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Paradox of a particle's trajectory moving on a string

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Abstract This paper deals with the paradoxical properties of the solution of string vibration under a moving mass. The solutions published to date are not simple enough and cannot be applied to investigations in the entire range of mass speeds, including the overcritical range. We propose a formulation of the problem that allows us to reduce the problem to a second-order matrix differential equation. Its solution is characteristic of all features of the critical, subcritical, and overcritical motion. Results exhibit discontinuity of the mass trajectory at the end support point, which has not been previously reported in the literature. The closed solution in the case of a massless string is analyzed and the discontinuity is proved. Numerical results obtained for an inertial string demonstrate similar features. Small vibrations are analyzed, which is why the effect discussed in the paper is of purely mathematical interest. However, the phenomenon results in complexity in discrete solutions.

Keywords Moving mass · Vibrations of string · Inertial load

1 Introduction

Inertial loads moving on strings and beams with sub- or supercritical speed are of special interest. Theoretical solutions are applied to many practical problems: train–track interaction, vehicle–bridge interaction, pantograph collectors in railways, magnetic rails, guideways in robotic solutions, etc. The problem has been widely treated in literature. Attempts to find a solution to this problem started in the middle of the 19th century. However, we do not yet have a complete closed analytical solution. The term describing the concentrated mass motion is the reason for difficulties. Differential equations with variable coefficients, which except for a few cases do not have analytical solutions, pose serious limitations on closed solutions. These types of equations are finally solved by numerical means.

In the literature numerous historical reviews concerning the moving-loads problem exist (for example, [1–3]). In most cases the moving massless constant force was considered as a moving load. This type of problem results in closed solutions. Unfortunately, the problem of inertial loads is still open. Saller [4] considered the moving mass for the first time. He proved, in spite of essential simplifications, the significant influence of the moving mass in beam dynamics. In the 1930s two important contributions for researchers working in the field of moving loads appeared. Inglis [5] applied simplifications and the solution was expressed with only the first term of the trigonometric series; the time function obeyed a second-order differential equation with variable coefficients. This equation was derived considering the acceleration under the moving mass, expressed by the so-called Renaudot formula. In fact this is the derivative with constant velocity, computed with the chain rule. The final solution of the differential equation with variable coefficients was proposed as an infinite series, which approaches the solution.

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Schallenkamp [6] proposed another approach to the problem of moving masses. However, his attempt allows us to describe the motion only under the moving mass. The method of separation of variables by the expansion of the unknown function into a sine Fourier series was applied. Boundary conditions in the beam were taken into account in a natural way. The ordinary differential equation that describes the motion under the moving mass was expressed in generalized coordinates by using the second Lagrange equations. The generalized force was derived from the principle of virtual work. Schallenkamp's approach is relatively complex and converges slowly since the final solution is expressed in terms of a triple infinite series.

The works of Inglis and Schallenkamp can be considered as the basis for the analysis of the problem of moving mass in successive works such as [7–9] amongst others. An excellent and important monograph in this field was written by Szcześniak [10], in which one can find hundreds of references concerning moving loads on beams and strings. In [11] the authors consider a simply supported beam modeled by Bernoulli–Euler theory. The equation of motion is written in integral–differential form with Green function terms. In order to compute this equation a dual numerical scheme has been used: a backward difference technique was applied to treat the time parameter while numerical integration was used for the spatial parameter. This solution approach, though applicable to higher velocities, still requires complex mathematical operations. Each solution enables us to determine displacements under the moving load only and does not give solutions for a wide range of the parameters x and t. Only one closed analytical solution can be found in the literature. Smith [12] proposed a purely analytical solution for an inertial moving load, however, only in the case of the massless string. The basic motion equation without the term that describes the string inertia was transformed to the hypergeometrical equation, which has an analytical solution in terms of infinite series. Frýba [13] applied the same approach and found a closed analytical solution for the particular case $\alpha' = 1$. However, the formula given in [13] contains mistakes.

Recent papers have contributed the analysis of complex problems of structures subjected to moving inertial loads [14] or oscillators [15–17]. Variable speed was analyzed in [18–20]. Equivalent mass influence is analyzed in [21]. An infinitely long string subjected to a uniformly accelerated point mass was also treated [22] and the analytical solution of the problem concerning the motion of an infinite string on a Winkler foundation subjected to an inertial load moving at a constant speed has been given [23].

In the paper we consider small vibrations of massless and massed strings subjected to a moving inertial load and propose an analytical–numerical solution of the problem. The final solution has the form of a matrix differential equation of second order. Numerical integration results in a solution for the full range of the velocity, including undercritical and overcritical regimes. It exhibits discontinuity of the mass trajectory at the end support point, a new feature which has not been reported in literature. The closed solution in the case of the massless string is analyzed and its discontinuity is proved mathematically. Fully numerical results obtained for the inertial string demonstrate a similar feature. Since small vibrations are analyzed, the discontinuity effect discussed in the paper is of purely mathematical interest.

Results are compared with approached numerical solutions obtained by the finite element method. The string is subjected to a moving oscillator. In the case of the rigid spring we approach the analytical solution. However, in the case of higher speed ($v > 0.2c$) the accuracy of the finite element method (FEM) solution is poor.

2 Analytical formulation

Let us consider a string of the length l, cross-sectional area A, mass density ρ, tensile force N, subjected to a mass m accompanied by a force P (Fig. 1), moving with a constant speed v. The motion equation of the string under a moving inertial load with a constant speed v has the form

$$-N\,\frac{\partial^2 u(x,t)}{\partial x^2} + \rho A\,\frac{\partial^2 u(x,t)}{\partial t^2} = \delta(x-vt)\,P - \delta(x-vt)\,m\,\frac{\partial^2 u(vt,t)}{\partial t^2}\,. \tag{1}$$

We impose boundary conditions

$$u(0,t) = 0 \quad u(l,t) = 0 \tag{2}$$

and initial conditions

$$u(x,0) = 0 \quad \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = 0\,. \tag{3}$$

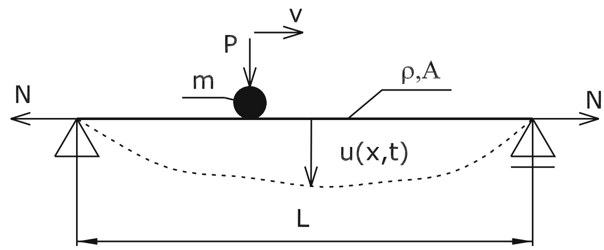

Fig. 1 Moving inertial load

In order to reduce the partial differential equation to an ordinary differential equation, we apply the Fourier sine integral transformation in a finite range (i.e., the finite length of the string) (4), (5)

$$V(j,t) = \int_0^l u(x,t)\,\sin\frac{j\pi x}{l}\,\mathrm{d}x \tag{4}$$

$$u(x,t) = \frac{2}{l}\sum_{j=1}^{\infty} V(j,t)\,\sin\frac{j\pi x}{l}\ . \tag{5}$$

We can present each of the functions as an infinite sum of sine functions (5) with corresponding coefficients (4). Then the expansion of the moving mass acceleration in a series takes the form

$$\frac{\partial^2 u(vt,t)}{\partial t^2} = \frac{2}{l}\sum_{k=1}^{\infty}\left[\ddot V(k,t)\,\sin\frac{k\pi vt}{l} + \frac{2k\pi v}{l}\,\dot V(k,t)\,\cos\frac{k\pi vt}{l} - \frac{k^2\pi^2 v^2}{l^2}\,V(k,t)\,\sin\frac{k\pi vt}{l}\right]. \tag{6}$$

The integral transformation (4) of Eq. (1) with consideration of (6) can be performed:

$$N\,\frac{j^2\pi^2}{l^2}\,V(j,t) + \rho A\,\ddot V(j,t) = P\,\sin\frac{j\pi ct}{l} - m\,\frac{\partial^2 u(vt,t)}{\partial t^2}\int_0^l \delta(x-vt)\,\sin\frac{j\pi x}{l}\,\mathrm{d}x\ . \tag{7}$$

The integral containing the delta Dirac function in the above equation is

$$\int_0^l \delta(x-vt)\,\sin\frac{j\pi x}{l}\,\mathrm{d}x = \sin\frac{j\pi vt}{l}\ . \tag{8}$$

Let us now consider (6) and (8):

$$\begin{aligned} N\,\frac{j^2\pi^2}{l^2}\,V(j,t) + \rho A\,\ddot V(j,t) &= P\,\sin\frac{j\pi vt}{l} - \frac{2m}{l}\sum_{k=1}^{\infty}\ddot V(k,t)\,\sin\frac{k\pi vt}{l}\,\sin\frac{j\pi vt}{l} \\ &\quad -\frac{2m}{l}\sum_{k=1}^{\infty}\frac{2k\pi v}{l}\,\dot V(k,t)\,\cos\frac{k\pi vt}{l}\,\sin\frac{j\pi vt}{l} \\ &\quad +\frac{2m}{l}\sum_{k=1}^{\infty}\frac{k^2\pi^2 v^2}{l^2}\,V(k,t)\,\sin\frac{k\pi vt}{l}\,\sin\frac{j\pi vt}{l}\ . \end{aligned} \tag{9}$$

Finally, the motion equation after Fourier transformation can be written

$$\begin{aligned} &\rho A\,\ddot V(j,t) + \alpha\sum_{k=1}^{\infty}\ddot V(k,t)\,\sin\omega_k t\,\sin\omega_j t + 2\alpha\sum_{k=1}^{\infty}\omega_k\,\dot V(k,t)\,\cos\omega_k t\,\sin\omega_j t \\ &+\Omega^2\,V(j,t) - \alpha\sum_{k=1}^{\infty}\omega_k^2\,V(k,t)\,\sin\omega_k t\,\sin\omega_j t = P\,\sin\omega_j t\ , \end{aligned} \tag{10}$$

where

$$\omega_k = \frac{k\pi v}{l}, \quad \omega_j = \frac{j\pi v}{l}, \quad \Omega^2 = N\frac{j^2\pi^2}{l^2}, \quad \alpha = \frac{2m}{l}. \tag{11}$$

The analytical solution for this problem does not exist. We must solve this final equation numerically. Thus we obtain a semi-analytical solution. Equation (10) is written in a matrix form, where **M**, **C**, and **K** are square matrices ($j, k = 1 \ldots n$):

$$\mathbf{M}\begin{bmatrix}\ddot{V}(1,t)\\ \ddot{V}(2,t)\\ \vdots\\ \ddot{V}(n,t)\end{bmatrix} + \mathbf{C}\begin{bmatrix}\dot{V}(1,t)\\ \dot{V}(2,t)\\ \vdots\\ \dot{V}(n,t)\end{bmatrix} + \mathbf{K}\begin{bmatrix}V(1,t)\\ V(2,t)\\ \vdots\\ V(n,t)\end{bmatrix} = \mathbf{P} \tag{12}$$

or

$$\mathbf{M}\ddot{\mathbf{V}} + \mathbf{C}\dot{\mathbf{V}} + \mathbf{K}\mathbf{V} = \mathbf{P}, \tag{13}$$

where

$$\mathbf{M} = \begin{bmatrix}\rho A & 0 & \cdots & 0\\ 0 & \rho A & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \rho A\end{bmatrix} + \alpha\begin{bmatrix}\sin\frac{1\pi vt}{l}\sin\frac{1\pi vt}{l} & \sin\frac{1\pi vt}{l}\sin\frac{2\pi vt}{l} & \cdots & \sin\frac{1\pi vt}{l}\sin\frac{n\pi vt}{l}\\ \sin\frac{2\pi vt}{l}\sin\frac{1\pi vt}{l} & \sin\frac{2\pi vt}{l}\sin\frac{2\pi vt}{l} & \cdots & \sin\frac{2\pi vt}{l}\sin\frac{n\pi vt}{l}\\ \vdots & \vdots & \ddots & \vdots\\ \sin\frac{n\pi vt}{l}\sin\frac{1\pi vt}{l} & \sin\frac{n\pi vt}{l}\sin\frac{2\pi vt}{l} & \cdots & \sin\frac{n\pi vt}{l}\sin\frac{n\pi vt}{l}\end{bmatrix}, \tag{14}$$

$$\mathbf{C} = 2\alpha\begin{bmatrix}\frac{1\pi v}{l}\sin\frac{1\pi vt}{l}\cos\frac{1\pi vt}{l} & \frac{2\pi v}{l}\sin\frac{1\pi vt}{l}\cos\frac{2\pi vt}{l} & \cdots & \frac{n\pi v}{l}\sin\frac{1\pi vt}{l}\cos\frac{n\pi vt}{l}\\ \frac{1\pi v}{l}\sin\frac{2\pi vt}{l}\cos\frac{1\pi vt}{l} & \frac{2\pi v}{l}\sin\frac{2\pi vt}{l}\cos\frac{2\pi vt}{l} & \cdots & \frac{n\pi v}{l}\sin\frac{2\pi vt}{l}\cos\frac{n\pi vt}{l}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{1\pi v}{l}\sin\frac{n\pi vt}{l}\cos\frac{1\pi vt}{l} & \frac{2\pi v}{l}\sin\frac{n\pi vt}{l}\cos\frac{2\pi vt}{l} & \cdots & \frac{n\pi v}{l}\sin\frac{n\pi vt}{l}\cos\frac{n\pi vt}{l},\end{bmatrix} \tag{15}$$

$$\mathbf{K} = \begin{bmatrix}\frac{1^2\pi^2}{l^2}N & 0 & \cdots & 0\\ 0 & \frac{2^2\pi^2}{l^2}N & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \frac{n^2\pi^2}{l^2}N\end{bmatrix} \tag{16}$$
$$-\alpha\begin{bmatrix}\frac{1^2\pi^2v^2}{l^2}\sin\frac{1\pi vt}{l}\sin\frac{1\pi vt}{l} & \frac{2^2\pi^2v^2}{l^2}\sin\frac{1\pi vt}{l}\sin\frac{2\pi vt}{l} & \cdots & \frac{n^2\pi^2v^2}{l^2}\sin\frac{1\pi vt}{l}\sin\frac{n\pi vt}{l}\\ \frac{1^2\pi^2v^2}{l^2}\sin\frac{2\pi vt}{l}\sin\frac{1\pi vt}{l} & \frac{2^2\pi^2v^2}{l^2}\sin\frac{2\pi vt}{l}\sin\frac{2\pi vt}{l} & \cdots & \frac{n^2\pi^2v^2}{l^2}\sin\frac{2\pi vt}{l}\sin\frac{n\pi vt}{l}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{1^2\pi^2v^2}{l^2}\sin\frac{n\pi vt}{l}\sin\frac{1\pi vt}{l} & \frac{2^2\pi^2v^2}{l^2}\sin\frac{n\pi vt}{l}\sin\frac{2\pi vt}{l} & \cdots & \frac{n^2\pi^2v^2}{l^2}\sin\frac{n\pi vt}{l}\sin\frac{n\pi vt}{l}\end{bmatrix},$$

$$\mathbf{P} = P\begin{bmatrix}\sin\frac{1\pi vt}{l}\\ \sin\frac{2\pi vt}{l}\\ \vdots\\ \sin\frac{n\pi vt}{l}\end{bmatrix}. \tag{17}$$

When the coefficients $V(j,t)$ are computed, the displacements of the string (5) can be found as a solution of (1). This solution has full range and we can calculate the displacement at each point of the string and and for all values of v. We see that, assuming $\rho = 0$ in (14), we have the formulation for the massless string.

3 Results

First we present the moderate convergence rate of the series which constitutes the solution (Fig. 2). We denote the wave speed in the unloaded string by c ($c^2 = N/\rho A$). Further figures will exhibit the vertical deflection of

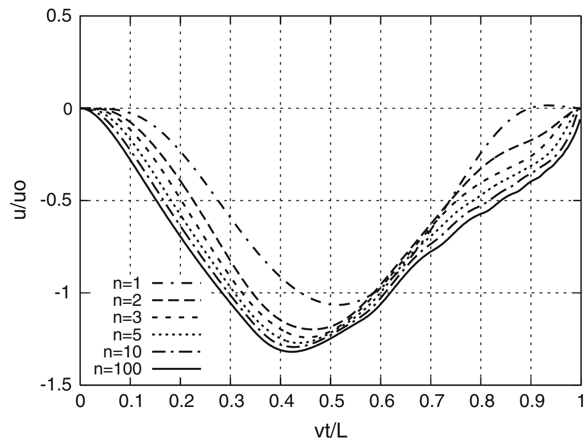

Fig. 2 Trigonometric series convergence for $v = 0.2c$

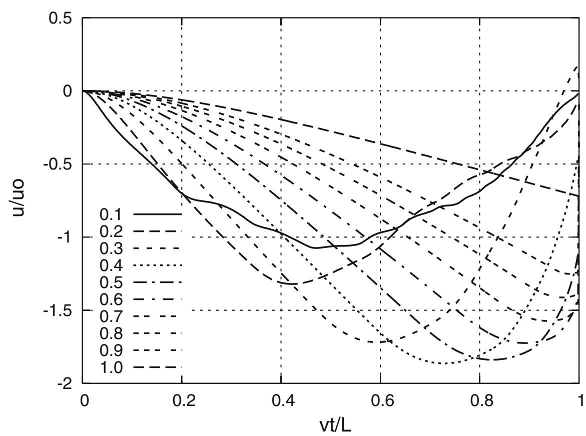

Fig. 3 Inertial string—displacements computed semi-analytically

the string u related to the deflection in the quasistatic mass motion in the middle of the span u_0. We can notice that the first term is already close to the exact solution. Three or five terms are sufficient for an accurate result in the engineering sense. We must emphasize here that higher mass speeds, for example, $0.9c$ or c, require as many as 100 terms and a short time step for the time integration of the differential equation, since the solution exhibits small jumps near the final support. The plot for the various velocity values v is shown in Fig. 3.

Let us look at the diagrams of the displacements of the string at the point under the mass. A diagram for various masses related to the string mass is shown in Fig. 4 for a speed of $v = 0.2c$. A more detailed presentation of the string motion is given in Fig. 5. We can notice the sharp edge of the wave and reflection from both supports. Moreover, the wave reflection from the traveling mass is clearly visible, especially for the case $v = 1.2c$. Both the mass trajectory and waves are depicted.

The convergence near the end point is depicted in Fig. 6. The mass trajectory is plotted for increasing numbers of term at a speed of $v = 0.5c$. We note that the function tends slowly to the jump at $x = l$. All characteristic lines are smooth. The convergence rate is low and, especially near $x = l$, the number of terms used must be at least then 50. In the high velocity range (in our case $v > 0.8$) a sufficiently short time step for the integration of (13) must be applied (even 10^{-5}) to avoid small oscillations of the solution in the last stage.

Supersonic motion of the mass results in zero displacement. In the diagram obtained numerically this value oscillates with low amplitude. The amplitude decreases with increasing numbers of terms in the sum (Fig. 7).

Analytical results were compared with numerical solutions obtained by the finite element method. The string was discretized by a set of 100 finite elements and was subjected to an oscillator moving over the span. Two autonomous systems were considered: a string subjected to a contact force between the oscillator spring and the string, and the oscillator itself, subjected to a force P applied to a mass and displacements determined

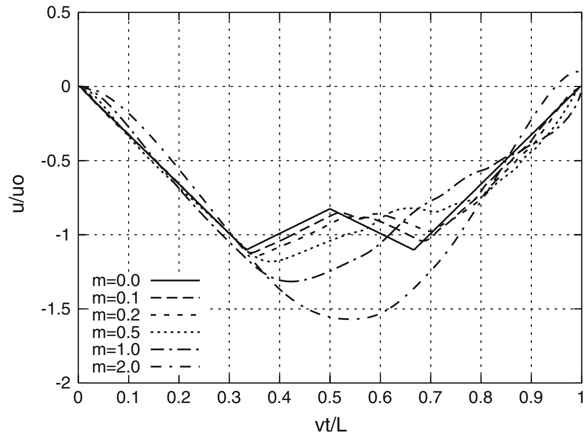

Fig. 4 Displacements under the mass for different mass values at a speed of $v = 0.2c$

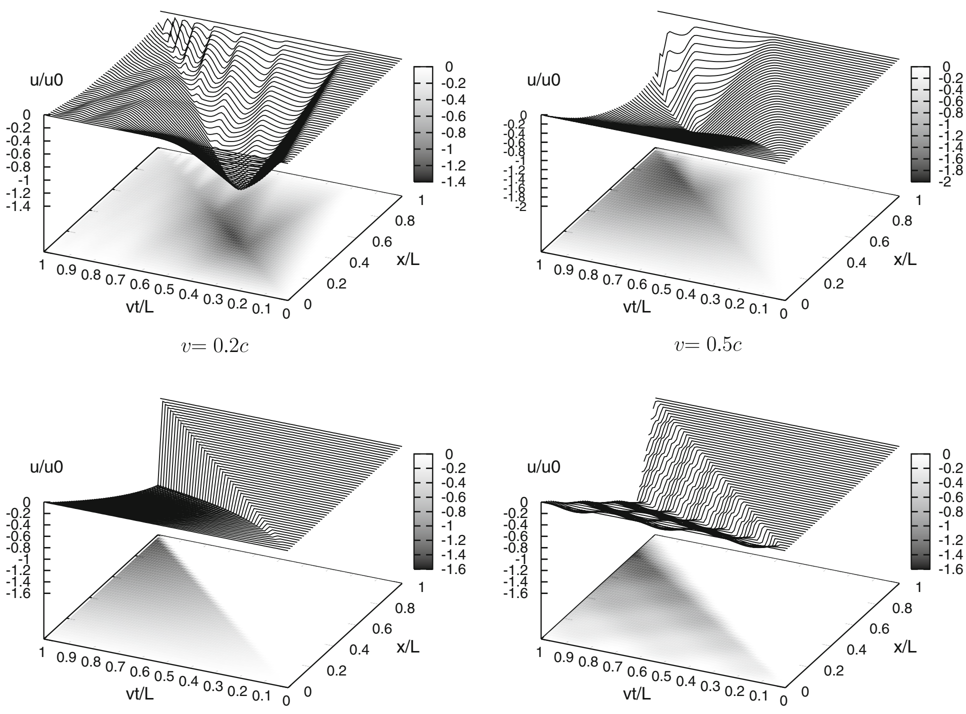

Fig. 5 Simulation of the string motion under the mass moving at $v = 0.2c$, $0.5c$, $1.0c$, and $1.2c$

from the string motion, applied to a spring. The oscillator spring stiffness was assumed to be high enough to simulate a rigid contact of the mass with the string. The results are depicted in Fig. 8.

4 Discontinuity of the solution

The advantages of the solution method presented in the paper allowed us to demonstrate an interesting feature of the solution near the end support. The resulting diagrams exhibit jumps of the mass displacement in time; let us consider the physical nature of these jumps. The simplest explanation is based on the force equilibrium (Fig. 9). We must remember that a constant string tension N is a fundamental assumption in our problem. Moreover, in Fig. 9 the horizontal force pushing the mass to maintain the speed v must be included in the scheme. At the final stage (as depicted in Fig. 9) the remaining distance d will be traversed in a time d/v,

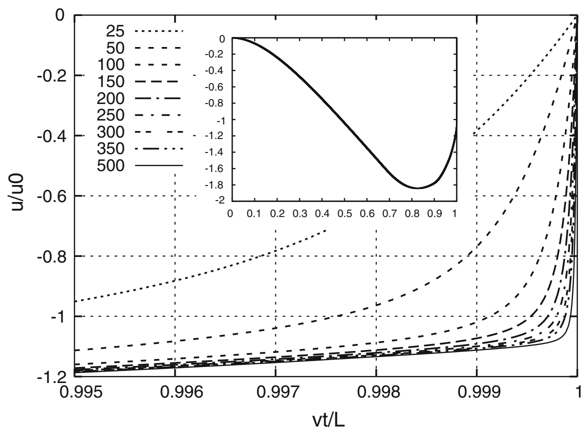

Fig. 6 The convergence of the mass trajectory traveling with $v = 0.5c$ near the end point, for various numbers of terms (25, 50, …, 500)

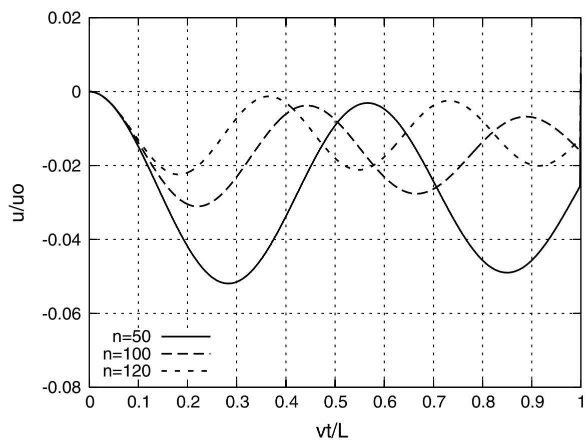

Fig. 7 Convergence of displacements under the mass at a speed of $v = 1.05c$

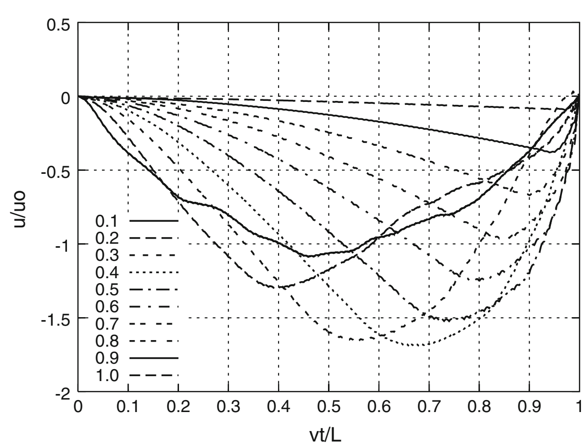

Fig. 8 Finite element solution: displacements of the string under the oscillator

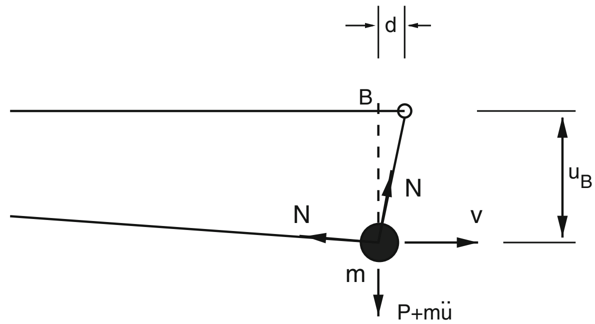

Fig. 9 Final stage of the moving mass

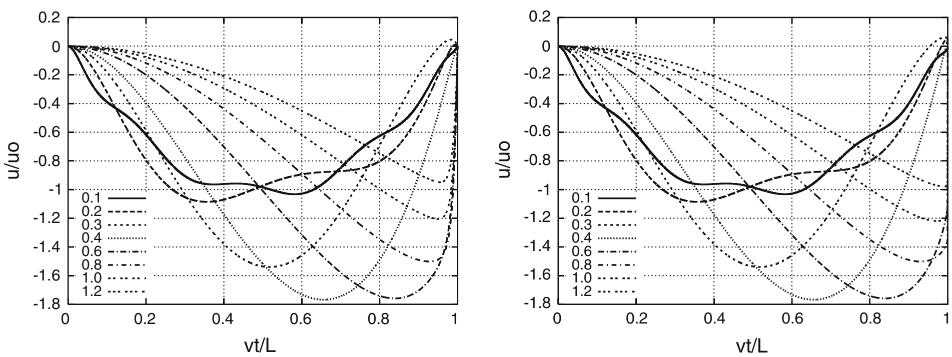

Fig. 10 Trajectories for the mass moving on a massless string for: lower number of terms in the sum (*left diagram*) and higher number of terms (*right diagram*)

during which the mass m must be lifted from the position u_B to zero. If the deflection u_B is high enough compared with other parameters the necessary acceleration applied to the mass must result in strong forces in the string $F \sim umv^2/d^2$. In such a case F can exceed N if m or v is sufficiently high. This violates our assumptions and the condition for the applicability of the equation for small vibrations $(\partial u/\partial x)^2 << 1$.

Let us consider a massless string, which is a particular case of our problem. The solution is given by a sum [13]

$$y(\tau) = \frac{4\alpha}{\alpha - 1}\,\tau\,(\tau - 1)\sum_{k=1}^{\infty}\prod_{i=1}^{k}\frac{(a+i-1)(b+i-1)}{c+i-1}\;\frac{\tau^k}{k!}, \tag{18}$$

where $\tau = vt/l > 0$ is the time parameter and $\alpha = Nl/(2mv^2) > 0$ determines the dimensionless parameter. The parameters a, b, and c are

$$a_{1,2} = \frac{3 \pm \sqrt{1+8\alpha}}{2} \qquad b_{1,2} = \frac{3 \mp \sqrt{1+8\alpha}}{2} \qquad c = 2. \tag{19}$$

In the case of $\alpha = 1$ the initial problem has a closed solution

$$u(\tau) = \frac{4}{3}\tau(1-\tau) - \frac{4}{3}\tau\,(1 + 2\tau\ln(1-\tau) - 2\ln(1-\tau))\,. \tag{20}$$

Here we consider the case of $\alpha \neq 1$. In Fig. 10 we can notice the strong influence of the precision on the solution near the end support. Let us consider the solution given by (18). The first term $\tau(\tau-1)$ is zero for $\tau = 1$.

$$\sum_{k=1}^{\infty}\prod_{i=1}^{k}\frac{(a+i-1)(b+i-1)}{c+i-1}\;\frac{\tau^k}{k!} \tag{21}$$

tends to ∞ if $\tau \to 1$. We have an indefinite solution at $\tau = 1^-$.

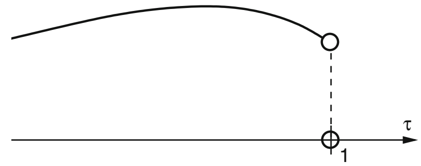

Fig. 11 Discontinuity of the function (18) at $\tau = 1$

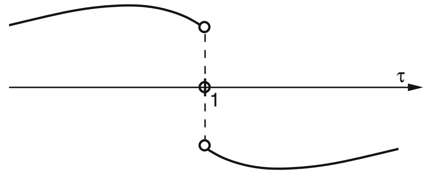

Fig. 12 Left and right limits at $\tau = 1$

The same result can be obtained on the base of Abel theorem. The power series can be written in the form

$$\sum_{k=1}^{\infty} A_k \,\tau^k\,, \qquad A_k = \prod_{i=1}^{k} \frac{(a+i-1)(b+i-1)}{(c+i-1)\;i}. \tag{22}$$

In this case $\lim_{\tau\to 1^-} A_k\,\tau^k = \infty$ and $y(1^-) = 0\cdot\infty$.

In the case $a + b < c$ the series (22) is convergent and there are no singularities. However, this is not our case. In the case of $a + b > c$ the series diverges (the sum tends to ∞). We have an indefinite value $0\cdot\infty$ while testing the function.

We can also perform another scheme of analysis. Below we will include the term $\tau(\tau - t)$ in the sum. Thus (18) can be reduced to the following form:

$$(1-\tau)\sum_{k=1}^{\infty}\frac{(a_k)\,(b_k)}{(c_k)}\,\frac{\tau^k}{k!} = \frac{a\,b\,\tau}{c} + \sum_{k=2}^{\infty}\frac{(a_{k-1})\,(b_{k-1})}{(c_{k-1})}\left(\frac{(a+k-1)(b+k-1)}{k(c+k-1)} - 1\right)\frac{\tau^k}{(k-1)!} \tag{23}$$

where

$$(a_k) = a\,(a\,+\,1)\cdots(a\,+\,k\,-\,1)$$
$$(b_k) = b\,(b\,+\,1)\cdots(b\,+\,k\,-\,1)$$
$$(c_k) = c\,(c\,+\,1)\cdots(c\,+\,k\,-\,1)$$

By using the Rabbe criterion one can show that for $a + b < c + 2$ the limit

$$\lim_{\tau\to 1}\left[(1-\tau)\sum_{k=1}^{\infty}\prod_{i=1}^{k}\frac{(a+i-1)(b+i-1)}{c+i-1}\,\frac{\tau^k}{k!}\right]$$

is finite. Now we can estimate the value of the sum (23). The sum of the first two to three terms, depending on the parameters, including $ab\tau/c$, is positive. The next terms are all positive. This proves that the sum (23) is finite and is greater than 0. The function (18) is depicted in Fig. 11.

The case $a + b = c + 1$ is particular (our set of parameters), for which the convergence is faster.

Let us look at the boundary condition at $\tau = 1$ (Fig. 12). We can say that it is fulfilled. We can imagine the symmetrical problem, with the mass moving from $\tau = 2$ towards $\tau = 1$ (with opposite direction of the force P). Then we have two analogous problems at $\tau = 1$. Both limits result in a zero value at $\tau = 1$:

$$\frac{1}{2}\left(\lim_{\tau\to 1^-} y(\tau) + \lim_{\tau\to 1^+} y(\tau)\right) = 0\,.$$

We can also consider the derivative $\mathrm{d}y/\mathrm{d}\tau$. The resulting formula can be derived. For negative P the result is

$$\lim_{\tau\to 1^-}\frac{\mathrm{d}y}{\mathrm{d}\tau} = \infty\,. \tag{24}$$

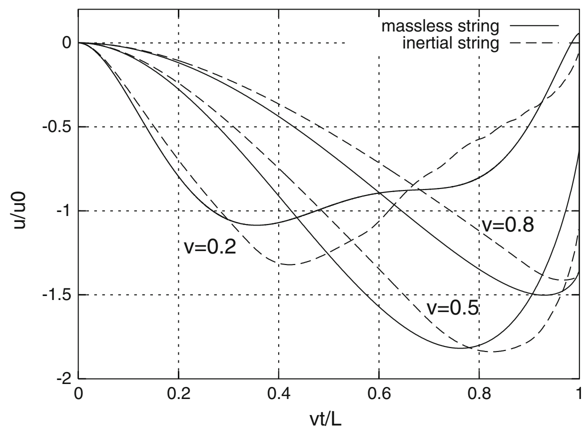

Fig. 13 Comparison of the particle's trajectory moving on a massless and inertial string

We can observe the same properties of the solution in the case of the inertial string. A comparative plot is presented in Fig. 13. We can emphasize that, for lower $m/\rho Al$ ratio, the coincidence of each pair of curves is greater. However, an analytical proof of discontinuity in the case of an inertial string cannot be obtained because of the numerical integration stage.

5 Conclusions

Herein we present a global analytical formulation for the vibration problem for a string, both massless and inertial, subjected to a moving mass. The numerical solution of the resulting second-order matrix differential equation is relatively simple and is valid for the whole range of speed v (subcritical, critical, and overcritical). The analysis of the results exhibits a jump of the mass in the neighborhood of the end support. The force acting on the mass is, however, limited to the tensile force N. Discontinuity of the mass trajectory at $x = l$ exists in the case $0 < v \leq c$. In the case of a massless string this discontinuity is proven mathematically. In the case $v > c$ there is no discontinuity, since for $x \geq vt$ the deflection $u(vt, t) = 0$.

Unfortunately, we cannot answer the question of whether the string is continuous in the case that the discontinuity of the particle's trajectory occurs. The shape of the massless or massed string is not determined in the analytical form; we can only expect such a discontinuity on the base of numerical results. The particle motion is continuous only in the trivial case of $m = 0$. The expression in parenthesis in (23) is equal to zero, and since $\alpha = 0$ in (18), finally $y(1^-) = 0$ and $y(1) = 0$.

We consider small vibrations. Discontinuity in this case is a feature of mathematical rather then practical interest. However, in various analytical or numerical investigations of problems with a traveling inertial load one may meet slow convergence of solutions in places where boundary conditions are imposed. Our analysis can explain anomalies in such cases.

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