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First passage failure of quasi non-integrable generalized Hamiltonian systems

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Abstract The first passage failure of quasi non-integrable generalized Hamiltonian systems is studied. First, the generalized Hamiltonian systems are reviewed briefly. Then, the stochastic averaging method for quasi non-integrable generalized Hamiltonian systems is applied to obtain averaged Itô stochastic differential equations, from which the backward Kolmogorov equation governing the conditional reliability function and the Pontryagin equation governing the conditional mean of the first passage time are established. The conditional reliability function and the conditional mean of first passage time are obtained by solving these equations together with suitable initial condition and boundary conditions. Finally, an example of power system under Gaussian white noise excitation is worked out in detail and the analytical results are confirmed by using Monte Carlo simulation of original system.

Keywords Quasi non-integrable generalized Hamiltonian system · Stochastic averaging · First passage failure

1 Introduction

In the past few decades, there has been considerable interest in the study of reliability of stochastic dynamical systems due to its practical significance in determining the performance of a wide range of systems. The first passage is the most important failure model in stochastic dynamics but it is very difficult to obtain the first passage probability. The known exact solution of the first passage problem is limited to the one-dimensional diffusion process. Therefore, several numerical methods such as generalized cell-mapping procedure [1], Monte Carlo simulation [2] and the singular perturbation [3] have been proposed to obtain the statistic of first passage problem for higher-dimensional stochastic systems in the literature. At present, a powerful approximate technique for analyzing the first passage problem of higher-dimensional stochastic systems is the combination approach of the stochastic averaging method and the diffusion process theory of the first passage time, which has been applied by many authors (for example, see [4–9] and the references therein).

In practice, many systems in science and engineering are of odd dimension, which can be modeled as stochastically excited and dissipated generalized Hamiltonian systems. Such systems may be classified into five groups based on the integrability and resonance of the associated generalized Hamiltonian systems. An n-dimensional quasi non-integrable generalized Hamiltonian system is a generalized Hamiltonian system

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with M Casimir functions $C_1, \ldots, C_M$ and one first integral H subject to lightly linear and (or) nonlinear dampings and weakly random excitations. To the authors' knowledge, the first passage failure of quasi non-integrable generalized Hamiltonian systems has not been studied so far.

In the present paper, the equations governing quasi non-integrable generalized Hamiltonian system are reduced to a set of averaged Itô stochastic differential equations by using the stochastic averaging method. Then, the backward Kolmogorov equation governing the conditional reliability function and the Pontryagin equation governing the conditional mean of the first passage time are derived from the averaged equations. Finally, a three-dimensional power system subject to Gaussian white noise excitation is taken as an example to illustrate the proposed procedure. The numerical results for the example are verified by using those from Monte Carlo simulation of original system.

2 Generalized Hamiltonian systems

An n-dimensional dynamical system governed by

$$\dot{x}_i = J'_{ij}(\mathbf{x})\,\partial H'/\partial x_j, \quad i, j = 1, \ldots, n \tag{1}$$

is called a generalized Hamiltonian system. In Eq. (1), $\mathbf{x} = [x_1, \ldots, x_n]^{\mathrm{T}}$ is a state vector; the dot denotes the derivative with respect to time t; $H' = H'(\mathbf{x})$ is the twice differentiable generalized Hamiltonian; $[J'_{ij}(\mathbf{x})]$ is an $n \times n$ anti-symmetric structural matrix , which satisfies the Jacobi identities [10] and, therefore, provides a generalized Poisson bracket

$$[F, G] = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial F}{\partial x_i} J'_{ij}(\mathbf{x}) \frac{\partial G}{\partial x_j} \tag{2}$$

for two dynamical quantities $F(\mathbf{x})$ and $G(\mathbf{x})$ in phase space.

A function $F = F(\mathbf{x})$ is called a first integral of system (1) if $[F, H] = 0$. A function $C = C(\mathbf{x})$ is called Casimir function if $[C, G] = 0$, where $G = G(\mathbf{x})$ is any real-valued function. Obviously, a Casimir function is a first integral of the system (usually generalized Hamiltonian systems). A generalized Hamiltonian system having only one first integral except Casimir functions is called the non-integrable generalized Hamiltonian system.

3 Stochastic averaging

Consider a stochastically excited and dissipative generalized Hamiltonian system governed by the following equations:

$$\dot{X}_i = J'_{ij}(\mathbf{X}) \frac{\partial H'}{\partial X_j} + \varepsilon d'_{ij}(\mathbf{X}) \frac{\partial H'}{\partial X_j} + \varepsilon^{1/2} f_{is}(\mathbf{X})\, W_s(t), \quad i, j = 1, \ldots, n;\ s = 1, \ldots, l \tag{3}$$

where $\mathbf{X} = [X_1, \ldots, X_n]^{\mathrm{T}}$; $[J'_{ij}(\mathbf{X})]$ is an $n \times n$ anti-symmetric structural matrix; $H'(\mathbf{X})$ is twice differentiable generalized Hamiltonian; $\varepsilon d'_{ij}(\mathbf{X})$ and $\varepsilon^{1/2} f_{is}(\mathbf{X})$ are the coefficients of dampings and the amplitudes of stochastic excitations, respectively; ε is a small positive parameter; $W_s(t)$ are Gaussian white noises with intensities $\mathrm{E}[W_s(t)W_z(t+\tau)] = 2D_{sz}\delta(\tau), s, z = 1, \ldots, l$.

Equation (3) can be modeled as Strationvich stochastic differential equations and then converted into Itô stochastic differential equations by adding the Wong-Zakai correction terms $D_{sz} f_{js} f_{iz}/\dot{X}_j$. Splitting the Wong-Zakai correction terms into conservative part and dissipative part, and combining the two parts with $J'_{ij}(\mathbf{X})\partial H'/\partial X_j$ and $d'_{ij}(\mathbf{X})\partial H'/\partial X_j$, respectively, Eq. (3) is converted into the following Itô equations:

$$dX_i = \left[J_{ij}(\mathbf{X}) \frac{\partial H}{\partial X_j} + \varepsilon d_{ij}(\mathbf{X}) \frac{\partial H}{\partial X_j}\right] dt + \varepsilon^{1/2} \sigma_{is}(\mathbf{X})\, dB_s(t) \quad i, j = 1, \ldots, n;\ s = 1, \ldots, l \tag{4}$$

where $[J_{ij}(\mathbf{X})]$ is a modified structural matrix; H is a modified Hamiltonian; $\varepsilon d_{ij}(\mathbf{X})$ is the coefficients of modified dampings; $B_s(t)$ are the standard Wiener processes and $\boldsymbol{\sigma}\boldsymbol{\sigma}^T = 2\mathbf{f}\mathbf{D}\mathbf{f}^{\mathrm{T}}$.

Assume that the generalized Hamiltonian system governed by Eq. (4) with $\varepsilon = 0$ is non-integrable. Then, the Eq. (4) describes a quasi non-integrable generalized Hamiltonian system. By using Itô differential rule, Eq. (4) can be converted into following equations:

$$\begin{aligned} dC_m &= \varepsilon\left(d_{ij}\frac{\partial C_m}{\partial x_i}\frac{\partial H}{\partial x_j} + \frac{1}{2}\sigma_{is}\sigma_{js}\frac{\partial^2 C_m}{\partial x_i \partial x_j}\right)dt + \varepsilon^{1/2}\sigma_{is}\frac{\partial C_m}{\partial x_i}dB_s(t) \\ dH &= \varepsilon\left(d_{ij}\frac{\partial H}{\partial x_i}\frac{\partial H}{\partial x_j} + \frac{1}{2}\sigma_{is}\sigma_{js}\frac{\partial^2 H}{\partial x_i \partial x_j}\right)dt + \varepsilon^{1/2}\sigma_{is}\frac{\partial H}{\partial x_i}dB_s(t) \\ dX_v &= \left(J_{vj}\partial H/\partial X_j + \varepsilon d_{vj}\partial H/\partial X_j\right)dt + \varepsilon^{1/2}\sigma_{vs}dB_s \\ &\quad m = 1,\ldots,M;\ i,j = 1,\ldots,n;\ v = M+2,\ldots,M+n;\ s = 1,\ldots,l \end{aligned} \tag{5}$$

In Eq. (5), $C_1,\ldots,C_M$ and H are slowly varying processes while $X_{M+1},\ldots,X_n$ are rapidly varying process. According to a theorem due to Khasminskii [11], $C_1,\ldots,C_M$ and H weakly converge to a $M+1$-dimensional diffusion process as $\varepsilon \to 0$, in a time interval $0 \le t \le T$, where $T \sim \mathrm{O}(\varepsilon^{-1})$. The limiting processes for $C_1,\ldots,C_M$ and H can be described by the following averaged Itô stochastic differential equations:

$$\begin{aligned} dC_m &= \varepsilon\bar{U}_m(C_1,\ldots,C_M,H)dt + \varepsilon^{1/2}\bar{\sigma}_{ms}(C_1,\ldots,C_M,H)dB_s(t) \\ dH &= \varepsilon\bar{U}_H(C_1,\ldots,C_M,H)dt + \varepsilon^{1/2}\bar{\sigma}_{Hs}(C_1,\ldots,C_M,H)dB_s(t) \\ &\quad m = 1,\ldots,M;\ s = 1,\ldots,l \end{aligned} \tag{6}$$

where

$$\begin{aligned} \bar{U}_m &= \frac{1}{T}\int_\Omega \left[\left(d_{ij}\frac{\partial C_m}{\partial x_i}\frac{\partial H}{\partial x_j} + \frac{1}{2}\sigma_{is}\sigma_{js}\frac{\partial^2 C_m}{\partial x_i \partial x_j}\right)\Big/ A\right]\mathrm{d}\mathbf{X}' \\ \bar{U}_H &= \frac{1}{T}\int_\Omega \left[\left(d_{ij}\frac{\partial H}{\partial x_i}\frac{\partial H}{\partial x_j} + \frac{1}{2}\sigma_{is}\sigma_{js}\frac{\partial^2 H}{\partial x_i \partial x_j}\right)\Big/ A\right]\mathrm{d}\mathbf{X}' \\ \bar{b}_{m_1 m_2} &= \bar{\sigma}_{m_1 s}\bar{\sigma}_{m_2 s} = \frac{1}{T}\int_\Omega \left[\left(\sigma_{is}\sigma_{js}\frac{\partial C_{m_1}}{\partial x_i}\frac{\partial C_{m_2}}{\partial x_j}\right)\Big/ A\right]\mathrm{d}\mathbf{X}' \\ \bar{b}_{m_1 H} &= \bar{\sigma}_{m_1 s}\bar{\sigma}_{Hs} = \frac{1}{T}\int_\Omega \left[\left(\sigma_{is}\sigma_{js}\frac{\partial C_{m_1}}{\partial x_i}\frac{\partial H}{\partial x_j}\right)\Big/ A\right]\mathrm{d}\mathbf{X}' \\ \bar{b}_{HH} &= \bar{\sigma}_{Hs}\bar{\sigma}_{Hs} = \frac{1}{T}\int_\Omega \left[\left(\sigma_{is}\sigma_{js}\frac{\partial H}{\partial x_i}\frac{\partial H}{\partial x_j}\right)\Big/ A\right]\mathrm{d}\mathbf{X}' \\ T &= \int_\Omega \left[1\Big/ A\right]\mathrm{d}\mathbf{X}' \\ \mathbf{X}' &= \left[X_{M+2},\ldots,X_n\right]^{\mathrm{T}} \\ \Omega &= \{(X_{M+2},\ldots,X_n)\,|C_m(\mathbf{X}) \le C_m;\ H(\mathbf{X}) \le H\} \\ &\quad m_1, m_2, m = 1,\ldots,M;\ s = 1,\ldots,l \end{aligned} \tag{7}$$

in which $\mathrm{A} = \left|\partial\left(C, H, \mathbf{X}'\right)/\partial\mathbf{X}\right|$ is absolute value of Jacobi determinant.

4 Backward Kolmogrov equation and Pontryagin equation

For most non-integrable generalized Hamiltonian systems, H represents the total energy while C_i may represent the energies of sub-systems. Suppose that C_i vary in interval $[C^i_{\min}, C^i_{\max})$, where $i = 1,\ldots M$, and H varies in interval $[H_{\min}, H_{\max})$. Accordingly, the system (6) is safe or in normal operation when $C_i < C^i_{\max}$

and $H < H_{\max}$ and the first passage failure occurs when $C_i \geq C^i_{\max}$ or $H \geq H_{\max}$, i.e., the safety domain Ω_s is the interior of a super-rectangle with boundaries consisting of Γ^i_1, Γ_2, Γ^i_3 and Γ_4,i.e.,

$$\begin{aligned}
&\Gamma^i_1 : C_i = C^i_{\max}, \quad H_{\min} \leq H < H_{\max}, \quad C^j_{\min} \leq C_j < C^j_{\max}, \quad i \neq j \\
&\Gamma_2 : C^i_{\min} \leq C_i < C^i_{\max}, \quad H = H_{\max} \\
&\Gamma^i_3 : C_i = C^i_{\min}, \quad H_{\min} \leq H < H_{\max}, \quad C^j_{\min} \leq C_j < C^j_{\max}, \quad i \neq j \\
&\Gamma_4 : C^i_{\min} \leq C_i < C^i_{\max}, \quad H = H_{\min}
\end{aligned} \tag{8}$$

where Γ^i_1 and Γ_2 are the absorbing boundaries while Γ^i_3 and Γ_4 are the reflecting boundaries.

The conditional reliability function, denoted by $R(t|\mathbf{C}_0, H_0)$, is defined as the probability of processes $[\mathbf{C}, H]^T$ being in the safety domain Ω_s within the time interval $0 < \tau \leq t$ given initial condition $[\mathbf{C}_0, H_0]^T$ being in the safety domain Ω_s, i.e.,

$$R\left(t|\mathbf{C}_0, H_0\right) = \text{Prob}\left\{\left(\mathbf{C}\left(\tau\right), H\left(\tau\right)\right) \in \Omega_s,\ \tau \in (0, t] \,|\, (\mathbf{C}_0, H_0) \in \Omega_s\right\} \tag{9}$$

It is the integral of the conditional transition probability density in Ω_s, which is the transition probability density of the sample functions which remain in safety domain Ω_s in time interval $(0, t]$. For diffusion process $[\mathbf{C}, H]^{\mathrm{T}}$, the conditional transient probability density is governed by the backward Kolmogorov equation with drift and diffusion coefficients defined by Eq. (7). Therefore, the conditional reliability function is governed by the following backward Kolmogorov equation:

$$\begin{aligned}
\frac{1}{\varepsilon}\frac{\partial R}{\partial t} = &\sum_{m=1}^{M}\left(\bar{U}_m\left(\mathbf{C}_0, H_0\right)\frac{\partial R}{\partial C_{0m}}\right) + \bar{U}_H\left(\mathbf{C}_0, H_0\right)\frac{\partial R}{\partial H_0} \\
&+\frac{1}{2}\sum_{m_1=1}^{M}\sum_{m_2=1}^{M}\left(\bar{b}_{m_1 m_2}\left(\mathbf{C}_0, H_0\right)\frac{\partial^2 R}{\partial \mathbf{C}_{0v_1}\partial \mathbf{C}_{0v_2}}\right) + \sum_{m_1=1}^{M}\bar{b}_{m_1 H}\left(\mathbf{C}_0, H_0\right)\frac{\partial^2 R}{\partial C_{0m_1}\partial H_0} \\
&+\frac{1}{2}\bar{b}_{HH}\left(\mathbf{C}_0, H_0\right)\frac{\partial^2 R}{\partial H_0^2}
\end{aligned} \tag{10}$$

in which the coefficients are defined by Eq. (11) with $[\mathbf{C}, H]^{\mathrm{T}}$ replaced by $[\mathbf{C}_0, H_0]^{\mathrm{T}}$. The associated boundary conditions are

$$R\left(t|\mathbf{C}_0, H_0\right) = 0 \quad \text{at } \Gamma^i_1 \text{ and } \Gamma_2 \tag{11}$$

$$R\left(t|\mathbf{C}_0, H_0\right) = \text{finite} \quad \text{at } \Gamma^i_3 \text{ and } \Gamma_4 \tag{12}$$

The initial condition is

$$R\left(0|\mathbf{C}_0, H_0\right) = 1, \quad [\mathbf{C}_0, H_0]^{\mathrm{T}} \in \Omega_1 \tag{13}$$

The conditional probability density of the first passage time can be derived from the conditional reliability function as follows:

$$p\left(\tau|\mathbf{C}_0, H_0\right) = -\left.\frac{\partial R\left(t|\mathbf{C}_0, H_0\right)}{\partial t}\right|_{t=\tau} \tag{14}$$

where τ is the first passage time.

Define the conditional mean of the first passage time of system (6)

$$\mu\left(\mathbf{C}_0, H_0\right) = \int_0^\infty T p\left(T|\mathbf{C}_0, H_0\right)\mathrm{d}T = \int_0^\infty R\left(T|\mathbf{C}_0, H_0\right)\mathrm{d}T \tag{15}$$

The Pontryagin equation governing the conditional mean of the first passage time can be derived from Eq. (10) in terms of relationship (15) as follows:

$$
\begin{aligned}
-\frac{1}{\varepsilon} = & \sum_{m=1}^{M}\left(\bar{U}_m(\mathbf{C}_0, H_0)\frac{\partial \mu}{\partial C_{0m}}\right) + \bar{U}_H(\mathbf{C}_0, H_0)\frac{\partial \mu}{\partial H_0} \\
& + \frac{1}{2}\sum_{m_1=1}^{M}\sum_{m_2=1}^{M}\left(\bar{b}_{m_1 m_2}(\mathbf{C}_0, H_0)\frac{\partial^2 \mu}{\partial C_{0v_1}\partial C_{0v_2}}\right) \\
& + \sum_{m_1=1}^{M}\bar{b}_{m_1 H}(\mathbf{C}_0, H_0)\frac{\partial^2 \mu}{\partial C_{0m_1}\partial H_0} + \frac{1}{2}\bar{b}_{HH}(\mathbf{C}_0, H_0)\frac{\partial^2 \mu}{\partial H_0^2}
\end{aligned} \tag{16}
$$

The associated boundary conditions are derived from Eqs. (11) and (12) in term of relationship (15). They are

$$\mu(\mathbf{C}_0, H_0) = 0 \quad \text{at } \Gamma_1^i \text{ and } \Gamma_2 \tag{17}$$

$$\mu(\mathbf{C}_0, H_0) = \text{finite} \quad \text{at } \Gamma_3^i \text{ and } \Gamma_4 \tag{18}$$

Note that the boundary conditions at Γ_3^i and Γ_4 are qualitative rather than quantitative and can be replaced by quantitative ones by using Eqs. (10) and (16), respectively, according to the limiting behavior of the drift and diffusion coefficients in Eqs. (10) and (16) at Γ_3^i and Γ_4, respectively.

The conditional reliability function can be obtained by solving the backward Kolmogorov equation (10) together with its boundary conditions (11) and (12) and initial condition (13). The conditional probability density of the first passage time is obtained by using Eq. (14). The conditional mean of the first passage time can be obtained either by using Eq. (15) or directly by solving the Pontryagin equation (16) together with its boundary conditions (17) and (18) numerically.

5 Example

Consider a single machine infinite power system with steam control governed by following equations:

$$
\begin{aligned}
\dot{\delta} &= \omega_0 \omega \\
\dot{\omega} &= \frac{P_m}{M} - \frac{D\omega}{M} - P_0(1 + \xi_1(t))\sin\delta \\
\dot{P}_m &= -\frac{P_m - P_m^0}{T_s} + \frac{\delta - \delta_0}{T_s} + u
\end{aligned} \tag{19}
$$

where δ is the rotor angle; ω is the rotor speed; P_m is the mechanical power; $\omega_0 = 2\pi f_0$; M is the inertia coefficient of a generator; D is the damping coefficient; $P_0(1 + \xi_1(t))$ is the load and its random perturbation; T_s is the time constant of steam volume; $\xi_1(t)$ is Gaussian white noise with intensity $D_1 = 0.25$, respectively; u is the steam control and here $u = 0$ is assumed.

Let $x_1 = \delta$, $x_2 = \omega$, $x_3 = \mathrm{P_m}$, $x_1^0 = \delta_0$, $x_3^0 = \mathrm{P_m^0}$, $a = 1/M$, $b = P_0$, $c = D/M\omega_0$, $d = 1/T_s$, $f = D/M$, $e = \omega_0$ can be rewritten as follows:

$$
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \left(\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & d/a \end{pmatrix}\right)\begin{pmatrix} -ax_3 + ab\sin x_1 \\ ex_2 \\ a(x_3 - x_3^0 - x_1 + x_1^0) \end{pmatrix} + \begin{pmatrix} 0 \\ -b\sin x_1 \\ 0 \end{pmatrix}\xi_1(t) \tag{20}
$$

Then, Eq. (20) can be converted into the following Itô stochastic differential equations:

$$
\begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix} = \left[J(x)\frac{\partial H}{\partial x} - R(x)\frac{\partial H}{\partial x}\right]dt + \begin{pmatrix} 0 \\ -b\sin x_1 \\ 0 \end{pmatrix} dB_1(t) \tag{21}
$$

where

$$J(x)=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix},\quad R(x)=\begin{pmatrix}0&0&0\\0&c&0\\0&0&d/a\end{pmatrix},\quad \frac{\partial H}{\partial x}=\begin{pmatrix}-ax_3+ab\sin x_1\\ ex_2\\ a(x_3-x_3^0-x_1+x_1^0)\end{pmatrix},$$

$$H(x_1,x_2,x_3)=\frac{1}{2}ex_2^2-ax_3\left(x_1-x_1^0\right)+\frac{1}{2}a\left(x_3-x_3^0\right)^2-ab\left(\cos x_1-\cos x_1^0\right) \tag{22}$$

in which $\mathbf{x}_e^s=[x_1^s,0,x_3^s]^{\mathrm{T}}$ is the stable equilibration point of system.

The generalized Hamiltonian system associated with system (21) has one Casimir function $C(x)=(x_3-x_3^0)^2/2$ and one first integral $H(x)$, where $H(x)$ is the total energy of the system (21) and $C(x)$ is the energy of steam system associated with the system (21).

By using the stochastic averaging method proposed in Sect. 3, the averaged Itô equations for C and H can be derived as follows:

$$\begin{aligned}dC&=\bar{m}_1(C,H)dt\\ dH&=\bar{m}_H(C,H)dt+\bar{\sigma}_H(C,H)dB_1(t)\end{aligned} \tag{23}$$

where

$$\begin{aligned}\bar{m}_1&=\frac{1}{T}\int_{x_{20}}^{x_{10}}\left[-d\left(x_3-x_3^0\right)\left(x_3-x_3^0-x_1+x_1^0\right)/x_2\right]\mathrm{d}x_1\\
\bar{m}_H&=\frac{1}{T}\int_{x_{20}}^{x_{10}}\left\{\left[-fex_2^2-da\left(x_3-x_3^0-x_1+x_1^0\right)^2+D_1eb^2\sin^2 x_1\right]/x_2\right\}\mathrm{d}x_1\\
\bar{b}_{HH}&=\tilde{\sigma}_{Hs}\tilde{\sigma}_{Hs}=\frac{1}{T}\int_{x_{20}}^{x_{10}}\left[\left(2D_1e^2b^2x_2^2\sin^2 x_1\right)/x_2\right]\mathrm{d}x_1\\
T&=\int_{x_{20}}^{x_{10}}[1/x_2]\mathrm{d}x_1\end{aligned} \tag{24}$$

In Eq. (24), $x_2=\sqrt{2(H-aC+ax_3(x_1-x_1^0)+ab(\cos x_1-\cos x_1^0))/e}$ and $x_3=\sqrt{2C}+x_3^0$. The upper and lower limits of integrals, $x_{10}(H,C)$ and $x_{20}(H,C)$, are the two roots of equation $-ax_3(x_1-x_1^0)-ab(\cos x_1-\cos x_1^0)=H-aC$, respectively.

For conservative system associated with system (21), there are two equilibration points: stable equilibration point $x_e^s=[x_1^s,0,x_3^s]^{\mathrm{T}}$ and unstable equilibration point $\mathbf{x}_e^u=[x_1^u,0,x_3^u]^{\mathrm{T}}$. From stable equilibration point we get the $H_{\min}=H(\mathbf{x}_e^s)=0$, while we can get the critical value $H_{\max}=H(\mathbf{x}_e^u)$ from unstable equilibration point. Besides, the energy of steam system, $C(x)$, has its normal limit, i.e., $C_{\min}\le C<C_{\max}$. Thus, the safety domain $\Omega_s$of system (21) is the interior of a rectangle with boundaries consisting of Γ_1, Γ_2, Γ_3 and Γ_4, i.e.,

$$\begin{aligned}&\Gamma_1: C=C_{\max},\ H_{\min}\le H<H_{\max}\\ &\Gamma_2: C_{\min}\le C<C_{\max},\ H=H_{\max}\\ &\Gamma_3: C=C_{\min},\ H_{\min}<H<H_{\max}\\ &\Gamma_4: C_{\min}\le C<C_{\max},\ H=H_{\min}\end{aligned} \tag{25}$$

where the boundaries Γ_1 and Γ_2 are absorbing boundaries while the boundaries Γ_3 and Γ_4 are reflecting boundaries (See Fig. 1a). However, since the electrical power should not be larger than the mechanical power [12], i.e., $b\sin x_1\le x_3$, the reflecting boundary Γ_4 should be replaced by the reflecting boundary Γ_5 as follows:

$$\Gamma_5: C_{\min}\le C<C_{\max},\ H_{\min}\le H<H_{\max} \tag{26}$$

where $H=H(x_1,x_2,x_3)$ is defined in Eq. (22) with $x_1=\sin^{-1}(x_3/b)$, $x_2=0$, $x_3=\sqrt{2C}+x_3^0$ (see Fig. 1b).

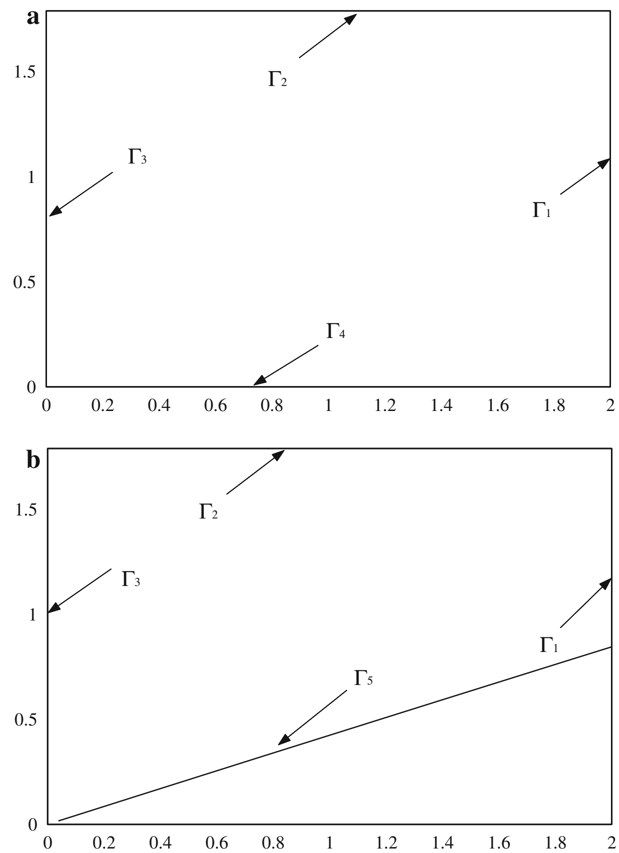

Fig. 1 The safety domain of system (20). Γ_1 and Γ_2 are the absorbing boundaries while Γ_4 and Γ_5 is the reflecting boundary. The parameters are taken in (**b**) as follows: $P_0 = 8$, $M = 2.0$ s, $x_1^0 = \pi/6$, $x_3^0 = 4$, $T_s = 4$ s, $a = 0.5$, $b = 8.0$, $c = 0.0001$, $d = 0.25$, $f = 0.0314$, $e = 100\pi$, $H_{\max} = 1.7889$, $H_{\min} = H_0 = 0.0$, $C_{\min} = C_0 = 0.0$, $C_{\max} = 2.0$

The conditional reliability function is governed by following backward Kolmogorov equation:

$$\frac{\partial R}{\partial t} = \bar{m}_1 \frac{\partial R}{\partial C_0} + \bar{m}_H \frac{\partial R}{\partial H_0} + \frac{\bar{b}_{HH}}{2} \frac{\partial^2 R}{\partial H_0^2} \tag{27}$$

where the coefficients are defined by Eq. (24) with $[\mathrm{C}, H]^{\mathrm{T}}$ replaced by $[\mathrm{C}_0, H_0]^{\mathrm{T}}$. The associated boundary conditions are

$$R\,(t|C_0, H_0) = 0 \quad \text{at } \Gamma_1 \text{ and } \Gamma_2 \tag{28}$$

$$R\,(t|C_0, H_0) = \text{finite} \quad \text{at } \Gamma_3 \text{ and } \Gamma_5 \tag{29}$$

The initial condition is defined by Eq. (13).

Note that the boundary conditions at Γ_3 and Γ_5 are qualitative and have to be replaced by quantitative ones for solving Eq. (27) numerically. They are

$$\frac{\partial R}{\partial t} = m_{200} \frac{\partial R}{\partial H_0} + \frac{b_{20}}{2} \frac{\partial^2 R}{\partial H_0^2} \tag{30}$$

where

$$m_{200} = \frac{1}{T}\int_{y_{20}}^{y_{10}} \left\{\left[-fex_2^2 - da\left(-x_1 + x_1^0\right)^2 + D_1 eb^2 \sin^2 x_1\right]\Big/ x_2\right\} \mathrm{d}x_1$$

$$b_{20} = \frac{1}{T}\int_{y_{20}}^{y_{10}} \left[\left(2D_1 e^2 b^2 x_2^2 \sin^2 x_1\right)\Big/ x_2\right] \mathrm{d}x_1 \tag{31}$$

$$T = \int_{y_{20}}^{y_{10}} \left[1\Big/ x_2\right] \mathrm{d}x_1$$

for Γ_3,

$$\frac{\partial R}{\partial t} = m_{10}\frac{\partial R}{\partial C_0} + m_{20}\frac{\partial R}{\partial H_0} \tag{32}$$

where

$$m_{10} = -d\left(x_3 - x_3^0\right)\left(x_3 - x_3^0 - x_1^{\min} + x_1^0\right)$$

$$m_{20} = -da\left(x_3 - x_3^0 - x_1^{\min} + x_1^0\right)^2 + D_1 eb^2 \sin^2 x_1^{\min} \tag{33}$$

$$x_{\min} = \sin^{-1}\left(\frac{x_3}{b}\right); \quad x_3 = \sqrt{2C_0} + x_3^0$$

for Γ_5. Note that in Eq. (31), $x_2 = \sqrt{2(H_0 + ax_3^0(x_1 - x_1^0) + ab(\cos x_1 - \cos x_1^0))/e}$, the upper and lower limits of integrals, $y_{10}(H_0)$ and $y_{20}(H_0)$, are the two roots of equation $-ax_3^0(x_1 - x_1^0) - ab(\cos x_1 - \cos x_1^0) = H_0$, respectively.

Similarly, the Pontryagin equation for the conditional mean of the first passage time of system (23) is derived from Eq. (27) as follows:

$$\bar{m}_1\frac{\partial \mu}{\partial C_0} + \bar{m}_H\frac{\partial \mu}{\partial H_0} + \frac{\bar{b}_{HH}}{2}\frac{\partial^2 \mu}{\partial H_0^2} = -1 \tag{34}$$

together with the boundary conditions

$$\mu\left(C_0, H_0\right) = 0 \quad \text{at } \Gamma_1 \text{ and } \Gamma_2 \tag{35}$$

$$\mu\left(C_0, H_0\right) = \text{finite} \quad \Gamma_3 \text{ and } \Gamma_5 \tag{36}$$

The qualitative conditions at Γ_3 and Γ_5 can be converted into the following quantitative ones:

$$m_{200}\frac{\partial \mu}{\partial H_0} + \frac{1}{2}b_{20}\frac{\partial^2 \mu}{\partial H_0^2} = -1 \tag{37}$$

for Γ_3,

$$-1 = m_{10}\frac{\partial \mu}{\partial C_0} + m_{20}\frac{\partial \mu}{\partial H_0} \tag{38}$$

for Γ_5.

The backward Kolmogorov equation (27) can be solved with boundary conditions(28), (30), (32)and initial condition (13) numerically while the Pontryagin equation (34) can be solved with boundary conditions (35), (37) and (38) by using successive over-relaxation method.

Some numerical results for the conditional reliability function, the conditional probability density and mean of the first-passage time are shown in Figs. 2, 3, 4, 5, and 6. To assess the validity and accuracy of the proposed procedure, the results from Monte Carlo simulation of system (20) are also obtained and shown in Figs. 2, 3, and 4 using symbols filled circle and filled square. It is seen that the analytical results agree quite well with simulation results and higher excitation intensity leads to lower reliability of system. Additional results of reliability function and mean first passage time are obtained by using the proposed method and shown in Figs. 5 and 6, respectively. Note that the reliability function and mean first passage time are monotonously decreasing function of initial energies.

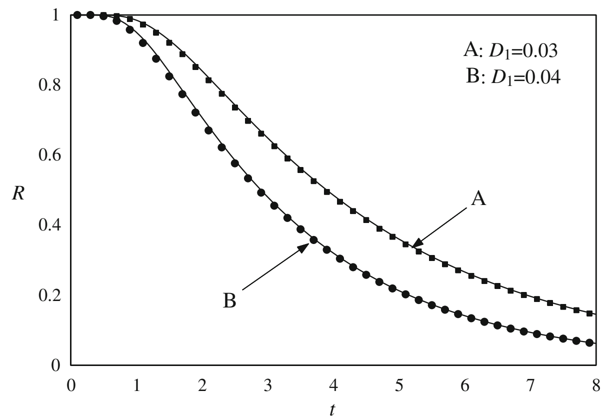

Fig. 2 The conditional reliability function of system (20) for zero initial condition. *Solid lines* denote the analytical results; *filled circle* and *filled square* denote the results from Monte Carlo simulation of original system (20)

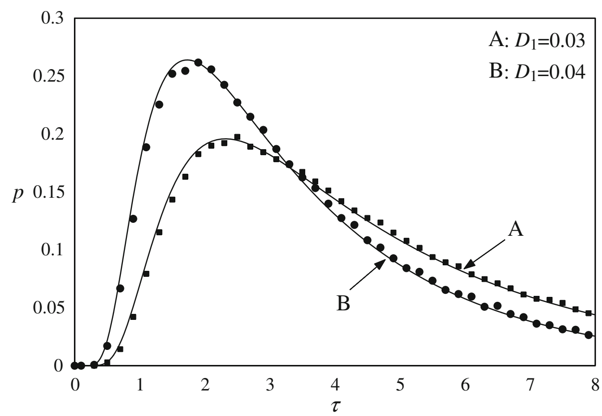

Fig. 3 The probability density of first passage time of system (20) for zero initial condition. The parameters are the same as those in Fig. 1b. *Solid lines* denote the analytical results; *filled circle* and *filled square* denote the results from Monte Carlo simulation of original system (20)

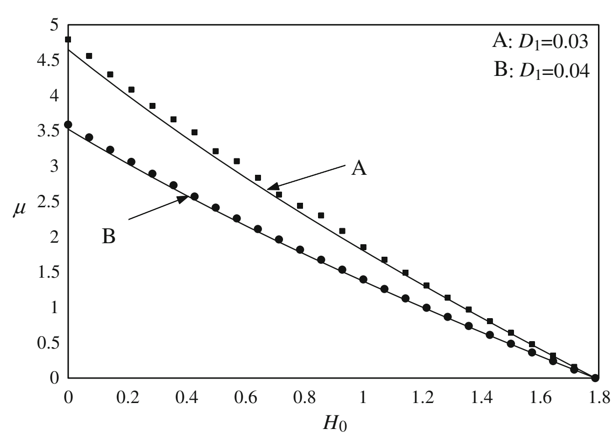

Fig. 4 The mean first passage time of system (20) as function of initial energy H_0 for $C_0 = 0.0$. The parameters are the same as those in Fig. 1b. *Solid lines* denote the analytical results; *filled circle* and *filled square* denote the results from Monte Carlo simulation of original system(20)

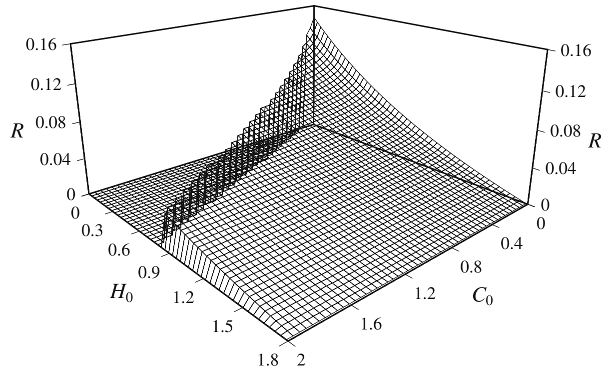

Fig. 5 The reliability function of system (20) at $t = 8$ (second) as function of H_0 and C_0 with $D_1 = 0.03$. The other parameters are the same as those in Fig. 1b

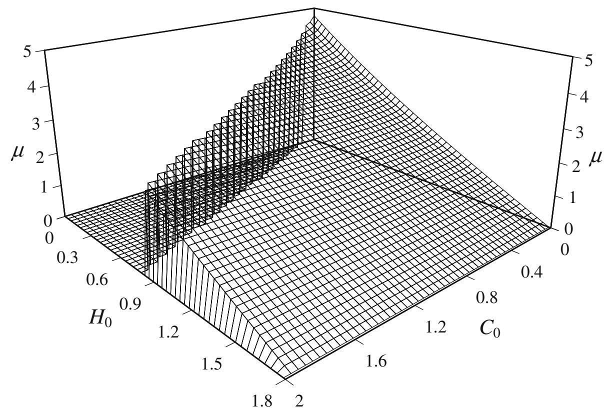

Fig. 6 The mean first passage time of system (20) as function of H_0 and C_0 with $D_1 = 0.03$. The other parameters are the same as those in Fig. 1b

6 Conclusions

In the present paper, the first passage failure of quasi non-integrable generalized Hamiltonian systems has been investigated. The stochastic averaging method has been applied to reduce the dimension of the original system. The backward Kolmogorov equation governing the conditional reliability function and the Pontryagin equation governing the conditional mean of the first passage time have been established from the averaged Itô equations and solved numerically. The numerical results obtained for one example of three-dimensional power system agree well with those from Monte Carlo simulation of original system.

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