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Motion of two interconnected mass points under action of non-symmetric viscous friction

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Abstract This paper deals with the analysis of a one-dimensional motion of two mass points in a resistive medium. The force of resistance is described by small non-symmetric viscous friction acting on each mass point. The magnitude of this force depends on the direction of motion. The mass points are interconnected with a kinematic constraint or with an elastic element. Using the averaging method the expressions for the stationary "on the average" velocity of the systems's motion as a single whole is found. In case of a small degree of non-symmetry an explicit expression for the stationary "on the average" velocity of the system is derived. For the other case we obtained algebraic equations for the corresponding stationary velocity.

Keywords Non-symmetric viscous friction · Averaging method · Stationary velocity · Locomotion

1 Introduction

The motion of technical system and living beings happens in resistive media. This resistance to motion that has to be overcome is of many different kinds—depending on speed, size, and the characteristics of the surrounding medium. Therefore, there exist various forms of locomotion in nature and technics. Non-pedal forms of locomotion mostly consist of a conversion of periodic internally driven motions into change of external position. Realization of this type of locomotion is possible in different ways [1].

A large series of papers is devoted to a motion at activity on a surface of contact of a dry friction. The works [2,3] were devoted to snakelike robots, which are modeled as a chain of rigid links connected by actuated revolute joints. Two-link and three-link robots have been investigated in detail. Chernousko [4] investigated the rectilinear motion of a body with a movable internal mass (moving along a straight line parallel to the line

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of the body motion) on a rough plane. Such mechanisms can be fixed in a prescribed position with high degree of accuracy (10^{-8} m) [5], which enable them to be used in high-precision positioning units.

The maximization of average velocity of the steady-state motion of the system actuated by a periodic motion of the internal masses is an important problem for programming control modes for such system. Optimal periodic velocity-controlled motions were founded in [6–8]. The authors in [7,8] considered motions with both dry-friction resistance, piecewise-linear resistance, and quadratic-law viscous resistance.

A number of papers dealt with the rectilinear motion of systems of bodies on a rough, where the bodies are joined by elastic elements in the case when the force of normal pressure is not changed, and the system is moved by forces that change harmonically acting between the bodies. The non-symmetry of the friction force, required for a motion in a given direction, is provided by the dependence of the friction coefficient on the sign of the velocity of constituent bodies of the system. This effect can be achieved if the contact surfaces of the robot are equipped with a special scaly (needle-shaped) plate with a required orientation of scales (needles). In [9,10] of a system of two bodies joined by an elastic element were considered. The motion is excited by harmonic force acting between the bodies.

The rectilinear motion of a vibration-driven robot with two internal masses along a rough horizontal plane with symmetric friction coefficients is considered in [11]. One of the internal masses moves relative to the body along the line of its motion, whereas the other mass moves along the normal one, which makes it possible to influence the normal reaction of the supporting surface. Both masses perform harmonic vibrations with the same frequency but shifted in phase. It is shown that by controlling the phase shift and the frequency of the vibrations of the internal masses one can change the direction of motion of the body and the average velocity of the steady state (velocity-periodic) motion of the robot.

In this paper we consider the rectilinear motion of two mass points in a resistive medium. The force of resistance is described by small non-symmetric (piecewise-linear) viscous friction acting on each mass point. The mass points are interconnected with a kinematic constraint or with an elastic element. The expression for the stationary “on the average” velocity of the system’s motion as a single whole is found.

2 Two mass points subjected to a kinematic constraint

We consider here the so-called asymmetric viscous friction defined by the relation

$$F(v) = -d(v)\,v, \tag{1}$$

where the coefficient of viscous friction $d(v)$ depends on the direction of motion:

$$d(v) = \begin{cases} d_-, & \text{if } v < 0, \\ d_+, & \text{if } v > 0. \end{cases} \tag{2}$$

The value $d(0)$ can be chosen arbitrarily since $F(0) = 0$ for any d.

$$d_- \geq d_+ \geq 0, \quad d_- \neq 0. \tag{3}$$

The dependence (1) is piecewise linear, which gives a satisfactory approximation of the real physical phenomenon.

We will consider several models of objects, the motion of which is influenced by a viscous friction force defined by expression (1).

Consider a system of two mass points moving along the same straight line Ox (see Fig. 1). Let m denote the mass of each mass point and let x_1 and x_2 denote their coordinates.

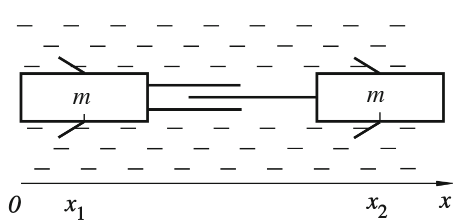

Fig. 1 Two mass points with kinematic constraints

The mass points are subjected to a kinematic constraint that specifies the time history of the distance $2l$ between the mass points, i.e.,

$$x_2 - x_1 = 2\,l(t). \tag{4}$$

We assume the quantity $l(t)$ changes harmonically:

$$l(t) = l_0 + b\,\sin\omega\,t. \tag{5}$$

Here, l_0 is the distance from the center of mass of the system to the center of mass of each of the mass points at the initial time instant, b is the amplitude, and ω is the frequency of the variation of the distance. The mass points are acted upon by the viscous friction forces $F(\dot{x}_1)$ and $F(\dot{x}_2)$, respectively, defined by relations (1) and (2). The law of motion of the center of mass for the system under consideration gives

$$2\,m\,\ddot{x}_C = F(\dot{x}_1) + F(\dot{x}_2), \tag{6}$$

where $x_C = \frac{1}{2}(x_1 + x_2)$ is the coordinate of the center of mass of the system. Let V denote the velocity of the center of mass of the system, i.e., $V = \frac{1}{2}(\dot{x}_1 + \dot{x}_2)$. Then, with reference to the constraint equation (4), the equation of motion (6) can be represented as

$$2\,m\,\dot{V} = F(V - \dot{l}) + F(V + \dot{l}). \tag{7}$$

Introducing the dimensionless variables (denoted by an asterisk), we obtain

$$t^* = \omega\,t, \quad V^* = \frac{V}{b\,\omega}, \quad d^*(V^*) = \frac{d(V)}{d_-}. \tag{8}$$

In terms of the dimensionless variables, Eq. (7), with reference to relation (5), becomes

$$\dot{V} = -\frac{1}{2}\,\frac{d_-}{m\,\omega}\,\left[d(V - \cos\,t)\cdot(V - \cos\,t) + d(V + \cos\,t)\cdot(V + \cos\,t)\right]. \tag{9}$$

In this equation, we preserve the notation used for the primary dimensional variables for the respective dimensionless quantities.

We then subject Eq. (9) to the initial condition

$$V(0) = V_0. \tag{10}$$

The dimensionless coefficient of friction $d(V)$, in accordance with (2), (3), and (8), takes the form

$$d(V) = \begin{cases} 1, & \text{if } V < 0, \\ \kappa, & \text{if } V > 0, \end{cases} \tag{11}$$

where $\kappa = \frac{d_+}{d_-}$, $0 \le \kappa \le 1$. Note that

$$\varepsilon = \frac{d_-}{m\,\omega}. \tag{12}$$

The parameter $\varepsilon = \frac{d_- b\,\omega}{m\,b\,\omega^2}$ characterizes the ratio of the maximum magnitude of the viscous friction force to the maximum excitation force acting on each of the mass points in the motion relative to the center of mass. In what follows, we assume the parameter ε to be small, i.e., $\varepsilon \ll 1$. The parameter ε may be small even for fairly large values of the viscous friction coefficient d_-, if, for example, the frequency ω is high. This is an additional support for considering the case of small ε.

Equation (9) subjected to the initial condition (10) is a standard form equation in terms of the method of averaging. This averaging method is based on the idea of separating the motion into a smooth progression (slow) and a frequent oscillation about the trend (rapid). Therefore, “slow” and “rapid” variables corresponding to these motions are introduced. The system of equations describing oscillation processes can be reduced to a system in the so-called standard form. If the right-hand sides fulfill sufficiently general assumptions this method can be applied. Then the difference between the solution of the averaged system and the solution of the exact system has an order of magnitude of ε on the time interval $[0,\, 1/\varepsilon]$, provided that both sets of equations are subjected to the same initial conditions [12].

Now, we apply the above-described procedure of averaging to Eq. (9), i.e., average the right-hand side with respect to the fast variable t, thus replacing this equation by

$$\dot{V} = -\frac{\varepsilon}{4\pi}\int_0^{2\pi}[d(V-\cos\,t)\cdot(V-\cos t)+d(V+\cos\,t)\cdot(V+\cos\,t)]\,\mathrm{d}t. \tag{13}$$

We preserve the letter V in the averaged equation to denote the velocity of the center of mass of the system. Denote

$$I_1 = \int_0^{2\pi} d(V-\cos\,t)\cdot(V-\cos\,t)\,\mathrm{d}t,\quad I_2 = \int_0^{2\pi} d(V+\cos\,t)\cdot(V+\cos\,t)\mathrm{d}t,$$

$$I_1 = \int_0^{\alpha}(V-\cos\,t)\,\mathrm{d}t + \kappa\int_{\alpha}^{2\pi-\alpha}(V-\cos\,t)\,\mathrm{d}t + \int_{2\pi-\alpha}^{2\pi}(V-\cos\,t)\mathrm{d}t,$$

where $\alpha = \arccos V$. The integration leads to the expression

$$I_1 = \begin{cases} 2\pi\,V, & \text{if } V < -1,\\ 2\pi\,\kappa\,V + 2\,(1-\kappa)\,(V\,\arccos V - \sqrt{1-V^2}), & \text{if } |V| \le 1,\\ 2\pi\,\kappa\,V, & \text{if } V > 1. \end{cases}$$

The Integral $I_2 = I_1$. Finally, Eq. (13) is reduced to the equation

$$\dot{V} = \begin{cases} -\varepsilon\,V, & \text{if } V < -1,\\ -\varepsilon\left[\kappa\,V + \frac{1}{\pi}\,(1-\kappa)\,(V\,\arccos V - \sqrt{1-V^2})\right], & \text{if } |V| \le 1,\\ -\varepsilon\,\kappa\,V, & \text{if } V > 1, \end{cases} \tag{14}$$

subject to the initial condition $V(0) = V_0$. As has already been mentioned, any unsteady motion approaches a steady-state motion as time passes. For that reason, we are interested in steady-state motions, i.e., in motions with a constant velocity. In fact, the velocity of the system will be constant "on average", since small high-frequency vibrations will be imposed on the constant velocity component.

Let

$$P(V,\kappa) = \kappa\,V + \frac{1}{\pi}\,(1-\kappa)\left(V\,\arccos V - \sqrt{1-V^2}\right). \tag{15}$$

Then, the issue of the existence of steady-state solutions of Eq. (14) for $\kappa \neq 0$ is reduced to the issue of the existence of real roots of the equation $P(V,\kappa) = 0$, where $P(V,\kappa)$ is defined by expression (15). Calculate the derivative P'_V of the function $P(V,\kappa)$ with respect to V for $|V| < 1$ to obtain

$$P'_V = \kappa + \frac{1}{\pi}\,(1-\kappa)\,\arccos V > 0. \tag{16}$$

Therefore, for fixed κ, the function $P(V,\kappa)$ increases over the interval $|V| < 1$. In addition, since the values $P(-1,\kappa) = -1$ and $P(0,\kappa) = -\frac{1}{\pi}(1-\kappa)$ are negative for $0 < \kappa < 1$, while the value $P(1,\kappa) = \kappa$ is positive, the equation $P(V,\kappa) = 0$ has a unique root $V = V_s$ for a fixed κ from the interval $0 < \kappa < 1$, and this root lies in the interval $0 < V_s < 1$. Now let $\kappa = 1$, which corresponds to the symmetric linear (rather than piecewise linear) viscous friction ($d_- = d_+$). Then, the equation $P(V,\kappa) = 0$ has the unique root $V_s = 0$, as follows from expression (15). This result is quite natural and expected, since in the case of the symmetric friction, preference cannot be given to any of the directions of motion.

We now assume that there is no friction for the forward motion ($d_+ = 0$), i.e., $\kappa = 0$. In this case, it follows from Eq. (14) that if $V_0 \le 1$ in the initial condition (10), then the steady-state solution of Eq. (14) is defined by the root of the equation $P(V,0) = 0$. In view of condition (15), this root is unique and is given by $V_s = 1$. If $V_0 > 1$, then, as follows from the last condition in Eq. (14), any solution is a steady-state solution to this equation.

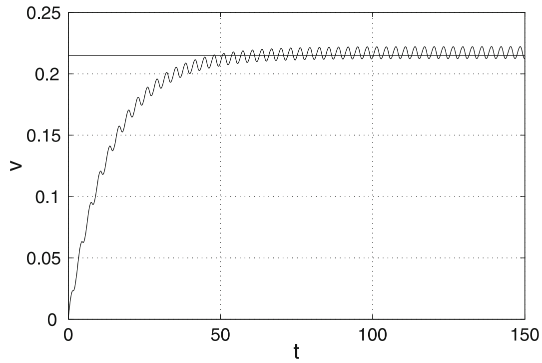

Fig. 2 Numerical solution of the exact equation of motion in comparison with the steady-solution $V_s = 0.217$.

To investigate the dependence of the steady-state velocity $V = V_s$ on κ, we will use the relation

$$\frac{d\,V_s}{d\,\kappa} = -\frac{P'_\kappa(V_s,\kappa)}{P'_V(V_s,\kappa)}. \tag{17}$$

We differentiate the function $P(V,\kappa)$ of (15) with respect to κ to obtain the expression for P'_κ for $|V| < 1$:

$$P'_\kappa = \frac{1}{\pi}\left[V\,(\pi - \arccos V) + \sqrt{1 - V^2}\right] > 0. \tag{18}$$

Based on the Eqs. (16)–(18), we conclude that $\frac{d\,V_s}{d\,\kappa} < 0$ and, hence, the value of the steady-state velocity decreases as κ increases.

We further investigate the stability of the steady-state solution found. The variational equation for $0 \le V_s < 1$ has the form

$$\delta\dot{V} = -P'_V(V_s,\kappa)\delta V.$$

Since $P'_V > 0$ for $0 \le V_s < 1$ according to (16), the steady-state solution $V = V_s$ is stable. For $\kappa = 0$, the stability of the steady-state solution follows from inequality (16) for $|V| \le 1$ and, in addition, from the fact that Eq. (14) for $V > 1$ has the form $\dot{V} = 0$. Figure 2 shows the result of the numerical solution of the exact Eq. (9) for $\varepsilon = 0.1$ and $\kappa = 0.5$. The steady-state solution of the averaged system (14) found from the equation $P(V,\kappa) = 0$ is given by $V_s = 0.217$. It is apparent from Fig. 2 that there is rather good agreement between the exact solution and the steady-state solution of the averaged system for chosen $\varepsilon = 0.1$. In conclusion, we consider the case where the coefficients of viscous friction d_- and d_+ are close to each other, i.e., $\delta = 1 - \kappa \ll 1$. In this case, it is possible to obtain an asymptotic expression for the velocity V_s as a function of κ. To that end, we seek the solution $V = V_s$ of the equation $P(V,\kappa) = 0$ in the form of a series in terms of powers of δ:

$$V_s = V_{s0} + \delta\,V_{s1} + \delta^2\,V_{s2} + \cdots\,. \tag{19}$$

By substituting the expansion of (19) into expression (15) for $P(V,\kappa)$ and expanding the functions that occur in this expression into a power series, we find $V_{s0} = 0$, $V_{s1} = \frac{1}{\pi}$, $V_{s2} = \frac{1}{2\pi}$. Finally, we obtain

$$V_s = \frac{1}{2\,\pi}\,(1-\kappa)\,(3-\kappa) + o(1-\kappa)^2. \tag{20}$$

Figure 3 shows a plot of the functions $V_s(\kappa)$ obtained from the equation $P(V,\kappa) = 0$ (curve 1) and from asymptotic expression (20) (curve 2). For $0.6 \le \kappa \le 1$, the plots almost coincide. Even for $\kappa = 0.5$ expression (20) gives $V_s = 0.199$, which is close to the steady-state solution $V_s = 0.217$ found from the equation $P(V,\kappa) = 0$.

We will now consider a number of other problems that can be reduced to the problem just solved.

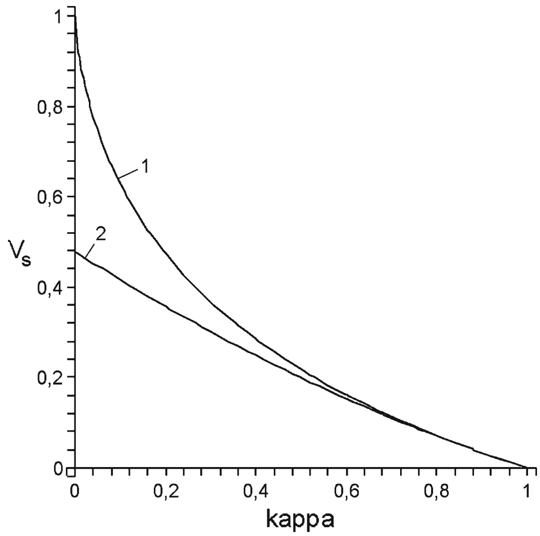

Fig. 3 Dependence V_s versus κ

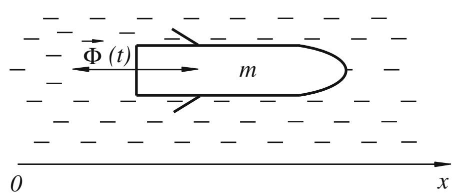

Fig. 4 A rigid body acted upon by a periodic force

3 A rigid body acted upon by a periodic force

We consider a rigid body of mass m that moves along a straight line Ox (see Fig. 4). The body is acted upon by the force of viscous friction $F(v)$ defined by Eq. (1), and the exciting harmonic force $\Phi(t)$ with amplitude B and frequency ω:

$$\Phi(t) = B\,\sin\omega\,t.$$

Then, the velocity V of the center of mass is governed by the equation

$$m\,\dot{V} = F(V) + B\,\sin\omega t. \tag{21}$$

Introducing the dimensionless variables (denoted by the asterisk), we obtain

$$t^* = \omega\,t, \quad V^* = \frac{V}{L\,\omega}, \quad d^*(V^*) = \frac{d(V)}{d_-}, \quad L = \frac{B}{m\,\omega^2}.$$

In terms of the dimensionless variables, Eq. (21) becomes

$$\dot{V} = -\varepsilon\,d(V) + \sin\,t. \tag{22}$$

The parameter $\varepsilon = \frac{d_-}{m\,\omega}$ coincides with that of Eq. (12). This parameter can be represented as $\varepsilon = \frac{d_-}{m\,\omega} = \frac{d_- L\,\omega}{m\,L\,\omega^2} = \frac{d_- L\,\omega}{B}$ and, hence, characterizes the ratio of the maximum magnitude of the viscous friction force to the maximum magnitude of the excitation force. We introduce the change of variables as

$$u = V + \cos t \tag{23}$$

to represent Eq. (23)

$$\dot{u} = -\varepsilon\,d(u - \cos t) \cdot (u - \cos t). \tag{24}$$

In what follows, we again assume the parameter ε to be small, which means that the force due to viscous friction is small as compared to the excitation force.

Equation (24) is a standard form equation, to which the procedure of averaging can be applied. The structure of Eq. (24) coincides with that of Eq. (9). The right-hand side of Eq. (9) is multiplied by the coefficient $\frac{\varepsilon}{2}$ but contains the sum of two terms that have identical averages. Therefore, one can make use of the result of the averaging of Eq. (9), in which it should be replaced by u, to obtain

$$\dot{u} = -\varepsilon \begin{cases} u, & \text{if } u < -1, \\ \kappa\, u + \frac{1}{\pi}(1-\kappa)\,(u \arccos u - \sqrt{1-u^2}), & \text{if } |u| \le 1, \\ \kappa\, u, & \text{if } u > 1. \end{cases} \tag{25}$$

All conclusions of the previous section regarding the steady-state solution of Eq. (14) remain valid for Eq. (25).

When returning to the variable V in accordance with expression (23), we obtain $V = u_s - \cos t$, where u_s is the steady-state solution of Eq. (25). Hence, u_s is the average velocity of the body in the steady-state motion, since the average of the function $\cos t$ over a period is equal to zero.

4 A rigid body with a moving internal mass

Consider a rigid body of mass m_0 (the main body) that moves along a straight line Ox. The body is acted upon by the force of viscous friction applied by the environment. The friction is expressed by Eq. (1). Inside the main body, there is an internal body of mass m_1 that interacts with the main body and moves relative to it along a straight line parallel to the axis Ox. Introducing the axis $C\xi$ attached to the main body that passes through the center of mass C of this body and is parallel to the axis Ox and denoting the coordinate of the center of mass of the main body relative to a inertial reference frame as x, the coordinate identifying the position of the internal body relative to the main body as ξ, and the coordinate of the center of mass of the system of two bodies relative to the fixed frame as x_C (see Fig. 5).

The motion equation of the internal body relative to the main body is assumed to be specified by a function $\xi(t)$. The equation of motion of the center of mass of the system has the form

$$m\,\ddot{x}_C = F(\dot{x}), \tag{26}$$

where $m = m_0 + m_1$ is the total mass of the system. We substitute the expression $x_C = \frac{m_0\,x + m_1\,(x+\xi)}{m}$ for the coordinate of the center of mass of the system into Eq. (26) to obtain

$$m\,\ddot{x} = F(\dot{x}) - m_1\,\ddot{\xi}. \tag{27}$$

Let the motion equation of the internal body be specified as follows:

$$\xi(t) = b\,\sin\omega\,t. \tag{28}$$

Then Eq. (27) becomes

$$\ddot{x} = F(\dot{x}) + m_1\,b\,\omega^2\,\sin\omega\,t. \tag{29}$$

By introducing the notation $V = \dot{x}$ and $B = m_1\,b\,\omega^2$ Eq. (29) is reduced to the relation that coincides with Eq. (21) and governs the motion of a body under the action of a harmonic force.

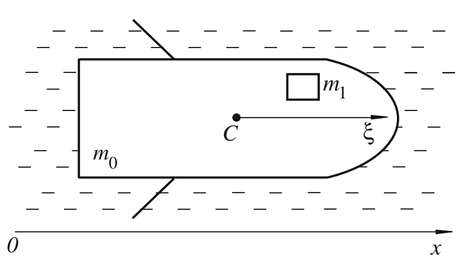

Fig. 5 A rigid body with a moving internal mass

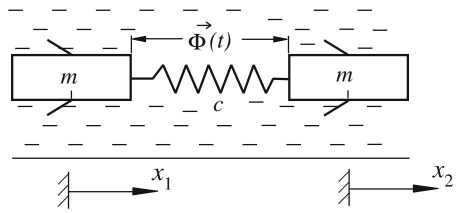

Fig. 6 Two bodies connected by a spring

5 Two bodies connected by a spring

Previously, we dealt with single-degree-of-freedom mechanical systems. We now consider a model that represents a mechanical system with two degrees of freedom.

Let two bodies, each of mass m, move along a straight line Ox, see Fig. 6. The bodies are connected by a spring of stiffness c. Each of the bodies is acted upon by the viscous friction force $F(v)$ defined by equation (1).

The motion of the system is excited by a harmonic force $\Phi(t)$ with amplitude B and frequency ω acting between the bodies:

$$\Phi(t) = B\,\sin\omega\,t.$$

The equations governing the motion are

$$\begin{aligned} m\,\ddot{x}_1 + c\,(x_1 - x_2) &= -\,F(\dot{x}_1) + B\,\sin\omega\,t,\\ m\,\ddot{x}_2 + c\,(x_2 - x_1) &= -\,F(\dot{x}_2) - B\,\sin\omega\,t, \end{aligned} \tag{30}$$

where x_1 and x_2 denote the coordinates of the centers of mass of the bodies relative to a inertial reference frame. Introducing dimensionless variables (again using an asterisk), we obtain

$$x_i^* = \frac{x_i}{L},\quad t^* = t\sqrt{\frac{c}{m}},\quad d^*(\dot{x}_i^*) = \frac{d(\dot{x}_i)}{d_-},\quad \nu = \omega\sqrt{\frac{m}{c}},\quad \varepsilon = \frac{d_-}{\sqrt{m\,c}},\quad \beta = \frac{B}{d_- L}\sqrt{\frac{m}{c}},$$

where L is a unit of length.

In terms of the dimensionless variables, the equations of motion become

$$\begin{aligned} \ddot{x}_1 + x_1 - x_2 &= -\varepsilon\,d(\dot{x}_1)\,\dot{x}_1 + \varepsilon\,\beta\,\sin\nu\,t,\\ \ddot{x}_2 + x_2 - x_1 &= -\varepsilon\,d(\dot{x}_2)\,\dot{x}_2 - \varepsilon\,\beta\,\sin\nu\,t. \end{aligned} \tag{31}$$

Here,

$$d(\dot{x}_i) = \begin{cases} 1, & \text{if } \dot{x}_i < 0,\\ \kappa, & \text{if } \dot{x}_i > 0, \end{cases} \tag{32}$$

where $\kappa = \frac{d_+}{d_-}$, $0 \le \kappa \le 1$ and $i = 1, 2$.

In what follows, we assume that ε is a small parameter, while the quantities β and $|x_2 - x_1|$ are on the following orders:

$$\varepsilon \ll 1,\quad \beta \sim 1,\quad |x_2 - x_1| \sim 1. \tag{33}$$

According to the definition of ε and β, the first two relations of (33) can be written as

$$\frac{d_-}{\sqrt{mc}} \ll 1,\quad \frac{B}{d_- L}\sqrt{\frac{m}{c}} \sim 1. \tag{34}$$

The relative motion of the system's bodies is governed by the equation

$$\ddot{\eta} + 2\,\eta = -2\,\varepsilon\,\beta\,\sin\nu t - \varepsilon\,[d(\dot{x_2})\,\dot{x_2} - d(\dot{x_1})\,\dot{x_1}]\,, \tag{35}$$

where $\eta = x_2 - x_1$.

This equation is obtained by subtracting the first equation of (31) from its second equation. We ignore the viscous friction (the second term in the right-hand side of (35)) to estimate the order of magnitude of the quantity $|x_2 - x_1|$. Then we construct a partial solution of the Eq. (35) that corresponds to the forced oscillations. If $\nu \neq \sqrt{2}$, this solution is

$$\eta = -\frac{2\varepsilon\beta}{2-\nu^2}\sin\nu t.$$

The maximum of the absolute value of this relation is given by

$$|\eta|_{\max} = \frac{2\varepsilon\beta}{|2-\nu^2|}. \tag{36}$$

By substituting the expressions for ε, β and ν into (36) we obtain

$$|\eta|_{\max} = \frac{2\beta}{L|2c-\omega^2 m|}.$$

We choose the length scale L such that $|\eta|_{\max} = 1$, i.e.,

$$L = \frac{2\beta}{|2c-\omega^2 m|},$$

to provide the relation $|x_2 - x_1| \sim 1$ in the dimensionless units.

With such a choice, the conditions (34), validating the utilization of the method of averaging, become

$$\frac{d_-}{\sqrt{mc}} \ll 1, \quad \frac{|2c-\omega^2 m|}{2d_-}\sqrt{\frac{m}{c}} \sim 1.$$

The natural (resonant) frequency corresponds to the relative oscillations of the system's bodies in the absence of friction and excitation ($\varepsilon = 0$). These oscillations are governed by (35) with zero right-hand side. The general solution of the equation for the relative oscillations has the form

$$\eta = A\cos\left(\sqrt{2}t + \theta_0\right),$$

where A (amplitude) and θ_0 (initial phase of the oscillations) are arbitrary constants. The natural oscillations occur with the cyclic frequency $\nu_n = \sqrt{2}$.

To enable the method of averaging to be applied to system (31), we reduce this system to a standard form. To that end, we introduce the change of variables that has already been utilized:

$$\begin{aligned} &x_1 = X - a\,\cos\varphi, \quad x_2 = X + a\,\cos\varphi, \\ &\dot{X} = V, \quad \dot{x}_1 = V + a\sqrt{2}\,\sin\varphi, \quad \dot{x}_2 = V - a\sqrt{2}\,\sin\varphi, \\ &\varphi = \sqrt{2}\,t + \theta, \end{aligned} \tag{37}$$

where X is the coordinate of the center of mass, V its velocity, and φ, a, ϑ are functions of time.

We rewrite the system (31) in the form

$$\begin{aligned} \ddot{X} &= -\frac{\varepsilon}{2}\left[d(\dot{x}_2)\dot{x}_2 + d(\dot{x}_1)\dot{x}_1\right], \\ \ddot{x}_2 - \ddot{x}_1 + 2(x_2 - x_1) &= -\varepsilon\left[d(\dot{x}_2)\dot{x}_2 - d(\dot{x}_1)\dot{x}_1\right] - 2\varepsilon\beta\sin\nu t, \end{aligned} \tag{38}$$

and from the relation (37) we have

$$\begin{aligned} x_2 - x_1 &= 2a\cos\varphi = 2a\cos\left(\sqrt{2}t + \vartheta\right), \\ \dot{x}_2 - \dot{x}_1 &= -2\sqrt{2}a\sin\varphi = -2\sqrt{2}\sin\left(\sqrt{2}t + \vartheta\right). \end{aligned} \tag{39}$$

From the first equation (39) we obtain

$$\dot{x}_2 - \dot{x}_1 = 2\dot{a}\cos\varphi - 2a\left(\sqrt{2} + \dot{\vartheta}\right)\sin\varphi. \tag{40}$$

Comparing (40) with the second equation (39) we find

$$-2\sqrt{2}a\sin\varphi = 2\dot{a}\cos\varphi - 2a\left(\sqrt{2}+\dot{\vartheta}\right)\sin\varphi$$

and from here

$$\dot{\vartheta} = \frac{\dot{a}}{a}\cdot\frac{\cos\varphi}{\sin\varphi}. \tag{41}$$

Differentiating the second equation (39) we obtain

$$\ddot{x}_2 - \ddot{x}_1 = -2\sqrt{2}\dot{a}\sin\varphi - 2\sqrt{2}a\left(\sqrt{2}+\dot{\vartheta}\right)\cos\varphi. \tag{42}$$

Then the second equation (38) takes the form

$$-2\sqrt{2}\dot{a}\sin\varphi - 2\sqrt{2}a\dot{\vartheta}\cos\varphi = -\varepsilon\left[d(\dot{x}_2)\dot{x}_2 - d(\dot{x}_1)\dot{x}_1\right] - 2\varepsilon\beta\sin\nu t. \tag{43}$$

We will consider the behavior of the system in the neighborhood of the resonance and assume that the difference of the excitation frequency ν from the resonant frequency $\nu_n = \sqrt{2}$ has on order of magnitude of ε, i.c.

$$\nu = \sqrt{2} + \varepsilon\varDelta,$$

where the constant parameter $\varDelta$ has an order of unity. For that, we introduce the new slow variable ξ:

$$\xi = \nu t - \varphi, \quad \dot{\xi} = \nu - \dot{\varphi} = -\dot{\theta} + \varepsilon\,\varDelta. \tag{44}$$

Using relations (38) and (41)–(44), we represent system (31) in a standard form:

$$\begin{aligned}
\dot{V} &= -\frac{\varepsilon}{2}\Big[d\left(V - a\sqrt{2}\,\sin\varphi\right)\cdot\left(V - a\sqrt{2}\,\sin\varphi\right)\\
&\quad + d\left(V + a\sqrt{2}\,\sin\varphi\right)\cdot\left(V + a\sqrt{2}\,\sin\varphi\right)\Big],\\
\dot{a} &= \frac{\varepsilon}{2\sqrt{2}}\sin\varphi\Big[d\left(V - a\sqrt{2}\,\sin\varphi\right)\cdot\left(V - a\sqrt{2}\,\sin\varphi\right)\\
&\quad - d\,(V + a\sqrt{2}\,\sin\varphi)\cdot\left(V + a\sqrt{2}\,\sin\varphi\right) + 2\,\beta\,\sin(\xi+\varphi)\Big],\\
\dot{\xi} &= -\frac{\varepsilon}{2\,a\sqrt{2}}\cos\varphi\Big[d\left(V - a\sqrt{2}\,\sin\varphi\right)\cdot\left(V - a\sqrt{2}\,\sin\varphi\right)\\
&\quad - d\left(V + a\sqrt{2}\,\sin\varphi\right)\cdot\left(V + a\sqrt{2}\,\sin\varphi\right) + 2\,\beta\,\sin(\xi+\varphi)\Big] + \varepsilon\,\varDelta.
\end{aligned} \tag{45}$$

Assuming $\varepsilon \ll 1$, we average the right-hand side of system (45) with respect to the fast variable φ to obtain

$$\begin{aligned}
\dot{V} &= \begin{cases} -\varepsilon\,a\,u\sqrt{2}, & u < -1,\\ -\frac{\varepsilon}{\sqrt{2}}\,a\left[u\,(1+\kappa) - \frac{2}{\pi}(1-\kappa)\,(u\,\arcsin\,u + \sqrt{1-u^2})\right], & |u| \le 1,\\ -\varepsilon\,\kappa\,a\,u\sqrt{2}, & u > 1,\end{cases}\\
\dot{a} &= \begin{cases} \frac{\varepsilon}{2}\,(-a + \frac{\beta}{\sqrt{2}}\,\cos\xi), & u < -1,\\ \frac{\varepsilon}{2}\Big[\frac{a}{\pi}\,(1-\kappa)\,(\arcsin u + u\sqrt{1-u^2}) & \\ \qquad -\frac{(1+\kappa)}{2}\,a + \frac{\beta}{\sqrt{2}}\,\cos\xi\Big], & |u| \le 1,\\ \frac{\varepsilon}{2}\,(-a\,\kappa + \frac{\beta}{\sqrt{2}}\,\cos\xi), & u > 1,\end{cases}\\
\dot{\xi} &= -\frac{\varepsilon}{2}\left(\frac{\beta}{a\sqrt{2}}\,\sin\xi - 2\,\varDelta\right),
\end{aligned} \tag{46}$$

where $u = \frac{V}{a\sqrt{2}}$.

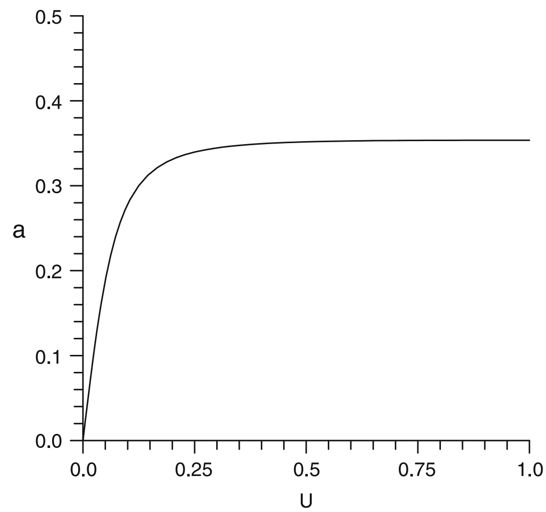

Fig. 7 Dependence of a_s on u

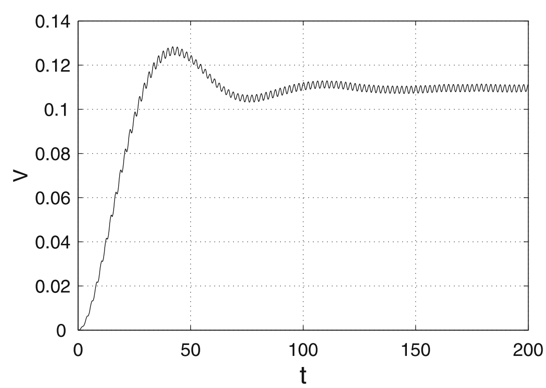

Fig. 8 Results of the numerical solution of the exact equation

We are interested in the solutions that correspond to the motion of the entire system with a constant velocity V. Then, it follows from system (46) that the amplitude a and the phase ξ are also constant.

The steady-state values of the variables u and a are defined (after eliminating ξ) from the system of transcendental equations

$$\begin{aligned} P(u,\kappa) &= u\,(1+\kappa) - \frac{2}{\pi}\,(1-\kappa)\,\left(u \text{ arcsin } u + \sqrt{1-u^2}\right) \\ &= 2\left[\kappa u + \frac{1}{\pi}(1-\kappa)\left(u \text{ arccos}\, u - \sqrt{1-u^2}\right)\right] = 0, \\ a &= \frac{\pi\,\beta\,|u|}{\sqrt{2\,(1-\kappa)^2\,(1-u^2)^3 + 8\,\pi^2\,\Delta^2\,u^2}}. \end{aligned} \tag{47}$$

The expression in square brackets relative u coincides with expression (15) relative V. Hence, Eq. (47) $P(u,\kappa)=0$ has the same roots, as the equation $P(V,\kappa)=0$, where $P(V,\kappa)$ is given by expression (15). Thus, all conclusions for a solution $V=V_s$ of an equation $P(V,\kappa)=0$ are kept for a solution $u=u_s$ of an Eq. (47).

Based on this u_s, we determine the stationary amplitude a from the second equation of (47). The dependence $a_s = a_s(u_s)$ is presented in Fig. 7. In the case $\kappa = 0.5$ we first find $u_s = 0.22$ from (47), and then $a_s = a_s(u_s)$. This yields $V_s = 0.33 \cdot 0.22 \cdot \sqrt{2} = 0.10$.

Thus, having found a unique value $u = u_s$ from the first equation of (47), we use the second equation to determine the amplitude $a = a_s$ and then find a unique value of the steady-state velocity $V_s = a_s\, u_s\,\sqrt{2}$.

Figure 8 shows the results of the numerical solution of the exact equation (31) for $\varepsilon = 0.1$, $\kappa = 0.5$, $\Delta = 1$, and $\beta = 1$.

Figure 8 shows a good agreement of the stationary velocity value $V_s = 0.10$ (from the averaged system equations) with the numerical solution of the exact equation.

If the coefficients of viscous friction d_- and d_+ are closed other, i.e., $\delta = 1 - \kappa \ll 1$ we obtain

$$u_s = \frac{1}{2\pi}(1-\kappa)(3-\kappa) + o(1-\kappa)^2, \quad a_s = \frac{1}{2\sqrt{2}} \cdot \frac{\beta}{\Delta} + o(1-\kappa).$$

Finally, the expression for $V_s = a_s u_s \sqrt{2}$ is

$$V_s = \frac{\beta}{4\pi\Delta}(1-\kappa)(3-\kappa) + o(1-\kappa)^2. \tag{48}$$

For $\beta = 1$, $\delta = 1$, $\kappa = 0.5$ the expression (48) gives $V_s = \frac{5}{16\pi} \approx 0.099$.

6 Conclusion

We have considered the rectilinear motion of two interconnected mass points in a resistive medium. Using the averaging method the dependence of the stationary velocity on the non-symmetry of the coefficient of viscosity was found. In case of a small degree of non-symmetry we derived an explicit expression for the stationary "on the average" velocity of the system. For the other case we obtained algebraic equations for the corresponding stationary velocity. It depends monotonously on the degree of non-symmetry.

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