Arch Appl Mech (2011) 81: 199–213
DOI 10.1007/s00419-010-0405-z

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Solution of free vibration equations of semi-rigid connected Reddy–Bickford beams resting on elastic soil using the differential transform method

Received: 31 October 2008 / Accepted: 6 January 2010 / Published online: 22 January 2010

Abstract The literature regarding the free vibration analysis of Bernoulli–Euler and Timoshenko beams under various supporting conditions is plenty, but the free vibration analysis of Reddy–Bickford beams with variable cross-section on elastic soil with/without axial force effect using the Differential Transform Method (DTM) has not been investigated by any of the studies in open literature so far. In this study, the free vibration analysis of axially loaded and semi-rigid connected Reddy–Bickford beam with variable cross-section on elastic soil is carried out by using DTM. The model has six degrees of freedom at the two ends, one transverse displacement and two rotations, and the end forces are a shear force and two end moments in this study. The governing differential equations of motion of the rectangular beam in free vibration are derived using Hamilton's principle and considering rotatory inertia. Parameters for the relative stiffness, stiffness ratio and nondimensionalized multiplication factor for the axial compressive force are incorporated into the equations of motion in order to investigate their effects on the natural frequencies. At first, the terms are found directly from the analytical solutions of the differential equations that describe the deformations of the cross-section according to the high-order theory. After the analytical solution, an efficient and easy mathematical technique called DTM is used to solve the governing differential equations of the motion. The calculated natural frequencies of semi-rigid connected Reddy–Bickford beam with variable cross-section on elastic soil using DTM are tabulated in several tables and figures and are compared with the results of the analytical solution where a very good agreement is observed.

Keywords Differential transform method · Elastic soil · Free vibration · Partial differential equation · Reddy–Bickford beam · Semi-rigid connection

1 Introduction

The cross-sectional displacements of Bernoulli–Euler beam theory are shown in Fig. 1a, and expresses as

$$u(x, z, t) = -z \cdot \frac{\partial w_0(x, t)}{\partial x} \tag{1}$$

$$w(x, z, t) = w_0(x, t) \tag{2}$$

where $w_0(x, t)$ is the lateral displacement of the beam neutral axis, z is the distance from the beam neutral axis.

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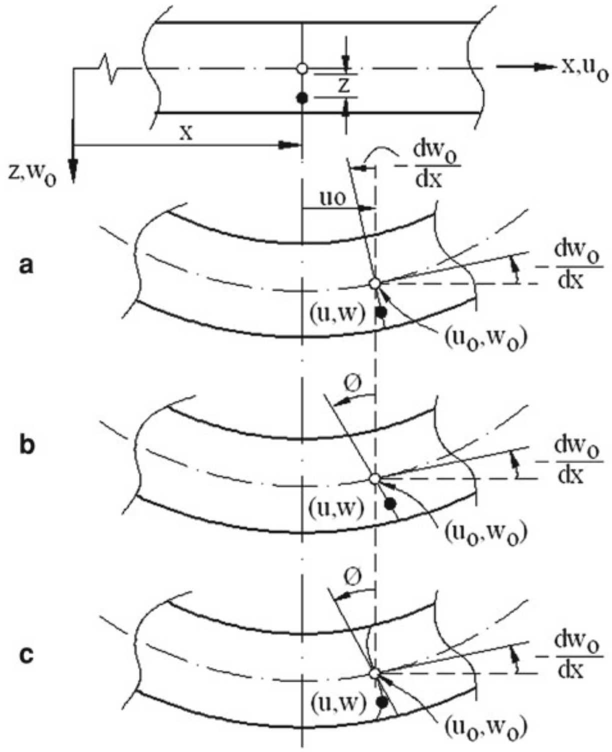

Fig. 1 Cross-section displacements in different beam theories [1]. **a** Bernoulli–Euler beam theory, **b** Timoshenko beam theory, **c** Reddy–Bickford beam theory

Timoshenko beam theory predicts a uniform shear distribution, so necessitating the use of a so-called shear factor [2–5]. The cross-sectional displacements of Timoshenko beam theory are shown in Fig. 1b and the equations for Timoshenko beam theory which relaxes the restriction on the angle of shearing deformations are

$$u(x, z, t) = z \cdot \phi(x, t) \tag{3}$$

$$w(x, z, t) = w_0(x, t) \tag{4}$$

where $\phi(x, t)$ represents the rotation of a normal to the axis of the beam. Han et al. presented a comprehensive study of Bernoulli–Euler, Rayleigh, Shear and Timoshenko beam theories [6].

The real shear deformation distribution is not uniform along the depth of the beam, so that Timoshenko beam theory is not recommended for composite beams, where the accurate determination of the shear stresses is required. Especially, it was found that the Timoshenko shear deformation theory has some major numerical problems such as locking in the numerical analysis for composite materials. The other problem was the need to supply an artificially derived shear correction factor. Although some remedies were devised, as a result, several higher-order theories have emerged. These theories, with small variations, are due to Bickford, Levinson, Heyliger and Reddy, Wang et al. and others all relax the restriction on the warping of the cross-section and allow variation in the longitudinal direction of the beam which is cubic [1,7–9].

In this paper, Reddy–Bickford beam theory is used, which seems a good compromise between accuracy and simplicity [1,7]. The cross-sectional displacements of Reddy–Bickford beam theory are shown in Fig. 1c and according to Reddy–Bickford beam theory, the displacements of the rectangular beam can be written as [1,10,11]:

$$u(x, z, t) = z \cdot \phi(x, t) - \gamma \cdot z^3 \cdot \left[\phi(x, t) + \frac{\partial w(x, t)}{\partial x}\right] \tag{5}$$

$$w(x, z, t) = w_0(x, t) \tag{6}$$

where $\gamma = \frac{4}{3 \cdot h^2}$; h is the height of the beam.

Previously, numerous researchers studied the behavior of the different beams supported by elastic foundations [12–15]. Esmailzadeh and Ohadi investigated vibration and stability analysis of non-uniform Timoshenko beams under axial and distributed tangential loads [16]. Yesilce and Catal calculated normalized natural frequencies of Timoshenko pile due to the different values of axial force using transfer matrix and considering rotatory inertia [17].

DTM was applied to solve linear and non-linear initial value problems and partial differential equations by many researches. The concept of DTM was first introduced by Zhou and he used DTM to solve both linear and non-linear initial value problems in electric circuit analysis [18]. Chen and Ho solved eigenvalue problems for the free and transverse vibration problems of a rotating twisted Timoshenko beam under axial loading by using DTM [19,20]. DTM was applied to solve a second order non-linear differential equation that describes the under damped and over damped motion of a system subject to external excitation by Jang and Chen [21]. After that, numerous researchers solved the first order both the linear and non-linear two-point boundary value problems, linear and non-linear initial value problems, eigenvalue problems and linear partial differential equations by using one or n-dimensional DTM [22–27]. Malih and Dang applied DTM to the free vibration of Bernoulli–Euler beams [28]. Bert and Zeng used DTM to investigate analysis of axial vibration of compound bars [29]. Özdemir and Kaya investigated flapwise bending vibration analysis of a rotating tapered cantilever Bernoulli–Euler beam by DTM [30]. In the other study, the out-of-plane free vibration analysis of a double tapered Bernoulli–Euler beam, mounted on the periphery of a rotating rigid hub is performed using DTM by Ozgumus and Kaya [31]. Catal suggested DTM for the free vibration analysis of both ends simply supported and one end fixed, the other end simply supported Timoshenko beams resting on elastic soil foundation [32,33]. Catal and Catal calculated the critical buckling loads of partially embedded Timoshenko pile in elastic soil by DTM [34]. Ho and Chen investigated the vibration problems of an axially loaded non-uniform spinning twisted Timoshenko beam by using DTM [35]. Bildik et al. expressed the definitions and operations of DTM and Adomian's decomposition method on different partial differential equations [36]. Free vibration analysis of a rotating, double tapered Timoshenko beam featuring coupling between flapwise bending and torsional vibrations is performed using DTM by Ozgumus and Kaya [37]. In the other study, Kaya and Ozgumus introduced DTM to analyze the free vibration response of an axially loaded, closed-section composite Timoshenko beam which features material coupling between flapwise bending and torsional vibrations due to ply orientation [38]. Ertürk and Momani presented a numerical comparison between DTM and Adomian's decomposition method for solving fourth-order boundary value problems [39]. DTM was applied to construct semi numerical-analytic solutions of linear sixth-order boundary value problems with two-point boundary value conditions by Ertürk [40]. Numerical solution to buckling analysis of Bernoulli–Euler beams and columns were obtained using DTM and harmonic differential quadrature for various support conditions considering the variation of flexural rigidity by Rajasekaran [41].

In this study, the free vibration analysis of a rectangular, two layered and semi-rigid connected Reddy–Bickford beam resting on elastic soil is performed. At the beginning of the study, the governing equations of motion are obtained applying Hamilton's principle and Winkler hypothesis and considering rotatory inertia. In the solution part, the equations of motion, including the parameters for the relative stiffness, stiffness ratio and nondimensionalized multiplication factor for the axial compressive force, are solved using analytical method and an efficient mathematical technique, called DTM. Finally, the natural frequencies of Reddy–Bickford beam are calculated and the effects of the parameters, mentioned above, are investigated by using the computer package, Matlab.

2 The mathematical model and formulation

Axially loaded Reddy–Bickford beam with variable cross-section, the rotation flexible ends and translation springs resting on elastic soil, is shown in Fig. 2. It is assumed that the elastic soil that the beam is on behaves due to Winkler hypothesis, the width along the beam length to be constant, the behavior of the material is linear-elastic and the small displacement theory is valid in this paper.

The relation between displacement function $w(x, t)$ of a beam on elastic soil and the distributed force $q(x, t)$ existing at the elastic soil under this beam can be written as:

$$q(x, t) = C_S \cdot w(x, t) \tag{7}$$

where $C_S = C_0 \cdot b$, C_0 is the modulus of subgrade reaction, b is the width of the beam.

Using Hamilton's principle and Eqs. (5) and (6) and considering rotatory inertia, the equations of motion for a rectangular and axially loaded Reddy–Bickford beam with variable cross-section on elastic soil can be written as:

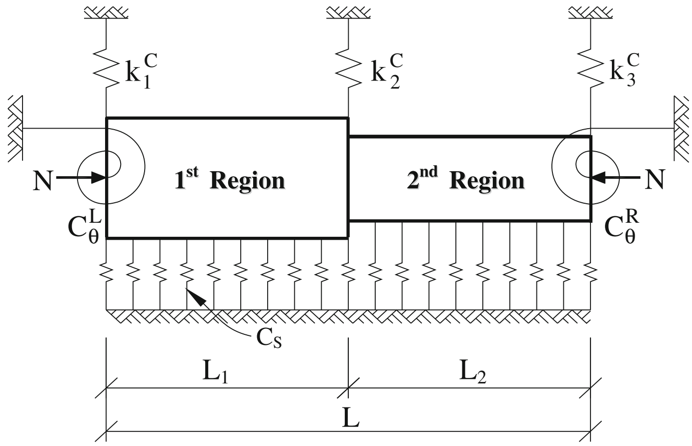

Fig. 2 Axially loaded and semi-rigid connected Reddy–Bickford beam

$$-\frac{68}{105}\cdot EI_j\cdot\frac{\partial^2\phi_j\left(x_j,t\right)}{\partial x_j^2}+\frac{16}{105}\cdot EI_j\cdot\frac{\partial^3 w_j\left(x_j,t\right)}{\partial x_j^3}+\frac{8}{15}\cdot GA_j\cdot\left[\phi_j\left(x_j,t\right)+\frac{\partial w_j\left(x_j,t\right)}{\partial x_j}\right]$$
$$=-\frac{68}{105}\cdot\frac{m_j\cdot I_j}{A_j}\cdot\frac{\partial^2\phi_j\left(x_j,t\right)}{\partial t^2}+\frac{16}{105}\cdot\frac{m_j\cdot I_j}{A_j}\cdot\frac{\partial^3 w_j\left(x_j,t\right)}{\partial x_j\cdot\partial t^2} \tag{8}$$

$$-m_j\cdot\frac{\partial^2 w_j\left(x_j,t\right)}{\partial t^2}+\frac{8}{15}\cdot GA_j\cdot\left[\frac{\partial\phi_j\left(x_j,t\right)}{\partial x_j}+\frac{\partial^2 w_j\left(x_j,t\right)}{\partial x_j^2}\right]+\frac{16}{105}\cdot EI_j\cdot\frac{\partial^3\phi_j\left(x_j,t\right)}{\partial x_j^3}$$
$$-N\cdot\frac{\partial^2 w_j\left(x_j,t\right)}{\partial x_j^2}-\frac{1}{21}\cdot EI_j\cdot\frac{\partial^4 w_j\left(x_j,t\right)}{\partial x_j^4}-C_S\cdot w_j\left(x_j,t\right)$$
$$=-\frac{1}{21}\cdot\frac{m_j\cdot I_j}{A_j}\cdot\frac{\partial^4 w_j\left(x_j,t\right)}{\partial x_j^2\cdot\partial t^2}+\frac{16}{105}\cdot\frac{m_j\cdot I_j}{A_j}\cdot\frac{\partial^3\phi_j\left(x_j,t\right)}{\partial x_j\cdot\partial t^2}$$
$$\left(0\le x_j\le L_j\right)\quad(j=1,2) \tag{9}$$

where $w_j(x_j,t)$ is displacement function for jth region of the beam; $\phi_j(x_j,t)$ represents the rotation of a normal to the axis for jth region of the beam; m_j is mass per unit length, A_j is the cross-section area, I_j is moment of inertia and x_j is the beam position for jth region; L is length of the beam; N is the axial compressive force; t is time variable; E, G are Young's modulus and shear modulus of the beam, respectively.

Assuming that the motion is harmonic we substitute for $w_j(z_j,t)$ and $\phi_j(z_j,t)$ the following:

$$w_j\left(z_j,t\right)=w_j\left(z_j\right)\cdot\sin\left(\omega\cdot t\right) \tag{10}$$

$$\phi_j\left(z_j,t\right)=\phi_j\left(z_j\right)\cdot\sin\left(\omega\cdot t\right)\quad(j=1,2) \tag{11}$$

and obtain a system of two coupled ordinary equation for each region as

$$-\frac{68}{105}\cdot\frac{EI_j}{L^2}\cdot\frac{\mathrm{d}^2\phi_j\left(z_j\right)}{\mathrm{d}z_j^2}+\frac{16}{105}\cdot\frac{EI_j}{L^3}\cdot\frac{\mathrm{d}^3 w_j\left(z_j\right)}{\mathrm{d}z_j^3}+\frac{8}{15}\cdot GA_j\cdot\left[\phi_j\left(z_j\right)+\frac{1}{L}\cdot\frac{\mathrm{d}w_j\left(z_j\right)}{\mathrm{d}z_j}\right]$$
$$=\frac{68}{105}\cdot\frac{m_j\cdot I_j\cdot\omega^2}{A_j}\cdot\phi_j\left(z_j\right)-\frac{16}{105}\cdot\frac{m_j\cdot I_j\cdot\omega^2}{A_j\cdot L}\cdot\frac{\mathrm{d}w_j\left(z_j\right)}{\mathrm{d}z_j} \tag{12}$$

$$m_j \cdot \omega^2 \cdot w_j\left(z_j\right) + \frac{8}{15} \cdot \frac{GA_j}{L} \cdot \left[\frac{\mathrm{d}\phi_j\left(z_j\right)}{\mathrm{d}z_j} + \frac{1}{L} \cdot \frac{\mathrm{d}^2 w_j\left(z_j\right)}{\mathrm{d}z_j^2}\right] + \frac{16}{105} \cdot \frac{EI_j}{L^3} \cdot \frac{\mathrm{d}^3\phi_j\left(z_j\right)}{\mathrm{d}z_j^3}$$
$$-\frac{1}{21} \cdot \frac{EI_j}{L^4} \cdot \frac{\mathrm{d}^4 w_j\left(z_j\right)}{\mathrm{d}z_j^4} - C_S \cdot w_j\left(z_j\right) - \frac{N}{L^2} \cdot \frac{\mathrm{d}^2 w_j\left(z_j\right)}{\mathrm{d}z_j^2}$$
$$= \frac{1}{21} \cdot \frac{m_j \cdot I_j \cdot \omega^2}{A_j \cdot L^2} \cdot \frac{\mathrm{d}^2 w_j\left(z_j\right)}{\mathrm{d}z_j^2} - \frac{16}{105} \cdot \frac{m_j \cdot I_j \cdot \omega^2}{A_j \cdot L} \cdot \frac{\mathrm{d}\phi_j\left(z_j\right)}{\mathrm{d}z_j} \tag{13}$$

where $z_j = \frac{x_j}{L}$ and ω is natural frequency of the beam.

It is assumed that the solution is

$$w_j\left(z_j\right) = C_j \cdot e^{i \cdot s_j \cdot z_j} \tag{14}$$
$$\phi_j\left(z_j\right) = P_j \cdot e^{i \cdot s_j \cdot z_j} \quad (j = 1, 2) \tag{15}$$

where $i = \sqrt{-1}$. Substituting Eqs. (14) and (15) into Eqs. (12) and (13) results in

$$\left(\frac{8}{15} \cdot GA_j - \frac{68}{105} \cdot \frac{m_j \cdot I_j \cdot \omega^2}{A_j} + \frac{68}{105} \cdot \frac{EI_j}{L^2} \cdot s_j^2\right) \cdot P_j$$
$$+ \left(\frac{8}{15} \cdot \frac{GA_j}{L} \cdot s_j \cdot i + \frac{16}{105} \cdot \frac{m_j \cdot I_j \cdot \omega^2}{A_j \cdot L} \cdot s_j \cdot i - \frac{16}{105} \cdot \frac{EI_j}{L^3} \cdot s_j^3 \cdot i\right) \cdot C_j = 0 \tag{16}$$

$$\left(\frac{8}{15} \cdot \frac{GA_j}{L} \cdot s_j \cdot i + \frac{16}{105} \cdot \frac{m_j \cdot I_j \cdot \omega^2}{A_j \cdot L} \cdot s_j \cdot i - \frac{16}{105} \cdot \frac{EI_j}{L^3} \cdot s_j^3 \cdot i\right) \cdot P_j$$
$$+ \left(m_j \cdot \omega^2 - \frac{8}{15} \cdot \frac{GA_j}{L^2} \cdot s_j^2 - \frac{1}{21} \cdot \frac{EI_j}{L^4} \cdot s_j^4 - C_S + \frac{N}{L^2} \cdot s_j^2 + \frac{1}{21} \cdot \frac{m_j \cdot I_j \cdot \omega^2}{A_j \cdot L^2} \cdot s_j^2\right) \cdot C_j = 0 \tag{17}$$

Eqs. (16) and (17) can be written in matrix form for the unknowns P_j and C_j as:

$$\begin{bmatrix} A_{11}^{(j)} & A_{12}^{(j)} \\ A_{21}^{(j)} & A_{22}^{(j)} \end{bmatrix} \cdot \begin{Bmatrix} P_j \\ C_j \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (j = 1, 2) \tag{18}$$

where

$$A_{11}^{(j)} = \frac{8}{15} \cdot GA_j - \frac{68}{105} \cdot \frac{m_j \cdot I_j \cdot \omega^2}{A_j} + \frac{68}{105} \cdot \frac{EI_j}{L^2} \cdot s_j^2 \tag{19a}$$

$$A_{12}^{(j)} = A_{21}^{(j)} = \frac{8}{15} \cdot \frac{GA_j}{L} \cdot s_j \cdot i + \frac{16}{105} \cdot \frac{m_j \cdot I_j \cdot \omega^2}{A_j \cdot L} \cdot s_j \cdot i - \frac{16}{105} \cdot \frac{EI_j}{L^3} \cdot s_j^3 \cdot i \tag{19b}$$

$$A_{22}^{(j)} = m_j \cdot \omega^2 - \frac{8}{15} \cdot \frac{GA_j}{L^2} \cdot s_j^2 - \frac{1}{21} \cdot \frac{EI_j}{L^4} \cdot s_j^4 - C_S + \frac{N}{L^2} \cdot s_j^2 + \frac{1}{21} \cdot \frac{m_j \cdot I_j \cdot \omega^2}{A_j \cdot L^2} \cdot s_j^2 \tag{19c}$$

and the non-trivial solution will be obtained when the determinant of the coefficient matrix will be zero for each region. Thus, we have a sixth-order equation with the unknowns for each region, resulting in six values and the general solution functions for each region can be written as:

$$w_j\left(z_j, t\right) = \left[C_{j1} \cdot e^{i \cdot s_{j1} \cdot z_j} + C_{j2} \cdot e^{i \cdot s_{j2} \cdot z_j} + C_{j3} \cdot e^{i \cdot s_{j3} \cdot z_j} + C_{j4} \cdot e^{i \cdot s_{j4} \cdot z_j} + C_{j5} \cdot e^{i \cdot s_{j5} \cdot z_j} + C_{j6} \cdot e^{i \cdot s_{j6} \cdot z_j}\right] \cdot \sin\left(\omega \cdot t\right) \tag{20}$$

$$\phi_j\left(z_j, t\right) = \left[P_{j1} \cdot e^{i \cdot s_{j1} \cdot z_j} + P_{j2} \cdot e^{i \cdot s_{j2} \cdot z_j} + P_{j3} \cdot e^{i \cdot s_{j3} \cdot z_j} + P_{j4} \cdot e^{i \cdot s_{j4} \cdot z_j} + P_{j5} \cdot e^{i \cdot s_{j5} \cdot z_j} + P_{j6} \cdot e^{i \cdot s_{j6} \cdot z_j}\right] \cdot \sin\left(\omega \cdot t\right) \quad (j = 1, 2) \tag{21}$$

The twenty-four constants, $C_{j1}, \ldots, C_{j6}$ and $P_{j1}, \ldots, P_{j6}$, will be found from Eqs. (16), (17) and boundary conditions.

For each region, the expression for bending rotation $w'_j(z_j, t)$ is given by

$$w'_j\left(z_j, t\right) = \frac{1}{L} \cdot \frac{dw_j\left(z_j\right)}{dz_j} \cdot \sin\left(\omega \cdot t\right) \quad (j = 1, 2) \tag{22}$$

For each region, the shear force function $Q_j(z_j, t)$ can be obtained by using Eqs. (20) and (21) as:

$$\begin{aligned} Q_j\left(z_j, t\right) = &\left[\frac{EI_j}{21 \cdot L^3} \cdot \frac{d^3 w_j\left(z_j\right)}{dz_j^3} + \left(-\frac{8 \cdot GA_j}{15 \cdot L} + \frac{N}{L} + \frac{m_j \cdot I_j \cdot \omega^2}{21 \cdot A_j \cdot L}\right) \cdot \frac{dw_j(z)}{dz_j}\right] \cdot \sin\left(\omega \cdot t\right) \\ &- \left[\frac{16 \cdot EI_j}{105 \cdot L^2} \cdot \frac{d^2 \phi_j\left(z_j\right)}{dz_j^2} + \left(\frac{8 \cdot GA_j}{15} + \frac{16 \cdot m_j \cdot I_j \cdot \omega^2}{105 \cdot A_j}\right) \cdot \phi_j\left(z_j\right)\right] \cdot \sin\left(\omega \cdot t\right) \quad (j = 1, 2) \end{aligned} \tag{23}$$

Similarly, the bending moment function $M_j\left(z_j, t\right)$ can be obtained by using Eqs. (20) and (21) as:

$$M_j\left(z_j, t\right) = \left(-\frac{EI_j}{21 \cdot L^2} \cdot \frac{d^2 w_j\left(z_j\right)}{dz_j^2} - N \cdot w_j(z_j) + \frac{16 \cdot EI_j}{105 \cdot L} \cdot \frac{d\phi_j\left(z_j\right)}{dz_j}\right) \cdot \sin\left(\omega \cdot t\right) \quad (j = 1, 2) \tag{24}$$

For each region, the higher-order moment function $M_{hj}\left(z_j, t\right)$ can be obtained as:

$$M_{hj}\left(z_j, t\right) = \left(\frac{16 \cdot EI_j}{105 \cdot L^2} \cdot \frac{d^2 w_j\left(z_j\right)}{dz_j^2} - \frac{68 \cdot EI_j}{105 \cdot L} \cdot \frac{d\phi_j\left(z_j\right)}{dz_j}\right) \cdot \sin\left(\omega \cdot t\right) \quad (j = 1, 2) \tag{25}$$

3 The differential transform method (DTM)

Partial differential equations are often used to describe engineering problems whose closed form solutions are very difficult to establish in many cases. Therefore, approximate numerical methods are often preferred. However, in spite of the advantages of these on hand methods and the computer codes that are based on them, closed form solutions are more attractive due to their implementation of the physics of the problem and their convenience for parametric studies. Moreover, closed form solutions have the capability and facility to solve inverse problem of determining and designing the geometry and characteristics of an engineering system and to achieve a prescribed behavior of the system. Considering the advantages of the closed form solutions mentioned above, DTM is introduced in this study as the solution method. DTM is a semi-analytic transformation technique based on Taylor series expansion and is a useful tool to obtain analytical solutions of the differential equations. Certain transformation rules are applied and the governing differential equations and the boundary conditions of the system are transformed into a set of algebraic equations in terms of the differential transforms of the original functions in DTM. The solution of these algebraic equations gives the desired solution of the problem. The different from high-order Taylor series method is; Taylor series method requires symbolic computation of the necessary derivatives of the data functions and is expensive for large orders. DTM is an iterative procedure to obtain analytic Taylor series solutions of differential equations [37].

A function $w(z)$, which is analytic in a domain D, can be represented by a power series with a center at $z = z_0$, any point in D. The differential transform of the function $w(z)$ is given by

$$W\left(k\right) = \frac{1}{k!} \cdot \left(\frac{d^k w\left(z\right)}{dz^k}\right)_{z=z_0} \tag{26}$$

where $w(z)$ is the original function and $W(k)$ is the transformed function. The inverse transformation is defined as:

$$w\left(z\right) = \sum_{k=0}^{\infty} \left(z - z_0\right)^k \cdot W\left(k\right) \tag{27}$$

From Eqs. (26) and (27) we get

$$w(z)=\sum_{k=0}^{\infty}\frac{(z-z_0)^k}{k!}\cdot\left(\frac{\mathrm{d}^k w(z)}{\mathrm{d}z^k}\right)_{z=z_0} \tag{28}$$

Equation (28) implies that the concept of the differential transformation is derived from Taylor's series expansion, but the method does not evaluate the derivatives symbolically. However, relative derivative are calculated by iterative procedure that are described by the transformed equations of the original functions. In real applications, the function $w(z)$ in Eq. (27) is expressed by a finite series and can be written as:

$$w(z)=\sum_{k=0}^{\bar{N}}(z-z_0)^k\cdot W(k) \tag{29}$$

Equation (29) implies that $\sum_{k=\bar{N}+1}^{\infty}(z-z_0)^k W(k)$ is negligibly small. Where $\bar{N}$ is series size and the value of $\bar{N}$ depends on the convergence of the eigenvalues.

Theorems that are frequently used in differential transformation of the differential equations and the boundary conditions are introduced in Tables 1 and 2, respectively.

3.1 Using differential transformation to solve motion equations

Equations (12) and (13) can be rewritten as follows:

$$\frac{\mathrm{d}^3 w_j(z_j)}{\mathrm{d}z_j^3}=\left(\frac{17\cdot L}{4}\right)\cdot\frac{\mathrm{d}^2\phi_j(z_j)}{\mathrm{d}z_j^2}-\left(\frac{7}{2}\cdot\beta_j+\frac{\lambda_j^4\cdot I_j}{A_j\cdot L^2}\right)\cdot\frac{\mathrm{d}w_j(z_j)}{\mathrm{d}z_j}+\left(\frac{17\cdot\lambda_j^4\cdot I_j}{4\cdot A_j\cdot L}-\frac{7}{2}\cdot\beta_j\cdot L\right)\cdot\phi_j(z_j) \tag{30}$$

$$\frac{\mathrm{d}^4 w_j(z_j)}{\mathrm{d}z_j^4}=\left(\frac{16\cdot L}{5}\right)\cdot\frac{\mathrm{d}^3\phi_j(z_j)}{\mathrm{d}z_j^3}+\left(\frac{56}{5}\cdot\beta_j-21\cdot N_r^{(j)}\cdot\pi^2-\frac{\lambda_j^4\cdot I_j}{A_j\cdot L^2}\right)\cdot\frac{\mathrm{d}^2 w_j(z_j)}{\mathrm{d}z_j^2}$$
$$+\left(\frac{56}{5}\cdot\beta_j\cdot L+\frac{16\cdot\lambda_j^4\cdot I_j}{5\cdot A_j\cdot L}\right)\cdot\frac{\mathrm{d}\phi_j(z_j)}{\mathrm{d}z_j}+21\cdot\left(\lambda_j^4-\alpha_j\right)\cdot w_j(z_j)\quad(j=1,2) \tag{31}$$

Table 1 DTM theorems used for equations of motion

Original function	Transformed function
$w(z)=u(z)\pm v(z)$	$W(k)=U(k)\pm V(k)$
$w(z)=a\cdot u(z)$	$W(k)=a\cdot U(k)$
$w(z)=\frac{\mathrm{d}^m u(z)}{\mathrm{d}z^m}$	$W(k)=\frac{(k+m)!}{k!}\cdot U(k+m)$
$w(z)=u(z)\cdot v(z)$	$W(k)=\sum_{r=0}^{k}U(r)\cdot V(k-r)$
$w(z)=z^m$	$W(k)=\delta(k-m)=\begin{cases}0 \text{ if } k\neq m\\ 1 \text{ if } k=m\end{cases}$

Table 2 DTM theorems used for boundary conditions

$z=0$		$z=1$	
Original boundary conditions	Transformed boundary conditions	Original boundary conditions	Transformed boundary conditions
$w(0)=0$	$W(0)=0$	$w(1)=0$	$\sum_{k=0}^{\infty}W(k)=0$
$\frac{\mathrm{d}w}{\mathrm{d}z}(0)=0$	$W(1)=0$	$\frac{\mathrm{d}w}{\mathrm{d}z}(1)=0$	$\sum_{k=0}^{\infty}k\cdot W(k)=0$
$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2}(0)=0$	$W(2)=0$	$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2}(1)=0$	$\sum_{k=0}^{\infty}k\cdot(k-1)\cdot W(k)=0$
$\frac{\mathrm{d}^3 w}{\mathrm{d}z^3}(0)=0$	$W(3)=0$	$\frac{\mathrm{d}^3 w}{\mathrm{d}z^3}(1)=0$	$\sum_{k=0}^{\infty}k\cdot(k-1)\cdot(k-2)\cdot W(k)=0$

where

$$\lambda_j = \sqrt[4]{\frac{m_j \cdot \omega^2 \cdot L^4}{EI_j}} \quad \text{(frequency factor of the } j\text{th region)} \tag{32a}$$

$$\alpha_j = \frac{C_S \cdot L^4}{EI_j} \quad \text{(relative stiffness of the } j\text{th region)} \tag{32b}$$

$$\beta_j = \frac{GA_j \cdot L^2}{EI_j} \quad \text{(stiffness ratio of the } j\text{th region)} \tag{32c}$$

$$N_r^{(j)} = \frac{N \cdot L^2}{\pi^2 \cdot EI_j} \quad \text{(nondimensionalized multiplication factor for the axial force of the } j\text{th region)} \tag{32d}$$

Because of the width along the beam length to be constant in this paper, the relationship between the parameters, mentioned above, can be written as:

$$\lambda_1 = \lambda_2 \cdot \sqrt[4]{\left(\frac{m_1}{m_2}\right) \cdot \left(\frac{h_2}{h_1}\right)^3} \tag{33a}$$

$$\alpha_1 = \alpha_2 \cdot \left(\frac{h_2}{h_1}\right)^3 \tag{33b}$$

$$\beta_1 = \beta_2 \cdot \left(\frac{h_2}{h_1}\right)^2 \tag{33c}$$

$$N_r^{(1)} = N_r^{(2)} \cdot \left(\frac{h_2}{h_1}\right)^3 \tag{33d}$$

The differential transform method is applied to Eqs. (30) and (31) by using the theorems introduced in Table 1 and the following expression are obtained:

$$W_j(k+3) = \left(\frac{17 \cdot L}{4}\right) \cdot \frac{\Phi_j(k+2)}{(k+3)} - \left(\frac{7}{2} \cdot \beta_j + \frac{\lambda_j^4 \cdot I_j}{A_j \cdot L^2}\right) \cdot \frac{W_j(k+1)}{(k+2) \cdot (k+3)} + \left(\frac{17 \cdot \lambda_j^4 \cdot I_j}{4 \cdot A_j \cdot L} - \frac{7}{2} \cdot \beta_j \cdot L\right) \cdot \frac{\Phi_j(k)}{(k+1) \cdot (k+2) \cdot (k+3)} \tag{34}$$

$$W_j(k+4) = \left(\frac{16 \cdot L}{5}\right) \cdot \frac{\Phi_j(k+3)}{(k+4)} + \left(\frac{56}{5} \cdot \beta_j - 21 \cdot N_r^{(j)} \cdot \pi^2 - \frac{\lambda_j^4 \cdot I_j}{A_j \cdot L^2}\right) \cdot \frac{W_j(k+2)}{(k+3) \cdot (k+4)} + \left(\frac{56}{5} \cdot \beta_j \cdot L + \frac{16 \cdot \lambda_j^4 \cdot I_j}{5 \cdot A_j \cdot L}\right) \cdot \frac{\Phi_j(k+1)}{(k+2) \cdot (k+3) \cdot (k+4)} + 21 \cdot \left(\lambda_j^4 - \alpha_j\right) \cdot \frac{W_j(k)}{(k+1) \cdot (k+2) \cdot (k+3) \cdot (k+4)} \quad (j = 1, 2) \tag{35}$$

where $W_j(k)$ and $\Phi_j(k)$ are the transformed functions of $w_j(z_j)$ and $\phi_j(z_j)$, respectively.

The behavior of the beam ends that are semi-rigid supported against rotation is modeled by an elastic spring. The left rotational spring constant (C_θ^L) and the right rotational spring constant (C_θ^R) shown in Fig. 2 can be obtained for the same value of the fixity factor as:

$$C_\theta^L = \frac{3 \cdot EI_1 \cdot f}{(1-f) \cdot L} \tag{36a}$$

$$C_\theta^R = \frac{3 \cdot EI_2 \cdot f}{(1-f) \cdot L} \tag{36b}$$

where f is fixity factor [42] and C_θ is the rotational restraining stiffness at the ends of the beam.

The boundary conditions and the transformed boundary conditions by applying the differential transform method, using the theorems introduced in Table 2 are presented in Table 3.

Substituting the transformed boundary conditions into Eqs. (34) and (35) and taking $W_1(0) = c_1$, $W_1(1) = c_2$ and $\Phi_1(0) = c_3$, the following matrix expression is obtained

$$\begin{bmatrix} A_{11}^{(\bar{N})}(\omega) & A_{12}^{(\bar{N})}(\omega) & A_{13}^{(\bar{N})}(\omega) \\ A_{21}^{(\bar{N})}(\omega) & A_{22}^{(\bar{N})}(\omega) & A_{23}^{(\bar{N})}(\omega) \\ A_{31}^{(\bar{N})}(\omega) & A_{32}^{(\bar{N})}(\omega) & A_{33}^{(\bar{N})}(\omega) \end{bmatrix} \cdot \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{37}$$

where c_1, c_2 and c_3 are constants and $A_{i1}^{(\bar{N})}(\omega)$, $A_{i2}^{(\bar{N})}(\omega)$ and $A_{i3}^{(\bar{N})}(\omega)(i = 1, 2, 3)$ are polynomials of ω corresponding $\bar{N}$.

In the last step, for non-trivial solution, equating the coefficient matrix that is given in Eq. (37) to zero one determines the natural frequencies of the vibrating system as is given in Eq. (38).

$$\begin{vmatrix} A_{11}^{(\bar{N})}(\omega) & A_{12}^{(\bar{N})}(\omega) & A_{13}^{(\bar{N})}(\omega) \\ A_{21}^{(\bar{N})}(\omega) & A_{22}^{(\bar{N})}(\omega) & A_{23}^{(\bar{N})}(\omega) \\ A_{31}^{(\bar{N})}(\omega) & A_{32}^{(\bar{N})}(\omega) & A_{33}^{(\bar{N})}(\omega) \end{vmatrix} = 0 \tag{38}$$

The ith estimated eigenvalue, $\omega_i^{(\bar{N})}$ corresponds to $\bar{N}$ and the value of $\bar{N}$ is determined as:

$$\left|\omega_i^{(\bar{N})} - \omega_i^{(\bar{N}-1)}\right| \leq \varepsilon \tag{39}$$

where $\omega_i^{(\bar{N}-1)}$ is the ith estimated eigenvalue corresponding to $(\bar{N} - 1)$ and ε is the small tolerance parameter. If Eq. (39) is satisfied, the ith estimated eigenvalue, $\omega_i^{(\bar{N})}$ is obtained.

Table 3 The boundary conditions and the transformed boundary conditions of two layered and semi-rigid connected Reddy–Bickford beam resting on elastic foundation

Boundary conditions	Transformed boundary conditions
$M_1(z_1 = 0) = -C_\theta^L \cdot \phi_1(z_1 = 0)$	$\bar{M}_1(0) + C_\theta^L \cdot \Phi_1(0) = 0$
$M_{h1}(z_1 = 0) = 0$	$\bar{M}_{h1}(0) = 0$
$T_1(z_1 = 0) = -k_1^C \cdot w_1(z_1 = 0)$	$\bar{T}_1(0) + k_1^C \cdot W_1(0) = 0$
$w_1\left(z_1 = \frac{L_1}{L}\right) = w_2(z_2 = 0)$	$\sum_{k=0}^{\bar{N}} z_1^k \cdot W_1(k) - W_2(0) = 0$
$w_1'\left(z_1 = \frac{L_1}{L}\right) = w_2'(z_2 = 0)$	$\sum_{k=1}^{\bar{N}} k \cdot z_1^{k-1} \cdot W_1(k) - W_2(1) = 0$
$\phi_1\left(z_1 = \frac{L_1}{L}\right) = \phi_2(z_2 = 0)$	$\sum_{k=0}^{\bar{N}} z_1^k \cdot \Phi_1(k) - \Phi_2(0) = 0$
$M_1\left(z_1 = \frac{L_1}{L}\right) = M_2(z_2 = 0)$	$\sum_{k=0}^{\bar{N}} z_1^k \cdot \bar{M}_1(k) - \bar{M}_2(0) = 0$
$M_{h1}\left(z_1 = \frac{L_1}{L}\right) = M_{h2}(z_2 = 0)$	$\sum_{k=0}^{\bar{N}} z_1^k \cdot \bar{M}_{h1}(k) - \bar{M}_{h2}(0) = 0$
$T_1\left(z_1 = \frac{L_1}{L}\right) - k_2^C \cdot w_2(z_2 = 0) = T_2(z_2 = 0)$	$\sum_{k=0}^{\bar{N}} z_1^k \cdot \bar{T}_1(k) - k_2^C \cdot W_2(0) - \bar{T}_2(0) = 0$
$M_2\left(z_2 = \frac{L_2}{L}\right) = -C_\theta^R \cdot \phi_2\left(z_2 = \frac{L_2}{L}\right)$	$\sum_{k=0}^{\bar{N}} z_2^k \cdot \bar{M}_2(k) + C_\theta^R \cdot \sum_{k=0}^{\bar{N}} z_2^k \cdot \Phi_2(k) = 0$
$M_{h2}\left(z_2 = \frac{L_2}{L}\right) = 0$	$\sum_{k=0}^{\bar{N}} z_2^k \cdot \bar{M}_{h2}(k) = 0$
$T_2\left(z_2 = \frac{L_2}{L}\right) = -k_3^C \cdot w_2\left(z_2 = \frac{L_2}{L}\right)$	$\sum_{k=0}^{\bar{N}} z_2^k \cdot \bar{T}_2(k) + k_3^C \cdot \sum_{k=0}^{\bar{N}} z_2^k \cdot W_2(k) = 0$

k_1^C, k_2^C and k_3^C are the translational spring constants, $\bar{M}_j(k)$, $\bar{M}_{hj}(k)$ and $\bar{T}_j(k)$ are the transformed functions of $M_j(z_j)$, $M_{hj}(z_j)$ and $T_j(z_j)$, respectively

4 Numerical analysis and discussions

For numerical analysis, axially loaded and semi-rigid connected Reddy–Bickford beam with variable cross-section shown in Fig. 2 is considered in the paper. Natural frequencies of the beam, ω_i $(i = 1, 2, \ldots, 6)$ are calculated by using computer program prepared in Matlab by authors. Natural frequencies are found by determining values for which the determinant of the coefficient matrix is equal to zero. There are various methods for calculating the roots of the frequency equation. One common used and simple technique is the secant method in which a linear interpolation is employed. The eigenvalues, the natural frequencies, are determined by a trial and error method based on interpolation and the bisection approach. One such procedure consists of evaluating the determinant for a range of frequency values, ω_i. When there is a change of sign between successive evaluations, there must be a root lying in this interval. The iterative computations are determined when the value of the determinant changed sign due to a change of 10^{-4} in the value of ω_i.

The numerical results of this paper are obtained based on a rectangular Reddy–Bickford beam with the following data as:

$$m_1 = 0.50968 \text{ kN s}^2/\text{m}; \quad m_2 = 0.25484 \text{ kN s}^2/\text{m}; \quad \frac{h_2}{h_1} = 0.50; \quad L = 3.0 \text{ m}; \quad L_1 = 2.0 \text{ m};$$
$$EI_1 = 1.900 \times 10^4 \text{ kN m}^2; \quad EI_2 = 2.375 \times 10^3 \text{ kN m}^2; \quad f = 0.50; \quad k_1^C = 1.71 \times 10^6 \text{ kN/m};$$
$$k_2^C = 1.42 \times 10^6 \text{kN/m}; \quad k_3^C = 1.1875 \times 10^6 \text{ kN/m}; \quad \beta_2 = 10, 11 \text{ and } 12; \quad N_r^{(2)} = 0.25 \text{ and } 0.50;$$
$$\alpha_2 = 1, 10, 100, 1000 \text{ and } 100000$$

The values of C_S are calculated due to relative stiffness values of the second region (α_2).

The frequency values of semi-rigid connected Reddy–Bickford beam on elastic soil obtained for the first three modes by using DTM are presented in Tables 4 and 5 being compared with the frequency values obtained by using analytical method for $\beta_2 = 10$ and the different values of relative stiffness (α_2) and nondimensionalized multiplication factor for the axial compressive force $(N_r^{(2)})$ of the second region. The fourth, fifth and sixth natural frequencies of semi-rigid connected Reddy–Bickford beam by using DTM are compared with the frequency values obtained by using analytical method in Tables 6 and 7 for $\beta_2 = 11$ and the different values of α_2 and $N_r^{(2)}$. For $\beta_2 = 12$ and $N_r^{(2)} = 0.25$, the variations of frequency factors (λ_2) due to relative stiffness (α_2) for the first three modes are presented in Fig. 3. Because of the other five couplings between the three values of β_2 and the two values of $N_r^{(2)}$ the variations of frequency factors (λ_2) are almost same, the variations of frequency factors (λ_2) for $\beta_2 = 12$ and $N_r^{(2)} = 0.25$ are presented in this paper.

As the axial compressive force acting to the beam is increased for the other variables (β_2 and α_2) are constant, the natural frequency values of semi-rigid connected Reddy–Bickford beam resting on elastic soil are decreased. This result indicates that, the increasing for the axial compressive force leads to reduction for Reddy–Bickford beam theory. This result is very important for the effect of axial force.

An increase is observed in natural frequency values of the first six modes of Reddy–Bickford beam for the conditions of β_2 and $N_r^{(2)}$ ratio being constant and the values of the relative stiffness are increased. This result indicates that, the increasing for the relative stiffness leads to augmentation in natural frequency values for Reddy–Bickford beam theory.

For the other variables ($N_r^{(2)}$ and α_2) are constant, as the stiffness ratio is increased, an increase is observed in natural frequency values of the first six modes of Reddy–Bickford beam. The increasing for the stiffness ratio leads to augmentation in natural frequency values for Reddy–Bickford beam theory.

In application of DTM, the natural frequency values of axially loaded and semi-rigid connected Reddy–Bickford beam with variable cross-section are calculated by increasing series size N. In Tables 4 and 5, convergences of the first three natural frequencies for $\beta_2 = 10$ are introduced and in Tables 6 and 7, the fourth, fifth and sixth natural frequencies of semi-rigid connected Reddy–Bickford beam by using DTM are compared with the frequency values obtained by using analytical method $\beta_2 = 11$. Here, it is seen that, when the series size is taken 38, the natural frequency values of the third mode can be appeared. Additionally, here it is seen that higher modes appear when more terms are taken into account in DTM applications. The sixth natural frequency values of semi-rigid connected Reddy–Bickford beam can be obtained, when the series size is taken 66. Thus, depending on the order of the required mode, one must try a few values for the term number at the beginning of the calculations in order to find the adequate number of terms.

Table 4 The first three natural frequencies of semi-rigid connected Reddy–Bickford beam on elastic soil for $\beta_2 = 10$ and $N_r^{(2)} = 0.25$

	$\bar{N}$	$\beta_2 = 10$ and $N_r^{(2)} = 0.25$														
		$\alpha_2 = 1$			$\alpha_2 = 10$			$\alpha_2 = 100$			$\alpha_2 = 1,000$			$\alpha_2 = 10,000$		
		ω_1 (rad/s)	ω_2 (rad/s)	ω_3 (rad/s)	ω_1 (rad/s)	ω_2 (rad/s)	ω_3 (rad/s)	ω_1 (rad/s)	ω_2 (rad/s)	ω_3 (rad/s)	ω_1 (rad/s)	ω_2 (rad/s)	ω_3 (rad/s)	ω_1 (rad/s)	ω_2 (rad/s)	ω_3 (rad/s)
Method	30	148.9721	325.7601	379.4185	150.7047	327.2311	379.9467	167.0432	341.0948	388.3812	282.4362	434.9655	482.6331	773.0897	844.4603	1006.0337
	32	148.9722	325.7930	379.1272	150.7049	327.2347	379.9343	167.0434	341.0992	388.0737	282.4363	434.9730	482.5824	773.0898	844.4040	1006.8380
	34	148.9722	325.7965	379.0933	150.7049	327.2350	379.9002	167.0434	341.0997	388.0378	282.4363	434.9739	482.5769	773.0898	844.3957	1006.9486
	36	148.9722	325.7969	379.0899	150.7049	327.2351	379.8966	167.0434	341.0998	388.0341	282.4363	434.9740	482.5763	773.0898	844.3950	1006.9619
	38	148.9722	325.7969	379.0896	150.7049	327.2351	379.8964	167.0434	341.0998	388.0338	282.4363	434.9740	482.5762	773.0898	844.3950	1006.9634
Analytic method		148.9722	325.7969	379.0896	150.7049	327.2351	379.8964	167.0434	341.0998	388.0338	282.4363	434.9740	482.5762	773.0898	844.3950	1006.9634

Table 5 The first three natural frequencies of semi-rigid connected Reddy–Bickford beam on elastic soil for $\beta_2 = 10$ and $N_r^{(2)} = 0.50$

	$\bar{N}$	$\beta_2 = 10$ and $N_r^{(2)} = 0.50$														
		$\alpha_2 = 1$			$\alpha_2 = 10$			$\alpha_2 = 100$			$\alpha_2 = 1,000$			$\alpha_2 = 10,000$		
		ω_1 (rad/s)	ω_2 (rad/s)	ω_3 (rad/s)	ω_1 (rad/s)	ω_2 (rad/s)	ω_3 (rad/s)	ω_1 (rad/s)	ω_2 (rad/s)	ω_3 (rad/s)	ω_1 (rad/s)	ω_2 (rad/s)	ω_3 (rad/s)	ω_1 (rad/s)	ω_2 (rad/s)	ω_3 (rad/s)
Method	30	136.9369	277.7857	356.8485	138.8251	279.5626	357.6331	156.4523	296.6915	365.8362	276.4189	416.0312	451.9703	770.9645	836.5320	992.1143
	32	136.9370	277.7999	356.8044	138.8253	279.5773	357.5889	156.4524	296.6936	365.4493	276.4190	416.0385	451.3651	770.9646	836.5208	992.3502
	34	136.9370	277.8014	356.7998	138.8253	279.5788	357.5842	156.4524	296.6937	365.4034	276.4190	416.0392	451.2905	770.9646	836.5196	992.3797
	36	136.9370	277.8015	356.7994	138.8253	279.5789	357.5839	156.4524	296.6938	365.3986	276.4190	416.0393	451.2823	770.9646	836.5195	992.3830
	38	136.9370	277.8015	356.7993	138.8253	279.5789	357.5838	156.4524	296.6938	365.3981	276.4190	416.0393	451.2815	770.9646	836.5195	992.3833
Analytic method		136.9370	277.8015	356.7993	138.8253	279.5789	357.5838	156.4524	296.6938	365.3981	276.4190	416.0393	451.2815	770.9646	836.5195	992.3833

Table 6 The fourth, fifth and sixth natural frequencies of semi-rigid connected Reddy–Bickford beam on elastic soil for $\beta_2 = 11$ and $N_r^{(2)} = 0.25$

Relative stiffness (α_2)	$\beta_2 = 11$ and $N_r^{(2)} = 0.25$					
	ω_4 (rad/s)		ω_5 (rad/s)		ω_6 (rad/s)	
	DTM ($\bar{N} = 44$)	Analytic Method	DTM ($\bar{N} = 52$)	Analytic method	DTM) ($\bar{N} = 66$)	Analytic method
1	674.5446	674.5446	953.5922	953.5922	1072.1511	1072.1511
10	674.9340	674.9340	954.1051	954.1051	1072.4151	1072.4151
100	678.8154	678.8154	959.2148	959.2148	1075.0565	1075.0565
1,000	716.4421	716.4421	1008.3401	1008.3401	1101.6521	1101.6521
10,000	1126.7674	1126.7674	1302.7454	1302.7454	1442.2163	1442.2163

Table 7 The fourth, fifth and sixth natural frequencies of semi-rigid connected Reddy–Bickford beam on elastic soil for $\beta_2 = 11$ and $N_r^{(2)} = 0.50$

Relative stiffness (α_2)	$\beta_2 = 11$ and $N_r^{(2)} = 0.50$					
	ω_4 (rad/s)		ω_5 (rad/s)		ω_6 (rad/s)	
	DTM ($\bar{N} = 44$)	Analytic method	DTM ($\bar{N} = 52$)	Analytic method	DTM ($\bar{N} = 66$)	Analytic method
1	651.2212	651.2212	872.9341	872.9341	1044.0406	1044.0406
10	651.6281	651.6281	873.5052	873.5052	1044.3011	1044.3011
100	655.6815	655.6815	879.1943	879.1943	1046.9040	1046.9040
1,000	694.8232	694.8232	934.0404	934.0404	1072.7779	1072.7779
10,000	1112.6172	1112.6172	1281.1587	1281.1587	1389.1712	1389.1712

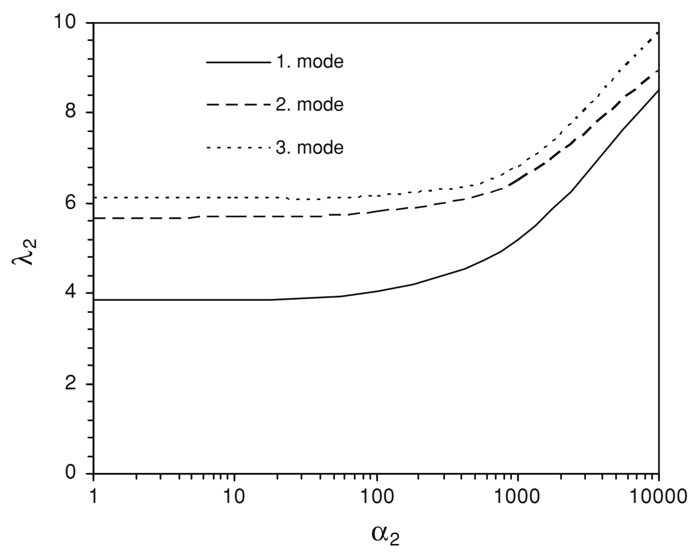

Fig. 3 Variation of frequency factors (λ_2) due to relative stiffness (α_2) for $\beta_2 = 12$ and $N_r^{(2)} = 0.25$

5 Conclusions

In this study, starting from the governing differential equations of motion in free vibration, analytical solution and DTM algorithm are developed by using Reddy–Bickford beam theory and the iterative-based computer programs are developed for solution of linear-homogeneous frequency equation set relating to free vibration of axially loaded and semi-rigid connected beam with variable cross-section resting on elastic soil. Variation in free vibration natural frequencies for the first six modes of the beam is investigated for the different values of the relative stiffness, stiffness ratio and nondimensionalized multiplication factor for the axial compressive force. The calculated natural frequencies of Reddy–Bickford beam on elastic soil by using DTM are compared with the results of the analytical solution. The essential steps of the DTM application includes transforming the governing equations of motion into algebraic equations, solving the transformed equations and then applying a process of inverse transformation to obtain any desired natural frequency. All the steps of the DTM are very straightforward and the application of the DTM to both the equations of motion and the boundary conditions seem to be very involved computationally. However, all the algebraic calculations are finished quickly using symbolic computational software. Besides all these, the analysis of the convergence of the results show that DTM solutions converge fast. When the results of the DTM are compared with the results of analytical method, very good agreement is observed.

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