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**ORIGINAL**

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# Macrohomogeneity condition in dynamics of micropolar media

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**Abstract** We determine the macrohomogeneity (Hill-Mandel type) condition in the dynamic response of inhomogeneous micropolar (Cosserat) materials. The setting calls for small deformation gradients and curvatures, but without restrictions on the constitutive behavior and without any requirements of spatial periodicity. The condition gives admissible boundary loadings, along with extra terms representing kinetic energy contributions of both classical type and micropolar type. The said loadings involve various combinations of average stresses and strains, along with couple-stresses and curvature-torsion tensors. If applied to a specific microstructure in a computational mechanics approach, these boundary loadings will allow one to determine scale-dependent homogenization toward a representative volume element (RVE) of an equivalent homogeneous micropolar medium in either elastic or inelastic settings. By restricting the continuum model to an inhomogeneous Cauchy continuum and/or a quasi-static setting, the macrohomogeneity condition simplifies to conventional versions.

**Keywords** Micropolar media · Cosserat media · Random media · Hill-Mandelcondition · Homogenization · Macrohomogeneity

## 1 Introduction

Consider a micropolar (Cosserat) elastic continuum, whose each point has two vectorial degrees of freedom: displacement ($u_i$) and rotation ($\varphi_i$). Tractions on any face with the unit outer normal vector $n_j$ involve generally asymmetric stress ($\tau_{ji}$) and couple-stress tensors ($\mu_{ji}$)

$$t_i = \tau_{ji} n_j \quad m_i = \mu_{ji} n_j \tag{1.1}$$

The kinematics of the continuum is governed by the relations involving the generally asymmetric strain ($\gamma_{ji}$) and curvature-torsion tensors ($\kappa_{ji}$)

$$\gamma_{ji} = u_{i,j} - e_{kji}\varphi_k \qquad \kappa_{ji} = \varphi_{i,j}. \tag{1.2}$$

We do not assume $\varphi_i = \frac{1}{2} e_{ijk} u_{k,j}$, which would imply a restricted (Koiter or couple-stress) continuum.

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The equations of motion are

$$\tau_{ji,j} = \rho \ddot{u}_i \qquad e_{ijk}\tau_{jk} + \mu_{ji,j} = I_{ij}\ddot{\varphi}_j. \tag{1.3}$$

In the setting of a kinematically infinitesimal theory relations (1.2) lead to

$$\dot{\gamma}_{ji} = \dot{u}_{i,j} - e_{kji}\dot{\varphi}_k \qquad \dot{\kappa}_{ji} = \dot{\varphi}_{i,j} \tag{1.4}$$

where the overdot indicates a material time derivative effectively equal to a partial time derivative.

Clearly, the setting is dynamic with small deformation gradients and curvatures but without any specific restrictions on the constitutive behavior, which can be elastic, viscoelastic, plastic and so on. Also we impose no requirements of spatial periodicity or homogeneity on the material. In other words, the continuum is spatially inhomogeneous (e.g., specified by random fields of stiffness tensors), thus modeling a spatially heterogeneous material (e.g. a granular matter). In order to homogenize this continuum—i.e. to find a spatially homogeneous micropolar medium with constant constitutive coefficients—one must first determine a macrohomogeneity (Hill-Mandel type) condition in dynamic, or at least static, response of such inhomogeneous micropolar media.

One can introduce a random micropolar medium $B$ as an ensemble of deterministic realizations

$$B = \{B(\omega); \omega \in \Omega\}. \tag{1.5}$$

where $\Omega$ is the sample space [1]. For instance, in the case of a linear elastic chiral material, $B(\omega)$ stands for three random fields of tensors $C^{(1)}_{ijkl}$, $C^{(2)}_{ijkl}$, and $C^{(3)}_{ijkl}$ appearing in Hooke's law

$$\begin{aligned} \tau_{ij} &= C^{(1)}_{ijkl}\gamma_{kl} + C^{(3)}_{ijkl}\kappa_{kl}, \\ \mu_{ij} &= C^{(3)}_{ijkl}\gamma_{kl} + C^{(2)}_{ijkl}\kappa_{kl}. \end{aligned} \tag{1.6}$$

In the case of a non-chiral micropolar material, we have $C^{(1)}_{ijkl}$ and $C^{(2)}_{ijkl}$, while in the case of a Cauchy material there is only $C^{(1)}_{ijkl}$ with its three well-known symmetries. Just like in other situations, the new Hill-Mandel condition is to hold for any mesoscale domain—i.e. a region $B$ of finite extent relative to a typical (microscale) grain size—in any specific realization $B(\omega)$ of a $\mathcal{B}$. We denote the volume of $B(\omega)$ by $V$ and its bounding surface $\partial B$ by $S$.

Note that the task at hand is different than the studies of the late nineties (begun in [2]) focused on finding a homogeneous micropolar continuum smoothing a spatially heterogeneous Cauchy material. The latter trades a highly detailed information on the Cauchy-type microstructure for a less detailed description via an equivalent, homogeneous Cosserat continuum. The present study aims to trade a highly detailed information on an inhomogeneous Cosserat microstructure for a description in terms of an equivalent, homogeneous Cosserat continuum. Here, one can think of just about any micropolar-type material—such as a granular mass, beam (or fiber) framework, stone or brick masonry [3–5]—and recognize that it is not perfectly periodic but always displays some disorder. The question which thus arises is: On what scale relative to the typical grain size is a homogeneous micropolar continuum attained? With this issue in mind, in the next section we generalize the original work by Li and Liu [6], while in the subsequent section we rely on the results of Onck [7].

## 2 Basic relations

In the setting of a kinematically infinitesimal theory, (1.1) is replaced by

$$\dot{\gamma}_{ji} = \dot{u}_{i,j} - e_{kji}\dot{\varphi}_k \qquad \dot{\kappa}_{ji} = \dot{\varphi}_{i,j} \tag{2.1}$$

where the overdot indicates a material time derivative equal to a partial time derivative.

Define the classical (i.e. associated with velocities) and micropolar (i.e. associated with rotations) kinetic energy densities

$$\begin{aligned} k_c &= \frac{1}{2}\rho \upsilon_i \upsilon_i \qquad \upsilon_i \equiv \dot{u}_i \\ k_m &= \frac{1}{2} I_{ij} w_i w_j \qquad w_i \equiv \dot{\varphi}_i \end{aligned} \tag{2.2}$$

and their volume averages

$$\overline{k_c} = \frac{1}{V}\int\limits_V k_c dV$$
$$\overline{k_m} = \frac{1}{V}\int\limits_V k_m dV \tag{2.3}$$

as well as the time rates of these averages

$$(\overline{k_c})^{\cdot} \equiv \frac{d}{dt}\left(\frac{1}{V}\int\limits_V k_c dV\right) = \frac{d}{dt}\left(\frac{1}{V}\int\limits_V \frac{1}{2}\rho \upsilon_i \upsilon_i dV\right)$$
$$(\overline{k_m})^{\cdot} \equiv \frac{1}{V}\int\limits_V k_m dV = \frac{d}{dt}\left(\frac{1}{V}\int\limits_V \frac{1}{2} I_{ij} w_i w_j dV\right) \tag{2.4}$$

We shall also need the volume averages of rates of kinetic energy densities

$$\overline{\dot{k}_c} \equiv \frac{1}{V}\int\limits_V \dot{k}_c dV = \frac{1}{V}\int\limits_V \left(\frac{1}{2}\rho\upsilon_i^2\right)^{\cdot} dV = \frac{1}{V}\int\limits_V \rho\upsilon_i\dot{\upsilon}_i dV \equiv \frac{1}{V}\int\limits_V \rho\dot{u}_i\ddot{u}_i dV$$
$$\overline{\dot{k}_m} \equiv \frac{1}{V}\int\limits_V \dot{k}_m dV = \frac{1}{V}\int\limits_V \left(\frac{1}{2}\rho\varphi_i^2\right)^{\cdot} dV = \frac{1}{V}\int\limits_V \rho w_i\dot{w}_i dV \equiv \frac{1}{V}\int\limits_V \rho\dot{\varphi}_i\ddot{\varphi}_i dV \tag{2.5}$$

## 3 Macrohomogeneity condition in the presence of inertia

Consider the volume average of the classical stress power

$$\begin{aligned}\overline{\tau_{ij}\dot{\gamma}_{ij}} &= \frac{1}{V}\int\limits_V \tau_{ji}\dot{\gamma}_{ji} dV = \frac{1}{V}\int\limits_V \tau_{ji}\left(\dot{u}_{i,j} - e_{kji}\dot{\varphi}_k\right) dV\\ &= \frac{1}{V}\int\limits_V \tau_{ji}\dot{u}_{i,j} dV - \frac{1}{V}\int\limits_V \tau_{ji} e_{kji}\dot{\varphi}_k dV\\ &= \frac{1}{V}\int\limits_V \left[\left(\tau_{ji}\dot{u}_i\right)_{,j} - \tau_{ji,j}\dot{u}_{i,j}\right] dV - e_{kji}\overline{\tau_{ji}\dot{\varphi}_k}\\ &= \overline{t_i\dot{u}_i} - \overline{\dot{k}_c} - e_{kji}\overline{\tau_{ji}\dot{\varphi}_k}\end{aligned} \tag{3.1}$$

whereby we employed (1.1)$_1$–(1.4)$_1$ and (2.5). Next, consider the product of volume averages

$$\begin{aligned}\overline{\tau_{ij}}\,\overline{\dot{\gamma}_{ij}} &= \overline{\tau_{ji}}\frac{1}{V}\int\limits_V \left(\dot{u}_{i,j} - e_{kji}\dot{\varphi}_k\right) dV\\ &= \overline{t_i}\,\overline{\dot{u}_i} - (\overline{k_c})^{\cdot} - e_{kji}\overline{\tau_{ji}}\,\overline{\dot{\varphi}_k}\end{aligned} \tag{3.2}$$

where in the last equality we used the first equation of motion. It follows that

$$\overline{\tau_{ji}\dot{\gamma}_{ji}} - \overline{\tau_{ji}}\overline{\dot{\gamma}_{ji}} = \frac{1}{V}\int\limits_{\partial V} \left(t_i - \overline{\tau_{ki}} n_k\right)\left(\dot{u}_i - \overline{\dot{u}_{i,j}} x_j\right) dS - \overline{\dot{k}_c} + (\overline{k_c})^{\cdot} \tag{3.3}$$

This also leads to

$$\overline{\tau_{ji}\dot{\gamma}_{ji}} - \overline{\tau_{ji}}\overline{\dot{\gamma}_{ji}} + \overline{\mu_{ji}\dot{\kappa}_{ji}} - \overline{\mu_{ji}}\overline{\dot{\kappa}_{ji}} = \left(\overline{\tau_{ji}\dot{u}_{i,j}} - \overline{\tau_{ji}}\,\overline{\dot{u}_{i,j}}\right) + \left(\overline{\mu_{ji}\dot{\kappa}_{i,j}} - \overline{\mu_{ki}}\,\overline{\dot{\kappa}_{i,j}}\right) - e_{kji}\left(\overline{\dot{\varphi}_k\tau_{ji}} - \overline{\dot{\varphi}_k}\,,\overline{\tau_{ji}}\right) \tag{3.4}$$

On the other hand, we derive

$$
\begin{aligned}
\overline{\mu_{ji}\dot{\kappa}_{ji}} - \overline{\mu_{ji}}\,\overline{\dot{\gamma}_{ji}} &= \frac{1}{V}\int\limits_V \mu_{ji}\dot{\varphi}_{i,j}dV - \overline{\mu_{ji}}\frac{1}{V}\int\limits_V \dot{\varphi}_{i,j}dV \\
&= \frac{1}{V}\int\limits_V \left[\left(\mu_{ji}\dot{\varphi}_i\right)_{,j} - \mu_{ji,j}\,\dot{\varphi}_i\right]dV - \overline{\mu_{ji}}\frac{1}{V}\int\limits_{\partial V} \dot{\varphi}_i n_j dS \\
&= \frac{1}{V}\int\limits_S \mu_{ki}\dot{\varphi}_i n_k dS - \frac{1}{V}\int\limits_V \left(I\ddot{\varphi}_i - e_{ijk}\mu_{jk}\right)\dot{\varphi}_i dV - \overline{\mu_{ji}}\frac{1}{V}\int\limits_S \dot{\varphi}_i n_j dS \\
&= \frac{1}{V}\int\limits_S \mu_{ki}\dot{\varphi}_i n_k dS - \overline{\dot{k}_m} + \frac{1}{V}\int\limits_V e_{ijk}\mu_{jk}\dot{\varphi}_i dV - \overline{\mu_{ji}}\frac{1}{V}\int\limits_S \dot{\varphi}_i n_j dS
\end{aligned}
\tag{3.5}
$$

while, separately, we determine

$$
\begin{aligned}
\frac{1}{V}\int\limits_S n_k\mu_{ki}\left(\overline{\dot{\varphi}_{i,j}}x_j\right)dS &= \overline{\dot{\varphi}_{i,j}}\frac{1}{V}\int\limits_S n_k\mu_{ki}x_j dS = \overline{\dot{\varphi}_{i,j}}\frac{1}{V}\int\limits_V \left(\mu_{ki}x_j\right)_{,k}dV \\
&= \overline{\dot{\varphi}_{i,j}}\frac{1}{V}\int\limits_V \left(\mu_{ki,k}\,x_j + \mu_{ki}\delta_{jk}\right)dV \\
&= \overline{\dot{\varphi}_{i,j}}\frac{1}{V}\int\limits_V \mu_{ki,k}\,x_j dV + \overline{\dot{\varphi}_{i,j}}\,\overline{\mu_{ji}} \\
&= \overline{\dot{\varphi}_{i,j}}\frac{1}{V}\int\limits_V \mu_{ki,k}\,x_j dV + \frac{1}{V}\int\limits_S n_k\overline{\mu_{ki}}\,\overline{\dot{\varphi}_{i,j}}x_j dS
\end{aligned}
\tag{3.6}
$$

On account of Eq. (3.6), we find

$$
-\frac{1}{V}\int\limits_S n_k\mu_{ki}\left(\overline{\dot{\varphi}_{i,j}}x_j\right)dS + \frac{1}{V}\int\limits_S n_k\mu_{ki}\left(\overline{\dot{\varphi}_{i,j}}x_j\right)dS + \frac{1}{V}\int\limits_V \mu_{ki,k}x_j\overline{\dot{\varphi}_{i,j}}dV
\tag{3.7}
$$

Upon adding (3.7)–(3.5), we obtain

$$
\begin{aligned}
\overline{\mu_{ji}\dot{\kappa}_{ji}} - \overline{\mu_{ji}}\,\overline{\dot{\gamma}_{ji}} &= \frac{1}{V}\int\limits_S \left(n_k\mu_{ki}\varphi_i - n_k\overline{\mu_{ki}}\varphi_i\right)dS - \frac{1}{V}\int\limits_S \mu_{ki,k}\varphi_i dS \\
&\quad + \frac{1}{\partial V}\int\limits_S \left(-n_k\mu_{ki}\overline{\dot{\varphi}_{i,j}}x_j + n_k\overline{\mu_{ki}}\,\overline{\dot{\varphi}_{i,j}}x_j\right)dS + \overline{\dot{\varphi}_{i,j}}\frac{1}{V}\int\limits_V \mu_{ki,k}x_j dV \\
&= \frac{1}{V}\int\limits_S \left(m_i - \overline{\mu_{ki}}n_k\right)\left(\dot{\varphi}_i - \overline{\dot{\varphi}_{i,j}}x_j\right)dS - \frac{1}{V}\int\limits_V \mu_{ki,k}\left(\dot{\varphi}_i - \overline{\dot{\varphi}_{i,j}}x_j\right)dV
\end{aligned}
\tag{3.8}
$$

Later on it will be useful to note that

$$
\begin{aligned}
-e_{ijk}\left(\overline{\mu_{jk}\dot{\varphi}_k} - \overline{\mu_{jk}}\dot{\varphi}_k\right) &= \frac{1}{V}\int\limits_V e_{kji}\mu_{ji}\varphi_k dV + e_{kji}\overline{\mu_{ji}}\overline{\dot{\varphi}_k} + \\
&\quad -\frac{1}{V}\int\limits_V \dot{\varphi}_k\left(I\ddot{\varphi}_k - \mu_{jk,j}\right)dV + e_{kji}\overline{\mu_{ji}}\overline{\dot{\varphi}_k} \\
&= -\overline{\dot{k}_m} + \frac{1}{V}\int\limits_V \dot{\varphi}_k\mu_{jk,j}dV + e_{kji}\overline{\mu_{ji}}\,\overline{\dot{\varphi}_k}
\end{aligned}
\tag{3.9}
$$

Substituting (3.3), (3.8), and (3.9) into (3.4) results in

$$\overline{\tau_{ji}\dot{\gamma}_{ji}} - \overline{\tau_{ji}}\overline{\dot{\gamma}_{ji}} + \overline{\mu_{ji}\dot{\kappa}_{ji}} - \overline{\mu_{ji}}\overline{\dot{\kappa}_{ji}} = \frac{1}{V}\int\limits_S (t_i - \overline{\tau_{ki}}n_k)\left(\dot{u}_i - \overline{\dot{u}_{i,j}}x_j\right) dS$$
$$+\frac{1}{V}\int\limits_S (m_i - \overline{\mu_{ki}}n_k)\left(\dot{\varphi}_i - \overline{\dot{\varphi}_{i,j}}x_j\right) dS + -\dot{\overline{k_c}} + (\overline{k_c})^{\cdot} - \dot{\overline{k_m}} + \frac{1}{V}\int\limits_V \mu_{ki,k}\overline{\dot{\varphi}_{i,j}}x_j dV + e_{kji}\overline{\mu_{ji}}\overline{\dot{\varphi}_k} \tag{3.10}$$

However, the last two terms on the right-hand side of the above equation are shown to be equal

$$\overline{\dot{\varphi}_{i,j}}\frac{1}{V}\int\limits_V \left[\left(\mu_{ki}x_j\right)_{,k} - \mu_{ki}\delta_{jk}\right] dV + \overline{\dot{\varphi}_k}\frac{1}{V}\int\limits_V e_{kji}\mu_{ji} dV$$
$$= \overline{\dot{\varphi}_{i,j}}\frac{1}{V}\int\limits_S \mu_{ki}x_j n_k dS - \overline{\mu_{ji}}\,\overline{\dot{\kappa}_{ji}} - \overline{\dot{\varphi}_k}\frac{1}{V}\int\limits_V \left(\mu_{jk,j} - I\ddot{\varphi}_k\right) dV$$
$$= \frac{1}{V}\int\limits_S \mu_{ki}n_k\left(\overline{\dot{\varphi}_{i,j}}x_j - \dot{\varphi}_i\right) dS - \overline{\mu_{ji}}\,\overline{\dot{\kappa}_{ji}} + (\overline{k_m})^{\cdot} = (\overline{k_m})^{\cdot} \tag{3.11}$$

As a result, we arrive at the macrohomogeneity condition

$$\overline{\tau_{ji}\dot{\gamma}_{ji}} - \overline{\tau_{ji}}\overline{\dot{\gamma}_{ji}} + \overline{\mu_{ji}\dot{\kappa}_{ji}} - \overline{\mu_{ji}}\,\overline{\dot{\kappa}_{ji}} = \frac{1}{V}\int\limits_S (t_i - \overline{\tau_{ki}}n_k)\left(\dot{u}_i - \overline{\dot{u}_{i,j}}x_j\right) dS$$
$$+\frac{1}{V}\int\limits_S (m_i - \overline{\mu_{ki}}n_k)\left(\dot{\varphi}_i - \overline{\dot{\varphi}_{i,j}}x_j\right) dS - \dot{\overline{k_c}} + (\overline{k_c})^{\cdot} - \dot{\overline{k_m}} + (\overline{k_m})^{\cdot} \tag{3.12}$$

As observed in [6] in the absence of kinetic energies, of all the four combinations of boundary conditions, this relation [i.e. (3.13) below] admits only one set, namely

$$\dot{u}_i = \overline{\dot{u}_{i,j}}x_j \qquad m_i = \overline{\mu_{ki}}n_k \qquad \forall \mathbf{x} \in \partial B \tag{3.13}$$

meaning that, say, no uniform traction boundary conditions ($t_i = \overline{\tau_{ki}} \quad n_k$) may be applied. This is bad news given the well-known lower bound character of that condition—as opposed to the upper bound character of $(3.13)_1$—in studies of scale-dependent homogenization and effective properties in mechanics of random Cauchy materials [1]. This issue motivates the next section.

Clearly, in the quasi-static case (3.12) reduces to

$$\overline{\tau_{ji}\dot{\gamma}_{ji}} - \overline{\tau_{ji}}\,\overline{\dot{\gamma}_{ji}} + \overline{\mu_{ji}\dot{\kappa}_{ji}} - \overline{\mu_{ji}}\,\overline{\dot{\kappa}_{ji}} = \frac{1}{V}\int\limits_S (t_i - \overline{\tau_{ki}}n_k)\left(\dot{u}_i - \overline{\dot{u}_{i,j}}x_j\right) dS$$
$$+ \frac{1}{V}\int\limits_S (m_i - \overline{\mu_{ki}}n_k)\left(\dot{\varphi}_i - \overline{\dot{\varphi}_{i,j}}x_j\right) dS \tag{3.14}$$

while in the non-micropolar (Cauchy) case we recover the classical Hill-Mandel condition

$$\overline{\sigma_{ji}\varepsilon_{ji}} - \overline{\sigma_{ji}}\,\overline{\varepsilon_{ij}} = \frac{1}{V}\int\limits_S (t_i - \overline{\sigma_{ki}}n_k)\left(\dot{u}_i - \overline{\varepsilon_{ij}}x_j\right) dS, \tag{3.15}$$

where $\tau_{ji}$ and $\gamma_{ji}$ have reverted, respectively, to the conventional (symmetric) tensors $\sigma_{ij}$ and $\varepsilon_{ij}$ of the classical continuum theory. Of course, one may also consider a non-micropolar (Cauchy) dynamic case where $-\dot{\overline{k_c}} + (\overline{k_c})^{\cdot}$ would have to be added on the RHS in (3.15).

## 4 Onck's homogenization of a cellular solid

In a separate study, Onck [7] considered homogenization of a micropolar cellular material, with the main result that the total rate of work done on a finite volume $V$ can be expressed in terms of either uniform kinematic or traction boundary conditions involving symmetric as well as anti-symmetric stress and strain, along with the couple-stress and curvature-torsion tensors. Since his study was done in quasi-static setting (i.e. without any inertia terms), we do not express his key equations in rate form here. Thus, in the case of kinematic boundary conditions, we have

$$u_i = \overline{\varepsilon_{ji}} x_j \quad \varphi_i = \frac{1}{2} e_{lji} \overline{\alpha_{jl}} + \overline{\kappa_{ji}} \left( x_j - X_j \right) \tag{4.1}$$

which involves a split of the strain tensor into symmetric and anti-symmetric parts

$$\overline{\gamma_{ji}} = \overline{\varepsilon_{ji}} + \overline{\alpha_{ji}} \quad \overline{\varepsilon_{ji}} \equiv \overline{\gamma_{(ji)}} \quad \overline{\alpha_{ji}} \equiv \overline{\gamma_{[ji]}} \tag{4.2}$$

where $X_j$ are the coordinates of an arbitrary reference point of the domain under consideration, ensuring objectivity of the results. Thus, the work

$$\overline{W} = \overline{t_i} \overline{u_i} + \overline{m_i}\, \overline{\varphi_i} = \overline{\tau_{ji}}\, \overline{\gamma_{ji}} + \overline{\mu_{ji}}\, \overline{\kappa_{ji}} \tag{4.3}$$

becomes

$$\overline{W} = \overline{\sigma_{ji}}\, \overline{\varepsilon_{ji}} + \overline{\beta_{ji}}\, \overline{\alpha_{ji}} + \overline{\mu_{ji}}\, \overline{\kappa_{ji}} \tag{4.4}$$

where

$$\begin{aligned}
\overline{\sigma_{ji}} &= \frac{1}{V} \int_S \left( f_i x_j + f_j x_i \right) ds \\
\overline{\beta_{ji}} &= \frac{1}{2V} e_{lji} \int_S m_i \, dS \\
\overline{\mu_{ji}} &= \frac{1}{V} \int_S m_i \left( x_j - X_j \right) dS
\end{aligned} \tag{4.5}$$

In the above we replace discrete summations over all the cellular surface elements of $V$ appearing in Onck's study by integrals over $\partial V$. Also in the above we use the Cauchy stress tensor split into its symmetric and anti-symmetric parts

$$\overline{\tau_{ji}} = \overline{\sigma_{ji}} + \overline{\beta_{ji}} \quad \overline{\sigma_{ji}} \equiv \overline{\tau_{(ji)}} \quad \overline{\beta_{ji}} \equiv \overline{\tau_{[ji]}} \tag{4.6}$$

Onck has also shown that the total rate of work done on a $dV$ continuum element in the quasi-static case (i.e., in the absence of inertia forces), under traction boundary conditions

$$t_i = \overline{\tau_{ji}} x_j \quad m_i = m_i^0 + \overline{\mu_{ji}} n_j \tag{4.7}$$

where

$$m_i^0 = -\frac{\overline{V_{jk}}}{S} e_{ijl} \beta_{ji} \quad \text{where} \quad \overline{V_{jk}} = \int_S x_j n_k ds \qquad S = \int_S ds^{(k)} \tag{4.8}$$

With $dV$ playing the role of $B$, in the above we again replace the discrete summations over all the surface elements of $B$ by the integrals over $\partial B = S$.

On account of (4.4), these relations lead to the following form of the macrohomogeneity condition for the dynamic case:

$$\overline{\tau_{ji}\dot{\gamma}_{ji}} - \overline{\tau_{ji}}\,\overline{\dot{\gamma}_{ji}} + \overline{\mu_{ji}\dot{\kappa}_{ji}} - \overline{\mu_{ji}}\,\overline{\dot{\kappa}_{ji}} = \frac{1}{V}\int_S \left(t_i - \overline{\tau_{ki}}n_k\right)\left(\dot{u}_i - \overline{\dot{\gamma}_{ji}}x_j\right)dS$$
$$+\frac{1}{V}\int_S \left(m_i - m_i^0 + M_{ji}n_j\right)\left[\dot{\varphi}_i - \frac{1}{2}e_{lji}\overline{\dot{\alpha}_{ji}} + \overline{\dot{\kappa}_{ji}}\left(x_j - X_j\right)\right]dS +$$
$$-\overline{\dot{k}_c} + (\overline{k_c})^{\cdot} - \overline{\dot{k}_m} + (\overline{k_m})^{\cdot} \tag{4.9}$$

Thus, one can now set up four different types of boundary conditions

$$t_i(\mathbf{x}) = \overline{\tau_{ki}}n_k \quad \text{and} \quad m_i(\mathbf{x}) = m_i^0 + \overline{\mu_{ji}}n_j \quad \forall\mathbf{x} \in \partial B \tag{4.10}$$

or

$$t_i(\mathbf{x}) = \overline{\tau_{ki}}n_k \quad \text{and} \quad \dot{\varphi}_i(\mathbf{x}) = \frac{1}{2}e_{lji}\overline{\dot{\alpha}_{ji}} + \overline{\dot{\kappa}_{ji}}\left(x_j - X_j\right) \quad \forall\mathbf{x} \in \partial B \tag{4.11}$$

or

$$m_i(\mathbf{x}) = m_i^0 + \overline{\mu_{ji}}n_j \quad \text{and} \quad \dot{u}_i(\mathbf{x}) = \overline{\dot{\varepsilon}_{ji}}x_j \quad \forall\mathbf{x} \in \partial B \tag{4.12}$$

or

$$\dot{u}_i(\mathbf{x}) = \overline{\dot{\varepsilon}_{ji}}x_j \quad \text{and} \quad \dot{\varphi}_i(\mathbf{x}) = \frac{1}{2}e_{lji}\overline{\dot{\alpha}_{ji}} + \overline{\dot{\kappa}_{ji}}\left(x_j - X_j\right) \quad \forall\mathbf{x} \in \partial B \tag{4.13}$$

In the case of static loading, the rates are removed from these relations.

By a reference to minimum potential and complementary energy principles [8], the first and the fourth of these would have a bounding character, while the second and third would likely result in some intermediate response. In addition, one may also consider mixed-orthogonal boundary conditions

$$\left(t_i(\mathbf{x}) - \overline{\tau_{ki}}n_k\right)\left(\dot{u}_i - \overline{\dot{\varepsilon}_{ji}}x_j\right) = 0$$

and

$$\left(m_i(\mathbf{x}) - m_i^0 + M_{ji}n_j\right)\left[\dot{\varphi}_i - \frac{1}{2}e_{lji}\overline{\dot{\alpha}_{ji}} + \overline{\dot{\kappa}_{ji}}(x_j - X_j)\right] = 0 \quad \forall\mathbf{x} \in \partial B \tag{4.14}$$

which would also lead to some intermediate responses [9]. Note here that, strictly speaking, an unambiguous way of writing $(4.14)_1$ involves orthogonal projections [10]

$$\left(u_i(\mathbf{x}) - u_i^0\right)n_i = 0 \text{ and } \left(\delta_{ij} - n_i n_j\right)\left[\tau_{ij}(\mathbf{u})n_j - t_i(\mathbf{x})\right] = 0 \quad \forall\mathbf{x} \in \partial B \tag{4.15}$$

whereby the same type of equation may be set up for $(4.14)_2$.

## 5 Closure

The condition (4.9) gives admissible boundary loadings, along with extra terms representing kinetic energy contributions of both classical and micropolar type. These loadings involve various combinations of average stresses and strains, along with couple-stresses and curvature-torsion tensors. The proposed boundary conditions may now be applied to a specific micropolar-type microstructure through a computational mechanics approach. These boundary loadings would then allow one to determine the scale-dependent homogenization (or "finite size scaling") toward a representative volume element (RVE) of an equivalent homogeneous micropolar medium in either elastic or inelastic settings, analogous to many studies of such scaling in Cauchy-type composites, see [1] and references therein for a review. It will be most interesting to determine whether the scaling of the classical response (in terms of the $\boldsymbol{\tau}(\boldsymbol{\gamma})$ relation) is faster or slower than the scaling of the micropolar response (in terms of the $\boldsymbol{\mu}(\boldsymbol{\kappa})$ relation) in specific types of random micropolar materials. By restricting the continuum model to an inhomogeneous Cauchy continuum and/or a quasi-static setting, the macrohomogeneity condition simplifies to conventional, well-known versions.

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