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Shear deformation effect in plates stiffened by parallel beams

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Abstract In this paper a general solution for the analysis of shear deformable stiffened plates subjected to arbitrary loading is presented. According to the proposed model, the arbitrarily placed parallel stiffening beams of arbitrary doubly symmetric cross section are isolated from the plate by sections in the lower outer surface of the plate, taking into account the arising tractions in all directions at the fictitious interfaces. These tractions are integrated with respect to each half of the interface width resulting two interface lines, along which the loading of the beams as well as the additional loading of the plate is defined. Their unknown distribution is established by applying continuity conditions in all directions at the interfaces. The utilization of two interface lines for each beam enables the nonuniform distribution of the interface transverse shear forces and the nonuniform torsional response of the beams to be taken into account. The analysis of both the plate and the beams is accomplished on their deformed shape taking into account second-order effects. The analysis of the plate is based on Reissner's theory, which may be considered as the standard thick plate theory with which all others are compared, while the analysis of the beams is performed employing the linearized second order theory taking into account shear deformation effect. Six boundary value problems are formulated and solved using the analog equation method (AEM), a BEM based method. The solution of the aforementioned plate and beam problems, which are nonlinearly coupled, is achieved using iterative numerical methods. The adopted model permits the evaluation of the shear forces at the interfaces in both directions, the knowledge of which is very important in the design of prefabricated ribbed plates. The effectiveness, the range of applications of the proposed method and the influence of shear deformation effect are illustrated by working out numerical examples with great practical interest.

Keywords Elastic stiffened plate · Reinforced plate with beams · Bending · Nonuniform torsion · Warping · Ribbed plate · Boundary element method · Slab-and-beam structure · Shear deformation · Reissner's theory

1 Introduction

Structural plate systems stiffened by beams are widely used in buildings, bridges, ships, aircrafts and machines. Stiffening of the plate is used to increase its load carrying capacity and to prevent buckling especially in case of in-plane loading. Moreover, for cases wherein the plate or the beams are not very "thin" or the stiffeners are closely spaced, the error incurred from the ignorance of the effect of shear deformation may be substantial,

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while the accuracy of a classical analysis decreases and the truthfulness of the results is lost with growing plate or beam thickness. The extensive use of the aforementioned plate structures necessitates a rigorous analysis.

The behavior of stiffened plates under static loading has been widely studied. Initial research efforts employed a restrictive approach wherein the stiffeners are assumed to be closely spaced and the stiffened plate is replaced by an equivalent isotropic/orthotropic plate of uniform thickness [12,16,18]. Subsequently, in more refined approximations the adopted models for the analysis of the plate–beams system isolated the beams from the plate and employed numerical methods for the solution of the arising plate and beam problems with most frequently used the finite element method [15,17,21,27], the boundary element method [3,6,7,14,22,23,28,29] or a combination of these methods [2,13]. In all these approximations shear deformation effect is ignored, while with the exception of [23] the shear longitudinal and/or transverse forces at the interfaces have been neglected.

Contrary to the extended literature concerning the analysis of plates reinforced with beams ignoring shear deformation effect, relatively little work has been done on the corresponding problem of shear deformable stiffened plates. The FEM has been employed by Biswal and Ghosh [1] using higher order shear deformation theory to analyze stiffened laminated plates. FEM has also been employed by Deb and Booton [4] for the analysis of Mindlin's shear distortion theory for bending of eccentrically stiffened plates subjected to transverse loading, while the BEM has been used by Wen et al. [31] by coupling the shear deformable plate formulation and the two-dimensional plane stress elasticity. Also in these latter research efforts the solution of the bending problem of stiffened plates is not general since either the analysis of the plate and the beams is performed on the undeformed shape ignoring second-order effects or the shear longitudinal or transverse forces at the interfaces have been neglected or the torsional and warping behavior of the stiffening beams has been ignored.

In this paper a general solution for the static analysis of plates stiffened by arbitrarily placed parallel beams of arbitrary doubly symmetric cross section subjected to arbitrary loading is presented taking into account shear deformation effect in both the plate and the beams. The employed structural model is an improved one of that presented by Sapountzakis and Mokos in [23], so that a nonuniform distribution of the interface transverse shear force and the nonuniform torsional response of the beams are taken into account. According to this model, the stiffening beams are isolated again from the plate by sections in the lower outer surface of the plate, taking into account the arising tractions in all directions at the fictitious interfaces. These tractions are integrated with respect to each half of the interface width resulting two interface lines, along which the loading of the beams as well as the additional loading of the plate is defined. The utilization of two interface lines for each beam enables the nonuniform torsional response of the beams to be taken into account as the angle of twist is indirectly equated with the corresponding plate slope. The unknown distribution of the aforementioned integrated tractions is established by applying continuity conditions in all directions at the two interface lines. The analysis of both the plate and the beams is accomplished on their deformed shape taking into account second-order effects. The analysis of the plate is based on Reissner's theory [19,20], which may be considered as the standard thick plate theory with which all others are compared, while the analysis of the beams is performed employing the linearized second order theory taking into account shear deformation effect. Six boundary value problems are formulated and solved using the analog equation method (AEM) [10], a BEM based method. The solution of the aforementioned plate and beam problems, which are nonlinearly coupled, is achieved using iterative numerical methods. The adopted model permits the evaluation of the shear forces at the interfaces in both directions, the knowledge of which is very important in the design of prefabricated ribbed plates. The effectiveness, the range of applications of the proposed method and the influence of shear deformation effect are illustrated by working out numerical examples with great practical interest.

2 Statement of the problem

Consider a plate of homogeneous, isotropic and linearly elastic material with modulus of elasticity E, shear modulus G and Poisson ratio μ, having constant thickness h_p and occupying the two dimensional multiply connected region Ω of the x, y plane bounded by the piecewise smooth $K+1$ curves $\Gamma_0, \Gamma_1, \ldots, \Gamma_{K-1}, \Gamma_K$, as shown in Fig. 1. The plate is stiffened by a set of $i = 1, 2, \ldots, I$ arbitrarily placed parallel beams of arbitrary doubly symmetric cross section of homogeneous, isotropic and linearly elastic material with modulus of elasticity E_b^i, shear modulus G_b^i and Poisson ratio μ_b^i, which may have either internal or boundary point supports. For the sake of convenience the x axis is taken parallel to the beams. The stiffened plate is subjected to the lateral load $g = g(x, y)$. For the analysis of the aforementioned problem a global coordinate system Oxy for the analysis of the plate and local coordinate ones $O^i x^i y^i$ corresponding to the centroid axes of each beam are employed as shown in Fig. 1.

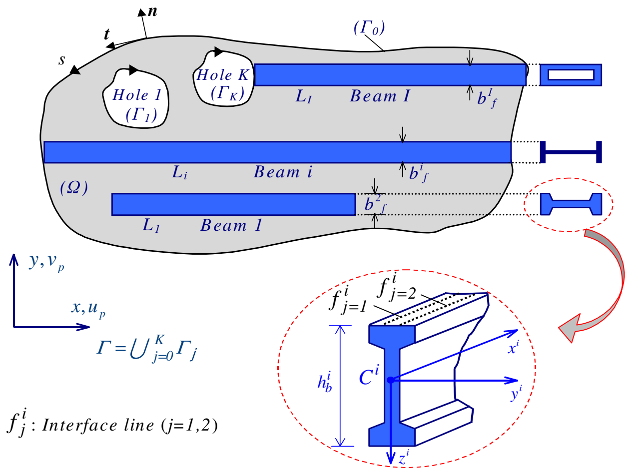

Fig. 1 Two dimensional region Ω occupied by the plate

The solution of the problem at hand is approached by an improved model of that proposed by Sapountzakis and Mokos in [23]. According to this model, the stiffening beams are isolated again from the plate by sections in its lower outer surface, taking into account the arising tractions at the fictitious interfaces (Fig. 2). Integration of these tractions along each half of the width of the i-th beam results in line forces per unit length in all directions in two interface lines, which are denoted by q_{xj}^i, q_{yj}^i and $q_{zj}^i \quad (j = 1, 2)$ encountering in this way the nonuniform distribution of the interface transverse shear forces q_y^i, which in the aforementioned model in [23] was ignored. The aforementioned integrated tractions result in the loading of the i-th beam as well as the additional loading of the plate. Their distribution is unknown and can be established by imposing displacement continuity conditions in all directions along the two interface lines, enabling in this way the nonuniform torsional response of the beams to be taken into account, which in the aforementioned model in [23] was also ignored.

The arising additional loading at the middle surface of the plate and the loading along the centroid axes of each beam can be summarized as follows:

a. In the plate (at the traces of the two interface lines $j = 1, 2$ of the i-th plate–beam interface)

  (i) A lateral line load q_{zj}^i.
  (ii) A lateral line load $\partial m_{pyj}^i/\partial x$ due to the eccentricity of the component q_{xj}^i from the middle surface of the plate. $m_{pyj}^i = q_{xj}^i h_p/2$ is the bending moment.
  (iii) A lateral line load $\partial m_{pxj}^i/\partial x$ due to the eccentricity of the component q_{yj}^i from the middle surface of the plate. $m_{pxj}^i = q_{yj}^i h_p/2$ is the bending moment.
  (iv) An inplane line body force q_{xj}^i at the middle surface of the plate.
  (v) An inplane line body force q_{yj}^i at the middle surface of the plate.

b. In each (i-th) beam ($O^i x^i y^i z^i$ system of axes)

  (i) A perpendicularly distributed line load q_{zj}^i along the beam centroid axis $O^i x^i$.
  (ii) A transversely distributed line load q_{yj}^i along the beam centroid axis $O^i x^i$.
  (iii) An axially distributed line load q_{xj}^i along the beam centroid axis $O^i x^i$.
  (iv) A distributed bending moment $m_{byj}^i = q_{xj}^i e_{zj}^i$ about $O^i y^i$ local beam centroid axis due to the eccentricities e_{zj}^i of the components q_{xj}^i from the beam centroid axis. $e_{z1}^i = e_{z2}^i = -h_b^i/2$ are the eccentricities.
  (v) A distributed bending moment $m_{bzj}^i = -q_{xj}^i e_{yj}^i$ about $O^i z^i$ local beam centroid axis due to the eccentricities e_{yj}^i of the components q_{xj}^i from the beam centroid axis. $e_{y1}^i = -b_f^i/4$, $e_{y2}^i = b_f^i/4$ are the eccentricities.

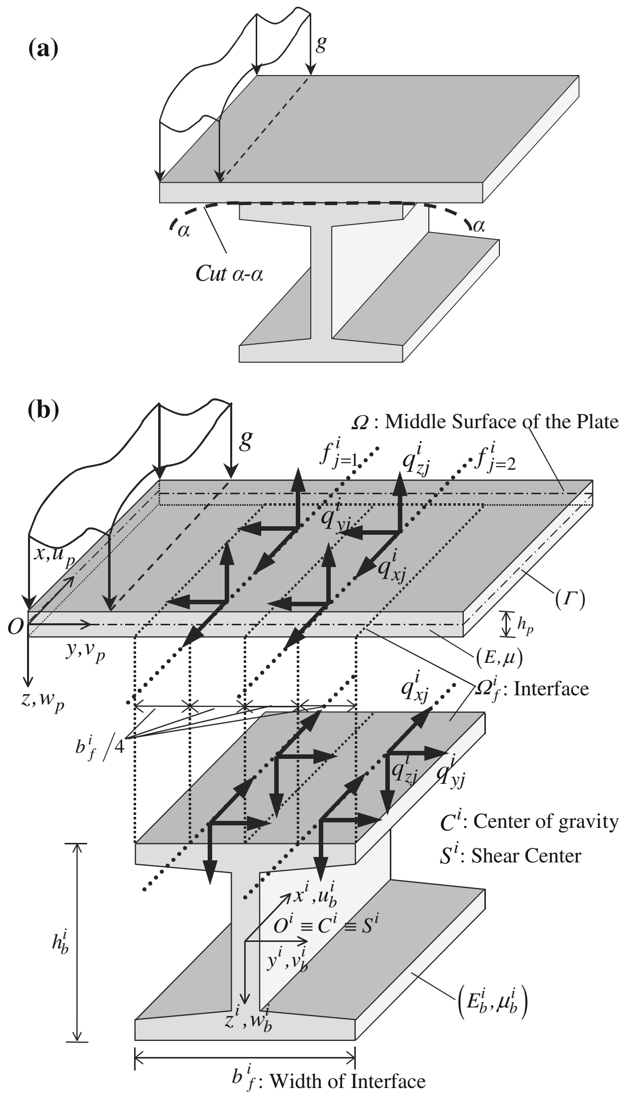

Fig. 2 Thin elastic plate stiffened by beams (**a**) and isolation of the beams from the plate (**b**)

(vi) A distributed twisting moment $m^i_{bxj} = q^i_{zj} e^i_{yj} - q^i_{yj} e^i_{zj}$ about $O^i x^i$ local beam shear center axis due to the eccentricities e^i_{zj}, e^i_{yj} of the components q^i_{yj}, q^i_{zj} from the beam shear center axis, respectively. $e^i_{z1} = e^i_{z1} = -h^i_b/2$ and $e^i_{y1} = -b^i_f/4$, $e^i_{y2} = b^i_f/4$ are the eccentricities.

The structural models and the aforementioned additional loading of the plate and the beams are shown in Fig. 3.

On the base of the above considerations the response of the plate and of the beams may be described by the following boundary value problems.

a. For the plate.

The plate undergoes transverse deflection and inplane deformation. Thus, for the transverse deflection, according to Reissner's theory, if $\psi = \psi(x, y)$ is a stress function satisfying the equation [30]

$$k\psi - \nabla^2 \psi = 0, \quad k = 10/h_p^2 \quad \text{in} \quad \Omega \tag{1}$$

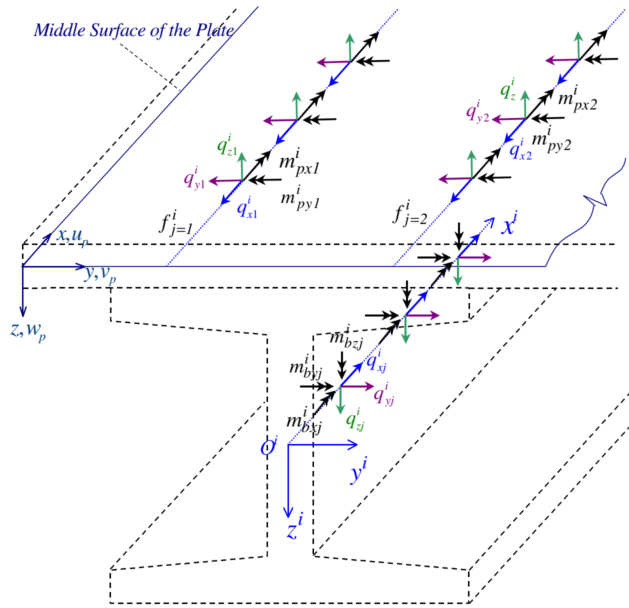

Fig. 3 Structural model and directions of the additional loading of the plate and the i-th beam

the average rotations ϕ_{px}, ϕ_{py} of the plate with respect to the axes y, x, respectively, taking into account the effect of shear deformation can be written as

$$\phi_{px} = -\frac{\partial w_p}{\partial x} + \frac{6}{5Gh_p} Q_{px} \quad \phi_{py} = -\frac{\partial w_p}{\partial y} + \frac{6}{5Gh_p} Q_{py} \tag{2a, b}$$

where the second term in the right hand side of these equations represents the additional angle of rotation due to shear deformation. Moreover, the stress resultants are given as

$$Q_{px} = -D\frac{\partial}{\partial x}\nabla^2 w_p + N_x\frac{\partial w_p}{\partial x} + N_{xy}\frac{\partial w_p}{\partial y} + \frac{\partial \psi}{\partial y} \tag{3a}$$

$$Q_{py} = -D\frac{\partial}{\partial y}\nabla^2 w_p + N_y\frac{\partial w_p}{\partial y} + N_{yx}\frac{\partial w_p}{\partial x} - \frac{\partial \psi}{\partial x} \tag{3b}$$

$$M_{px} = -D\left(\nabla^2 w_p + (\mu - 1)\frac{\partial^2 w_p}{\partial y^2}\right) + \frac{2}{k}\frac{\partial Q_{px}}{\partial x} - \frac{g\mu}{k(1-\mu)} \tag{3c}$$

$$M_{py} = -D\left(\nabla^2 w_p + (\mu - 1)\frac{\partial^2 w_p}{\partial x^2}\right) + \frac{2}{k}\frac{\partial Q_{py}}{\partial y} - \frac{g\mu}{k(1-\mu)} \tag{3d}$$

$$M_{pxy} = D(1-\mu)\frac{\partial^2 w_p}{\partial x \partial y} - \frac{1}{k}\left(\frac{\partial Q_{px}}{\partial y} + \frac{\partial Q_{py}}{\partial x}\right) \tag{3e}$$

and the equation of equilibrium employing the linearized second order theory can be written as

$$D\nabla^4 w_p - \left(N_x\frac{\partial^2 w_p}{\partial x^2} + 2N_{xy}\frac{\partial^2 w_p}{\partial x \partial y} + N_y\frac{\partial^2 w_p}{\partial y^2}\right) = g - \frac{h_p^2}{10}\frac{2-\mu}{1-\mu}\nabla^2 g$$
$$-\sum_{i=1}^{I}\left(\sum_{j=1}^{2}\left(q^i_{zj} - \frac{h_p^2}{10}\frac{2-\mu}{1-\mu}\nabla^2 q^i_{zj} - \frac{\partial m^i_{pxj}}{\partial y} + \frac{\partial m^i_{pyj}}{\partial x} - q^i_{xj}\frac{\partial w^i_{pj}}{\partial x} - q^i_{yj}\frac{\partial w^i_{pj}}{\partial y}\right)\delta^i_j\left(y - y_j\right)\right)$$
$$\text{in} \quad \Omega \tag{4}$$

where $w_p = w_p(x, y)$ is the transverse deflection of the plate; $D = Eh_p^3/12(1-v^2)$ is its flexural rigidity; $N_x = N_x(x, y)$, $N_y = N_y(x, y)$, $N_{xy} = N_{xy}(x, y)$ are the membrane forces per unit length of the plate cross section and $\delta(y - y_i)$ is the Dirac's delta function in the y direction.

The governing differential equations (1), (4) are also subjected to the pertinent boundary conditions of the problem, which are given as

$$\alpha_{p1} w_p + \alpha_{p2} Q_{pn} = \alpha_{p3} \tag{5a}$$

$$\beta_{p1} \phi_{pn} + \beta_{p2} M_{pn} = \beta_{p3} \quad \text{on} \quad \Gamma \tag{5b}$$

$$\gamma_{p1} \phi_{pt} + \gamma_{p2} M_{pnt} = \gamma_{p3} \tag{5c}$$

where $a_{pl}, \beta_{pl}, \gamma_{pl}$ $(l = 1, 2, 3)$ are given functions specified on the boundary Γ, Q_{pn}, M_{pn}, M_{pnt} are the shear force, the bending moment and the twisting moment along the boundary, respectively and ϕ_{pn}, ϕ_{pt} are the average rotations of the plate with respect to the axes t, n, respectively. The boundary conditions (5a, b, c) are the most general boundary conditions for the thick plate problem including also the elastic support. It is apparent that all types of the conventional boundary conditions (clamped, simply supported, free or guided edge) can be derived form these equations by specifying appropriately the functions a_{pl}, β_{pl} and γ_{pl} (e.g. for a clamped edge it is $a_{p1} = \beta_{p1} = \gamma_{p1} = 1, a_{p2} = a_{p3} = \beta_{p2} = \beta_{p3} = \gamma_{p2} = \gamma_{p3} = 0$). In accordance with Eqs. (5) the boundary stress resultants $Q_{pn}, Q_{pt}, M_{pn}, M_{pnt}$ using intrinsic coordinates n, s [8,11] are given as

$$Q_{pn} = -D\frac{\partial}{\partial n}\nabla^2 w_p + N_n \frac{\partial w_p}{\partial n} + N_{nt}\frac{\partial w_p}{\partial s} + \frac{\partial \psi}{\partial s} \tag{6a}$$

$$Q_{pt} = -D\frac{\partial}{\partial s}\nabla^2 w_p + N_t \frac{\partial w_p}{\partial s} + N_{nt}\frac{\partial w_p}{\partial n} - \frac{\partial \psi}{\partial n} \tag{6b}$$

$$\begin{aligned} M_{pn} = & -D\left[\nabla^2 w_p + (\mu - 1)\left(\frac{\partial^2 w_p}{\partial s^2} + \kappa \frac{\partial w_p}{\partial n}\right)\right] + \frac{2}{k}\left[\frac{\partial^2 \psi}{\partial s \partial n} - \kappa \frac{\partial \psi}{\partial s}\right] - \frac{g\mu}{k(1-\mu)} \\ & + \frac{2}{k}\left[-g + \frac{1}{k}\frac{(2-\mu)}{(1-\mu)}\nabla^2 g + D\frac{\partial^2}{\partial s^2}\left(\nabla^2 w_p\right) + D\kappa \frac{\partial}{\partial n}\left(\nabla^2 w_p\right)\right] \\ & + \frac{2}{k}\left[-N_{nt}\left(\frac{\partial^2 w_p}{\partial s \partial n} - \kappa \frac{\partial w_p}{\partial s}\right) - N_t\left(\frac{\partial^2 w_p}{\partial s^2} + \kappa \frac{\partial w_p}{\partial n}\right) + \frac{\partial N_n}{\partial n}\frac{\partial w_p}{\partial n} + \frac{\partial N_{nt}}{\partial n}\frac{\partial w_p}{\partial s}\right] \end{aligned} \tag{6c}$$

$$\begin{aligned} M_{pt} = & -D\left[\mu \nabla^2 w_p - (\mu - 1)\left(\frac{\partial^2 w_p}{\partial s^2} + \kappa \frac{\partial w_p}{\partial n}\right)\right] \\ & - \frac{2}{k}\left[D\frac{\partial^2}{\partial s^2}\left(\nabla^2 w_p\right) + D\kappa \frac{\partial}{\partial n}\left(\nabla^2 w_p\right) + \frac{\partial^2 \psi}{\partial s \partial n} - \kappa \frac{\partial \psi}{\partial s}\right] - \frac{g\mu}{k(1-\mu)} \\ & + \frac{2}{k}\left[\frac{\partial N_t}{\partial s}\frac{\partial w_p}{\partial s} + N_t\left(\frac{\partial^2 w_p}{\partial s^2} + \kappa \frac{\partial w_p}{\partial n}\right) + \frac{\partial N_{nt}}{\partial s}\frac{\partial w_p}{\partial n} + N_{nt}\left(\frac{\partial^2 w_p}{\partial s \partial n} - \kappa \frac{\partial w_p}{\partial s}\right)\right] \end{aligned} \tag{6d}$$

$$\begin{aligned} M_{pnt} = & (1-\mu) D\left[\frac{\partial^2 w_p}{\partial s \partial n} - \kappa \frac{\partial w_p}{\partial s}\right] \\ & - \frac{1}{k}\left[-2D\frac{\partial^2}{\partial s \partial n}\left(\nabla^2 w_p\right) + 2D\kappa \frac{\partial}{\partial s}\left(\nabla^2 w_p\right) + 2\frac{\partial^2 \psi}{\partial s^2} + 2\kappa \frac{\partial \psi}{\partial n} - k\psi\right] \\ & - \frac{1}{k}\left[\frac{\partial N_n}{\partial s}\frac{\partial w_p}{\partial n} + N_n\left(\frac{\partial^2 w_p}{\partial s \partial n} - \kappa \frac{\partial w_p}{\partial s}\right) + \frac{\partial N_{nt}}{\partial s}\frac{\partial w_p}{\partial s} + N_{nt}\left(\frac{\partial^2 w_p}{\partial s^2} + \kappa \frac{\partial w_p}{\partial n}\right)\right] \\ & - \frac{1}{k}\left[\frac{\partial N_t}{\partial n}\frac{\partial w_p}{\partial s} + N_t\left(\frac{\partial^2 w_p}{\partial s \partial n} - \kappa \frac{\partial w_p}{\partial s}\right) + \frac{\partial N_{nt}}{\partial n}\frac{\partial w_p}{\partial n} + N_{nt}\frac{\partial^2 w_p}{\partial n^2}\right] \end{aligned} \tag{6e}$$

in which $\kappa = \kappa(s)$ is the curvature of the boundary; $\partial/\partial s$ and $\partial/\partial n$ denote differentiation with respect to the arc length s of the boundary and the outward normal n to it, respectively.

Since linearized plate bending theory is considered, the components of the membrane forces N_x, N_y, N_{xy} are given as

$$N_x = C\left(\frac{\partial u_p}{\partial x} + \mu \frac{\partial v_p}{\partial y}\right) \tag{7a}$$

$$N_y = C\left(\mu \frac{\partial u_p}{\partial x} + \frac{\partial v_p}{\partial y}\right) \tag{7b}$$

$$N_{xy} = C\frac{1-\mu}{2}\left(\frac{\partial u_p}{\partial y} + \frac{\partial v_p}{\partial x}\right) \tag{7c}$$

where $C = Eh_p/\left(1-\mu^2\right)$; $u_p = u_p(x, y)$, $v_p = v_p(x, y)$ are the displacement components of the middle surface of the plate arising from the line body forces $q^i_{xj}, q^i_{yj} (i = 1, 2, \ldots, I), (j = 1, 2)$. These displacement components are established by solving independently the plane stress problem, which is described by the following boundary value problem (Navier's equations of equilibrium)

$$\nabla^2 u_p + \frac{1+\mu}{1-\mu}\frac{\partial}{\partial x}\left[\frac{\partial u_p}{\partial x} + \frac{\partial v_p}{\partial y}\right] - \frac{1}{Gh_p}\sum_{i=1}^{I}\left(\sum_{j=1}^{2} q^i_{xj}\delta^i_j\,(y-y_i)\right) = 0 \tag{8a}$$

$$\nabla^2 v_p + \frac{1+\mu}{1-\mu}\frac{\partial}{\partial y}\left[\frac{\partial u_p}{\partial x} + \frac{\partial v_p}{\partial y}\right] - \frac{1}{Gh_p}\sum_{i=1}^{I}\left(\sum_{j=1}^{2} q^i_{yj}\delta^i_j\,(y-y_i)\right) = 0 \quad \text{in} \quad \Omega \tag{8b}$$

$$\delta_{p1}u_{pn} + \delta_{p2}N_n = \delta_{p3} \tag{9a}$$

$$\varepsilon_{p1}u_{pt} + \varepsilon_{p2}N_t = \varepsilon_{p3} \quad \text{on} \quad \Gamma \tag{9b}$$

in which $G = E/2(1+\nu)$ is the shear modulus of the plate; N_n, N_t and u_{pn}, u_{pt} are the boundary membrane forces and displacements in the normal and tangential directions to the boundary, respectively; $\delta_{pl}, \varepsilon_{pl} \quad (l = 1, 2, 3)$ are functions specified on the boundary Γ.

b. For each (i-th) beam.

Each beam undergoes transverse deflection with respect to z^i and y^i axes, axial deformation along x^i axis and nonuniform angle of twist along x^i axis. Thus, for the transverse deflection with respect to z^i axis the equation of equilibrium employing the linearized second order theory and taking into account shear deformation effect can be written as [24]

$$\begin{aligned} E^i_b I^i_y\left(1 + \frac{N^i_b}{G^i_b A^i_z}\right)\frac{\partial^4 w^i_b}{\partial x^{i4}} &= \sum_{j=1}^{2}\left(q^i_{zj} - q^i_{xj}\frac{\partial w^i_b}{\partial x^i} + N^i_{bj}\frac{\partial^2 w^i_b}{\partial x^{i2}} - \frac{\partial m^i_{byj}}{\partial x^i}\right) \\ &-\frac{E^i_b I^i_y}{G^i_b A^i_z}\sum_{j=1}^{2}\left(\frac{\partial^2 q^i_{zj}}{\partial x^{i2}} - 3q^i_{xj}\frac{\partial^3 w^i_b}{\partial x^{i3}} - 3\frac{\partial q^i_{xj}}{\partial x^i}\frac{\partial^2 w^i_b}{\partial x^{i2}} - \frac{\partial^2 q^i_{xj}}{\partial x^{i2}}\frac{\partial w^i_b}{\partial x^i}\right) \end{aligned} \quad \text{in} \quad L^i, i = 1, 2, \ldots, I \tag{10}$$

subjected to the following boundary conditions

$$a^{zi}_1 w^i_b + a^{zi}_2 R^i_{bz} = a^{zi}_3 \tag{11a}$$

$$\beta^{zi}_1\theta^i_{by} + \beta^{zi}_2 M^i_{by} = \beta^{zi}_3 \quad \text{at the beam ends} \quad x^i = 0, L^i \tag{11b}$$

where $w^i_b = w^i_b\left(x^i\right)$ is the transverse deflection of the i-th beam with respect to z^i axis; I^i_{by} is its moment of inertia with respect to y^i axis; $N^i_{bj} = N^i_{bj}\left(x^i\right)$ are the axial forces at the x^i centroid axis arising from the line body forces q^i_{xj}; $a^{zi}_l, \beta^{zi}_l \quad (l = 1, 2, 3)$ are coefficients specified at the boundary of the i-th beam; $\theta^i_{by}, R^i_{bz}, M^i_{by}$ are the slope, the reaction and the bending moment at the i-th beam ends, respectively given as

$$\theta^i_{by} = -\frac{E^i_b I^i_{by}}{G^i_b A^i_z}\sum_{j=1}^{2}\left(1 + \frac{N^i_{bj}}{G^i_b A^i_z}\right)\frac{\partial^3 w^i_b}{\partial x^{i3}} - \frac{\partial w^i_b}{\partial x^i} \tag{12}$$

$$R^i_{bz} = -E^i_b I^i_{by}\sum_{j=1}^{2}\left(1 + \frac{N^i_{bj}}{G^i_b A^i_z}\right)\frac{\partial^3 w^i_b}{\partial x^{i3}} + \sum_{j=1}^{2}\left(N^i_{bj}\frac{\partial w^i_b}{\partial x^i}\right) \tag{13}$$

$$M^i_{by} = -E^i_b I^i_{by}\sum_{j=1}^{2}\left(1 + \frac{N^i_{bj}}{G^i_b A^i_z}\right)\frac{\partial^2 w^i_b}{\partial x^{i2}} \tag{14}$$

It is apparent that all types of the conventional boundary conditions (clamped, simply supported, free or guided edge) can be derived from Eqs. (11a, b) by specifying appropriately the coefficients a_l^{zi}, β_l^{zi} (e.g. for a simply supported end it is $a_1^{zi}=\beta_2^{zi}=1$, $a_2^{zi}=a_3^{zi}=\beta_1^{zi}=\beta_3^{zi}=0$).

Similarly, the $v_b^i=v_b^i\left(x^i\right)$ transverse deflection with respect to y^i axis must satisfy the following boundary value problem

$$E_b^i I_z^i\left(1+\frac{N_b^i}{G_b^i A_y^i}\right)\frac{\partial^4 v_b^i}{\partial x^{i4}}=\sum_{j=1}^{2}\left(q_{yj}^i-q_{xj}^i\frac{\partial v_b^i}{\partial x^i}+N_{bj}^i\frac{\partial^2 v_b^i}{\partial x^{i2}}-\frac{\partial m_{bzj}^i}{\partial x^i}\right)$$
$$-\frac{E_b^i I_z^i}{G_b^i A_y^i}\sum_{j=1}^{2}\left(\frac{\partial^2 q_{yj}^i}{\partial x^{i2}}-3q_{xj}^i\frac{\partial^3 v_b^i}{\partial x^{i3}}-3\frac{\partial q_{xj}^i}{\partial x^i}\frac{\partial^2 v_b^i}{\partial x^{i2}}-\frac{\partial^2 q_{xj}^i}{\partial x^{i2}}\frac{\partial v_b^i}{\partial x^i}\right)\quad \text{in} \quad L^i,\ i=1,2,\ldots,I \tag{15}$$

$$a_1^{yi}v_b^i+a_2^{yi}R_{by}^i=a_3^{yi} \tag{16a}$$
$$\beta_1^{yi}\theta_{bz}^i+\beta_2^{yi}M_{bz}^i=\beta_3^{yi}\quad \text{at the beam ends}\quad x^i=0,L^i \tag{16b}$$

where I_{bz}^i is the moment of inertia of the i-th beam with respect to y^i axis; a_l^{yi}, β_l^{yi} $(l=1,2,3)$ are coefficients specified at its boundary; θ_{bz}^i, R_{by}^i, M_{bz}^i are the slope, the reaction and the bending moment at the i-th beam ends, respectively given as

$$\theta_{bz}^i=-\frac{E_b^i I_{bz}^i}{G_b^i A_y^i}\sum_{j=1}^{2}\left(1+\frac{N_{bj}^i}{G_b^i A_y^i}\right)\frac{\partial^3 v_b^i}{\partial x^{i3}}-\frac{\partial v_b^i}{\partial x^i} \tag{17}$$

$$R_{by}^i=-E_b^i I_{bz}^i\sum_{j=1}^{2}\left(1+\frac{N_{bj}^i}{G_b^i A_y^i}\right)\frac{\partial^3 v_b^i}{\partial x^{i3}}+\sum_{j=1}^{2}\left(N_{bj}^i\frac{\partial v_b^i}{\partial x^i}\right) \tag{18}$$

$$M_{bz}^i=E_b^i I_{bz}^i\sum_{j=1}^{2}\left(1+\frac{N_{bj}^i}{G_b^i A_y^i}\right)\frac{\partial^2 v_b^i}{\partial x^{i2}} \tag{19}$$

In Eqs. (10), (12)–(14), (15), (17)–(19) $G_b^i A_y^i$, $G_b^i A_z^i$ are the shear rigidities of the Timoshenko's beam theory, where

$$A_y^i=\kappa_y^i A^i=\frac{1}{a_y^i}A^i \quad A_z^i=\kappa_z^i A^i=\frac{1}{a_z^i}A^i \tag{20a,b}$$

are the shear areas with respect to y, z axes, respectively with κ_y^i, κ_z^i the shear correction factors, a_y^i, a_z^i the shear deformation coefficients and A^i the cross section area of the i-th stiffening beam.

Since linearized beam bending theory is considered the axial deformation u_b^i of the beam arising from the arbitrarily distributed axial forces $q_{xj}^i (i=1,2,\ldots I)$, $(j=1,2)$ is described by solving independently the boundary value problem

$$E_b^i A_b^i\frac{\partial^2 u_b^i}{\partial x^{i2}}=-\sum_{j=1}^{2}q_{xj}^i\quad \text{in} L^i,\ i=1,2,\ldots,I \tag{21}$$

$$\gamma_1^{xi}u_b^i+\gamma_2^{xi}N_b^i=\gamma_3^{xi}\quad \text{at the beam ends} x^i=0,L^i \tag{22}$$

where N_b^i is the axial reaction at the i-th beam ends given as

$$N_b^i=\sum_{j=1}^{2}N_{bj}^i=E_b^i A_b^i\frac{\partial u_b^i}{\partial x_i} \tag{23}$$

Finally, the nonuniform angle of twist with respect to x^i shear center axis has to satisfy the following boundary value problem [25]

$$E_b^i I_{bw}^i \frac{\partial^4 \theta_{bx}^i}{\partial x^{i4}} - G_b^i I_{bx}^i \frac{\partial^2 \theta_{bx}^i}{\partial x^{i2}} = \sum_{j=1}^{2} m_{bxj}^i \quad \text{in} \quad L^i, i = 1, 2, \ldots, I \tag{24}$$

$$a_1^{xi} \theta_{bx}^i + a_2^{xi} M_{bx}^i = a_3^{xi} \tag{25a}$$

$$\beta_1^{xi} \frac{\partial \theta_{bx}^i}{\partial x^i} + \beta_2^{xi} M_{bw}^i = \beta_3^{xi} \quad \text{at the beam ends } x_i = 0, L_i \tag{25b}$$

where $\theta_{bx}^i = \theta_{bx}^i \left(x^i\right)$ is the variable angle of twist of the i-th beam along the x^i shear center axis; $G_b^i = E_b^i / 2(1 + \mu_b^i)$ is its shear modulus; I_{bw}^i, I_{bx}^i are the warping and torsion constants of the i-th beam cross section, respectively given as

$$I_{bw}^i = \int_{A^i} \left(\varphi_S^P\right)^2 dA^i \tag{26a}$$

$$I_{bx}^i = \int_{A^i} \left(\left(y^i\right)^2 + \left(z^i\right)^2 + y^i \frac{\partial \varphi_S^P}{\partial z^i} - z^i \frac{\partial \varphi_S^P}{\partial y^i} \right) dA^i \tag{26b}$$

with $\varphi_S^P \left(y^i, z^i\right)$ the primary warping function with respect to the shear center S of the A^i beam cross section; $a_l^{xi}, \beta_l^{xi} \quad (l = 1, 2, 3)$ are coefficients specified at the boundary of the i-th beam; $\frac{\partial \theta_{bx}^i}{\partial x^i}$ denotes the rate of change of the angle of twist and it can be regarded as the torsional curvature; M_{bx}^i is the twisting moment and M_{bw}^i is the warping moment due to the torsional curvature at the boundary of the i-th beam given as

$$M_{bx}^i = M_{bx}^{iP} + M_{bx}^{iS} \tag{27a}$$

$$M_{bw}^i = -E_b^i I_{xw}^i \frac{\partial^2 \theta_{bx}^i}{\partial x^{i2}} \tag{27b}$$

In Eq. (27a) M_{bx}^{iP} is the primary twisting moment resulting from primary shear stress distribution and M_{bx}^{iS} is the secondary twisting moment resulting from secondary shear stress distribution due to warping given as [25]

$$M_{bx}^{iP} = G_b^i I_{bx}^i \frac{\partial \theta_{bx}^i}{\partial x^i} \tag{28a}$$

$$M_{bx}^{iS} = -E_b^i I_{bw}^i \frac{\partial^3 \theta_{bx}^i}{\partial x^{i3}} \tag{28b}$$

The boundary conditions (25a,b) are the most general linear torsional boundary conditions for the beam problem including also the elastic support. It is apparent that all types of the conventional torsional boundary conditions (clamped, simply supported, free or guided edge) can be derived form these equations by specifying appropriately the coefficients $a_l^{xi}, \beta_l^{xi} \quad (l = 1, 2, 3)$ (e.g. for a clamped edge it is $a_1^{xi} = \beta_1^{xi} = 1, a_2^{xi} = a_3^{xi} = \beta_2^{xi} = \beta_3^{xi} = 0$).

Equations (1), (4), (8a), (8b), (10), (15), (21) and (24) constitute a set of eight coupled partial differential equations including fourteen unknowns, namely $\psi, w_p, u_p, v_p, w_b^i, v_b^i, u_b^i, \theta_{bx}^i, q_{x1}^i, q_{y1}^i, q_{z1}^i, q_{x2}^i, q_{y2}^i, q_{z2}^i$. Six additional equations are required, which result from the displacement continuity conditions in the directions of z^i, x^i and y^i local axes along the two interface lines of each (i-th) plate–beam interface. These conditions can be expressed as

In the direction of z^i local axis:

$$w_{p1}^i - w_b^i = -\frac{b_f^i}{4} \theta_{bx}^i \quad \text{along interface line } 1(f_{j=1}^i) \tag{29a}$$

$$w_{p2}^i - w_b^i = \frac{b_f^i}{4} \theta_{bx}^i \quad \text{along interface line } 2(f_{j=2}^i) \tag{29b}$$

In the direction of x^i local axis:

$$u_{p1}^i - u_b^i = \frac{h_p}{2}\frac{\partial w_{p1}^i}{\partial x} + \frac{h_b^i}{2}\frac{\partial w_b^i}{\partial x^i} + \frac{b_f^i}{4}\frac{\partial v_b^i}{\partial x^i} + \left(\phi_S^{iP}\right)_{f1}\frac{\partial \theta_{bx}^i}{\partial x^i} \quad \text{along interface line 1} \quad (f_{j=1}^i) \tag{29c}$$

$$u_{p2}^i - u_b^i = \frac{h_p}{2}\frac{\partial w_{p2}^i}{\partial x} + \frac{h_b^i}{2}\frac{\partial w_b^i}{\partial x^i} - \frac{b_f^i}{4}\frac{\partial v_b^i}{\partial x^i} + \left(\phi_S^{iP}\right)_{f2}\frac{\partial \theta_{bx}^i}{\partial x^i} \quad \text{along interface line 2} \quad (f_{j=2}^i) \tag{29d}$$

In the direction of y^i local axis:

$$v_{p1}^i - v_b^i = \frac{h_p}{2}\frac{\partial w_{p1}^i}{\partial y} + \frac{h_b^i}{2}\theta_{bx}^i \quad \text{along interface line } 1(f_{j=1}^i) \tag{29e}$$

$$v_{p2}^i - v_b^i = \frac{h_p}{2}\frac{\partial w_{p2}^i}{\partial y} + \frac{h_b^i}{2}\theta_{bx}^i \quad \text{along interface line } 2(f_{j=2}^i) \tag{29f}$$

where [5]

$$u_{pj}^i = -\frac{h_p}{2}\frac{\partial w_{pj}^i}{\partial x} + \frac{3}{2}\frac{Q_{px}}{G} \tag{30a}$$

$$v_{pj}^i = -\frac{h_p}{2}\frac{\partial w_{pj}^i}{\partial y} + \frac{3}{2}\frac{Q_{py}}{G} \tag{30b}$$

while [24]

$$\theta_{by}^i = -\frac{\partial w_b^i}{\partial x^i} + \frac{1}{G_b^i A_z^i}\left(-E_b^i I_y^i \frac{\partial^3 w_b^i}{\partial x^{i3}} - \frac{E_b^i I_y^i}{G_b^i A_z^i}\left(\frac{\partial q_{zj}^i}{\partial x^i} + N_{bj}^i \frac{\partial^3 w_b^i}{\partial x^{i3}} - 2q_{xj}^i \frac{\partial^2 w_b^i}{\partial x^{i2}} - \frac{\partial q_{zj}^i}{\partial x^i}\frac{\partial w_b^i}{\partial x^i}\right) + m_{byj}^i\right) \tag{31a}$$

$$\theta_{bz}^i = \frac{\partial v_b^i}{\partial x^i} - \frac{1}{G_b^i A_y^i}\left(-E_b^i I_z^i \frac{\partial^3 v_b^i}{\partial x^{i3}} - \frac{E_b^i I_z^i}{G_b^i A_y^i}\left(\frac{\partial q_{yj}^i}{\partial x^i} + N_{bj}^i \frac{\partial^3 v_b^i}{\partial x^{i3}} - 2q_{xj}^i \frac{\partial^2 v_b^i}{\partial x^{i2}} - \frac{\partial q_{yj}^i}{\partial x^i}\frac{\partial v_b^i}{\partial x^i}\right) + m_{bzj}^i\right) \tag{31b}$$

and $\left(\phi_S^{iP}\right)_{fj}$ is the value of the primary warping function with respect to the shear center S of the beam cross section at the point of the j-th interface line of the i-th plate–beam interface f. In all the aforementioned equations the values of the primary warping function $\varphi_S^{iP}(y^i, z^i)$ should be set having the appropriate algebraic sign corresponding to the local beam axes.

It is worth here noting that the coupling of the aforementioned equations is nonlinear due to the terms including the unknown q_{xj}^i and q_{yj}^i interface forces.

3 Numerical solution

The numerical solution of the boundary value problems described by Eqs. (4–5a, b, c), (8a,b–9a, b), (10–11a, b), (15–16a, b), (21–22) and (24–25a, b) will be accomplished employing the AEM [10]. This method is applied for the aforementioned problems as follows.

3.1 For the plate transverse deflection w_p

Let w_p be the sought solution of the boundary value problem described by Eqs. (4) and (5a, b, c). Applying the biharmonic operator to this function yields

$$\nabla^4 w = p_{pz}(x, y) \tag{32}$$

Equation (32) indicates that the solution of the original problem can be obtained as the deflection of a plate with unit flexural rigidity subjected to a flexural fictitious load $p_{pz}(x, y)$ under the same boundary conditions. The fictitious load is unknown. However, it can be established using BEM as follows.

The solution of Eq. (32) is written in integral form as [23]

$$w_p(x,y) = \frac{1}{2\pi}\int\int_{\Omega} \Lambda_{p4}(r)\, p_{pz}\, d\Omega$$
$$-\frac{1}{2\pi}\int_{\Gamma}\left[\Lambda_{p1}(r)\, w_p + \Lambda_{p2}(r)\frac{\partial w_p}{\partial n} + \Lambda_{p3}(r)\nabla^2 w_p + \Lambda_{p4}(r)\frac{\partial \nabla^2 w_p}{\partial n}\right] ds \tag{33}$$

where the kernels $\Lambda_{pi}(r)$, $(i = 1, 2, 3, 4)$ are given as

$$\Lambda_{p1}(r) = -\frac{\cos\phi}{r} \tag{34a}$$
$$\Lambda_{p2}(r) = \ln r + 1 \tag{34b}$$
$$\Lambda_{p3}(r) = -\frac{1}{4}(2r\ln r + r)\cos\phi \tag{34c}$$
$$\Lambda_{p4}(r) = \frac{1}{4}r^2\ln r \tag{34d}$$

where ϕ is the angle between vector $\boldsymbol{r}$ and the outward normal to the boundary $\boldsymbol{n}$. Notice that in Eq. (33) for the domain integral it is $r = |Q - P| = \left[(\xi - x)^2 + (n - y)^2\right]^{1/2}$, P, Q points inside the plate, whereas for the line integrals $r = |q - P| = \left[(\zeta - x)^2 + (\psi - y)^2\right]^{1/2}$, q point at the boundary of the plate. Applying the operator ∇^2 to both sides of the integral representation (33) the Laplasian of the plate deflection is obtained as

$$\nabla^2 w_p = \frac{1}{2\pi}\int\int_{\Omega} \Lambda_{p2}(r)\, p_{pz}\mathrm{d}\Omega - \frac{1}{2\pi}\int_{\Gamma}\left[\Lambda_{p1}(r)\nabla^2 w_p + \Lambda_{p2}(r)\frac{\partial \nabla^2 w_p}{\partial n}\right]\mathrm{d}s \tag{35}$$

Moreover, applying the Green identity for the harmonic operator for the function $\psi = \psi(x, y)$ and the fundamental solution of Eq. (1), the solution of Eq. (1) is obtained in integral form as [8]

$$\psi(x,y) = \frac{1}{2\pi}\int_{\Gamma}\left[\Lambda_{p5}(\rho)\frac{\partial\psi}{\partial n} + \Lambda_{p6}(\rho)\,\psi\right]\mathrm{d}s \tag{36}$$

where the kernels $\Lambda_{pi}(r)$, $(i = 5, 6)$ are given as

$$\Lambda_{p5}(r) = K_0(\rho) \tag{37a}$$
$$\Lambda_{p6}(r) = \frac{1}{l}K_1(\rho)\cos\phi \tag{37b}$$

with $K_0(\rho)$, $K_1(\rho)$ the zero- and first- order modified Bessel functions of the second kind, respectively; $\rho = r/l$, $l = \sqrt{1/k}$, $r = |Q - P|$.

The integral representations (33), (35), (36) written for the boundary points on Γ together with the boundary conditions (5a, b, c) can be employed to express the unknown boundary quantities w_p, $\partial w_p/\partial n$, $\nabla^2 w_p$, $\partial\nabla^2 w_p/\partial n$, ψ, $\partial\psi/\partial n$ in terms of p_{pz}. Eliminating these quantities from the discretized counterpart of Eq. (33) applied to all nodal points in the interior of the plate region Ω yields

$$\{w_p\} = [F_p]\{p_{pz}\} \tag{38}$$

where $[F_p]$ is an $M \times M$ known flexibility matrix, where M is the number of the plate internal nodal points; $\{w_p\}$, $\{p_{pz}\}$ are $M \times 1$ column matrices including the nodal values of the functions w_p, p_{pz}, respectively.

Moreover, the discretized counterpart of the derivatives of the plate deflection when applied to all nodal points in the interior of the plate, after elimination of the boundary quantities yield

$$\{w_{p,x}\} = [F_{px}]\{p_{pz}\} \tag{39a}$$
$$\{w_{p,y}\} = [F_{py}]\{p_{pz}\} \tag{39b}$$
$$\{w_{p,xx}\} = [F_{pxx}]\{p_{pz}\} \tag{39c}$$
$$\{w_{p,yy}\} = [F_{pyy}]\{p_{pz}\} \tag{39d}$$
$$\{w_{p,xy}\} = [F_{pxy}]\{p_{pz}\} \tag{39e}$$

where $[F_{px}], [F_{py}], [F_{pxx}], [F_{pyy}], [F_{pxy}]$ are known $M \times M$ coefficient matrices.

The final step of AEM is to apply Eq. (4) to the M nodal points inside Ω. This yields

$$\begin{aligned} &D\{p_{pz}\} - \left([\{N_x\}]_{\mathrm{dg}} \cdot [F_{pxx}] + 2[\{N_{xy}\}]_{\mathrm{dg}} \cdot [F_{pxy}] + [\{N_y\}]_{\mathrm{dg.}}[F_{pyy}]\right)\{p_{pz}\} \\ &\quad = \{g\} + \{Lg\} - [Z]\{q_z\} - [Z]\{Lq_z\} + [Z][X_y]\{q_y\} - [Z][X_x]\{q_x\} + [[Z]\{q_x\}]_{\mathrm{dg.}}[F_{px}]\{p_{pz}\} \\ &\qquad + [[Z]\{q_y\}]_{\mathrm{dg.}}[F_{py}]\{p_{pz}\} \end{aligned} \tag{40}$$

where $\{g\}$ is an $M \times 1$ column matrix including the values of the external load; $\{Lg\}$ and $\{Lq_z\}$ are known $M \times 1$ column matrices including the values of the $\frac{h_p^2}{10}\frac{2-\mu}{1-\mu}\nabla^2 g$ and $\frac{h_p^2}{10}\frac{2-\mu}{1-\mu}\nabla^2 q_{zj}^i$ quantities, respectively; $\{p_{pz}\}$ is an $M \times 1$ column matrix including the nodal values of the function p_{pz}; $[\{N_x\}]_{\mathrm{dg.}}, [\{N_{xy}\}]_{\mathrm{dg.}}$ and $[\{N_y\}]_{\mathrm{dg.}}$ are unknown diagonal $M \times M$ matrices including the values of the inplane forces; $\{q_x\}^{\mathrm{T}} = \{\{q_{x1}\}\{q_{x2}\}\}$, $\{q_y\}^{\mathrm{T}} = \{\{q_{y1}\}\{q_{y2}\}\}$ and $\{q_z\}^{\mathrm{T}} = \{\{q_{z1}\}\{q_{z2}\}\}$ are vectors with $2L$ elements including the unknown $q_{xj}^i, q_{yj}^i, q_{zj}^i (j = 1, 2)$ interface forces; $2L$ is the total number of the nodal points at the interfaces; $[Z]$ is a position $M \times 2L$ matrix which converts the vectors $\{q_x\}, \{q_y\}, \{q_z\}$ into corresponding ones with length M; the symbol $[]_{\mathrm{dg.}}$ indicates a diagonal $M \times M$ matrix with the elements of the included column matrix and $[F_{px}], [F_{py}], [F_{pxx}], [F_{pyy}], [F_{pxy}]$ are known $M \times M$ coefficient flexibility matrices [25]. The matrices $[X_x], [X_y]$ result after approximating the bending moment derivatives of m_{pyj}^i, m_{pxj}^i, respectively using appropriately central, backward, or forward differences. Their dimensions are $2L \times 2L$.

3.2 For the plate inplane displacement components u_p, v_p.

The boundary value problem described by Eqs. (8a, b), (9a, b) is solved numerically employing the boundary element method [9]. Using the same boundary discretization and solving the inplane plate problem for each nodal interface point separately for $q_{xj}^i = 1.0$ and $q_{yj}^i = 1.0 (j = 1, 2)$, the descretized M values of the nodal membrane forces for homogeneous boundary conditions (9a, b) ($\delta_{p3} = \varepsilon_{p3} = 0$) are expressed as follows

$$\{N_x\} = [G_{dx}^x]\{q_x\} + [G_{dx}^y]\{q_y\} \tag{41a}$$
$$\{N_{xy}\} = [G_{dxy}^x]\{q_x\} + [G_{dxy}^y]\{q_y\} \tag{41b}$$
$$\{N_y\} = [G_{dy}^x]\{q_x\} + [G_{dy}^y]\{q_y\} \tag{41c}$$

while the descretized L values of the nodal displacement components of the middle surface of the plate are given as

$$\{u_{p1}\} = [F_{d1}^{xx}]\{q_{x1}\} + [F_{d1}^{xy}]\{q_{y1}\} \tag{42a}$$
$$\{u_{p2}\} = [F_{d2}^{xx}]\{q_{x2}\} + [F_{d2}^{xy}]\{q_{y2}\} \tag{42b}$$
$$\{v_{p1}\} = [F_{d1}^{yx}]\{q_{x1}\} + [F_{d1}^{yy}]\{q_{y1}\} \tag{42c}$$
$$\{v_{p2}\} = [F_{d2}^{yx}]\{q_{x2}\} + [F_{d2}^{yy}]\{q_{y2}\} \tag{42d}$$

where $[G_{dx}^x], [G_{dx}^y], [G_{dxy}^x], [G_{dxy}^y], [G_{dy}^x]$ and $[G_{dy}^y]$ are known matrices with dimensions $M \times 2L$ and $[F_{d1}^{xx}], [F_{d2}^{xx}], [F_{d1}^{xy}], [F_{d2}^{xy}], [F_{d1}^{yx}], [F_{d2}^{yx}], [F_{d1}^{yy}], [F_{d2}^{yy}]$ are known flexibility matrices with dimensions $L \times L$.

3.3 For the beam transverse displacements w_b^i, v_b^i and for the angle of twist $\theta_{\tilde{x}}^i$

The numerical solution of the boundary value problems described by Eqs. (10–11a, b), (15–16a, b) and (24–25a, b) is similar.

Let w_b^i be the sought solution of the boundary value problem described by Eqs. (10) and (11a, b). Differentiating this function four times yields

$$\frac{d^4 w_b^i}{dx^{i4}} = p_{bz}\left(x^i\right) \tag{43}$$

Equation (43) indicates that the solution of the original problem can be obtained as the deflection of a beam with unit flexural rigidity subjected to a flexural fictitious load $p_{bz} = p_{bz}\left(x^i\right)$ under the same boundary conditions. The fictitious load is unknown. Following the formulation developed in [23] application of the Eq. (10) to the L nodal points in the interior of the beams yields

$$\begin{aligned}&\left(E_b^i I_{by}^i\left(1+\frac{N_b^i}{G_b^i A_z^i}\right)[I]-\left[\left\{N_b^i\right\}\right]_{\mathrm{dg.}}\left[F_{bxx}^z\right]+[\{q_{x1}\}+\{q_{x2}\}]_{\mathrm{dg.}}\left[F_{bx}^z\right]\right.\\&\quad-\left(\frac{E_b^i I_y^i}{G_b^i A_z^i}\right)\left(\left(3\left[\{q_{x1}\}+\{q_{x2}\}\right]_{\mathrm{dg.}}\left[F_{bxxx}^z\right]\right)+\left(3\left[\{dq_{x1}\}+\{dq_{x2}\}\right]_{\mathrm{dg.}}\left[F_{bxx}^z\right]\right)\right.\\&\quad\left.\left.+\left(3\left[\{ddq_{x1}\}+\{ddq_{x2}\}\right]_{\mathrm{dg.}}\left[F_{bx}^z\right]\right)\right)\right)\{p_{bz}\}\\&\quad=\{q_{z1}\}+\{q_{z2}\}+[X_{bx}]\left(\{q_{x1}\}+\{q_{x2}\}\right)-\left(\frac{E_b^i I_y^i}{G_b^i A_z^i}\right)\left[\{ddq_{z1}\}+\{ddq_{z2}\}\right]\end{aligned} \tag{44}$$

while similarly application of the Eq. (15) gives

$$\begin{aligned}&\left(E_b^i I_{bz}^i\left(1+\frac{N_b^i}{G_b^i A_y^i}\right)[I]-\left[\left\{N_b^i\right\}\right]_{\mathrm{dg.}}\left[F_{bxx}^y\right]+[\{q_{x1}\}+\{q_{x2}\}]_{\mathrm{dg.}}\left[F_{bx}^y\right]\right.\\&\quad-\left(\frac{E_b^i I_z^i}{G_b^i A_y^i}\right)\left(\left(3\left[\{q_{x1}\}+\{q_{x2}\}\right]_{\mathrm{dg.}}\left[F_{bxxx}^y\right]\right)+\left(3\left[\{dq_{x1}\}+\{dq_{x2}\}\right]_{\mathrm{dg.}}\left[F_{bxx}^y\right]\right)\right.\\&\quad\left.\left.+\left(3\left[\{ddq_{x1}\}+\{ddq_{x2}\}\right]_{\mathrm{dg.}}\left[F_{bx}^y\right]\right)\right)\right)\left\{p_{by}\right\}\\&\quad=\left\{q_{y1}\right\}+\left\{q_{y2}\right\}+\left[X_{by}\right]\left(\left\{q_{y1}\right\}+\left\{q_{y2}\right\}\right)-\left(\frac{E_b^i I_z^i}{G_b^i A_y^i}\right)\left[\left\{ddq_{y1}\right\}+\left\{ddq_{y2}\right\}\right]\end{aligned} \tag{45}$$

and for the angle of twist θ_{bx}^i application of the Eq. (24) yields

$$\begin{aligned}&\left(E_b^i I_{bw}^i\,[I]-G_b^i I_{bx}^i\,[I]\left[F_{bxx}^t\right]\right)\{p_{bx}\}=\left[e_{y1}\right]\{q_{z1}\}+\left[e_{y2}\right]\{q_{z2}\}\\&\quad-\left[e_{z1}\right]\left\{q_{y1}\right\}-\left[e_{z2}\right]\left\{q_{y2}\right\}\end{aligned} \tag{46}$$

where $\left[\left\{N_b^i\right\}\right]_{\mathrm{dg.}}$ is an unknown diagonal $L \times L$ matrix including the values of the axial forces; the symbol $[]_{\mathrm{dg.}}$ indicates a diagonal $L \times L$ matrix with the elements of the included column matrix. The matrices $[X_{bx}]$, $\left[X_{by}\right]$ result after approximating the derivatives of m_{byj}^i, m_{bzj}^i using appropriately central, backward, or forward differences. Their dimensions are also $L \times L$. Moreover, $\{p_{bz}\}$, $\left\{p_{by}\right\}$, $\{p_{bx}\}$, $\left\{q_{xj}\right\}$, $\left\{dq_{xj}\right\}$, $\left\{ddq_{xj}\right\}$, $\left\{q_{yj}\right\}$, $\left\{dq_{yj}\right\}$, $\left\{ddq_{yj}\right\}$, $\left\{q_{zj}\right\}$, $\left\{dq_{zj}\right\}$, $\left\{ddq_{zj}\right\}$ $(j = 1, 2)$ are $L \times 1$ column matrices including the values of the fictitious flexural, torsional loading, the interface forces and their derivatives, $\left[F_{bx}^y\right]$, $\left[F_{bxx}^y\right]$, $\left[F_{bxxx}^y\right]$, $\left[F_{bx}^z\right]$, $\left[F_{bxx}^z\right]$, $\left[F_{bxxx}^z\right]$, $\left[F_{bxx}^t\right]$ are $L \times L$ flexibility coefficient matrices, while $\left[e_{y1}\right]$, $\left[e_{y2}\right]$, $\left[e_{z1}\right]$, $\left[e_{z2}\right]$ are diagonal $L \times L$ matrices including the values of the eccentricities e_{yj}^i, e_{zj}^i of the components q_{zj}^i, q_{yj}^i with respect to the i-th beam shear center axis, respectively.

3.4 For the axial deformation u_b^i

Following the same procedure as in the previous sections, the descretized L values of the nodal axial forces and the nodal displacements at the beam centroid axis for homogeneous boundary conditions (22) ($\gamma_3^{xi} = 0$) can be expressed as

$$\left\{N_b^i\right\} = \left[G_b^x\right](\{q_{x1}\} + \{q_{x2}\}) \tag{47a}$$

$$\left\{u_b^i\right\} = \left[F_b^x\right](\{q_{x1}\} + \{q_{x2}\}) \tag{47b}$$

where $\left[G_b^x\right]$, $\left[F_b^x\right]$ are known $L \times L$ matrices.

Equations (40), (44), (45) and (46) after elimination of the quantities N_x, N_y, N_{xy}, N_b^i using Eqs. (41a, b, c), (47a) together with continuity conditions (29a–f) which employing Eqs. (41a–d), (47b) and after discretization at the L nodal points at the interfaces are written as

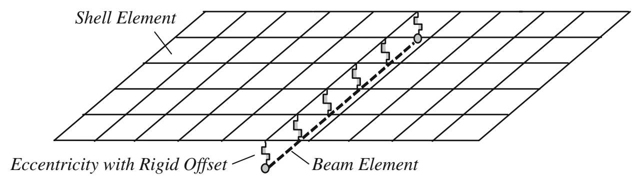

Fig. 4 Shell-beam model using rigid offsets for the analysis of the stiffened plates

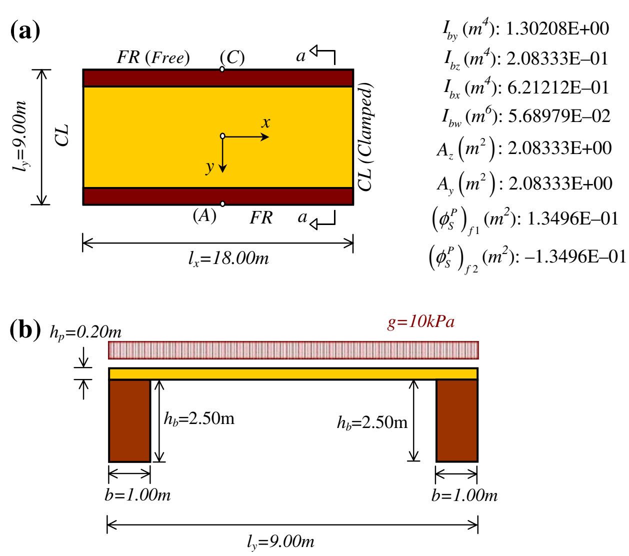

Fig. 5 Plan view (**a**) and section a–a (**b**) of the stiffened plate of Example 1

Table 1 Deflections w_p (mm) of the stiffened plate of Example 1

	Including inplane forces		Ignoring inplane forces	
	With shear deformation (Present study)	Without Shear deformation	With Shear deformation	Without shear deformation
Center	5.8588	5.8107 (0.82%)	6.3718 (8.76%)	6.1971 (5.77%)
Middle of the free edges A, C	0.2671	0.2319 (13.14%)	0.6748 (152.67%)	0.4964 (85.86%)

$$[Y_1]\left[F_p\right]\left\{p_{pz}\right\} - \left[F_b^z\right]\{p_{bz}\} = -\frac{b_f^i}{4}\left[F_b^t\right]\{p_{bx}\} \tag{48a}$$

$$[Y_2]\left[F_p\right]\left\{p_{pz}\right\} - \left[F_b^z\right]\{p_{bz}\} = \frac{b_f^i}{4}\left[F_b^t\right]\{p_{bx}\} \tag{48b}$$

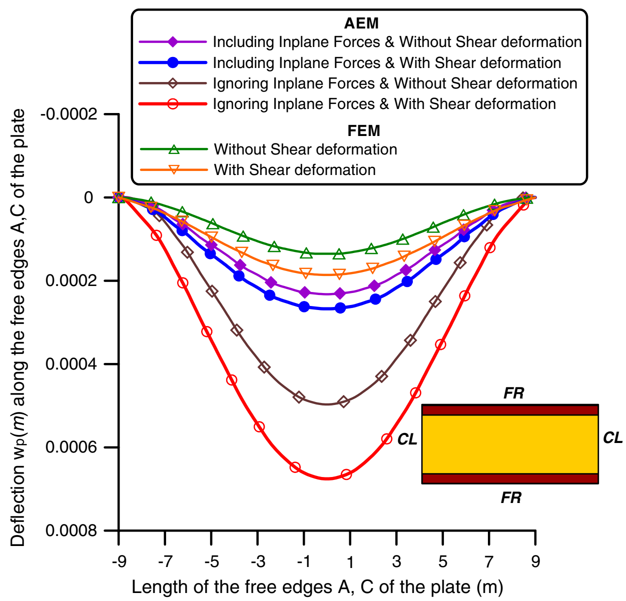

Fig. 6 Deflections along the free edges of the stiffened plate of Example 1 using the proposed (AEM) method and a FEM solution [26]

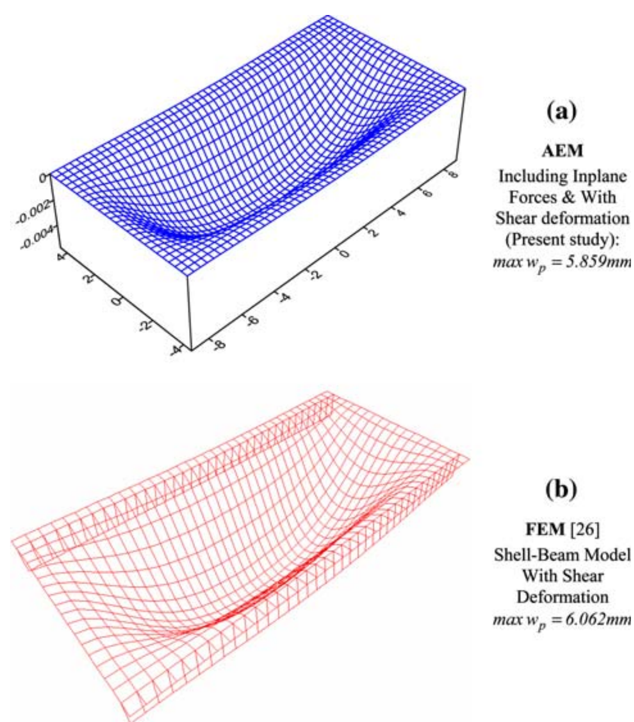

Fig. 7 Deflection surface of the stiffened plate of Example 1

$$\begin{aligned}&\left[F_{d1}^{xx}\right]\{q_{x1}\}+\left[F_{d1}^{xy}\right]\{q_{y1}\}-\left[F_{b}^{x}\right](\{q_{x1}\}+\{q_{x2}\})\\&=\frac{h_p}{2}\left[Y_1\right]\left[F_{px}\right]\{p_{pz}\}+\frac{h_b^i}{2}\left[F_{bx}^{z}\right]\{p_{bz}\}+\frac{b_f^i}{4}\left[F_{bx}^{y}\right]\{p_{by}\}+\left(\phi_S^P\right)_{f_1^i}\left[F_b^t\right]\{p_{bx}\}\end{aligned}\tag{49a}$$

$$\begin{aligned}&\left[F_{d2}^{xx}\right]\{q_{x2}\}+\left[F_{d2}^{xy}\right]\{q_{y2}\}-\left[F_{b}^{x}\right](\{q_{x1}\}+\{q_{x2}\})\\&=\frac{h_p}{2}\left[Y_2\right]\left[F_{px}\right]\{p_{pz}\}+\frac{h_b^i}{2}\left[F_{bx}^{z}\right]\{p_{bz}\}-\frac{b_f^i}{4}\left[F_{bx}^{y}\right]\{p_{by}\}+\left(\phi_S^P\right)_{f_2^i}\left[F_b^t\right]\{p_{bx}\}\end{aligned}\tag{49b}$$

Table 2 Bending moments M_{by}^i, M_{bz}^i (kNm), warping moment M_{bw}^i (kNm^2) and axial force N_b^i (kN) of the beams $i = 1, 2$ of the stiffened plate of Example 1

	Including inplane forces		Ignoring inplane forces	
	With shear deformation (Present study)	Without shear deformation	With Shear deformation	Without shear deformation
Bending moment M_{by}^i (kNm)				
Midspan	552.09	624.37	1250.29	1190.64
Edges	−1001.04	−1089.37	−1885.09	−1936.71
Bending moment M_{bz}^i (kNm)				
Midspan	75.69	77.06	0.00	0.00
Edges	−253.74	−284.86	0.00	0.00
Warping moment due to the torsional curvature M_{bw}^i (kNm^2)				
Midspan	18.65	18.72	18.88	18.88
Edges	−63.47	−60.65	−90.73	−91.03
Axial force N_b^i (kN)				
Midspan	416.58	390.94	0.00	0.00
Edges	−687.18	−568.36	0.00	0.00

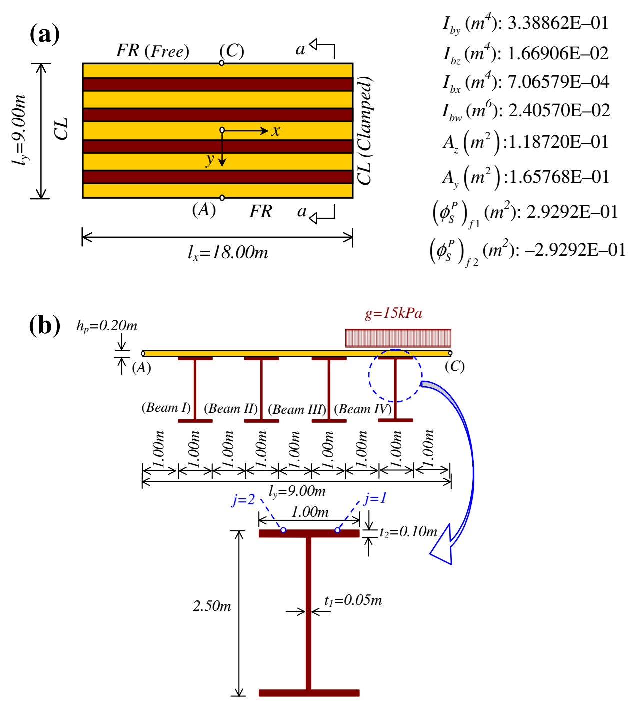

Fig. 8 Plan view (**a**) and section a–a (**b**) of the stiffened plate of Example 2

$$\left[F_{d1}^{yx}\right]\{q_{x1}\}+\left[F_{d1}^{yy}\right]\left\{q_{y1}\right\}-\left[F_b^y\right]\{p_{bz}\}=\frac{h_p}{2}\left[Y_1\right]\left[F_{py}\right]\left\{p_{pz}\right\}+\frac{h_p}{2}\left[F_b^t\right]\{p_{bx}\} \tag{50a}$$

$$\left[F_{d2}^{yx}\right]\{q_{x2}\}+\left[F_{d2}^{yy}\right]\left\{q_{y2}\right\}-\left[F_b^y\right]\{p_{bz}\}=\frac{h_p}{2}\left[Y_2\right]\left[F_{py}\right]\left\{p_{pz}\right\}+\frac{h_p}{2}\left[F_b^t\right]\{p_{bx}\} \tag{50b}$$

constitute a non-linear system of ten equations with respect to $\{q_{x1}\}$, $\{q_{x2}\}$, $\left\{q_{y1}\right\}$, $\left\{q_{y2}\right\}$, $\{q_{z1}\}$, $\{q_{z2}\}$ (interface forces) and $\left\{p_{pz}\right\}$, $\{p_{bx}\}$, $\left\{p_{by}\right\}$, $\{p_{bz}\}$ (fictitious loading of plate and beams). This system is solved using iterative numerical methods. Note that $\left[F_p\right]$ is known $M \times M$ coefficient flexibility matrix, $\left[F_b^z\right]$, $\left[F_b^y\right]$, $\left[F_b^t\right]$ are known flexibility coefficient matrices with dimensions $L \times L$, while $[Y_1]$, $[Y_2]$ are position $L \times M$ matrices which convert the matrices $\left[F_p\right]$, $\left[F_{px}\right]$, $\left[F_{py}\right]$ into corresponding ones with dimensions $L \times M$, appropriately referring to the nodal points of the two interface lines $f_{j=1}^i$, $f_{j=2}^i$, respectively.

Finally, it is worth noting that beams placed along the boundary of the plate are treated as every other stiffening beam, since the lines of action of the integrated interface force components q_{xj}^i, q_{yj}^i and q_{zj}^i $(j=1,2)$ will also be internal ones, taking special care during the numerical evaluation of the line integrals in order to avoid their "near singular integral behavior". According to this, boundary elements that are very close to each other (distance smaller than their length) are divided in sub elements, in each of which Gauss integration is applied [9].

Table 3 Deflections w_p (mm) of the stiffened plate of Example 2

	Including inplane forces		Ignoring inplane forces	
	With shear deformation (present study)	Without shear deformation	With Shear deformation	Without shear deformation
Center	−2.7048E−02	−3.5496E−03	−2.7528E − 02	−5.1296E−03
Middle of the free edge A	−6.1369E−02	−6.8369E−04	−4.8232E − 02	1.6856E−04
Middle of the free edge C	7.4194E−01	9.3331E−01	1.0525E + 00	1.2419E+00

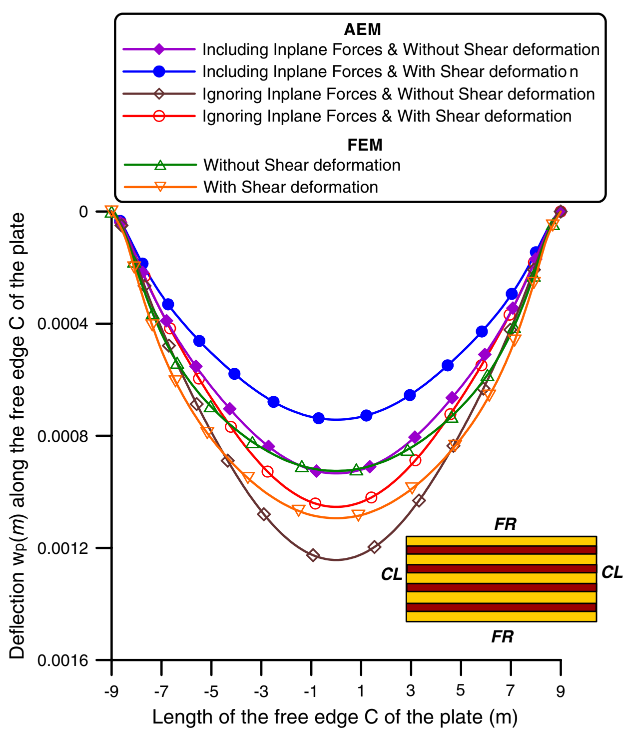

Fig. 9 Deflections along the free edge C of the stiffened plate of Example 2 using the proposed (AEM) method and a FEM solution [26]

4 Numerical examples

On the basis of the analytical and numerical procedures presented in the previous sections, a computer program has been written and representative examples have been studied to demonstrate the efficiency and the range of applications of the developed method. Moreover, in some cases in order to demonstrate the effectiveness of the proposed method its numerical results are compared with those obtained from a FEM solution using a commercial code [26] and employing a shell-beam model (modeling the plate with shell elements and the beam with beam ones ignoring torsional warping effects) using rigid offsets for the plate–beam connection, as this is shown in Fig. 4.

Example 1 A concrete $C20/25(E = 2.9 \times 10^7$ kPa, $\mu = 0.0)$ rectangular plate with dimensions $a \times b = 18.0 \times 9.0$ m subjected to a uniform load $g = 10$ kN/m^2 and stiffened by two concrete $C20/25(E_b = 2.9 \times 10^7$ kPa,

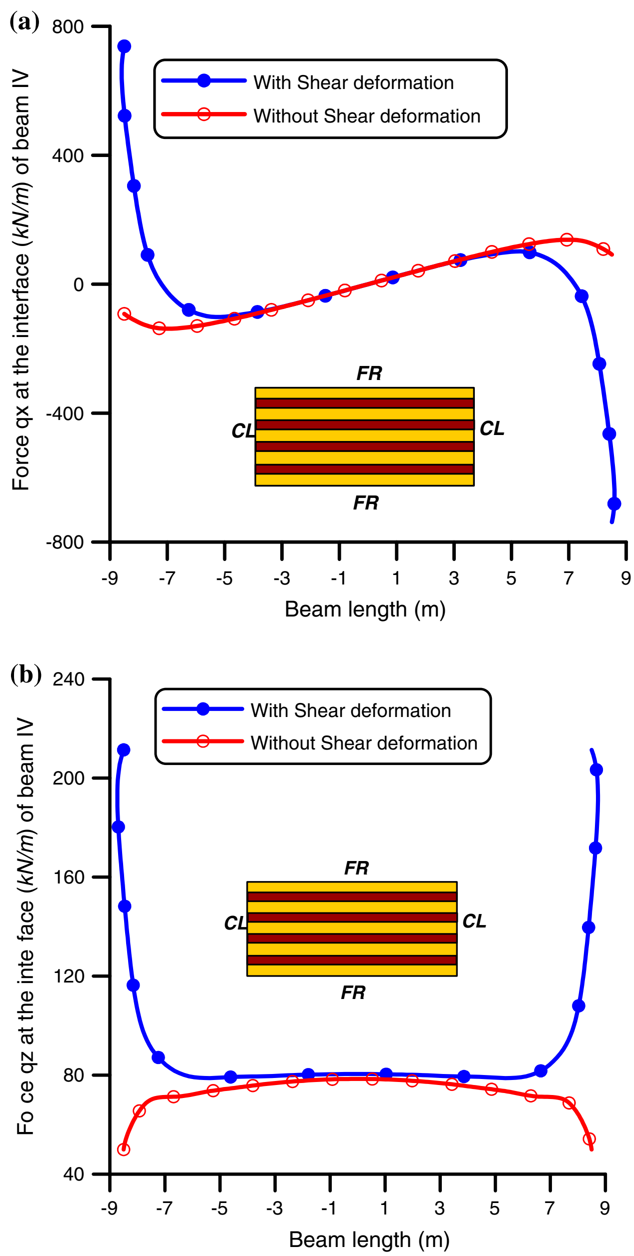

Fig. 10 Interface forces q_x^{IV}, q_z^{IV} of the beam IV of the stiffened plate of Example 2

$\mu_b = 0.0$) rectangular beams of 1.0 m width placed at its free sides (Fig. 5) has been studied. The plate is clamped along its small edges, while the other two edges are free according to both its transverse and inplane boundary conditions, while the beams are also clamped at their edges according to their transverse, axial and torsional boundary conditions. The obtained deflections of the stiffened plate in Table 1 at its center and at the middle of the free edges A and C (Fig. 5) and in Fig. 6 along the free edges, taking into account or ignoring inplane forces and shear deformation effect are presented demonstrating the significant influence of the inplane forces and the effect of shear deformation especially at the middle of the free edges (numbers in parentheses in the table indicate % discrepancy). In Figs. 6 and 7 the obtained values of the deflection surface of the stiffened plate are compared with those obtained from a FEM solution using a commercial code [26] as explained before. Moreover, in Table 2 the bending moments M_{by}^i, M_{bz}^i, the warping moment M_{bw}^i and the axial force N_b^i of the beams $i = 1, 2$ of the stiffened plate at midspan and at the beam edges are presented taking into account or ignoring inplane forces and shear deformation effect. From the aforementioned table as it is observed the magnitude of the transverse bending moment M_{bz}^i and the warping moment M_{bw}^i is considerable and should not be neglected as it usually happens in the analysis of projects. It is worth here noting that the evaluation of the bending moments M_{by}^i, M_{bz}^i is accomplished employing the relations

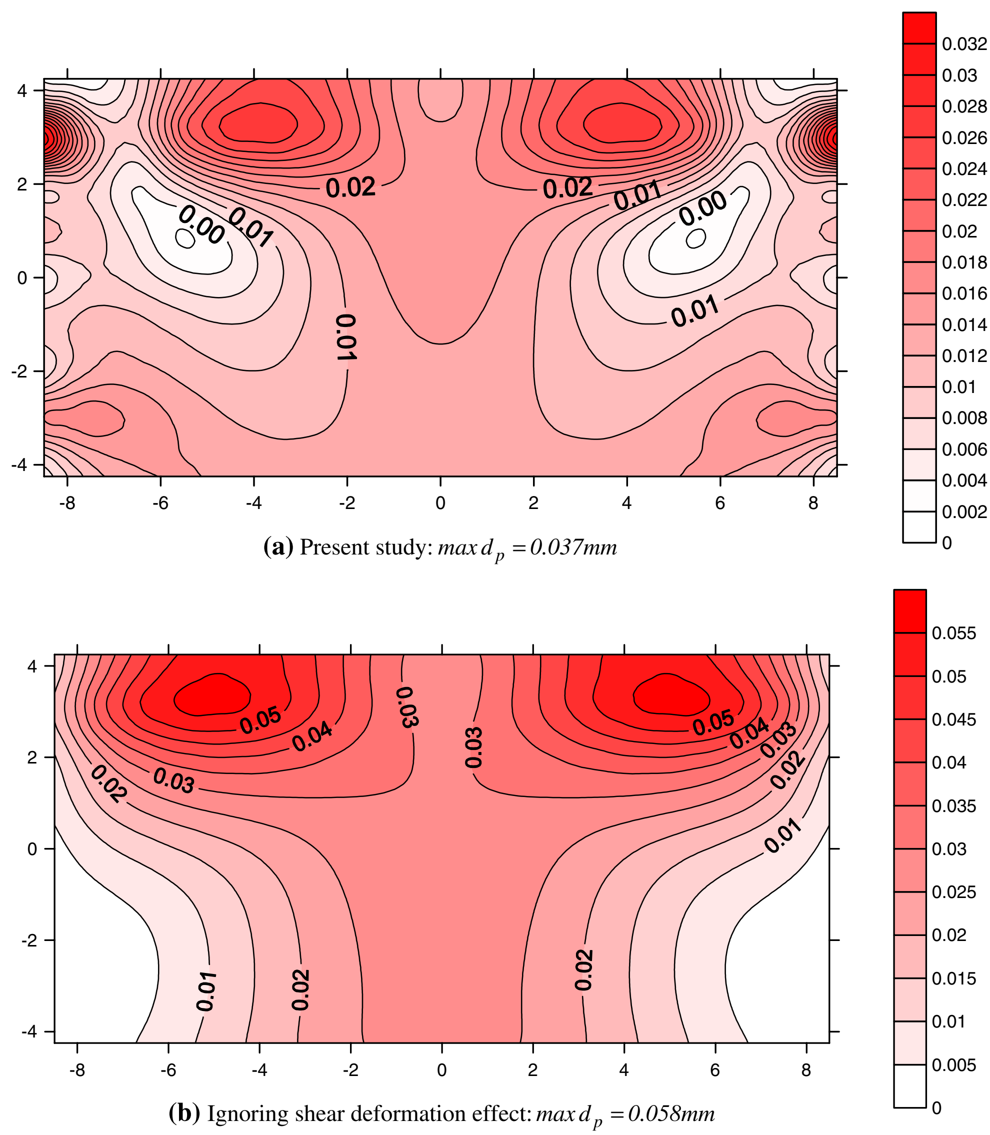

Fig. 11 Contour lines of the total displacement component d_p (mm) of the stiffened plate of Example 2 taking into account (**a**) or ignoring (**b**) shear deformation effect

$$M_{by}^i = -E_b^i I_{by}^i \frac{\partial^2 w_b^i}{\partial x^{i2}} - \frac{E_b^i I_{by}^i}{G_b^i A_z^i} \sum_{j=1}^{2} \left(q_{zj}^i - q_{xj}^i \frac{\partial w_b^i}{\partial x^i} + N_{bj}^i \frac{\partial^2 w_b^i}{\partial x^{i2}} \right) \tag{51a}$$

$$M_{bz}^i = -E_b^i I_{bz}^i \frac{\partial^2 v_b^i}{\partial x^{i2}} - \frac{E_b^i I_{bz}^i}{G_b^i A_y^i} \sum_{j=1}^{2} \left(q_{yj}^i - q_{xj}^i \frac{\partial v_b^i}{\partial x^i} + N_{bj}^i \frac{\partial^2 v_b^i}{\partial x^{i2}} \right) \tag{51b}$$

while the warping moment M_{bw}^i is evaluated using Eq. (27b).

Example 2 A concrete $C20/25(E = 2.9 \times 10^7$ kPa, $\mu = 0.0)$ rectangular plate with dimensions $a \times b = 18.0 \times 9.0$ m subjected to eccentric uniform load $g = 15\,k\text{N/m}^2$ and stiffened by four steel ($E_b^i = 2.1 \times 10^8$ kPa, $\mu_b^i = 0.3$) I-section beams symmetrically placed (Fig. 8) has been studied. The plate is clamped along its small edges, while the other two edges are free according to both its transverse and inplane boundary conditions, while the beams are also clamped at their edges according to their transverse, axial and torsional boundary conditions. The obtained deflections of the stiffened plate in Table 3 at its center and at the middle of the free edges A and C (Fig. 8) and in Fig. 9 along the free edge C, taking into account or ignoring inplane forces and shear deformation effect are presented demonstrating again the conclusions already drawn from Example 1, while the results of Fig. 9 are compared with those obtained from a FEM solution using a commercial code [26] as explained before. In Fig. 10 the obtained interface forces q_x^{IV}, q_z^{IV} of the beam IV (Fig. 8) of the stiffened plate taking into account or ignoring shear deformation effect are presented demonstrating the effect of shear deformation at the beams ends. Moreover, in Fig. 11 the total displacement component $d_p = \sqrt{\left(u_p^2 + v_p^2\right)}$ taking into account or ignoring shear deformation effect are presented. From the aforementioned figures the conclusions already drawn from Example 1 are verified.

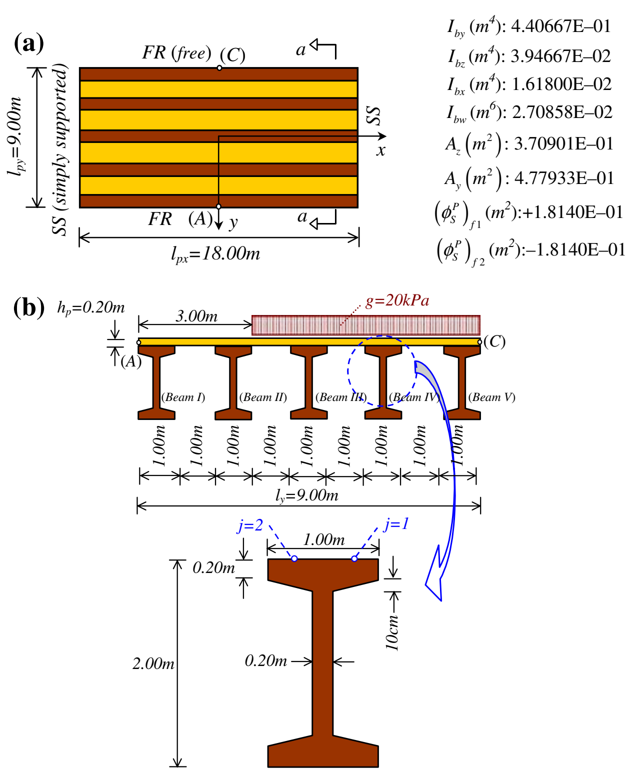

Fig. 12 Plan view (**a**) and section a–a (**b**) of the stiffened plate of Example 3

Example 3 A concrete $C20/25(E = 2.9 \times 10^7$ kPa, $\mu = 0.0)$ rectangular plate with dimensions $a \times b = 18.0 \times 9.0$ m subjected to eccentric uniform loading $g = 20$ kN/m^2 and stiffened by five concrete $C35/40(E_b = 3.35 \times 10^7$ kPa, $\mu_b = 0.2)$ I-section beams symmetrically placed (Fig. 12) forming a bridge deck has been studied. The plate along its small edges is simply supported according to its transverse boundary conditions, clamped according to its inplane ones, while the other two edges are free according to both its transverse and inplane boundary conditions. The beams at their edges are also simply supported according to their bending boundary conditions and clamped according to their axial and torsional ones. In Table 4 the obtained deflections of the stiffened plate at its center and at the middle of the free edges A and C (Fig. 12) and in Fig. 13 the deflection surfaces are presented taking into account or ignoring inplane forces and shear deformation effect. Finally, in Fig. 14 the obtained deflections w_b^i, v_b^i of each stiffening beam are presented taking into account or ignoring shear deformation effect. From the aforementioned table and figures the conclusions already drawn are once again verified.

Table 4 Deflections w_p (mm) of the stiffened plate of Example 3

	Including inplane forces		Ignoring inplane forces	
	With shear deformation (present study)	Without shear deformation	With shear deformation	Without shear deformation
Center	1.1094E + 00	8.9581E−01	7.2096E + 00	7.1152E + 00
Middle of the free edge A	−9.3156E − 02	−6.6141E−02	2.5232E − 01	1.8170E − 01
Middle of the free edge C	9.6674E − 01	7.1705E−01	7.3523E + 00	7.2615E + 00

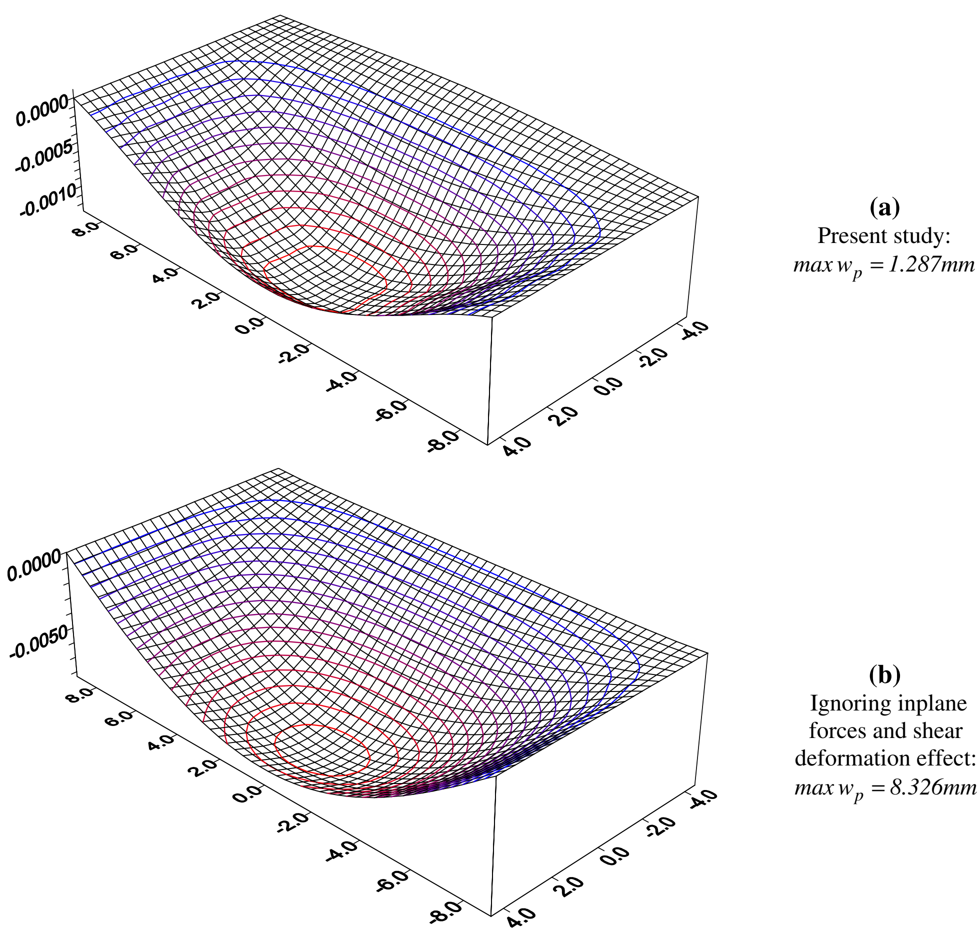

Fig. 13 Deflection surface of the stiffened plate of Example 3 taking into account (**a**) or ignoring (**b**) inplane forces and shear deformation effect

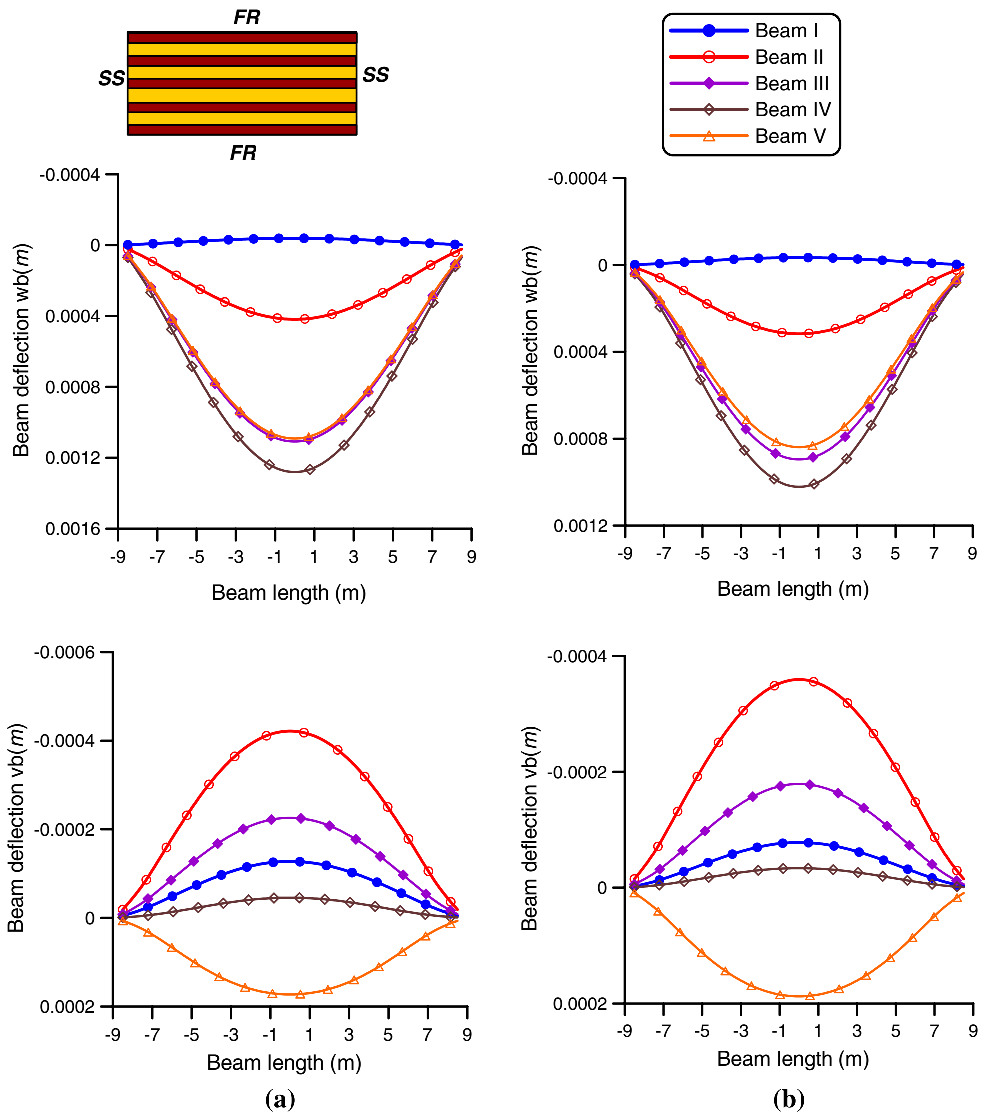

Fig. 14 Deflections w_b^i, v_b^i of the beams of the stiffened plate of Example 3 taking into account (**a**) or ignoring (**b**) shear deformation effect

5 Concluding remarks

A general solution for the analysis of shear deformable stiffened plates subjected to arbitrary loading is presented. The main conclusions that can be drawn from this investigation are

a. The proposed model permits the study of a stiffened plate subjected to an arbitrary loading, while both the number and the placement of the nonintersecting stiffening beams are also arbitrary (eccentric beams are also included).
b. The proposed model permits the evaluation of both the longitudinal and the transverse inplane shear forces at the interfaces between the plate and the beams, the knowledge of which is very important in the design of prefabricated plate beams structures (estimation of shear connectors in both directions).
c. The evaluated lateral deflections of the plate–beams system are found to exhibit considerable discrepancy from those of other models, which neglect inplane and axial forces and deformations.
d. In some cases, the influence of the shear deformation effect to the deflections at midspan, to the interface forces at the beam ends and to the stress resultants is remarkable and should not be neglected.
e. The magnitude of the transverse bending moment and of the warping moment is considerable and should not be neglected as it usually happens in the analysis of projects.

f. In some cases the discrepancy of the normal stresses arising from the ignorance of either the inplane forces and deformations or the shear deformation effect necessitates the inclusion of these effects in the analysis of stiffened plates.

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