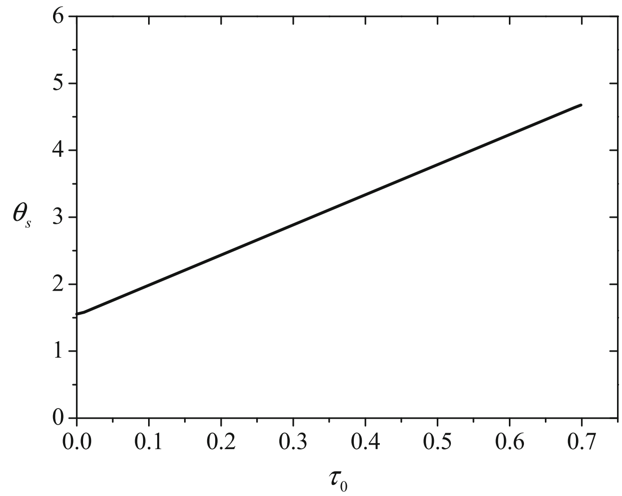

Fig. 4 The plot of θ_s versus τ_0 with parameters (47)

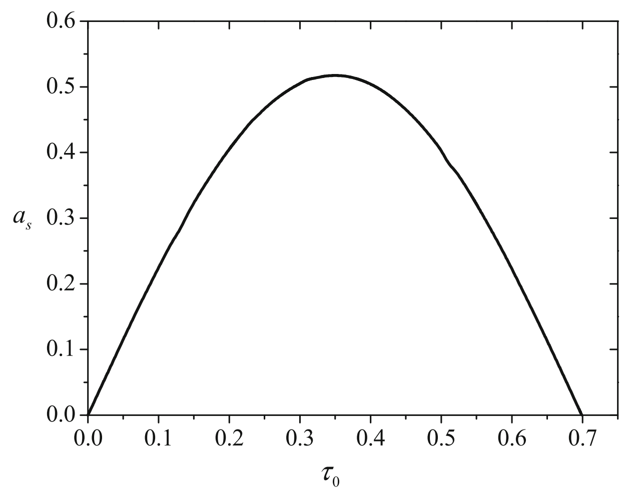

Fig. 5 The plot of a_s versus τ_0 with parameters (47)

From equation (37), the steady-state value u_s is not related to the value of τ_0 and can be obtained when k is fixed. Then, the steady-state value θ_s can be determined through equation (43). The dependence $\theta_s = \theta_s(\tau_0)$ is presented in Fig. 4, which shows that θ_s is a monotonic increasing function with respect to τ_0 from 0 to $T = 0.69$ s. After that, substituting the steady-state values u_s, θ_s, as well as the value of τ_0 into equation (44), the steady-state value of the amplitude a can be obtained. In Fig. 5, the relation between τ_0 and a_s is presented. By contrast with $\theta_s(\tau_0)$, $a_s(\tau_0)$ is not monotonic on τ_0, instead, a maximal value of a_s exists inside the range of τ_0 from 0 to T, which corresponds to the optimal situation that will be discussed later.

With the values of u_s and a_s determined, the average steady-state velocity V_s can be obtained via equation (46). Taking $k = 0.5$ as an example, the steady-state value u_s equals to 0.22 through equation(37). For $\tau_0 \in [0, T]$, the approximate average steady-state velocities are obtained and illustrated in Fig. 6 with solid line. Using 4th-order Runge–Kutta method, we simulate the exact equations of motion (2) with zero initial displacements and zero initial velocities of bodies 1 and 2. The numerical average steady-state velocities are shown in Fig. 6 with symbols. Figure 6 reflects that the analytical results based on method of averaging are in good agreement with those obtained from numerical simulation for almost all the values of τ_0, except for the case that the time shift between the excitations in different modules is close to zero.

Specially, three values of τ_0, i.e., $\tau_0 = 0.1$ s, $\tau_0 = 0.3$ s and $\tau_0 = 0.55$ s, corresponding to the three different cases described in section 2.3, are taken out as examples. For $\tau_0 = 0.1$ s, which belongs to case 1, the dimensionless results give

$$a_s \approx 0.22, \quad V_s \approx 0.06082.$$

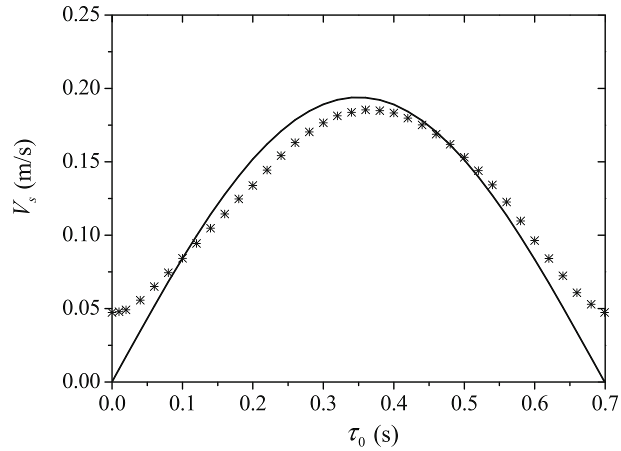

Fig. 6 The average steady-state velocity V_s resulting from numerical simulation (*symbols*) of the exact equations of motion and the approximate solution (*solid curve*) with parameters (47)

In the dimensional units,

$$V_s \approx 0.08427\,\text{m/s}. \tag{52}$$

For $\tau_0 = 0.3$ s, which belongs to case 2, the calculation results in the dimensionless values

$$a_s \approx 0.50, \quad V_s \approx 0.1365,$$

and the dimensional ones

$$V_s \approx 0.1892\,\text{m/s}. \tag{53}$$

For $\tau_0 = 0.55$ s, which belongs to case 3, the dimensionless results yield

$$a_s \approx 0.32, \quad V_s \approx 0.08692,$$

which in the dimensional units corresponds to

$$V_s \approx 0.1204\,\text{m/s}. \tag{54}$$

The average steady-state velocity obtained by numerical means are 0.0843 m/s, 0.1765 m/s and 0.1288 m/s for $\tau_0 = 0.1$ s, $\tau_0 = 0.3$ s and $\tau_0 = 0.55$ s, respectively, which are all in good agreement with the values of (52), (53) and (54).

Figure 7 shows the time histories of the dimensional variable V for $\tau_0 = 0.1$ s, $\tau_0 = 0.3$ s, and $\tau_0 = 0.55$ s. It is apparent from these plots that after a transient, the motion enter into a steady-state mode with a constant average steady-state value of V. Periodic oscillations are imposed on the constant average steady-state velocity.

Moreover, so as to verify the last constraint in (20), the dimensionless amplitudes of the relative motions are given: 0.52 for $\tau_0 = 0.1$ s, 1.04 for $\tau_0 = 0.3$ s, and 0.63 for $\tau_0 = 0.55$ s. These results meet the constraint $|x_1 - x_2| \sim O(10^0)$ in an acceptable accuracy. However, as is shown in Fig. 6, when the value of τ_0 is taken near zero or T, errors between the analytical results and numerical results are big. This is because the values of $|x_1 - x_2|$ are fairly small around $\tau_0 = 0$ and $\tau_0 = T$, which contradict the necessary constraint $|x_1 - x_2| \sim O(10^0)$ for the method of averaging. What need points out is, the big errors around the value of $\tau_0 = 0$ s and $\tau_0 = T$ do not mean that the original system does not have a solution with steady-state motion, but that such solution cannot be acquired by first-order approximation of method of averaging.

From Figs. 6 and 7, we notice that with the change of the value of parameter τ_0, the steady-state velocity changes obviously. For $\tau_0 = 0.1$ s, the system oscillates with a relatively small value of average steady-state velocity, while with a large amplitude of oscillation. When adding the value of τ_0, the average steady-state velocity increases, and the velocity amplitude decrease simultaneously. However, if the value of τ_0 keeps increasing, the average steady-state velocity does not monotonously increase, instead, it drops down, and the velocity amplitude increases meanwhile. Thus, a critical value of parameter τ_0 must exist between the region $[0,\ T]$. At this critical point, the motion has the maximal magnitude of average steady-state velocity as well as the minimal value of the velocity amplitude, which stands for a motion with little retroversion and higher efficiency. Such motion is the optimal one and is what we are interested. Next, the critical value of the parameter τ_0 as well as the optimal situation will be studied.

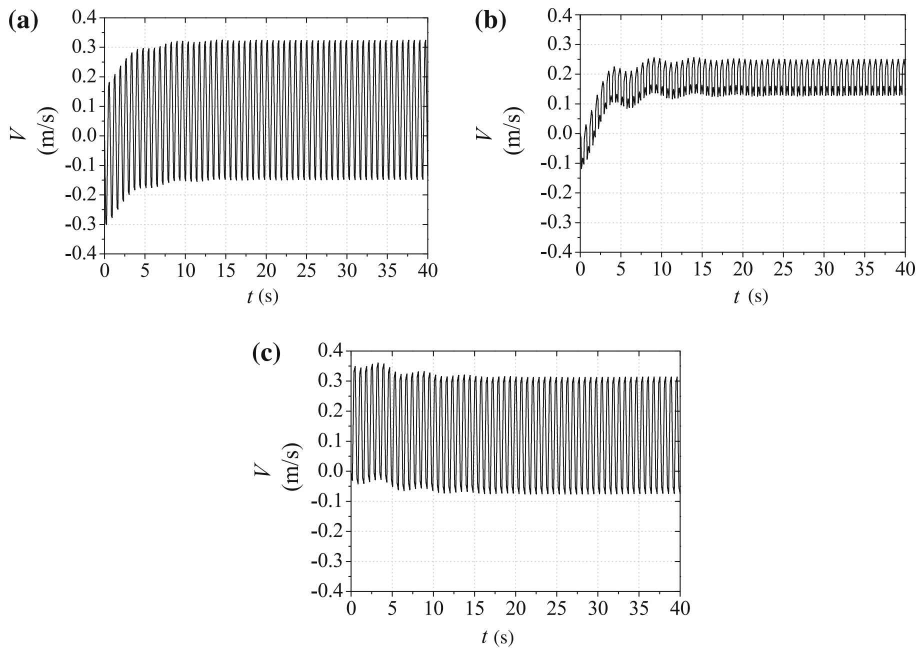

Fig. 7 The time history of the velocity V: **a** $\tau_0 = 0.1$ s; **b** $\tau_0 = 0.3$ s; **c** $\tau_0 = 0.55$ s

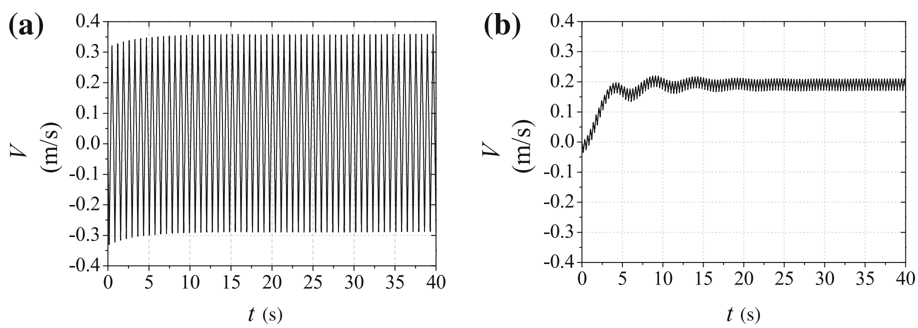

Fig. 8 The time history of the velocity V: **a** zero time shift $\tau_0 = 0$ s; **b** optimal situation $\tau_0 = 0.35$ s

4.2 Optimal situation

Observing Figs. 5 and 6 again, one know that at the extreme point, a maximal value of a_s is reached, which corresponds to the maximal average steady-state velocity. By numerical means, the value of τ_0 at the extreme point is 0.35 s, which is the critical value we need. In accordance with the theoretical analysis in section 3.2, it is this optimal value $(\tau_0)_{\mathrm{opt}} = 0.35$ s that keeps valid for all the $k \in [0, 1]$. At $\tau_0 = 0.35$ s, the optimal situation is achieved, the steady-state value a_s reaches to its maximum, and so is the average steady-state velocity. Through method of averaging, the dimensionless maximal average steady-state velocity is 0.1400, and the dimensional one is 0.1940 m/s for the averaged system. Via numerical simulation, the time histories of the velocity V for $\tau_0 = 0$ s and the optimal value $\tau_0 = 0.35$ s are shown in Fig. 8a, b, respectively. It reads that for $\tau_0 = 0.35$ s, the dimensional maximal average steady-state velocity is 0.1848 m/s, which coincides with the analytical result in high accuracy. By contrast, the dimensional steady-state velocity for zero time shift is only 0.04728 m/s. Making a comparisons between these two situations, one may find that the average steady-state velocity increases significantly due to our control over the time shift between the two internal excitations. On the other hand, the velocity amplitude has a remarkable decline to 0.042 m/s for the optimal situation with $\tau_0 = 0.35$ s from 0.75 m/s for the zero time shift situation.

Figure 9 shows the time histories of the velocity of each body in the cases of $\tau_0 = 0$ s and $\tau_0 = 0.35$ s. It is clearly that when there is no time shift, modules 1 and 2 move synchronously, with a low velocity. No elastic force is generated on the elastic element, that is, the motions of modules 1 and 2 do not influence each other at

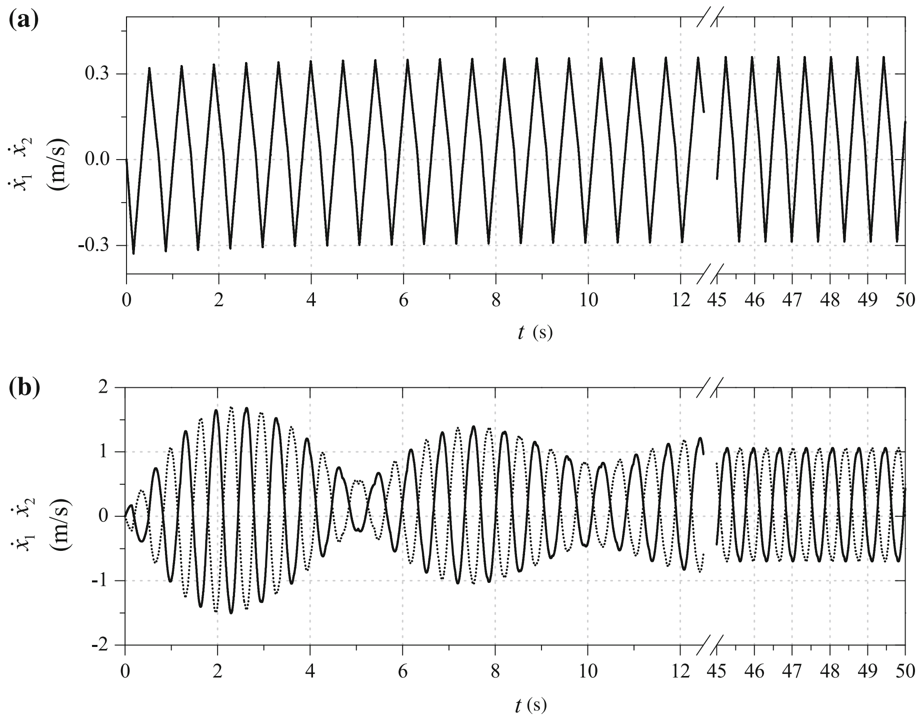

Fig. 9 The time histories of the velocity $\dot{x}_1$ (*solid line*) and $\dot{x}_2$ (*dot line*): **a** zero time shift $\tau_0 = 0$ s; **b** optimal situation $\tau_0 = 0.35$ s

all. While when the optimal value $\tau_0 = 0.35$ s is applied, modules 1 and 2 move with a half-cycle difference and with a higher velocities. At the time when one of the body moves backward with the maximal backward velocity, the other body just moves forward with the maximal forward velocity. Thus, an elastic force is generated owing to the deformation of the elastic element, and it seems that one body "pushes forward" the other.

5 Conclusion

System with movable internal masses is a kind of vibration-driven system, which has absorbed lots of studies around the world. Such system is simple in design and is promisingly to be used in micro-robots.

In this paper, the motion of a two-module vibration-driven system is studied. The system consists of two rigid bodies connected by an elastic element and is excited by internal acceleration-controlled masses inside the respective bodies. The resistance force between the system and the environment is assumed to be anisotropic linear. In the case of small coefficients of resistance and weak excitations, method of averaging is adopted to deal with the non-smooth factors in both the internal excitation and the external resistance. A series of algebraic equations is obtained to determine an approximate value of the average steady-state velocity of the system as a whole. It is shown that the magnitudes of average steady-state velocity of the entire system can be controlled by changing the time shift between the motions of the two internal masses. With such control, motion without retroversion can be achieved and the efficiency of the system improves significantly. In order to achieve a motion with maximal average steady-state velocity, the optimal value of the time shift is determined, and the optimal situation is studied. For a system with specific design parameters, numerical simulations validate our analytical results based on method of averaging.

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