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Mergen H. Ghayesh · Siavash Kazemirad · Mohammad A. Darabi · Pamela Woo

Thermo-mechanical nonlinear vibration analysis of a spring-mass-beam system

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Abstract Thermo-mechanical vibrations of a simply supported spring-mass-beam system are investigated analytically in this paper. Taking into account the thermal effects, the nonlinear equations of motion and internal/external boundary conditions are derived through Hamilton's principle and constitutive relations. Under quasi-static assumptions, the equations governing the longitudinal motion are transformed into functions of transverse displacements, which results in three integro-partial differential equations with coupling terms. These are solved using the direct multiple-scale method, leading to closed-form solutions for the mode functions, nonlinear natural frequencies and frequency–response curves of the system. The influence of system parameters on the linear and nonlinear natural frequencies, mode functions, and frequency–response curves is studied through numerical parametric analysis. It is shown that the vibration characteristics depend on the mid-plane stretching, intra-span spring, point mass, and temperature change.

Keywords The method of multiple timescales · Nonlinear vibration · Spring-mass-beam system

1 Introduction

Various engineering devices and machine components can be modeled as a flexible beam with nonlinear characteristics. In some of these applications, the beam is subjected to several *concentrated elements* such as point masses, non-ideal additional supports, springs, spring-mass systems, etc., at several locations along the beam length [1–8]; see references [9–11] for a review. On the other hand, in many applications, these systems are often subjected to vibration under thermal (temperature gradient induced) [12–18] and dynamical loadings. Therefore, these structures require a more complex analysis than a simple continuum, due to the simultaneous presence of additional concentrated elements and dynamic and thermal loads.

The vibrations of beams have been investigated for many years and were reviewed, for example, by Nayfeh and Mook [9]. In a fundamental work by Eisley [19], the nonlinear vibrations of beams and rectangular plates were investigated using a single-mode approximation. An approximate solution for the large amplitude vibration response of beams was obtained in [20] using the Ritz–Galerkin method. Further work on the subject involved, for example, the inclusion of rotary inertia and transverse shear in modeling of solid and sandwich beams [21], and large deformation nonlinearity [22], where perturbation and shooting techniques were employed to solve the equations of motion. These investigations were then extended for beams with concentrated elements in [23–27]. One of the first complete studies on this subject was by Dowell [23] for a

M. H. Ghayesh (✉) · S. Kazemirad · P. Woo
Department of Mechanical Engineering, McGill University, Montreal, QC H3A 2K6, Canada
E-mail: mergen.hajghayesh@mail.mcgill.ca
Tel.: +1-514-3986290

M. A. Darabi
Department of Mechanical Engineering, University of Guilan, P.O. Box 1841, Rasht, Iran

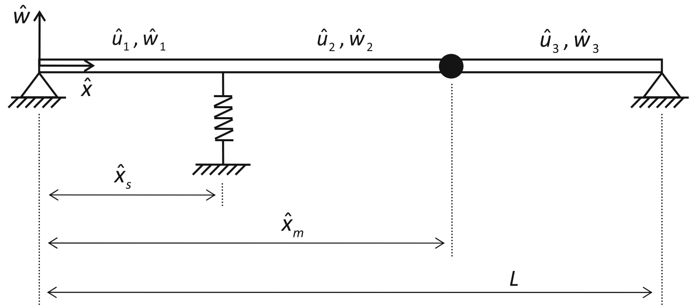

Fig. 1 A spring-mass-beam system under thermal loading

nonlinear beam-spring-mass system. Birman [24], on the other hand, investigated the dynamic response of the system resting on a nonlinear foundation. These studies were extended in [25,26] for a system with flexible boundary conditions and a geometric nonlinearity.

As is well known, temperature fields develop thermal stress due to thermal expansion or contraction which influences the dynamic behavior of mechanical systems. The literature on thermal vibrations of mechanical structures is quite abundant; here, only a few papers are reviewed. Treyssede [13] studied the nonlinear vibrations of thermally prestressed Euler–Bernoulli beams, where the effects of imperfections and prebending were investigated. The vibration and thermal buckling of sandwich beams consisting of a viscoelastic core and composite facing were studied in [14]. The finite-element method was used in [15] to treat the active control of thermo-mechanical vibrations of a composite beam. Additional contributions were made in [16] to investigate the free and forced thermal vibrations of a FGM Timoshenko beam, in [17] to deal with the simultaneous presence of magnetic fields and thermal loads, and in [18] to study the thermal effects on the large amplitude vibration of Timoshenko beams.

The present study focuses on vibration analysis of a spring-mass-beam system in the simultaneous presence of dynamical and thermal loadings. The beam is considered as a three-part system, and the equations of motion are derived using Hamilton's principle, which renders additionally the appropriate internal/external boundary conditions. The equations of motion form a set of nonlinear partial differential equations with nonlinear, time-dependent, and coupled internal boundary conditions. These are solved analytically via the method of multiple timescales, which provides direct insight into the basic relationships between the system parameters and the vibration response. The dependencies are shown via a numerical parametric study in the last section.

2 Model development and formulation

Shown in Fig. 1 is a simply supported beam carrying a concentrated mass and subjected to an additional nonlinear spring, which are located at different positions along the length of the beam. This system is additionally subjected to a uniform temperature rise.

The system consists of a beam of length L, constant density ρ, cross-sectional area A, and Young's modulus E, subjected to a uniform temperature rise of ΔT. The spring, with linear and nonlinear stiffness coefficients of $\hat{k}_1$ and $\hat{k}_2$, and the point mass $\hat{M}$ are attached to the beam at distances $\hat{x}_s$ and $\hat{x}_m$ from the left-end of the beam, respectively. The beam is considered as a three-part system, i.e., $0 < \hat{x} < \hat{x}_s$, $\hat{x}_s < \hat{x} < \hat{x}_m$ and $\hat{x}_m < \hat{x} < L$. Also, $\hat{w}_1$, $\hat{w}_2$, and $\hat{w}_3$ represent the respective transverse displacement for these three spans; $\hat{u}_1$, $\hat{u}_2$, and $\hat{u}_3$ are the corresponding longitudinal displacements. Properties before the spring $(0 < \hat{x} \leq \hat{x}_s)$ are noted with subscript 1, between the spring and point mass $(\hat{x}_s < \hat{x} \leq \hat{x}_m)$ with 2, and after the point mass $(\hat{x}_m < \hat{x} < L)$ with 3.

The assumptions in deriving the equations of motion are as follows: (1) Euler–Bernoulli beam theory is considered; (2) only the planar displacements are considered; (3) the nonlinearity type is geometric, due to the stretching of the mid-plane of the beam; (4) the maximum order of nonlinearity is cubic; (5) the mass is assumed to be a point mass; (6) the spring force acting on the beam is nonlinear due to the geometric nonlinearity; (7) the spring is attached to the centerline of the beam.

The strain (in Lagrangian description) in the beam for each span is given by

$$\varepsilon_j(\hat{x},\hat{t}) = \frac{\partial \hat{u}_j}{\partial \hat{x}} + \frac{1}{2}\left(\frac{\partial \hat{w}_j}{\partial \hat{x}}\right)^2, \qquad j = 1, 2, 3. \tag{1}$$

Considering this, the expression for the variation in potential strain energy of the beam, together with the spring, takes the form

$$\begin{aligned}
\int_{\hat{t}_1}^{\hat{t}_2} \delta\pi \, \mathrm{d}\hat{t} = EI \int_{\hat{t}_1}^{\hat{t}_2} \Bigg\{ & \frac{\partial^2 \hat{w}_1}{\partial \hat{x}^2}\delta\left(\frac{\partial \hat{w}_1}{\partial \hat{x}}\right)\bigg|_0^{\hat{x}_s} - \frac{\partial^3 \hat{w}_1}{\partial \hat{x}^3}\delta\hat{w}_1\bigg|_0^{\hat{x}_s} + \int_0^{\hat{x}_s} \frac{\partial^4 \hat{w}_1}{\partial \hat{x}^4}\delta\hat{w}_1 \, \mathrm{d}\hat{x} \\
& + \frac{\partial^2 \hat{w}_2}{\partial \hat{x}^2}\delta\left(\frac{\partial \hat{w}_2}{\partial \hat{x}}\right)\bigg|_{\hat{x}_s}^{\hat{x}_m} - \frac{\partial^3 \hat{w}_2}{\partial \hat{x}^3}\delta\hat{w}_2\bigg|_{\hat{x}_s}^{\hat{x}_m} + \int_{\hat{x}_s}^{\hat{x}_m} \frac{\partial^4 \hat{w}_2}{\partial \hat{x}^4}\delta\hat{w}_2 \, \mathrm{d}\hat{x} \\
& + \frac{\partial^2 \hat{w}_3}{\partial \hat{x}^2}\delta\left(\frac{\partial \hat{w}_3}{\partial \hat{x}}\right)\bigg|_{\hat{x}_m}^{L} - \frac{\partial^3 \hat{w}_3}{\partial \hat{x}^3}\delta\hat{w}_3\bigg|_{\hat{x}_m}^{L} + \int_{\hat{x}_m}^{L} \frac{\partial^4 \hat{w}_3}{\partial \hat{x}^4}\delta\hat{w}_3 \, \mathrm{d}\hat{x} \Bigg\} \mathrm{d}\hat{t} \\
& + A\int_{\hat{t}_1}^{\hat{t}_2} \Bigg\{ \left[E\left(\frac{\partial \hat{u}_1}{\partial \hat{x}} + \frac{1}{2}\left(\frac{\partial \hat{w}_1}{\partial \hat{x}}\right)^2\right) - \gamma\right]\delta\hat{u}_1\bigg|_0^{\hat{x}_s} + \frac{\partial \hat{w}_1}{\partial \hat{x}}\left[E\left(\frac{\partial \hat{u}_1}{\partial \hat{x}} + \frac{1}{2}\left(\frac{\partial \hat{w}_1}{\partial \hat{x}}\right)^2\right) - \gamma\right]\delta\hat{w}_1\bigg|_0^{\hat{x}_s} \\
& + \left[E\left(\frac{\partial \hat{u}_2}{\partial \hat{x}} + \frac{1}{2}\left(\frac{\partial \hat{w}_2}{\partial \hat{x}}\right)^2\right) - \gamma\right]\delta\hat{u}_2\bigg|_{\hat{x}_s}^{\hat{x}_m} + \frac{\partial \hat{w}_2}{\partial \hat{x}}\left[E\left(\frac{\partial \hat{u}_2}{\partial \hat{x}} + \frac{1}{2}\left(\frac{\partial \hat{w}_2}{\partial \hat{x}}\right)^2\right) - \gamma\right]\delta\hat{w}_2\bigg|_{\hat{x}_s}^{\hat{x}_m} \\
& + \left[E\left(\frac{\partial \hat{u}_3}{\partial \hat{x}} + \frac{1}{2}\left(\frac{\partial \hat{w}_3}{\partial \hat{x}}\right)^2\right) - \gamma\right]\delta\hat{u}_3\bigg|_{\hat{x}_m}^{L} + \frac{\partial \hat{w}_3}{\partial \hat{x}}\left[E\left(\frac{\partial \hat{u}_3}{\partial \hat{x}} + \frac{1}{2}\left(\frac{\partial \hat{w}_3}{\partial \hat{x}}\right)^2\right) - \gamma\right]\delta\hat{w}_3\bigg|_{\hat{x}_m}^{L} \Bigg\} \mathrm{d}\hat{t} \\
& - A\int_{\hat{t}_1}^{\hat{t}_2}\int_0^{\hat{x}_s} \Bigg\{ \frac{\partial}{\partial \hat{x}}\left[E\left(\frac{\partial \hat{u}_1}{\partial \hat{x}} + \frac{1}{2}\left(\frac{\partial \hat{w}_1}{\partial \hat{x}}\right)^2\right) - \gamma\right]\delta\hat{u}_1 \\
& + \frac{\partial}{\partial \hat{x}}\left[\frac{\partial \hat{w}_1}{\partial \hat{x}}\left(E\left(\frac{\partial \hat{u}_1}{\partial \hat{x}} + \frac{1}{2}\left(\frac{\partial \hat{w}_1}{\partial \hat{x}}\right)^2\right) - \gamma\right)\right]\delta\hat{w}_1 \Bigg\} \mathrm{d}\hat{x}\, \mathrm{d}\hat{t} \\
& - A\int_{\hat{t}_1}^{\hat{t}_2}\int_{\hat{x}_s}^{\hat{x}_m} \Bigg\{ \frac{\partial}{\partial \hat{x}}\left[E\left(\frac{\partial \hat{u}_2}{\partial \hat{x}} + \frac{1}{2}\left(\frac{\partial \hat{w}_2}{\partial \hat{x}}\right)^2\right) - \gamma\right]\delta\hat{u}_2 \\
& + \frac{\partial}{\partial \hat{x}}\left[\frac{\partial \hat{w}_2}{\partial \hat{x}}\left(E\left(\frac{\partial \hat{u}_2}{\partial \hat{x}} + \frac{1}{2}\left(\frac{\partial \hat{w}_2}{\partial \hat{x}}\right)^2\right) - \gamma\right)\right]\delta\hat{w}_2 \Bigg\} \mathrm{d}\hat{x}\, \mathrm{d}\hat{t} \\
& - A\int_{\hat{t}_1}^{\hat{t}_2}\int_{\hat{x}_m}^{L} \Bigg\{ \frac{\partial}{\partial \hat{x}}\left[E\left(\frac{\partial \hat{u}_3}{\partial \hat{x}} + \frac{1}{2}\left(\frac{\partial \hat{w}_3}{\partial \hat{x}}\right)^2\right) - \gamma\right]\delta\hat{u}_3 \\
& + \frac{\partial}{\partial \hat{x}}\left[\frac{\partial \hat{w}_3}{\partial \hat{x}}\left(E\left(\frac{\partial \hat{u}_3}{\partial \hat{x}} + \frac{1}{2}\left(\frac{\partial \hat{w}_3}{\partial \hat{x}}\right)^2\right) - \gamma\right)\right]\delta\hat{w}_3 \Bigg\} \mathrm{d}\hat{x}\, \mathrm{d}\hat{t} \\
& + \int_{\hat{t}_1}^{\hat{t}_2} \left(\hat{k}_1\hat{w}_1 \;\; \delta\hat{w}_1\big|_{\hat{x}_s} + \hat{k}_2\hat{w}_1^3 \; \delta\hat{w}_1\big|_{\hat{x}_s}\right)\mathrm{d}\hat{t},
\end{aligned} \tag{2}$$

where $\gamma = E\alpha_T\Delta T$, in which α_T is the thermal expansion coefficient, and π denotes the potential energy.

The variation in the kinetic energy of the beam with the end mass can be expressed as follows

$$\int_{\hat{t}_1}^{\hat{t}_2} \delta T \, \mathrm{d}\hat{t} = -\rho A \int_{\hat{t}_1}^{\hat{t}_2} \left[\int_0^{\hat{x}_s} \frac{\partial^2 \hat{w}_1}{\partial \hat{t}^2} \delta \hat{w}_1 \, \mathrm{d}\hat{x} + \int_{\hat{x}_s}^{\hat{x}_m} \frac{\partial^2 \hat{w}_2}{\partial \hat{t}^2} \delta \hat{w}_2 \, \mathrm{d}\hat{x} + \int_{\hat{x}_m}^{L} \frac{\partial^2 \hat{w}_3}{\partial \hat{t}^2} \delta \hat{w}_3 \, \mathrm{d}\hat{x} \right] \mathrm{d}\hat{t}$$
$$-\hat{M} \int_{\hat{t}_1}^{\hat{t}_2} \frac{\partial^2 \hat{w}_2}{\partial \hat{t}^2} \delta \hat{w}_2 \Big|_{x_m} \mathrm{d}\hat{t} \,, \tag{3}$$

where T is the kinetic energy. In Eq. 3, under the quasi-static assumption, the longitudinal inertial terms were not considered.

The application of Hamilton's principle gives

$$\delta \int_{\hat{t}_1}^{\hat{t}_2} (T - \pi) \, \mathrm{d}\hat{t} = 0, \tag{4a}$$

or

$$\rho A \frac{\partial^2 \hat{w}_j}{\partial \hat{t}^2} + EI \frac{\partial^4 \hat{w}_j}{\partial \hat{x}^4} = A \left\{ \frac{\partial^2 \hat{w}_j}{\partial \hat{x}^2} \left[E \left(\frac{\partial \hat{u}_j}{\partial \hat{x}} + \frac{1}{2} \left(\frac{\partial \hat{w}_j}{\partial \hat{x}} \right)^2 \right) - \gamma \right] \right.$$
$$\left. + \frac{\partial \hat{w}_j}{\partial \hat{x}} \frac{\partial}{\partial \hat{x}} \left[E \left(\frac{\partial \hat{u}_j}{\partial \hat{x}} + \frac{1}{2} \left(\frac{\partial \hat{w}_j}{\partial \hat{x}} \right)^2 \right) - \gamma \right] \right\}, \quad j = 1, 2, 3, \tag{4b}$$

$$\frac{\partial}{\partial \hat{x}} \left[E \left(\frac{\partial \hat{u}_j}{\partial \hat{x}} + \frac{1}{2} \left(\frac{\partial \hat{w}_j}{\partial \hat{x}} \right)^2 \right) - \gamma \right] = 0, \quad j = 1, 2, 3, \tag{4c}$$

$$\text{at } \hat{x} = 0: \quad \hat{w}_1 = \frac{\partial^2 \hat{w}_1}{\partial \hat{x}^2} = 0, \quad \hat{u}_1 = 0, \tag{4d}$$

$$\text{at } \hat{x} = L: \quad \hat{w}_3 = \frac{\partial^2 \hat{w}_3}{\partial \hat{x}^2} = 0, \quad \hat{u}_3 = 0, \tag{4e}$$

$$\text{at } \hat{x} = \hat{x}_s: \ \hat{w}_1 = \hat{w}_2, \ \frac{\partial \hat{w}_1}{\partial \hat{x}} = \frac{\partial \hat{w}_2}{\partial \hat{x}}, \ \frac{\partial^2 \hat{w}_1}{\partial \hat{x}^2} = \frac{\partial^2 \hat{w}_2}{\partial \hat{x}^2},$$
$$EI \left[\frac{\partial^3 \hat{w}_1}{\partial \hat{x}^3} - \frac{\partial^3 \hat{w}_2}{\partial \hat{x}^3} \right] - \left(\hat{k}_1 \hat{w}_1 + \hat{k}_2 \hat{w}_1^3 \right) = 0, \ \frac{\partial \hat{u}_1}{\partial \hat{x}} = \frac{\partial \hat{u}_2}{\partial \hat{x}}, \tag{4f}$$

$$\text{at } \hat{x} = \hat{x}_m: \ \hat{w}_2 = \hat{w}_3, \ \frac{\partial \hat{w}_2}{\partial \hat{x}} = \frac{\partial \hat{w}_3}{\partial \hat{x}}, \ \frac{\partial^2 \hat{w}_2}{\partial \hat{x}^2} = \frac{\partial^2 \hat{w}_3}{\partial \hat{x}^2},$$
$$EI \left[\frac{\partial^3 \hat{w}_2}{\partial \hat{x}^3} - \frac{\partial^3 \hat{w}_3}{\partial \hat{x}^3} \right] - \hat{M} \frac{\partial^2 \hat{w}_2}{\partial \hat{t}^2} = 0, \ \frac{\partial \hat{u}_2}{\partial \hat{x}} = \frac{\partial \hat{u}_3}{\partial \hat{x}}. \tag{4g}$$

Simplifying and adding Eq. 4c while employing the boundary conditions yields

$$\frac{\partial \hat{u}_j}{\partial \hat{x}} + \frac{1}{2} \left(\frac{\partial \hat{w}_j}{\partial \hat{x}} \right)^2 = \frac{1}{2L} \left[\int_0^{\hat{x}_s} \left(\frac{\partial \hat{w}_1}{\partial \hat{x}} \right)^2 \mathrm{d}\hat{x} + \int_{\hat{x}_s}^{\hat{x}_m} \left(\frac{\partial \hat{w}_2}{\partial \hat{x}} \right)^2 \mathrm{d}\hat{x} + \int_{\hat{x}_m}^{L} \left(\frac{\partial \hat{w}_3}{\partial \hat{x}} \right)^2 \mathrm{d}\hat{x} \right], \quad j = 1, 2, 3. \tag{5}$$

Inserting this into Eq. 4b and writing the transverse motion-related terms of Eq. 4d–4g gives

$$\frac{\partial^2 w_j}{\partial t^2} + \frac{\partial^4 w_j}{\partial x^4} + \gamma_T \frac{\partial^2 w_j}{\partial x^2} = \frac{1}{2}\varepsilon \frac{\partial^2 w_j}{\partial x^2}\left[\int_0^{x_s}\left(\frac{\partial w_1}{\partial x}\right)^2 \mathrm{d}x + \int_{x_s}^{x_m}\left(\frac{\partial w_2}{\partial x}\right)^2 \mathrm{d}x + \int_{x_m}^{1}\left(\frac{\partial w_3}{\partial x}\right)^2 \mathrm{d}x\right]$$
$$+\varepsilon F_j \cos(\Omega t), \quad j = 1, 2, 3, \tag{6a}$$

$$\text{at } x = 0\colon\ w_1 = \frac{\partial^2 w_1}{\partial x^2} = 0\,, \tag{6b}$$

$$\text{at } x = 1\colon\ w_3 = \frac{\partial^2 w_3}{\partial x^2} = 0\,, \tag{6c}$$

$$\text{at } x = x_s\colon\ w_1 = w_2,\ \frac{\partial w_1}{\partial x} = \frac{\partial w_2}{\partial x},\ \frac{\partial^2 w_1}{\partial x^2} = \frac{\partial^2 w_2}{\partial x^2},\ \frac{\partial^3 w_1}{\partial x^3} - \frac{\partial^3 w_2}{\partial x^3} - k_1 w_1 - k_2 \varepsilon w_1^3 = 0, \tag{6d}$$

$$\text{at } x = x_m\colon\ w_2 = w_3,\ \frac{\partial w_2}{\partial x} = \frac{\partial w_3}{\partial x},\ \frac{\partial^2 w_2}{\partial x^2} = \frac{\partial^2 w_3}{\partial x^2},\ \frac{\partial^3 w_2}{\partial x^3} - \frac{\partial^3 w_3}{\partial x^3} - M\frac{\partial w_2}{\partial t^2} = 0, \tag{6e}$$

with the following dimensionless parameters, defined as

$$\begin{aligned} &x = \frac{\hat{x}}{L},\ x_s = \frac{\hat{x}_s}{L},\ x_m = \frac{\hat{x}_m}{L},\ w_1 = \frac{\hat{w}_1}{\sqrt{\varepsilon} r},\ w_2 = \frac{\hat{w}_2}{\sqrt{\varepsilon} r},\ w_3 = \frac{\hat{w}_3}{\sqrt{\varepsilon} r},\ t = \frac{1}{L^2}\sqrt{\frac{EI}{\rho A}}\hat{t},\\ &k_1 = \frac{\hat{k}_1 L^3}{EI},\ k_2 = \frac{\hat{k}_2 r^2 L^3}{EI},\ M = \frac{\hat{M}}{\rho A L},\ \gamma_T = \frac{A\gamma L^2}{EI}, \end{aligned} \tag{7}$$

where r is the radius of gyration of the cross-sectional area, and ε is a small parameter in order to ensure that the nonlinear terms are small compared with the linear ones. Furthermore, the $\varepsilon F_j \cos(\Omega t)$ $(j = 1, 2, 3)$ terms are distributed harmonic forces exerted on the ith span. These were not in the derivations from the beginning for simplicity.

3 The application of the method of multiple timescales

In the two timescale-expansion form of the method of multiple timescales [28–45], a uniformly valid expansion is sought in the following form

$$w_j\,(x, t; \varepsilon) = w_{j0}\,(x, T_0, T_1) + \varepsilon w_{j1}\,(x, T_0, T_1) + O(\varepsilon^2), \quad j = 1, 2, 3, \tag{8}$$

where w_{10}, w_{20}, and w_{30} are functions describing the unperturbed motion of the beam; w_{11}, w_{21}, and w_{31} represent the corrections to the linear response; $T_0 = t$ and $T_1 = \varepsilon t$ represent the fast and the slow timescales, respectively, and $O(\varepsilon^2)$ denotes terms of order ε^2 and higher. The method of multiple timescales will determine how the fundamental motions w_{10}, w_{20}, and w_{30} are corrected by ε-terms (small perturbations).

The chain rule in time differentiation provides the relations

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial}{\partial T_0} + \varepsilon\frac{\partial}{\partial T_1},\\ \frac{\partial^2}{\partial t^2} &= \frac{\partial^2}{\partial T_0^2} + 2\varepsilon\frac{\partial^2}{\partial T_0 \partial T_1} + O(\varepsilon^2). \end{aligned} \tag{9}$$

Inserting Eq. 8 into Eq. 6a–6e, using Eq. 9, as well as equating coefficients of like powers of ε, the equations of order 1 and ε and the corresponding internal/external boundary conditions are obtained as

$$O(\varepsilon^0):$$

$$\frac{\partial^2 w_{j0}}{\partial T_0^2} + \frac{\partial^4 w_{j0}}{\partial x^4} + \gamma_T \frac{\partial^2 w_{j0}}{\partial x^2} = 0, \quad j = 1, 2, 3, \tag{10a}$$

$$\text{at } x = 0: \quad w_{10} = \frac{\partial^2 w_{10}}{\partial x^2} = 0, \tag{10b}$$

$$\text{at } x = 1: \quad w_{30} = \frac{\partial^2 w_{30}}{\partial x^2} = 0, \tag{10c}$$

$$\text{at } x = x_s: \quad w_{10} = w_{20}, \quad \frac{\partial w_{10}}{\partial x} = \frac{\partial w_{20}}{\partial x}, \quad \frac{\partial^2 w_{10}}{\partial x^2} = \frac{\partial^2 w_{20}}{\partial x^2}, \quad \frac{\partial^3 w_{10}}{\partial x^3} - \frac{\partial^3 w_{20}}{\partial x^3} - k_1 w_{10} = 0, \tag{10d}$$

$$\text{at } x = x_m: \quad w_{20} = w_{30}, \quad \frac{\partial w_{20}}{\partial x} = \frac{\partial w_{30}}{\partial x}, \quad \frac{\partial^2 w_{20}}{\partial x^2} = \frac{\partial^2 w_{30}}{\partial x^2}, \frac{\partial^3 w_{20}}{\partial x^3} - \frac{\partial^3 w_{30}}{\partial x^3} - M \frac{\partial^2 w_{20}}{\partial T_0^2} = 0, \tag{10e}$$

and

$$O(\varepsilon^1):$$

$$\frac{\partial^2 w_{j1}}{\partial T_0^2} + \frac{\partial^4 w_{j1}}{\partial x^4} + \gamma_T \frac{\partial^2 w_{j1}}{\partial x^2} = -2 \frac{\partial^2 w_{j0}}{\partial T_0 \partial T_1} + \frac{1}{2} \frac{\partial^2 w_{j0}}{\partial x^2} \left[\int_0^{x_s} \left(\frac{\partial w_{10}}{\partial x} \right)^2 \mathrm{d}x + \int_{x_s}^{x_m} \left(\frac{\partial w_{20}}{\partial x} \right)^2 \mathrm{d}x + \int_{x_m}^{1} \left(\frac{\partial w_{30}}{\partial x} \right)^2 \mathrm{d}x \right] + F_j \cos(\Omega t), \quad j = 1, 2, 3, \tag{11a}$$

$$\text{at } x = 0: \ w_{11} = \frac{\partial^2 w_{11}}{\partial x^2} = 0, \tag{11b}$$

$$\text{at } x = 1: \ w_{31} = \frac{\partial^2 w_{31}}{\partial x^2} = 0, \tag{11c}$$

$$\text{at } x = x_s: \ w_{11} = w_{21}, \quad \frac{\partial w_{11}}{\partial x} = \frac{\partial w_{21}}{\partial x}, \quad \frac{\partial^2 w_{11}}{\partial x^2} = \frac{\partial^2 w_{21}}{\partial x^2}, \quad \frac{\partial^3 w_{11}}{\partial x^3} - \frac{\partial^3 w_{21}}{\partial x^3} - k_1 w_{11} - k_2 w_{10}^3 = 0, \tag{11d}$$

$$\text{at } x = x_m: \ w_{21} = w_{31}, \quad \frac{\partial w_{21}}{\partial x} = \frac{\partial w_{31}}{\partial x}, \quad \frac{\partial^2 w_{21}}{\partial x^2} = \frac{\partial^2 w_{31}}{\partial x^2}, \quad \frac{\partial^3 w_{21}}{\partial x^3} - \frac{\partial^3 w_{31}}{\partial x^3} - M \left(\frac{\partial^2 w_{21}}{\partial T_0^2} + 2 \frac{\partial^2 w_{20}}{\partial T_0 \partial T_1} \right) = 0. \tag{11e}$$

The general solution for Eq. 10a may be expressed in the following form

$$w_{j0}(x, T_0, T_1) = \sum_{n=1}^{\infty} \left[\left(X_n(T_1) \mathrm{e}^{i\omega_n T_0} + \bar{X}_n(T_1) \mathrm{e}^{-i\omega_n T_0} \right) \tilde{w}_{jn}(x) \right], \quad j = 1, 2, 3, \tag{12}$$

where X_n are the complex-valued amplitude functions of slow timescale, i the imaginary unit, and $\tilde{w}_{1n}$, $\tilde{w}_{2n}$, and $\tilde{w}_{3n}$ represent the nth linear mode functions for the three different spans.

Substituting Eq. 12 into Eq. 10a–10e and balancing the coefficients of $e^{i\omega_n T_0}$ results in

$$\frac{d^4\tilde{w}_{jn}}{dx^4} + \gamma_T \frac{d^2\tilde{w}_{jn}}{dx^2} - \omega_n^2 \tilde{w}_{jn} = 0, \quad j = 1, 2, 3, \tag{13a}$$

$$\text{at } x = 0\text{: } \tilde{w}_{1n} = \frac{d^2\tilde{w}_{1n}}{dx^2} = 0, \tag{13b}$$

$$\text{at } x = 1\text{: } \tilde{w}_{3n} = \frac{d^2\tilde{w}_{3n}}{dx^2} = 0, \tag{13c}$$

$$\text{at } x = x_s\text{: } \tilde{w}_{1n} = \tilde{w}_{2n}, \quad \frac{d\tilde{w}_{1n}}{dx} = \frac{d\tilde{w}_{2n}}{dx}, \quad \frac{d^2\tilde{w}_{1n}}{dx^2} = \frac{d^2\tilde{w}_{2n}}{dx^2}, \quad \frac{d^3\tilde{w}_{1n}}{dx^3} - \frac{d^3\tilde{w}_{2n}}{dx^3} - k_1\tilde{w}_{1n} = 0, \tag{13d}$$

$$\text{at } x = x_m\text{: } \tilde{w}_{2n} = \tilde{w}_{3n}, \quad \frac{d\tilde{w}_{2n}}{dx} = \frac{d\tilde{w}_{3n}}{dx}, \quad \frac{d^2\tilde{w}_{2n}}{dx^2} = \frac{d^2\tilde{w}_{3n}}{dx^2}, \quad \frac{d^3\tilde{w}_{2n}}{dx^3} - \frac{d^3\tilde{w}_{3n}}{dx^3} + M\omega_n^2\tilde{w}_{2n} = 0. \tag{13e}$$

Since Eq. 13a–13e form a set of linear ordinary differential equations, the solutions are assumed to take the following form

$$\tilde{w}_{1n}(x) = c_{1n}\cosh(\beta_{1n}x) + c_{2n}\sinh(\beta_{1n}x) + c_{3n}\sin(\beta_{2n}x) + c_{4n}\cos(\beta_{2n}x), \tag{14a}$$
$$\tilde{w}_{2n}(x) = d_{1n}\cosh(\beta_{1n}x) + d_{2n}\sinh(\beta_{1n}x) + d_{3n}\sin(\beta_{2n}x) + d_{4n}\cos(\beta_{2n}x), \tag{14b}$$
$$\tilde{w}_{3n}(x) = e_{1n}\cosh(\beta_{1n}x) + e_{2n}\sinh(\beta_{1n}x) + e_{3n}\sin(\beta_{2n}x) + e_{4n}\cos(\beta_{2n}x), \tag{14c}$$

where $\beta_{1n} = \sqrt{0.5\left(-\gamma_T + \sqrt{\gamma_T^2 + 4\omega_n^2}\right)}$, $\beta_{2n} = \sqrt{0.5\left(\gamma_T + \sqrt{\gamma_T^2 + 4\omega_n^2}\right)}$, and c_{in}, d_{in}, e_{in} $\big(i = 1, 2, 3, 4;$ $n = 1, 2, \ldots, N\big)$, N being the number of modes, are different constants.

Inserting Eq. 14a–14c into Eq. 13b–13e gives

$$[M]_{12\times12}\ [c_{1n}\ \ c_{2n}\ \ c_{3n}\ c_{4n}\ \ d_{1n}\ \ d_{2n}\ \ d_{3n}\ d_{4n}\ e_{1n}\ e_{2n}\ e_{3n}\ e_{4n}]^T = [0]_{12\times1}, \tag{15}$$

where $[M]_{12\times12}$ is called the coefficient matrix.

This article considers the system near the nth resonance; for the general case where there are no internal resonances, it is sufficient to retain only the nth term in Eq. 12, i.e.,

$$w_{j0}(x, T_0, T_1) = \left[X_n(T_1)\,e^{i\omega_n T_0} + \bar{X}_n(T_1)\,e^{-i\omega_n T_0}\right]\tilde{w}_{jn}(\xi), \quad j = 1, 2, 3. \tag{16}$$

Inserting this into Eq. 11a–11e gives

$$O\left(\varepsilon^1\right):$$
$$\begin{aligned}
&\frac{\partial^2 w_{j1}}{\partial T_0^2} + \frac{\partial^4 w_{j1}}{\partial x^4} + \gamma_T\frac{\partial^2 w_{j1}}{\partial x^2} = -2\left(i\omega_n\frac{dX_n}{dT_1}e^{i\omega_n T_0} - i\omega_n\frac{d\bar{X}_n}{dT_1}e^{-i\omega_n T_0}\right)\tilde{w}_{jn}\\
&\quad + \left(X_n^3 e^{3i\omega_n T_0} + 3X_n^2\bar{X}_n e^{i\omega_n T_0} + 3\bar{X}_n^3 X_n e^{-i\omega_n T_0} + \bar{X}_n^3 e^{-3i\omega_n T_0}\right)\\
&\quad \times\frac{1}{2}\frac{d^2\tilde{w}_{jn}}{dx^2}\left(\int_0^{x_s}\left(\frac{d\tilde{w}_{1n}}{dx}\right)^2 dx + \int_{x_s}^{x_m}\left(\frac{d\tilde{w}_{2n}}{dx}\right)^2 dx + \int_{x_m}^{1}\left(\frac{d\tilde{w}_{3n}}{dx}\right)^2 dx\right)\\
&\quad + \frac{F_j}{2}\left(e^{i\Omega T_0} + e^{-i\Omega T_0}\right), \quad j = 1, 2, 3,
\end{aligned} \tag{17a}$$

$$\text{at } x = 0\text{: } w_{11} = \frac{\partial^2 w_{11}}{\partial x^2} = 0, \tag{17b}$$

$$\text{at } x = 1\text{: } w_{31} = \frac{\partial^2 w_{31}}{\partial x^2} = 0, \tag{17c}$$

$$\text{at } x = x_s:\ w_{11} = w_{21},\quad \frac{\partial w_{11}}{\partial x} = \frac{\partial w_{21}}{\partial x},\ \frac{\partial^2 w_{11}}{\partial x^2} = \frac{\partial^2 w_{21}}{\partial x^2},$$
$$\frac{\partial^3 w_{11}}{\partial x^3} - \frac{\partial^3 w_{21}}{\partial x^3} - k_1 w_{11} = k_2 \tilde{w}_{1n}^3 \left(X_n^3 \mathrm{e}^{3i\omega_n T_0} + 3X_n^2 \bar{X}_n \mathrm{e}^{i\omega_n T_0} + 3\bar{X}_n^2 X_n \mathrm{e}^{-i\omega_n T_0} + \bar{X}_n^3 \mathrm{e}^{-3i\omega_n T_0} \right), \tag{17d}$$

$$\text{at } x = x_m:\ w_{21} = w_{31},\quad \frac{\partial w_{21}}{\partial x} = \frac{\partial w_{31}}{\partial x},\ \frac{\partial^2 w_{21}}{\partial x^2} = \frac{\partial^2 w_{31}}{\partial x^2},\ \frac{\partial^3 w_{21}}{\partial x^3} - \frac{\partial^3 w_{31}}{\partial x^3} - M\frac{\partial^2 w_{21}}{\partial T_0^2}$$
$$= 2M\tilde{w}_{2n} \left(i\omega_n \frac{\mathrm{d}X_n}{\mathrm{d}T_1} \mathrm{e}^{i\omega_n T_0} - i\omega_n \frac{\mathrm{d}\bar{X}_n}{\mathrm{d}T_1} \mathrm{e}^{-i\omega_n T_0} \right). \tag{17e}$$

4 The solvability condition, nonlinear natural frequencies, and limit cycles

Equations 17a–17e form a set of coupled, nonlinear partial differential equations whose homogeneous parts possess nontrivial solutions. Therefore, they have a set of solutions if only if the *solvability condition* [26,42] is satisfied.

Fulfilling the solvability condition for the primary resonance case in Eqs. 17a–17g, $\Omega = \omega_n + \varepsilon\sigma_n$ (where σ_n is the nth detuning parameter), results in

$$X_n^2 \bar{X}_n + i\alpha_{1n} \frac{\mathrm{d}X_n}{\mathrm{d}T_1} = \frac{1}{2}\alpha_{2n} \mathrm{e}^{i\sigma_n T_1}, \tag{18a}$$

where

$$\alpha_{1n} = \frac{2M\omega_n \tilde{w}_{2n}^2(x_m) + 2\omega_n \left(\int_0^{x_s} \tilde{w}_{1n}^2 \mathrm{d}x + \int_{x_s}^{x_m} \tilde{w}_{2n}^2 \mathrm{d}x + \int_{x_m}^1 \tilde{w}_{3n}^2 \mathrm{d}x \right)}{3k_2 \tilde{w}_{1n}^4(x_s) - \frac{3}{2} \left(\int_0^{x_s} \left(\frac{\mathrm{d}\tilde{w}_{1n}}{\mathrm{d}x}\right)^2 \mathrm{d}x + \int_{x_s}^{x_m} \left(\frac{\mathrm{d}\tilde{w}_{2n}}{\mathrm{d}x}\right)^2 \mathrm{d}x + \int_{x_m}^1 \left(\frac{\mathrm{d}\tilde{w}_{3n}}{\mathrm{d}x}\right)^2 \mathrm{d}x \right) \left(\int_0^{x_s} \frac{\mathrm{d}^2\tilde{w}_{1n}}{\mathrm{d}x^2} \tilde{w}_{1n} \mathrm{d}x + \int_{x_s}^{x_m} \frac{\mathrm{d}^2\tilde{w}_{2n}}{\mathrm{d}x^2} \tilde{w}_{2n} \mathrm{d}x + \int_{x_m}^1 \frac{\mathrm{d}^2\tilde{w}_{3n}}{\mathrm{d}x^2} \tilde{w}_{3n} \mathrm{d}x \right)}, \tag{18b}$$

$$\alpha_{2n} = \frac{F_1 \int_0^{x_s} \tilde{w}_{1n} \mathrm{d}x + F_2 \int_{x_s}^{x_m} \tilde{w}_{2n} \mathrm{d}x + F_3 \int_{x_m}^1 \tilde{w}_{3n} \mathrm{d}x}{3k_2 \tilde{w}_{1n}^4(x_s) - \frac{3}{2} \left(\int_0^{x_s} \left(\frac{\mathrm{d}\tilde{w}_{1n}}{\mathrm{d}x}\right)^2 \mathrm{d}x + \int_{x_s}^{x_m} \left(\frac{\mathrm{d}\tilde{w}_{2n}}{\mathrm{d}x}\right)^2 \mathrm{d}x + \int_{x_m}^1 \left(\frac{\mathrm{d}\tilde{w}_{3n}}{\mathrm{d}x}\right)^2 \mathrm{d}x \right) \left(\int_0^{x_s} \frac{\mathrm{d}^2\tilde{w}_{1n}}{\mathrm{d}x^2} \tilde{w}_{1n} \mathrm{d}x + \int_{x_s}^{x_m} \frac{\mathrm{d}^2\tilde{w}_{2n}}{\mathrm{d}x^2} \tilde{w}_{2n} \mathrm{d}x + \int_{x_m}^1 \frac{\mathrm{d}^2\tilde{w}_{3n}}{\mathrm{d}x^2} \tilde{w}_{3n} \mathrm{d}x \right)}. \tag{18c}$$

In order to determine the nonlinear natural frequencies of the system, α_{2n} (which is related to the forcing amplitude) is set to zero, then Eq. 8a can be rewritten as

$$X_n^2 \bar{X}_n + i\alpha_{1n} \frac{\mathrm{d}X_n}{\mathrm{d}T_1} = 0. \tag{19}$$

This equation is a nonlinear ordinary differential equation in terms of the slow timescale. In order to solve this equation, one may consider a transformation of the following form

$$X_n(T_1) = \frac{1}{2} a_n \mathrm{e}^{i\theta_n}, \tag{20}$$

with $a_n = a_n(T_1)$ and $\theta_n = \theta_n(T_1)$ being real-valued functions.

Inserting Eq. 20 into Eq. 19 and separating the real and imaginary parts yields

$$\alpha_{1n} \frac{\mathrm{d}a_n}{\mathrm{d}T_1} = 0, \tag{21a}$$
$$\frac{1}{4} a_n^3 - \alpha_{1n} a_n \frac{\mathrm{d}\theta_n}{\mathrm{d}T_1} = 0. \tag{21b}$$

Solving Eq. 21a yields

$$a_n = a_{0(n)}, \tag{22}$$

where $a_{0(n)}$ is a constant value; there is no energy dissipation in the model, and hence, the amplitude of the response does not decay with time.

Solving Eq. 21b using Eq. 22 gives

$$\frac{d\theta_n}{dT_1} = \frac{1}{4\alpha_{1n}} a_{0(n)}^2. \tag{23}$$

Substituting Eq. 23 into Eq. 20 and inserting the resulting equation into Eq. 16(a–c), the nth nonlinear natural frequency of the system is obtained as

$$(\omega_{nl})_n = \frac{\varepsilon}{4\alpha_{1n}} a_{0(n)}^2 + \omega_n. \tag{24}$$

In order to obtain a relation between the detuning parameter and the vibration amplitude of the resonant case, one may substitute Eq. 20 into Eq. 18a. Separating the real and imaginary parts of the resulting equation, and writing $\gamma_n = \sigma_n T_1 - \theta_n$ yields

$$\alpha_{1n} \frac{da_n}{dT_1} - \alpha_{2n} \sin \gamma_n = 0, \tag{25a}$$

$$\frac{1}{4} a_n^3 - \alpha_{1n} a_n \frac{d\theta_n}{dT_1} - \alpha_{2n} \cos \gamma_n = 0. \tag{25b}$$

The steady-state forced response is obtained by setting $\frac{d\gamma_n}{dT_1} = \frac{da_n}{dT_1} = 0$ ($a_n = a_{0n}$, where a_{0n} is a constant) in Eqs. 25a, 25b. Eliminating γ_n from the resulting equations yields the following

$$(\sigma_1)_{1,2} = \frac{1}{\alpha_{1n} a_{0n}} \left(\frac{1}{4} a_{0n}^3 \mp |\alpha_{2n}| \right), \tag{26}$$

which relates the amplitude of the nth resonant amplitude to the corresponding detuning parameter.

5 Numerical results

The objective of this section is to describe the effects of system parameters (such as the spring and mass locations, mass value, linear and nonlinear stiffness coefficients of the spring, as well as the thermal coefficient factor) on the linear and nonlinear natural frequencies and frequency–responses of the system.

The linear natural frequencies of the system as a function of various system parameters are presented in Tables 1, 2, 3 and 4. These analytical results are compared with those obtained numerically via eight-mode Galerkin discretization, using the simply supported beam eigenfunctions as appropriate comparison functions. Reviewing these tables leads to several conclusions: (1) changing the spring-mass location along the beam length increases the first linear natural frequency up to the mid-span of the beam, with a decrease thereafter; (2) the linear natural frequencies increase with increasing linear stiffness coefficient of the spring; (3) increasing the value of the concentrated mass decreases the linear natural frequencies of the system (Table 3); (4) as the thermal coefficient factor increases, the linear natural frequencies decrease.

Table 1 The first, second, and third natural frequencies of the system as a function of the spring-mass location; $k_1 = 100$, $M = 0.5$, $\gamma_T = 1$

$x_s = x_m$	ω_1		ω_2		ω_3	
	Analytical method	Galerkin's technique	Analytical method	Galerkin's technique	Analytical method	Galerkin's technique
0.1	9.848	9.849	34.068	34.074	71.439	71.504
0.2	10.726	10.726	30.780	30.787	75.495	75.542
0.3	11.435	11.435	32.084	32.091	86.665	86.678
0.4	11.848	11.848	36.013	36.018	81.492	81.552
0.5	11.980	11.980	38.975	38.975	71.660	71.763
0.6	11.848	11.848	36.013	36.018	81.492	81.552
0.7	11.435	11.435	32.084	32.091	86.665	86.678
0.8	10.726	10.726	30.780	30.787	75.495	75.542
0.9	9.848	9.849	34.068	34.074	71.439	71.504

Table 2 The first, second, and third natural frequencies of the system as a function of the spring coefficient; $x_s = 0.4$, $x_m = 0.6$, $M = 0.5$, $\gamma_T = 1$

k_1	ω_1		ω_2		ω_3	
	Analytical method	Galerkin's technique	Analytical method	Galerkin's technique	Analytical method	Galerkin's technique
20	7.969	7.969	36.130	36.136	81.423	81.484
40	8.970	8.971	36.505	36.512	81.446	81.507
60	9.828	9.828	36.883	36.890	81.468	81.529
80	10.577	10.578	37.263	37.270	81.491	81.552
100	11.242	11.243	37.664	37.652	81.514	81.575
120	11.838	11.840	38.027	38.036	81.537	81.598
140	12.377	12.380	38.410	38.420	81.560	81.621
160	12.868	12.871	38.793	38.804	81.584	81.644
180	13.317	13.320	39.177	39.188	81.607	81.668
200	13.729	13.733	39.559	39.571	81.631	81.691

Table 3 The first, second, and third natural frequencies of the system as a function of the value of the concentrated mass; $x_s = 0.4$, $x_m = 0.6$, $k_1 = 100$, $\gamma_T = 1$

M	ω_1		ω_2		ω_3	
	Analytical method	Galerkin's technique	Analytical method	Galerkin's technique	Analytical method	Galerkin's technique
0.1	14.603	14.605	39.113	39.115	86.110	86.119
0.2	13.501	13.503	38.557	38.562	84.358	84.382
0.3	12.608	12.610	38.163	38.169	83.122	83.161
0.4	11.868	11.869	37.870	37.877	82.211	82.262
0.5	11.242	11.243	37.664	37.652	81.514	81.575
0.6	10.704	10.705	37.465	37.474	80.965	81.035
0.7	10.235	10.237	37.319	37.329	80.522	80.599
0.8	9.823	9.824	37.199	37.209	80.158	80.240
0.9	9.456	9.457	37.098	37.108	79.852	79.940

Table 4 The first, second, and third natural frequencies of the system as a function of the value of γ_T; $x_s = 0.4$, $x_m = 0.6$, $k_1 = 100$, $M = 0.5$

γ_T	ω_1		ω_2		ω_3	
	Analytical method	Galerkin's technique	Analytical method	Galerkin's technique	Analytical method	Galerkin's technique
0	11.463	11.465	38.090	38.098	81.976	82.037
0.25	11.408	11.410	37.979	37.987	81.860	81.922
0.50	11.353	11.354	37.868	37.876	81.745	81.806
0.75	11.297	11.299	37.756	37.764	81.630	81.691
1	11.242	11.243	37.664	37.652	81.514	81.575
1.25	11.185	11.187	37.532	37.540	81.399	81.459
1.50	11.129	11.131	37.419	37.427	81.283	81.343
1.75	11.072	11.074	37.306	37.314	81.167	81.227
2	11.015	11.017	37.193	37.201	81.050	81.110

Next, the influence of the system parameters on the first nonlinear natural frequency was studied (Figs. 2, 3, 4, 5, 6). Several conclusions may be drawn from these plots. First, the system is elastic, and therefore, the nonlinear natural frequencies are independent of time. Second, for a given first mode amplitude, changing the spring-beam location along the beam length from the left-end to the mid-span results in a larger value for the first nonlinear frequency of the system (Fig. 2), whereas increasing the value of the concentrated mass has an opposite effect. Third, for a given first mode amplitude, increasing the linear stiffness coefficient of the spring causes the first nonlinear natural frequency to increase (Fig. 4). Fourth, increasing the nonlinear stiffness coefficient of the spring leads to an increase in the curvature of the first nonlinear natural frequency, as seen in Fig. 5. Lastly, for a given first mode amplitude, as the thermal coefficient factor γ_T is increased, the first nonlinear natural frequency decreases, as seen in Fig. 6.

Finally, the influence of the system parameters on the frequency–response curves of the system (Figs. 7, 8, 9, 10) was examined. Several conclusions can be drawn from these figures: (1) as M is increased,

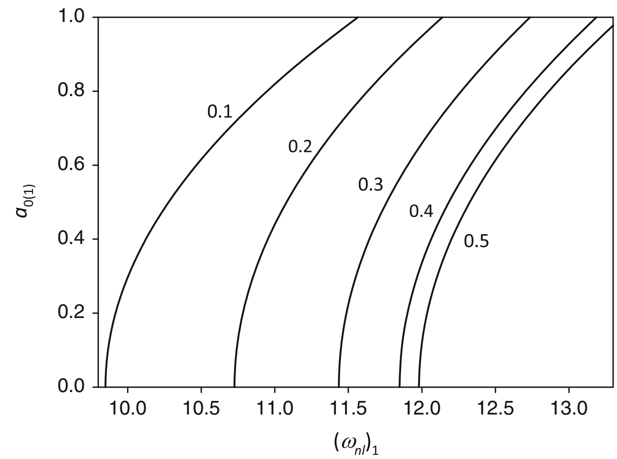

Fig. 2 The first mode amplitude as a function of the first nonlinear natural frequency of the system for different spring-mass locations ($x_s = x_m$ values are indicated on the curves); $k_1 = 100$, $k_2 = 10$, $M = 0.5$, and $\gamma_T = 1$

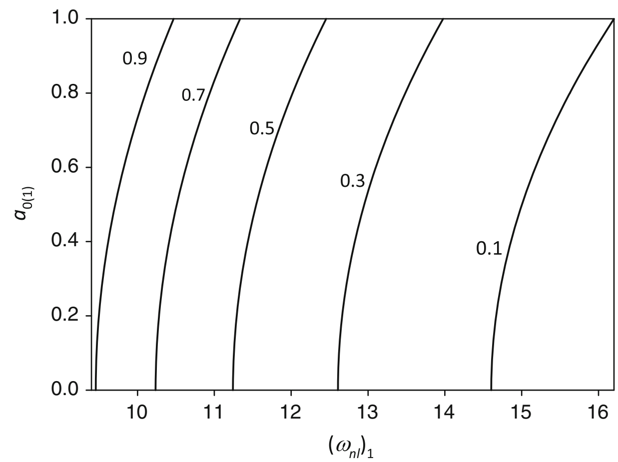

Fig. 3 The first mode amplitude as a function of the first nonlinear natural frequency of the system for a selection of the point-mass values (M values are indicated on the curves); $k_1 = 100$, $k_2 = 10$, $x_s = 0.4$, $x_m = 0.6$, and $\gamma_T = 1$

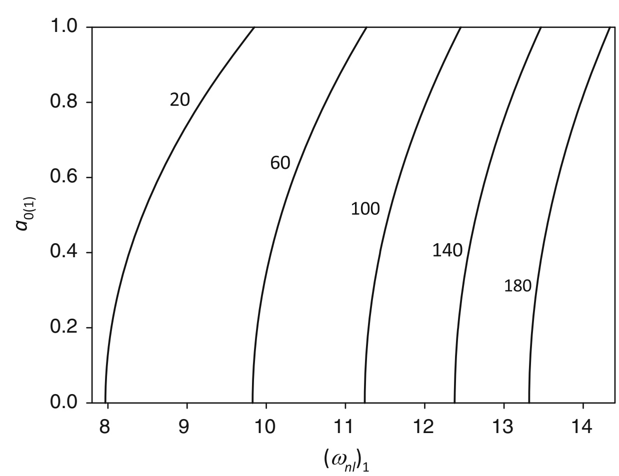

Fig. 4 The first mode amplitude as a function of the first nonlinear natural frequency of the system for several values of the linear stiffness coefficient of the spring (k_1 values are indicated on the curves); $k_2 = 10$, $x_s = 0.4$, $x_m = 0.6$, $M = 0.5$, and $\gamma_T = 1$

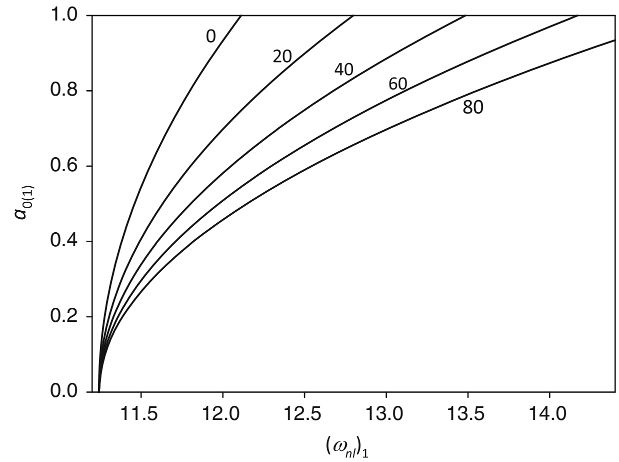

Fig. 5 The first mode amplitude as a function of the first nonlinear natural frequency of the system for a selection of the nonlinear stiffness coefficient of the spring (k_2 values are indicated on the curves); $k_1 = 100$, $x_s = 0.4$, $x_m = 0.6$, $M = 0.5$, and $\gamma_T = 1$

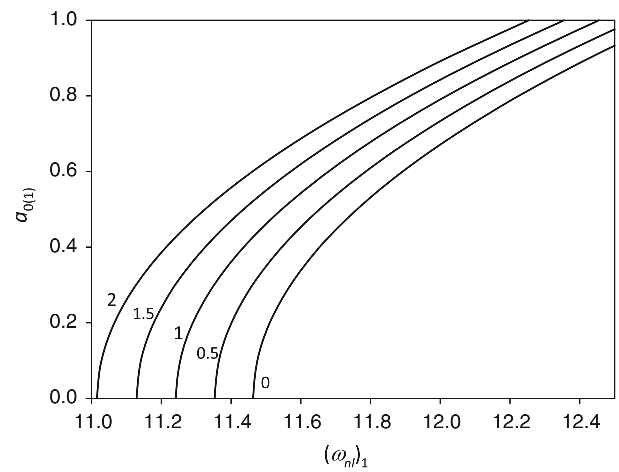

Fig. 6 The first mode amplitude as a function of the first nonlinear natural frequency of the system for a selection of γ_T (γ_T values are indicated on the curves); $k_1 = 100$, $k_2 = 10$, $x_s = 0.4$, $x_m = 0.6$, and $M = 0.5$

the curvature and hence the nonlinearity of the system decrease (Fig. 7); (2) increasing γ_T causes the curve to bend slightly more to the right and the hardening behavior of the system increases (Fig. 8); (3) as k_2 is increased, the curves bend more to the right, and hence, the jump phenomenon is affected and the hardening behavior of the system increases (Fig. 9); (4) increasing the forcing amplitude causes the response amplitude to increase, as seen in Fig. 10.

6 Conclusions

In this study, the thermo-mechanical vibrations of a simply supported spring-mass-beam system are investigated analytically. The beam in considered as a three-part system and Hamilton's principle along with constitutive relations are used to derive the nonlinear equations of motion and internal/external boundary conditions, while taking into account the thermal effects. The equations of motion form a set of nonlinear partial differential equations with nonlinear, time-dependent, and coupled internal boundary conditions. Under quasi-static assumptions, these equations are converted into a set of integro-partial differential equations. These PDEs are then solved via the method of multiple timescales, which results in the vibration response of the system.

The influence of system parameters on its vibration response is investigated through a numerical parametric study. The most important conclusions are summarized as: (1) for a given first mode amplitude, either changing the spring-beam location along the beam length from the left-end to the mid-span or increasing the

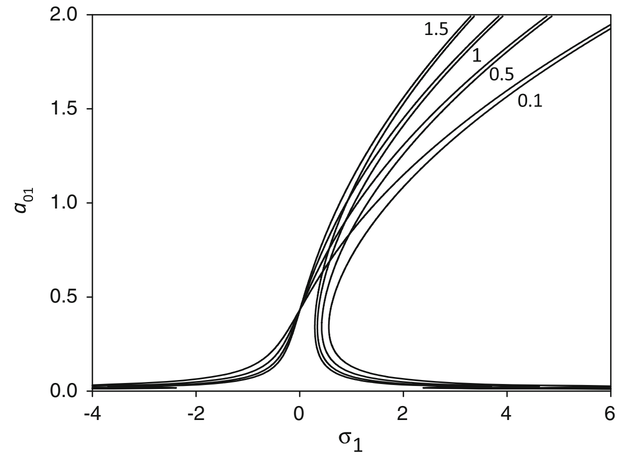

Fig. 7 The first frequency–response curve of the system for several values of the concentrated mass (M values are indicated on the curves); $F_1 = F_2 = F_3 = 5$, $k_1 = 100$, $k_2 = 10$, $x_s = 0.4$, $x_m = 0.6$, and $\gamma_T = 1$.

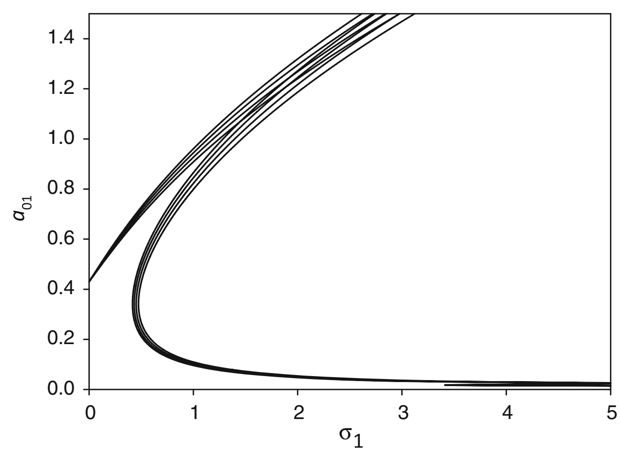

Fig. 8 The first frequency–response curve of the system for several values of γ_T (γ_T values are 0.0, 2.0, 4.0, and 6.0 from left to right); $F_1 = F_2 = F_3 = 5$, $k_1 = 100$, $k_2 = 10$, $M = 0.5$, $x_s = 0.4$, and $x_m = 0.6$

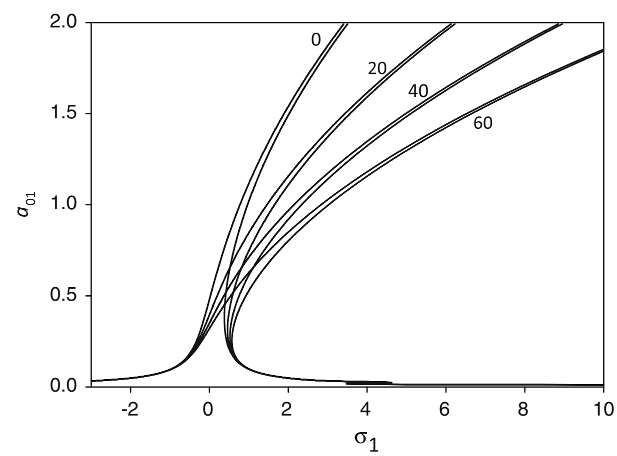

Fig. 9 The first frequency–response curve of the system for several values of the nonlinear stiffness coefficient of the spring (k_2 values are indicated on the curves); $F = F_1 = F_2 = F_3 = 5$, $k_1 = 100$, $M = 0.5$, $x_s = 0.4$, $x_m = 0.6$, and $\gamma_T = 1$

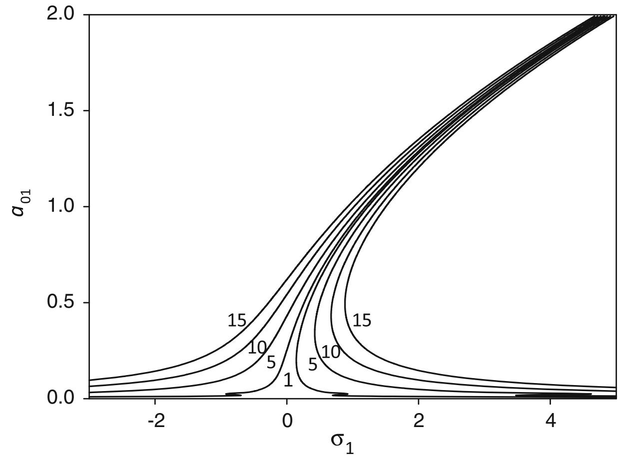

Fig. 10 The first frequency–response curve of the system for several values of the forcing amplitude, $F = F_1 = F_2 = F_3$ (F values are indicated on the curves); $k_1 = 100$, $k_2 = 10$, $M = 0.5$, $x_s = 0.4$, $x_m = 0.6$, and $\gamma_T = 1$

linear and nonlinear stiffness coefficients of the spring causes the first nonlinear natural frequency to increase; (2) on the other hand, increasing the value of the concentrated mass and the thermal coefficient factor γ_T has an opposite effect; (3) increasing γ_T and k_2 causes the frequency–response curves of the system to bend more to the right and the hardening behavior of the system increases, whereas increasing M has an opposite effect; (4) increasing the forcing amplitude causes the response amplitude to increase.

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