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First-passage failure of strongly nonlinear oscillators under combined harmonic and real noise excitations

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Abstract First-passage failure of strongly nonlinear oscillators under combined harmonic and real noise excitations is studied. The motion equation of the system is reduced to a set of averaged Itô stochastic differential equations by stochastic averaging in the case of resonance. Then, the backward Kolmogorov equation governing the conditional reliability function and a set of generalized Pontryagin equations governing the conditional moments of first-passage time are established. Finally, the conditional reliability function and the conditional probability density and mean first-passage time are obtained by solving the backward Kolmogorov equation and Pontryagin equation with suitable initial and boundary conditions. The procedure is applied to Duffing–van der Pol system in resonant case and the analytical results are verified by Monte Carlo simulation.

Keywords First-passage failure · Combined harmonic and real noise excitations · Duffing–van der Pol system · Stochastic averaging · Monte Carlo simulation

1 Introduction

First-passage problem has a long history, and it emerges from a wide range of stochastic phenomena, such as neuron firing, chemical reaction rates, the triggering of stock options and stochastic structural dynamics, etc. For stochastically excited mechanical or structural systems, first-passage failure happens when the state of the system leaves certain domain of state space (safe or admissible domain) for the first time and the machine or structure is disabled. The first-passage problem is related to the reliability and the life of mechanical or structural systems under random excitation. Thus, it is significant to evaluate the probability and/or statistics of the first-passage time.

However, the first-passage problem is among the most difficult problems in the theory of stochastic dynamics. At present, a mathematical exact solution is possible only if the random phenomenon in question can be treated as a diffusion process. For homogeneous diffusion processes, the conditional reliability function, namely, the probability that the response remains within the safety domain with a given initial state in it, is governed by the backward Kolmogorov equation and the moments of the first-passage time is governed by

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generalized Pontryagin equations. Known exactly analytical solutions are limited to one-dimensional case [1,2]. For higher dimensional systems, an approach to evaluate the probability and/or statistics of the first-passage time is numerical solution of the backward Kolmogorov equation for reliability function or the Pontryagin equation for mean first-passage time by using finite element method [3], finite difference method [4] or generalized cell mapping method [5]. Another way is Monte Carlo simulation (MCS) [6], especially the so-called Importance Sampling procedure [7] and controlled MCS [8], or Markov chain Monte Carlo [9,10].

A powerful technique for studying the first-passage time of higher dimensional systems is the combination of the stochastic averaging method and diffusion process method of first-passage time. The combination of the classical stochastic averaging method and the diffusion process method for first-passage time has been applied by many researchers to SDOF stochastic systems [11–13]. Recently, the combination of stochastic averaging method for a quasi-Hamiltonian system and the diffusion process method for first-passage time has been applied to study the first-passage time of MDOF strongly nonlinear stochastic systems [14].

Physical and engineering systems are often subjected to combined harmonic and random excitations. Such combined excitations arise in the study of stochastic resonance [15], seismic analysis of rotating machinery [16], trains crossing bridge-type structures during earthquakes or wind storms, dams excited by harmonic fluid motion combined with seismic activity [17], and uncoupled flapping motion of rotor blades of a helicopter in forward flight under the effect of atmospheric turbulence [18]. Linear and quasi-linear systems under combined harmonic and white-noise or wide-band random excitations have been studied by using the classical stochastic averaging method for obtaining the conditions of moment stability [19,20] or for obtaining the response probability density [21,22]. Recently, the method of multiple time scale has also been used to study the frequency response of quasi-linear system excited by combined deterministic and random excitations [23]. For a strongly nonlinear oscillator excited by combined harmonic and Gaussian white noise, the stochastic averaging method [24] and the method of multiple scales [25] have been adopted for obtaining the response statistics, and the generalized cell mapping with digraph (GCMD) method has been utilized to analyze crisis [26] and stochastic bifurcation [27]. So far, little work has been done on the first-passage failure of the nonlinear system excited by such combined excitation [17]. By using stochastic averaging method, Zhu and Wu have studied first-passage failure [28] and its feedback minimization [29] of strongly nonlinear oscillator under combined harmonic and white noise excitations.

Gaussian white noise is too ideal to exist in practice. So far, first passage-failure of strongly nonlinear oscillator under combined deterministic and real noise excitations has not been studied. In the present paper, the first-passage failure of strongly nonlinear oscillator under combined harmonic and real noise excitations is studied. The real noise possesses wide-band spectral density. After stochastic averaging based on generalized harmonic function, the two-dimensional non-homogeneous diffusion process of displacement and velocity with degenerate diffusion matrix is reduced to two-dimensional homogeneous diffusion process of amplitude and phase with non-degenerate diffusion matrix and only slowly varying processes are retained in the averaged equations. In the averaging, resonant and non-resonant cases are distinguished. This distinction is significant since harmonic function plays important role in resonant case while it can be neglected in the first approximation in non-resonant case. The approach is applied to Duffing–van der Pol oscillator under combined external harmonic and/or parametric wide-band random excitations. The effects of nonlinearity intensity, excitation intensity and initial amplitude on the probability and statistics of first passage time are examined. The analytical results are verified by using Monte Carlo simulation.

2 Generalized harmonic functions

Consider the free vibration of a non-linear conservative oscillator. The motion equation is

$$\ddot{x} + g(x) = 0. \tag{1}$$

The Hamiltonian (total energy) of the oscillator is

$$H = \frac{1}{2}\dot{x}^2 + V(x), \tag{2}$$

where

$$V(x) = \int_0^x g(u)\,\mathrm{d}u \tag{3}$$

is the potential energy. Assume that oscillator (1) has a family of periodic solutions surrounding elliptic equilibrium point $(b, 0)$ in phase plane $(x, \dot{x})$. The periodic solutions can be expressed as

$$x(t) = a\cos\varphi(t) + b, \tag{4}$$

$$\dot{x}(t) = -a\nu(a, \varphi)\sin\varphi(t), \tag{5}$$

where

$$\varphi(t) = \tau(t) + \theta, \tag{6}$$

$$\nu(a, \varphi) = \frac{\mathrm{d}\tau}{\mathrm{d}t} = \sqrt{\frac{2[V(a+b) - V(a\cos\varphi + b)]}{a^2\sin^2\varphi}}, \tag{7}$$

a and b are constants and related to H as follows:

$$V(a+b) = V(-a+b) = H. \tag{8}$$

$\cos\varphi(t)$ and $\sin\varphi(t)$ are called generalized harmonic functions [30]. Obviously, a and $\nu(a, \varphi)$ are the amplitude and instantaneous frequency of the oscillator respectively, and θ is the initial phase angle. Expanding ν^{-1} (a, φ) in Eq. (7) into Fourier series

$$\nu^{-1}(a, \varphi) = C_0(a) + \sum_{n=1}^{\infty} C_n(a)\cos n\varphi, \tag{9}$$

Integrating Eq. (9) with respect to φ from 0 to 2π yields average period

$$T(a) = 2\pi C_0(a), \tag{10}$$

and average frequency

$$\omega(a) = \frac{1}{C_0(a)}. \tag{11}$$

Thus, in average the following approximate relation can be used

$$\varphi(t) \approx \omega(a)t + \theta. \tag{12}$$

3 Stochastic averaging

Consider a strongly nonlinear conservative oscillator subject to lightly linear and (or) nonlinear damping and weakly external and (or) parametric excitations of harmonic function and wide-band real noises. The motion equation of the system is of the form

$$\ddot{X} + g(X) = \varepsilon f(X, \dot{X}, \Omega t) + \varepsilon^{1/2} h_k(X, \dot{X})\xi_k(t), \quad k = 1, 2, \ldots, r \tag{13}$$

where ε is a small parameter; εf denotes light damping and weakly external and (or) parametric harmonic excitation with frequency Ω; $\varepsilon^{1/2} h_k \xi_k(t)$ represent weakly external and (or) parametric random excitations and the repeated subscript represents summation; $\xi_k(t)$ are real noises with zero mean and correlation functions $R_{kl}(\tau)$ or spectral densities $S_{kl}(\omega)$.

When ε is very small, the response of system (13) can be considered as random spread of periodic motion of system (1). The solution can be assumed of the following form:

$$X(t) = A\cos\Phi(t) + B, \quad \dot{X}(t) = -A\nu(A, \Phi)\sin\Phi(t) \tag{14}$$

where

$$\Phi(t) = \tau(t) + \Theta(t), \tag{15}$$

$$\nu(A, \Phi) = \frac{\mathrm{d}\tau}{\mathrm{d}t} = \sqrt{\frac{2[V(A+B) - V(A\cos\Phi + B)]}{A^2\sin^2\Phi}}, \tag{16}$$

and A, Φ, τ and ν are all random processes. Treating Eq. (14) as a generalized van der Pol transformation from X, $\dot{X}$ to A, Θ, one can obtain the following equations for A and Θ:

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} &= \varepsilon F_1(A,\Phi,\Omega t) + \varepsilon^{1/2} U_{1k}(A,\Phi)\xi_k(t),\\ \frac{\mathrm{d}\Theta}{\mathrm{d}t} &= \varepsilon F_2(A,\Phi,\Omega t) + \varepsilon^{1/2} U_{2k}(A,\Phi)\xi_k(t),\end{aligned} \tag{17}$$

where

$$\begin{aligned}F_1 &= \frac{-A}{g(A+B)(1+h)} f(A\cos\Phi + B, -A\nu(A,\Phi)\sin\Phi, \Omega t)\nu(A,\Phi)\sin\Phi,\\ F_2 &= \frac{-1}{g(A+B)(1+h)} f(A\cos\Phi + B, -A\nu(A,\Phi)\sin\Phi, \Omega t)\nu(A,\Phi)(\cos\Phi + h),\\ U_{1k} &= \frac{-A}{g(A+B)(1+h)} h_k(A\cos\Phi + B, -A\nu(A,\Phi)\sin\Phi)\nu(A,\Phi)\sin\Phi,\\ U_{2k} &= \frac{-1}{g(A+B)(1+h)} h_k(A\cos\Phi + B, -A\nu(A,\Phi)\sin\Phi)\nu(A,\Phi)(\cos\Phi + h),\\ h &= \frac{dB}{dA} = \frac{g(-A+B)+g(A+B)}{g(-A+B)-g(A+B)}.\end{aligned} \tag{18}$$

According to Stratonovich–Khasminskii limit theorem [31,32], A and Θ converge weakly to two-dimensional diffusive Markov processes in a time interval of ε^{-1} order as $\varepsilon \to 0$, which can be represented by the following Itô stochastic differential equations

$$\begin{aligned}\mathrm{d}A &= \varepsilon S_1(A,\Phi,\Omega t)\,\mathrm{d}t + \varepsilon^{1/2} G_{1k}(A,\Phi)\,\mathrm{d}B_k(t),\\ \mathrm{d}\Theta &= \varepsilon S_2(A,\Phi,\Omega t)\,\mathrm{d}t + \varepsilon^{1/2} G_{2k}(A,\Phi)\,\mathrm{d}B_k(t),\end{aligned} \tag{19}$$

where $B_k(t)$ are independent unit Wiener processes,

$$\begin{aligned}S_i &= F_i + \int_{-\infty}^{0} \left(\left.\frac{\partial U_{ik}}{\partial A}\right|_t U_{1l}|_{t+\tau} R_{kl}(\tau) + \left.\frac{\partial U_{ik}}{\partial \Phi}\right|_t U_{2l}|_{t+\tau} R_{kl}(\tau)\right)\mathrm{d}\tau\\ b_{ij} &= G_{ik}G_{jk} = \int_{-\infty}^{\infty} \left(U_{ik}|_t \, U_{jl}\big|_{t+\tau} R_{kl}(\tau)\right)\mathrm{d}\tau, \quad i,j = 1,2;\; k,l = 1,\ldots,r\end{aligned} \tag{20}$$

Note that there are two procedures of averaging. One is stochastic averaging and the other is deterministic time averaging. The deterministic time averaging procedure will be discussed later. To complete the stochastic averaging, U_{ik} is expanded into Fourier series with respect to Φ as follows:

$$U_{ik} = U_{ik0}(A) + \sum_{n=1}^{\infty} U_{ikn}^{(c)} \cos n\Phi + U_{ikn}^{(s)} \sin n\Phi \tag{21}$$

Substituting the approximate average relationship between Φ and Θ of the form of Eq. (12) into Eq. (21) and completing the integration with respect to τ, one obtains the following averaged drift and diffusion coefficients:

$$\begin{aligned}&S_i(A,\Phi,\Omega t)\\ &= F_i(A,\Phi,\Omega t) + H_i(A,\Phi)\\ &= F_i + \pi \sum_{n=1}^{\infty} \left\{ \left[U_{1l0} \left(\frac{\mathrm{d}U_{ik0}}{\mathrm{d}A} + \frac{\mathrm{d}U_{ikn}^{(c)}}{\mathrm{d}A}\cos n\Phi + \frac{\mathrm{d}U_{ikn}^{(s)}}{\mathrm{d}A}\sin n\Phi \right)\right.\right.\\ &\quad \left.\left. + n\left(U_{ikn}^{(s)}\cos n\Phi - U_{ikn}^{(c)}\sin n\Phi\right) U_{2l0}\right] S_{kl}(0)\right\}\end{aligned}$$

$$+\pi\sum_{m,n=1}^{\infty}\left\{\left[\left(\frac{dU_{ik0}}{dA}+\frac{dU_{ikn}^{(c)}}{dA}\cos n\Phi+\frac{dU_{ikn}^{(s)}}{dA}\sin n\Phi\right)\left(U_{1lm}^{(c)}\cos m\Phi+U_{1lm}^{(s)}\sin m\Phi\right)\right.\right.$$
$$\left.+n\left(U_{ikn}^{(s)}\cos n\Phi-U_{ikn}^{(c)}\sin n\Phi\right)\left(U_{2lm}^{(c)}\cos m\Phi+U_{2lm}^{(s)}\sin m\Phi\right)\right]S_{kl}(m\omega(A))$$
$$+\left[\left(\frac{dU_{ik0}}{dA}+\frac{dU_{ikn}^{(c)}}{dA}\cos n\Phi+\frac{dU_{ikn}^{(s)}}{dA}\sin n\Phi\right)\left(U_{1lm}^{(s)}\cos m\Phi-U_{1lm}^{(c)}\sin m\Phi\right)\right.$$
$$\left.\left.+n\left(U_{ikn}^{(s)}\cos n\Phi-U_{ikn}^{(c)}\sin n\Phi\right)\left(U_{2lm}^{(s)}\cos m\Phi-U_{2lm}^{(c)}\sin m\Phi\right)\right]I_{kl}(m\omega(A))\right\} \tag{22}$$

where

$$S_{kl}(\omega)=\frac{1}{\pi}\int_{-\infty}^{0}R_{kl}(\tau)\cos\omega\tau\,d\tau$$
$$I_{kl}(\omega)=\frac{1}{\pi}\int_{-\infty}^{0}R_{kl}(\tau)\sin\omega\tau\,d\tau \tag{23}$$

System (13) has harmonic excitation and two cases can be classified: resonant case and non-resonant case. In non-resonant case, the harmonic excitation has no effect on the first approximation of the response. Thus, we are interested in resonant case, namely,

$$\frac{\Omega}{\omega(A)}=\frac{q}{p}+\varepsilon\sigma, \tag{24}$$

where p and q are relatively prime positive small integers and $\varepsilon\sigma$ is small detuning parameter. In this case, multiplying Eq. (24) by t and utilizing the approximate relation (12) yield

$$\Omega t=\frac{q}{p}\Phi+\varepsilon\sigma\tau-\frac{q}{p}\Theta \tag{25}$$

Introduce new angle variable

$$\Gamma=\varepsilon\sigma\tau-\frac{q}{p}\Theta \tag{26}$$

which Γ is a measure of the phase difference between the response and harmonic excitation. Then, Eq. (25) can be rewritten as

$$\Omega t=\frac{q}{p}\Phi+\Gamma. \tag{27}$$

Using Itô differential formula, one can obtain the following Itô stochastic differential equations for A, Φ and Γ:

$$\begin{aligned}dA&=\varepsilon S_1(A,\Phi,\Gamma)dt+\varepsilon^{1/2}G_{1k}(A,\Phi)dB_k(t),\\ d\Gamma&=\varepsilon\left\{\sigma\omega(A)-\left(\frac{q}{p}\right)S_2(A,\Phi,\Gamma)\right\}dt-\varepsilon^{1/2}\frac{q}{p}G_{2k}(A,\Phi)dB_k(t),\\ d\Phi&=[\omega(A)+\varepsilon S_2]dt+\varepsilon^{1/2}G_{2k}(A,\Phi)dB_k(t).\end{aligned} \tag{28}$$

Obviously, A and Γ are slowly varying processes while Φ is rapidly varying process. Expanding F_i into Fourier series

$$F_i(A,\Phi,\Gamma)=F_{i0}(A,\Gamma)+\sum_{n=1}^{\infty}\left[F_{in}^{(c)}\cos n\Phi+F_{in}^{(s)}\sin n\Phi\right], \tag{29}$$

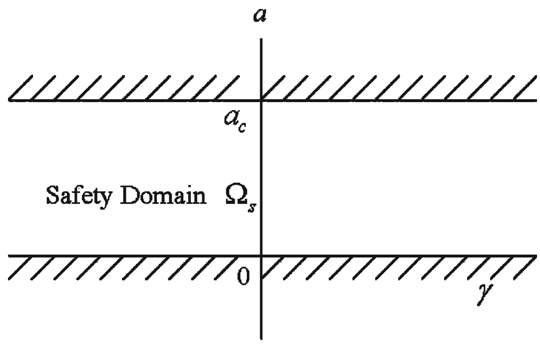

Fig. 1 Safety domain

and completing the time averaging with respect to Φ lead to the finally averaged Itô stochastic differential equations:

$$\begin{aligned} \mathrm{d}A &= \varepsilon \bar{m}_1(A, \Gamma)\mathrm{d}t + \varepsilon^{1/2}\bar{\sigma}_{1k}(A)\mathrm{d}B_k(t), \\ \mathrm{d}\Gamma &= \varepsilon \bar{m}_2(A, \Gamma)\mathrm{d}t + \varepsilon^{1/2}\bar{\sigma}_{2k}(A)\mathrm{d}B_k(t), \quad k = 1, \ldots, r \end{aligned} \tag{30}$$

where

$$\begin{aligned} \bar{m}_1(A, \Gamma) &= \langle F_1(A, \Phi, \Gamma) + H_1(A, \Phi)\rangle_\Phi = F_{10}(A, \Gamma) + \bar{H}_1(A) \\ \bar{m}_2(A, \Gamma) &= \left\langle [F_2(A, \Phi, \Gamma) + H_2(A, \Phi)]\left(-\frac{q}{p}\right) + \sigma\omega(A)\right\rangle_\Phi \\ &= \left(F_{20}(A, \Gamma) + \bar{H}_2(A)\right)\left(-\frac{q}{p}\right) + \sigma\omega(A) \\ \bar{b}_{ij}(A) &= \bar{\sigma}_{ik}\bar{\sigma}_{jk} = 2\pi U_{ik0}U_{jl0}S_{kl}(0) \\ &\quad + \pi\sum_{n=1}^{\infty}\left[\left(U_{ikn}^{(c)}U_{jln}^{(c)} + U_{ikn}^{(s)}U_{jln}^{(s)}\right)S_{kl}(n\omega(A)) + \left(U_{ikn}^{(c)}U_{jln}^{(s)} - U_{ikn}^{(s)}U_{jln}^{(c)}\right)I_{kl}(n\omega(A))\right] \\ \bar{H}_i(A) &= \pi\frac{\mathrm{d}U_{ik0}}{\mathrm{d}A}U_{1l0}S_{kl}(0) \\ &\quad + \frac{\pi}{2}\sum_{n=1}^{\infty}\left\{\left[\left(\frac{\mathrm{d}U_{ikn}^{(c)}}{\mathrm{d}A}U_{1ln}^{(c)} + \frac{\mathrm{d}U_{ikn}^{(s)}}{\mathrm{d}A}U_{1ln}^{(s)}\right) + n\left(U_{ikn}^{(s)}U_{2lm}^{(c)} - U_{ikn}^{(c)}U_{2ln}^{(s)}\right)\right]S_{kl}(n\omega(A))\right. \\ &\quad \left. + \left[\left(\frac{\mathrm{d}U_{ikn}^{(c)}}{\mathrm{d}A}U_{1ln}^{(s)} - \frac{\mathrm{d}U_{ikn}^{(s)}}{\mathrm{d}A}U_{1ln}^{(c)}\right) + n\left(U_{ikn}^{(s)}U_{2ln}^{(s)} + U_{ikn}^{(c)}U_{2ln}^{(c)}\right)\right]I_{kl}(n\omega(A))\right\} \\ &\quad i, j = 1, 2; \quad k, l = 1, \ldots, r; \end{aligned} \tag{31}$$

and $\langle\cdot\rangle_\Phi$ denotes the averaging with respect to Φ from 0 to 2π.

4 Backward Kolmogorov equation and generalized Pontryagin equations

$A(t)$ is the displacement amplitude of system (13). It is reasonable to assume that the first-passage failure occurs once $A(t)$ exceeds certain critical value a_c for the first time. In phase plane (a, γ), the safe domain Ω_s is inside of the two parallel lines $a = 0$ and $a = a_c$ (see Fig. 1). The conditional reliability function, denoted by $R(t|a_0, \gamma_0)$, is defined as the probability of $[A(t), \Gamma(t)]$ being in safely domain Ω_s within interval $(0, t]$ given initial state (a_0, γ_0) being in Ω_s, i.e.,

$$R(t|a_0, \gamma_0) = P\{(A(\tau), \Gamma(\tau)) \in \Omega_s, \tau \in (0, t]|\,(a_0, \gamma_0) \in \Omega_s\}. \tag{32}$$

It is the integral of the conditional transition probability density in Ω_s. The conditional transition probability density is the transition probability density of the sample functions which remain in safety domain Ω_s in all time interval $(0, t]$. For diffusion process $[A, \Gamma]^{\mathrm{T}}$, the conditional transition probability density is governed by the backward Kolmogorov equation with drift and diffusion coefficients defined by Eq. (31). Thus, the conditional reliability function is governed by the following backward Kolmogorov equation:

$$\frac{\partial R}{\partial t} = \kappa_1 \frac{\partial R}{\partial a_0} + \kappa_2 \frac{\partial R}{\partial \gamma_0} + \frac{1}{2}\chi_{11}\frac{\partial^2 R}{\partial a_0^2} + \chi_{12}\frac{\partial^2 R}{\partial a_0 \partial \gamma_0} + \frac{1}{2}\chi_{22}\frac{\partial^2 R}{\partial \gamma_0^2}, \tag{33}$$

where

$$\begin{aligned} \kappa_i &= \kappa_i(a_0, \gamma_0) = \varepsilon\, \bar{m}_i(A, \Gamma)|_{A=a_0, \Gamma=\gamma_0}, \\ \chi_{ij} &= \chi_{ij}(a_0) = \varepsilon \bar{b}_{ij} = \varepsilon \bar{\sigma}_{ir}(A)\bar{\sigma}_{jr}(A)\big|_{A=a_0}. \end{aligned} \tag{34}$$

The initial condition associated with Eq. (33) is

$$R(0|a_0, \gamma_0) = 1, \quad a_0 < a_c \tag{35}$$

and the boundary conditions are

$$R(t|0, \gamma_0) = \text{finite}, \tag{36}$$

$$R(t|a_c, \gamma_0) = 0, \tag{37}$$

$$R(t|a_0, \gamma_0 + 2n\pi) = R(t|a_0, \gamma_0). \tag{38}$$

The conditional probability of first-passage failure is

$$P_f(t|a_0, \gamma_0) = 1 - R(t|a_0, \gamma_0). \tag{39}$$

The conditional probability density of the first-passage time T is then the derivative of $P_f(t|a_0, \gamma_0)$, i.e.,

$$p(T|a_0, \gamma_0) = \left.\frac{\partial P_f}{\partial t}\right|_{t=T} = -\left.\frac{\partial R}{\partial t}\right|_{t=T}. \tag{40}$$

The conditional moments of the first passage time are defined as

$$T_n(a_0, \gamma_0) = \int_0^\infty T^n p(T|a_0, \gamma_0)\mathrm{d}T = n\int_0^\infty T^{n-1} R(T|a_0, \gamma_0)\mathrm{d}T, \quad n = 1, 2, \ldots. \tag{41}$$

It can be shown by using Eqs. (33), (40) and (41) that the conditional moments of the first-passage time are governed by the following generalized Pontryagin equations:

$$\frac{1}{2}\chi_{11}\frac{\partial^2 T_n}{\partial a_0^2} + \chi_{12}\frac{\partial^2 T_n}{\partial a_0 \partial \gamma_0} + \frac{1}{2}\chi_{22}\frac{\partial^2 T_n}{\partial \gamma_0^2} + \kappa_1\frac{\partial T_n}{\partial a_0} + \kappa_2\frac{\partial T_n}{\partial \gamma_0} = -nT_{n-1}, \qquad n = 1, 2, \ldots, \tag{42}$$

where κ_i and χ_{ij} are defined by Eq. (34). The boundary conditions associated with Eq. (42) are obtained from Eqs. (36)–(38) as

$$T_n(0, \gamma_0) = \text{finite}, \tag{43}$$

$$T_n(a_c, \gamma_0) = 0, \tag{44}$$

$$T_n(a_0, \gamma_0 + 2n\pi) = T_n(a_0, \gamma_0). \tag{45}$$

For $n = 1$, T_1 is the mean first-passage time and Eq. (42) is reduced to Pontryagin equation

$$\frac{1}{2}\chi_{11}\frac{\partial^2 T_1}{\partial a_0^2} + \chi_{12}\frac{\partial^2 T_1}{\partial a_0 \partial \gamma_0} + \frac{1}{2}\chi_{22}\frac{\partial^2 T_1}{\partial \gamma_0^2} + \kappa_1\frac{\partial T_1}{\partial a_0} + \kappa_2\frac{\partial T_1}{\partial \gamma_0} = -1 \tag{46}$$

To obtain the probability and statistics of the first-passage time, one have to solve backward Kolmogorov equation (33) with initial and boundary conditions (35)–(38), or to solve generalized Pontryagin equations (42) with boundary conditions (43)–(45). Generally, they can be solved only numerically. The backward Kolmogorov equation can be solved by using the implicit finite difference method of alternate direction type. And the Pontryagin equation can be solved by using finite difference method.

5 Numerical example

To illustrate the procedure developed above, consider a Duffing–van der Pol oscillator subject to external harmonic excitation and external and parametric wide-band real noise excitations. Duffing–van der Pol oscillator is a typical strongly nonlinear system studied by many researchers. It can represent the motion of a thin panel under supersonic air flow. It can also describe the dynamics of a single-model laser with a saturable absorber [25]. The motion equation of the system is of the form

$$\ddot{X} + (\beta_1 + \beta_2 X^2)\dot{X} + \omega_0^2 X + \alpha X^3 = E\cos\Omega t + \xi_1(t) + X\xi_2(t) \tag{47}$$

where $\beta_1, \beta_2, \omega 0, \alpha, E, \Omega$ are constants; $\xi_i(t)(i = 1, 2)$ are independently stationary and ergodic processes with zero mean and rational spectral densities

$$S_i(\omega) = \frac{D_i}{\pi}\frac{1}{\omega^2 + \omega_i^2}, \quad i = 1, 2 \tag{48}$$

in which ω_i and D_i are constants. $\xi_i(t)$ can be regarded as the output of the following first order linear filter

$$\dot{\xi}_i + \omega_i \xi_i = W_i(t), \quad i = 1, 2 \tag{49}$$

where $W_i(t)$ are Gaussian white noises in the sense of Stratonovich with intensities $2D_i$. It is assumed that β_i and E are of the same order of ε.

For this system,

$$\begin{aligned} V(x) &= \omega_0^2 x^2/2 + \alpha x^4/4, \\ g(x) &= \mathrm{d}V/\mathrm{d}x = \omega_0^2 x + \alpha x^3, \\ b &= h = 0 \end{aligned} \tag{50}$$

and

$$\begin{aligned} \nu(a,\varphi) &= \left[\left(\omega^2 + 3\alpha a^2/4\right)(1 + \lambda\cos 2\varphi)\right]^{1/2} \\ \lambda &= \alpha a^2/4\left(\omega^2 + 3\alpha a^2/4\right). \end{aligned} \tag{51}$$

$\nu(a,\varphi)$ can be approximated by the following finite sum with a relative error less than 0.03%:

$$\nu(a,\varphi) = b_0(a) + b_2(a)\cos 2\varphi + b_4(a)\cos 4\varphi + b_6(a)\cos 6\varphi \tag{52}$$

where

$$\begin{aligned} b_0(a) &= \left(\omega_0^2 + 3\alpha a^2/4\right)^{1/2}\left(1 - \lambda^2/16\right), \quad b_2(a) = \left(\omega_0^2 + 3\alpha a^2/4\right)^{1/2}\left(\lambda/2 + 3\lambda^3/64\right), \\ b_4(a) &= \left(\omega_0^2 + 3\alpha a^2/4\right)^{1/2}\left(-\lambda^2/16\right), \quad b_6(a) = \left(\omega_0^2 + 3\alpha a^2/4\right)^{1/2}\left(\lambda^3/64\right), \end{aligned} \tag{53}$$

The averaged frequency $\omega(a) = b_0(a)$.

Using the generalized van der Pol transformations (14), we can convert Eq. (47) into

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t} &= F_1(A,\Phi,\Omega t) + U_{11}(A,\Phi)\xi_1(t) + U_{12}(A,\Phi)\xi_2(t), \\ \frac{\mathrm{d}\Theta}{\mathrm{d}t} &= F_2(A,\Phi,\Omega t) + U_{21}(A,\Phi)\xi_1(t) + U_{22}(A,\Phi)\xi_2(t), \end{aligned} \tag{54}$$

where

$$\begin{aligned} F_1 &= -\frac{A}{g(A)}\left[\left(\beta_1 + \beta_2 A\cos^2\Phi\right)A\nu(A,\Phi)\sin\Phi + E\cos\Omega t\right]\nu(A,\Phi)\sin\Phi, \\ F_2 &= -\frac{1}{g(A)}\left[\left(\beta_1 + \beta_2 A\cos^2\Phi\right)A\nu(A,\Phi)\sin\Phi + E\cos\Omega t\right]\nu(A,\Phi)\cos\Phi, \\ U_{11} &= -\frac{A}{g(A)}\nu(A,\Phi)\sin\Phi, \quad U_{12} = -\frac{A^2}{g(A)}\nu(A,\Phi)\sin\Phi\cos\Phi, \\ U_{21} &= -\frac{1}{g(A)}\nu(A,\Phi)\cos\Phi, \quad U_{22} = -\frac{A}{g(A)}\nu(A,\Phi)\cos^2\Phi. \end{aligned} \tag{55}$$

Consider primary external resonance case. In this case,

$$\frac{\Omega}{\omega(a)} = 1 + \varepsilon\sigma \tag{56}$$

where $\varepsilon\sigma$ is a small detuning parameter. Introducing the new variable

$$\Gamma = \varepsilon\sigma\tau - \Theta \tag{57}$$

and completing the stochastic averaging procedure lead to the following averaged Itô equations:

$$\begin{aligned} \mathrm{d}A &= \bar{m}_1(A,\Gamma)\mathrm{d}t + \bar{\sigma}_{11}(A)\mathrm{d}B_1(t) + \bar{\sigma}_{12}(A)\mathrm{d}B_2(t), \\ \mathrm{d}\Gamma &= \bar{m}_2(A,\Gamma)\mathrm{d}t + \bar{\sigma}_{21}(A)\mathrm{d}B_1(t) + \bar{\sigma}_{22}(A)\mathrm{d}B_2(t), \end{aligned} \tag{58}$$

where the drift and diffusion coefficients are given in Appendix.

The backward Kolmogorov equation associated with Itô Eqs. (58) is

$$\frac{\partial R}{\partial t} = \kappa_1 \frac{\partial R}{\partial a_0} + \kappa_2 \frac{\partial R}{\partial \gamma_0} + \frac{1}{2}\chi_{11}\frac{\partial^2 R}{\partial a_0^2} + \frac{1}{2}\chi_{22}\frac{\partial^2 R}{\partial \gamma_0^2} \tag{59}$$

where κ_i, χ_{ij} are defined as

$$\begin{aligned} \kappa_i &= \kappa_i(a_0,\gamma_0) = \bar{m}_i(A,\Gamma)|_{A=a_0,\Gamma=\gamma_0}, \\ \chi_{ii} &= \chi_{ii}(a_0) = \bar{b}_{ii} = \bar{\sigma}_{ir}(A)\bar{\sigma}_{ir}(A)|_{A=a_0}, \quad i,r = 1,2 \end{aligned} \tag{60}$$

The associated two boundary conditions are Eqs. (36)–(38) and initial condition is Eq. (35). The backward Kolmogorov equation can be solved by using finite difference method with modified standard Thomas algorithm. Then the conditional probability density of first-passage time of system (47) can be obtained from the conditional reliability function by using Eq. (40).

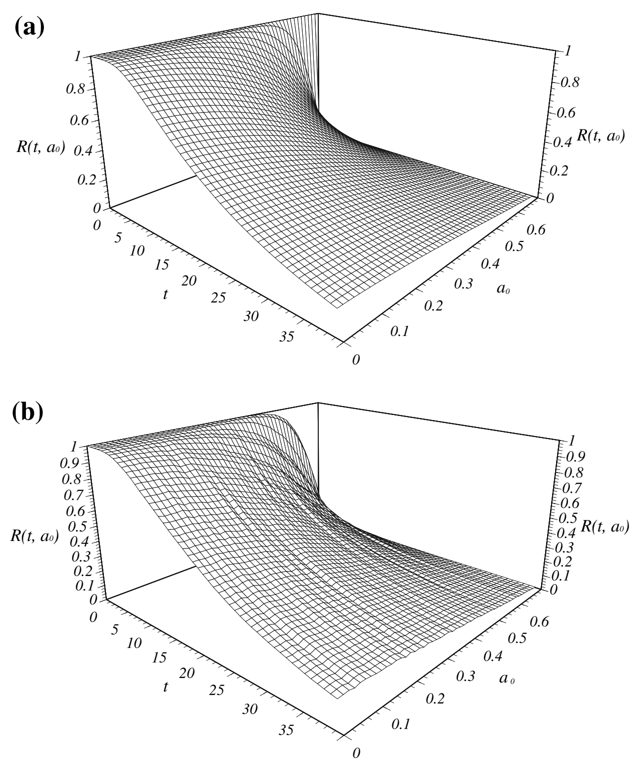

Fig. 2 Reliability function $R(t, a_0)$ of system (47) in primary external resonance. $\omega_0 = 1.0, \alpha = 1, \omega = 1.1, \Omega = 1.1, E = 0.01, a_c = 0.7, \gamma_0 = 1.25664, \beta_1 = -0.01, \beta_2 = 0.01, \omega_1 = \omega_2 = 30, D_1 = 10, D_2 = 3$. **a** Analytical result; **b** result from Monte Carlo simulation

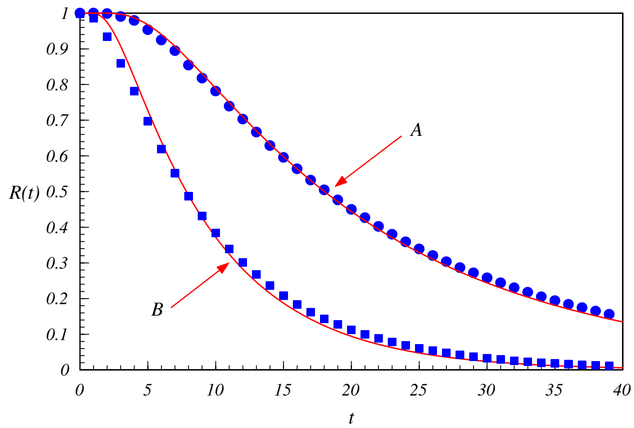

Fig. 3 Reliability function $R(t)$ of system (47) in primary external resonance. $\omega_0 = 1.0, \alpha = 1, \Omega = 1.1, E = 0.01, a_c = 0.7, a_0 = 0.168, \gamma_0 = 1.25664, \beta_1 = -0.01, \beta_2 = 0.01, \omega_1 = \omega_2 = 30, D_2 = 3; A : D_1 = 10; B : D_1 = 25$

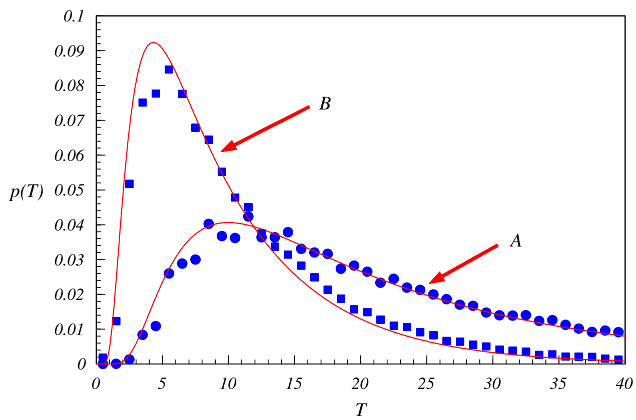

Fig. 4 Probability density of first-passage time of system (47) in primary external resonance. The parameters are the same as those in Fig. 3

The mean first-passage time of system (47) can be obtained either from the conditional reliability function by using Eq. (41) or from solving the following Pontryagin equation

$$\frac{1}{2}\chi_{11}\frac{\partial^2 T_1}{\partial a_0^2} + \frac{1}{2}\chi_{22}\frac{\partial^2 T_1}{\partial \gamma_0^2} + \kappa_1\frac{\partial T_1}{\partial a_0} + \kappa_2\frac{\partial T_1}{\partial \gamma_0} = -1 \tag{61}$$

with boundary conditions

$$T_1(0, \gamma_0) = \text{finite}, \tag{62}$$

$$T_1(a_c, \gamma_0) = 0, \tag{63}$$

$$T_1(a_0, \gamma_0 + 2n\pi) = T_1(a_0, \gamma_0). \tag{64}$$

Some numerical results are shown in Figs. 2, 3, and 4, where solid line denotes analytical results while square and circle results from Monte Carlo simulation of original system (47). It is seen that the analytical results are in rather good agreement with those from Monte Carlo simulation.

It is obviously shown in Figs. 2 and 3 that the reliability function is a monotonously decreasing function of time. It implies that the system will fail with probability one when $t \to \infty$. This is physically meaningful because of the external force exciting the system. Moreover, larger strength of external random excitation will induce larger failure probability, which is indicated in Fig. 3. And, larger strength of external random excitation

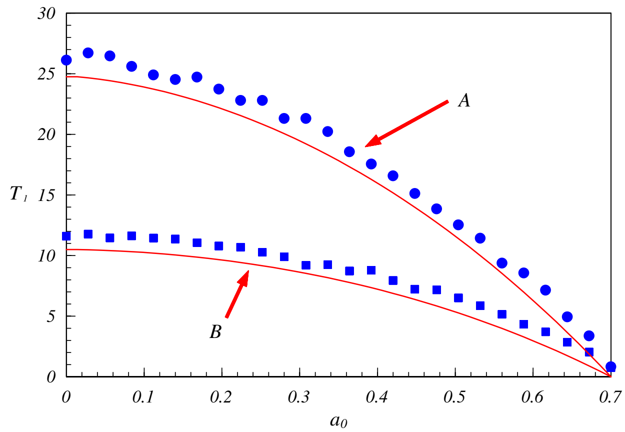

Fig. 5 Mean first-passage time of system (47) in primary external resonance. The parameter values are the same as those in Fig. 3 except that a_0 is a variable

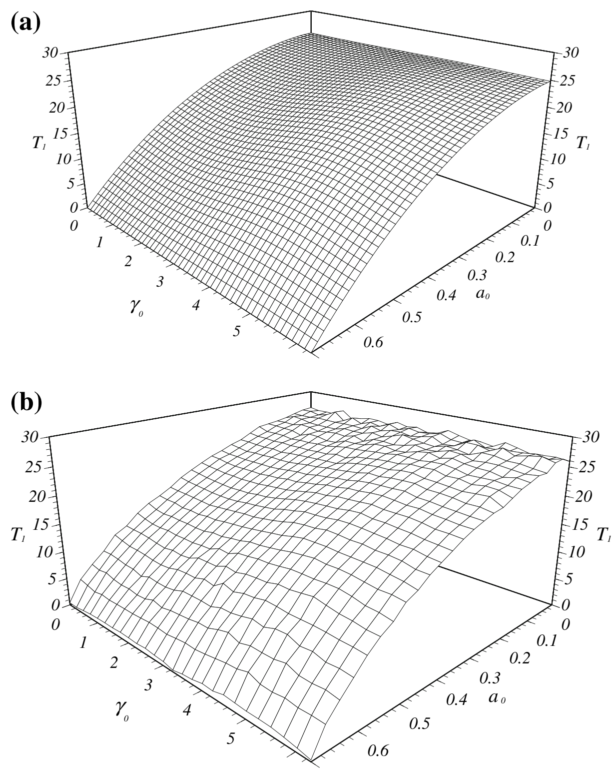

Fig. 6 Mean first-passage time of system (47) in primary external resonance. The parameter values are the same as those in Fig. 2 except that γ_0 is a variable. **a** Analytical result; **b** result from Monte Carlo simulation

will induce shorter mean first-passage time. This is verified by Fig. 5. Furthermore, Figs. 5 and 6 show that when the initial amplitude is more close to the critical value a_c, mean first-passage time is shorter. This is because the reliability is lower and the first-passage happens more easily when the initial state is more close to a_c, which can be seen from Fig. 2.

So, we can conclude that that all the results depend strongly on initial amplitude, the excitation intensity and nonlinearity intensity (see Figs. 7, 8). Even if the nonlinearity intensity is high, the proposed method still works well (see Figs. 7, 8). The reliability function is a monotonously decreasing function of time and the mean first-passage time is a monotonously decreasing function of initial amplitude. This observation is significant in the studying stochastic optimal control of the system with objectives of maximum reliability and maximum mean first passage time.

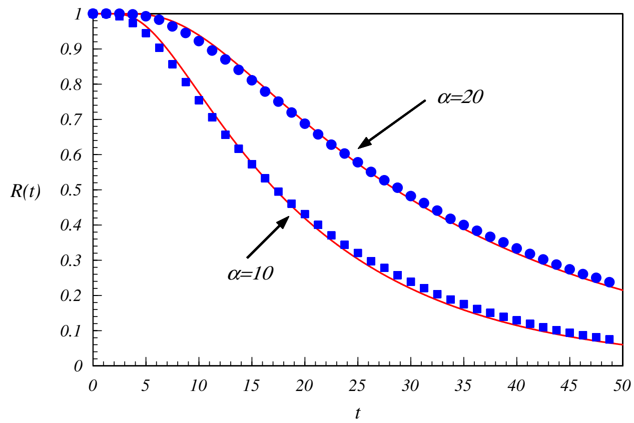

Fig. 7 Reliability function $R(t)$ of system (47) in primary external resonance. $\omega_0 = 1.0$, $\Omega = 1.1$, $E = 0.01$, $a_c = 0.7$, $a_0 = 0.168$, $\gamma_0 = 1.25664$, $\beta_1 = -0.01$, $\beta_2 = 0.01$, $\omega_1 = \omega_2 = 30$, $D_1 = 30$, $D_2 = 30$

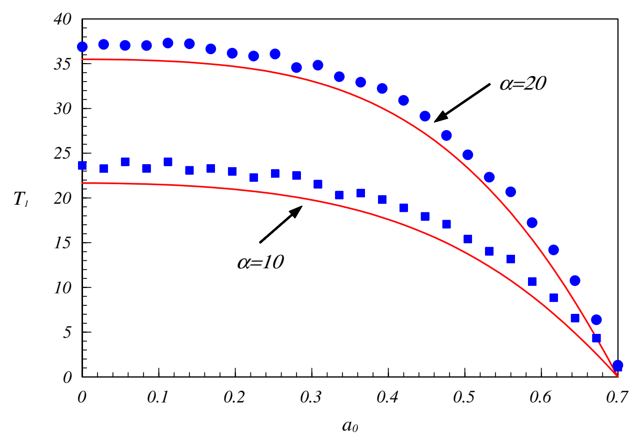

Fig. 8 Mean first-passage time of system (47) in primary external resonance. The parameter values are the same as those in Fig. 7 except that a_0 is a variable

6 Conclusions

In the present paper a new procedure for estimating first-passage time of strongly nonlinear oscillators under combined harmonic and real noise excitations has been proposed. By stochastic averaging, the motion equation is reduced to the averaged Itô equations for homogenous diffusion processes $A(t)$ and $\Gamma(t)$. The backward Kolmogorov equation for the conditional reliability function and the generalized Pontryagin equations for moments of first-passage time are derived from the averaged Itô equations. Duffing–van der Pol system is taken as an example to show the validity of this method. The analytical results are well verified by Monte Carlo simulation. The results show that reliability function is a monotonously decreasing function of time and mean first-passage time is a monotonously decreasing function of initial amplitude. All the results depend strongly on the excitation intensity, nonlinearity intensity and initial amplitude. The procedure can work well even if the nonlinearity intensity is high. It should be pointed that the proposed method can be extended to multi-degree-of-freedom (MDOF) strongly nonlinear systems subject to combined harmonic and wide-band noise excitations, if the stiffness terms are not coupled. In this case of MDOF, the presentation of generalized harmonic function is straightforward, no matter whether the damping terms are coupled or not. And external resonance and internal resonance should be considered. The corresponding backward Kolmogorov equation and Pontryagin equation are of higher dimension so it is more difficult to solve them. This will be our future work.

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Appendix

The drift and diffusion coefficients in Eq. (58) are

$$\bar{m}_1 = \bar{F}_{10}(A,\Gamma) + \bar{H}_1(A)$$
$$\bar{m}_2 = \bar{F}_{20}(A,\Gamma)$$
$$\bar{F}_{10}(A,\Gamma) = E\sin\Gamma(2b_0(A) - b_2(A))\Big/4\big(\alpha A^2+\omega_0^2\big) - A\big[\beta_1\big(16\omega_0^2+10\alpha A^2\big) + A^2\beta_2\big(4\omega_0^2+3\alpha A^2\big)\big]\Big/32\big(\alpha A^2+\omega_0^2\big)$$
$$\bar{F}_{20}(A,\Gamma) = \Omega - b_0(A) + E\cos\Gamma(2b_0(A)+b_2(A))\Big/4A\big(\alpha A^2+\omega_0^2\big)$$
$$\bar{H}_1(A) = \bar{m}_{11} + \bar{m}_{12} + \bar{m}_{13} + \bar{m}_{14}$$
$$\bar{m}_{11} = \bar{m}_{111}S_1(\omega(A)) + \bar{m}_{113}S_1(3\omega(A)) + \bar{m}_{115}S_1(5\omega(A)) + \bar{m}_{117}S_1(7\omega(A))$$
$$\bar{m}_{13} = \bar{m}_{131}S_1(\omega(A)) + \bar{m}_{133}S_1(3\omega(A)) + \bar{m}_{135}S_1(5\omega(A)) + \bar{m}_{137}S_1(7\omega(A))$$
$$\bar{m}_{12} = \bar{m}_{122}S_2(2\omega(A)) + \bar{m}_{124}S_2(4\omega(A)) + \bar{m}_{126}S_2(6\omega(A)) + \bar{m}_{128}S_2(8\omega(A))$$
$$\bar{m}_{14} = \bar{m}_{142}S_2(2\omega(A)) + \bar{m}_{144}S_2(4\omega(A)) + \bar{m}_{146}S_2(6\omega(A)) + \bar{m}_{148}S_2(8\omega(A))$$
$$\bar{m}_{111} = \pi[b_2(A) - 2b_0(A)] \times \left\{2\alpha A[2b_0(A) - b_2(A)] + (A^2\alpha+\omega_0^2)\left(\frac{db_2(A)}{dA} - 2\frac{db_0(A)}{dA}\right)\right\}\Big/\Big[8\big(A^2\alpha+\omega_0^2\big)^3\Big]$$
$$\bar{m}_{113} = \pi[b_2(A) - b_4(A)] \times\left\{2\alpha A[b_4(A) - b_2(A)] + (A^2\alpha+\omega_0^2)\left(\frac{db_2(A)}{dA} - \frac{db_4(A)}{dA}\right)\right\}\Big/\Big[8\big(A^2\alpha+\omega_0^2\big)^3\Big]$$
$$\bar{m}_{115} = \pi[b_4(A) - b_6(A)] \times\left\{2\alpha A[b_6(A) - b_4(A)] + (A^2\alpha+\omega_0^2)\left(\frac{db_4(A)}{dA} - \frac{db_6(A)}{dA}\right)\right\}\Big/\Big[8\big(A^2\alpha+\omega_0^2\big)^3\Big]$$
$$\bar{m}_{117} = \pi b_6(A)\left\{-2\alpha Ab_6(A) + (A^2\alpha+\omega_0^2)\frac{db_6(A)}{dA}\right\}\Big/\Big[8\big(A^2\alpha+\omega_0^2\big)^3\Big]$$
$$\bar{m}_{122} = \pi A[2b_0(A) - b_4(A)]\left\{[2b_0(A) - b_4(A)]\big(A^2\alpha-\omega_0^2\big) - A\big(A^2\alpha+\omega_0^2\big)\left(2\frac{db_0(A)}{dA} - \frac{db_4(A)}{dA}\right)\right\}\Big/\Big[32\big(A^2\alpha+\omega_0^2\big)^3\Big]$$
$$\bar{m}_{124} = \pi A[b_2(A) - b_6(A)]\left\{[b_6(A) - b_2(A)]\big(A^2\alpha-\omega_0^2\big) + A\big(A^2\alpha+\omega_0^2\big)\left(\frac{db_2(A)}{dA} - \frac{db_6(A)}{dA}\right)\right\}\Big/\Big[32\big(A^2\alpha+\omega_0^2\big)^3\Big]$$
$$\bar{m}_{126} = \pi Ab_4(A)\left\{b_4(A)\big(A^2\alpha-\omega_0^2\big) + A\big(A^2\alpha+\omega_0^2\big)\frac{db_4(A)}{dA}\right\}\Big/\Big[32\big(A^2\alpha+\omega_0^2\big)^3\Big]$$
$$\bar{m}_{128} = \pi Ab_6(A)\left\{b_6(A)\big(A^2\alpha-\omega_0^2\big) + A\big(A^2\alpha+\omega_0^2\big)\frac{db_6(A)}{dA}\right\}\Big/\Big[32\big(A^2\alpha+\omega_0^2\big)^3\Big]$$
$$\bar{m}_{131} = -\pi\big[b_2^2(A) - 4b_0^2(A)\big]\Big/\Big[8A\big(A^2\alpha+\omega_0^2\big)^2\Big]$$
$$\bar{m}_{133} = -3\pi\big[b_4^2(A) - b_2^2(A)\big]\Big/\Big[8A\big(A^2\alpha+\omega_0^2\big)^2\Big]$$
$$\bar{m}_{135} = -5\pi\big[b_6^2(A) - b_4^2(A)\big]\Big/\Big[8A\big(A^2\alpha+\omega_0^2\big)^2\Big]$$
$$\bar{m}_{137} = 7\pi b_6^2(A)\Big/\Big[8A\big(A^2\alpha+\omega_0^2\big)^2\Big]$$
$$\bar{m}_{142} = \pi A[2b_0(A) - b_4(A)][2b_0(A) + 2b_2(A) + b_4(A)]\Big/\Big[16\big(A^2\alpha+\omega_0^2\big)^2\Big]$$
$$\bar{m}_{144} = \pi A[b_2(A) - b_6(A)][b_2(A) + 2b_4(A) + b_6(A)]\Big/\Big[8\big(A^2\alpha+\omega_0^2\big)^2\Big]$$
$$\bar{m}_{146} = 3\pi Ab_4(A)[b_4(A) + 2b_6(A)]\Big/\Big[16\big(A^2\alpha+\omega_0^2\big)^2\Big]$$
$$\bar{m}_{148} = \pi Ab_6^2(A)\Big/\Big[4\big(A^2\alpha+\omega_0^2\big)^2\Big]$$

$$\bar{b}_{11} = \bar{\sigma}_{11}(A)\bar{\sigma}_{11}(A) + \bar{\sigma}_{12}(A)\bar{\sigma}_{12}(A) = \bar{b}_{111} + \bar{b}_{112}$$
$$\bar{b}_{22} = \bar{\sigma}_{21}(A)\bar{\sigma}_{21}(A) + \bar{\sigma}_{22}(A)\bar{\sigma}_{22}(A) = \bar{b}_{221} + \bar{b}_{222}$$
$$\bar{b}_{ij} = \bar{\sigma}_{i1}(A)\bar{\sigma}_{j1}(A) + \bar{\sigma}_{i2}(A)\bar{\sigma}_{j2}(A) = 0, \quad i \neq j$$
$$\bar{b}_{111} = \bar{b}_{1111}S_1(\omega(A)) + \bar{b}_{1113}S_1(3\omega(A)) + \bar{b}_{1115}S_1(5\omega(A)) + \bar{b}_{1117}S_1(7\omega(A))$$
$$\bar{b}_{221} = \bar{b}_{2211}S_1(\omega(A)) + \bar{b}_{2213}S_1(3\omega(A)) + \bar{b}_{2215}S_1(5\omega(A)) + \bar{b}_{2217}S_1(7\omega(A))$$
$$\bar{b}_{112} = \bar{b}_{1122}S_2(2\omega(A)) + \bar{b}_{1124}S_2(4\omega(A)) + \bar{b}_{1126}S_2(6\omega(A)) + \bar{b}_{1128}S_2(8\omega(A))$$
$$\bar{b}_{222} = \bar{b}_{2220}S_2(0) + \bar{b}_{2222}S_2(2\omega(A)) + \bar{b}_{2224}S_2(4\omega(A)) + \bar{b}_{2226}S_2(6\omega(A)) + \bar{b}_{2228}S_2(8\omega(A))$$
$$\bar{b}_{1111} = \pi[b_2(A) - 2b_0(A)]^2 \Big/ 4\left(\alpha A^2 + \omega_0^2\right)^2$$
$$\bar{b}_{1113} = \pi[b_2(A) - b_4(A)]^2 \Big/ 4\left(\alpha A^2 + \omega_0^2\right)^2$$
$$\bar{b}_{1115} = \pi[b_4(A) - b_6(A)]^2 \Big/ 4\left(\alpha A^2 + \omega_0^2\right)^2$$
$$\bar{b}_{1117} = \pi b_6^2(A) \Big/ 4\left(\alpha A^2 + \omega_0^2\right)^2$$
$$\bar{b}_{1122} = \pi A^2[b_4(A) - 2b_0(A)]^2 \Big/ 16\left(\alpha A^2 + \omega_0^2\right)^2$$
$$\bar{b}_{1124} = \pi A^2[b_2(A) - b_6(A)]^2 \Big/ 16\left(\alpha A^2 + \omega_0^2\right)^2$$
$$\bar{b}_{1126} = \pi A^2 b_4^2(A) \Big/ 16\left(\alpha A^2 + \omega_0^2\right)^2$$
$$\bar{b}_{1128} = \pi A^2 b_6^2(A) \Big/ 16\left(\alpha A^2 + \omega_0^2\right)^2$$
$$\bar{b}_{2211} = \pi[2b_0(A) + b_2(A)]^2 \Big/ \left[4A^2\left(\alpha A^2 + \omega_0^2\right)^2\right]$$
$$\bar{b}_{2213} = \pi[b_2(A) + b_4(A)]^2 \Big/ \left[4A^2\left(\alpha A^2 + \omega_0^2\right)^2\right]$$
$$\bar{b}_{2215} = \pi[b_4(A) + b_6(A)]^2 \Big/ \left[4A^2\left(\alpha A^2 + \omega_0^2\right)^2\right]$$
$$\bar{b}_{2217} = \pi b_6^2(A) \Big/ \left[4A^2\left(\alpha A^2 + \omega_0^2\right)^2\right]$$
$$\bar{b}_{2220} = \pi[2b_0(A) + b_2(A)]^2 \Big/ \left[8\left(\alpha A^2 + \omega_0^2\right)^2\right]$$
$$\bar{b}_{2222} = \pi[2b_0(A) + 2b_2(A) + b_4(A)]^2 \Big/ \left[16\left(\alpha A^2 + \omega_0^2\right)^2\right]$$
$$\bar{b}_{2224} = \pi[b_2(A) + 2b_4(A) + b_6(A)]^2 \Big/ \left[16\left(\alpha A^2 + \omega_0^2\right)^2\right]$$
$$\bar{b}_{2226} = \pi[b_4(A) + 2b_6(A)]^2 \Big/ \left[16\left(\alpha A^2 + \omega_0^2\right)^2\right]$$
$$\bar{b}_{2228} = \pi b_6^2(A) \Big/ \left[16\left(\alpha A^2 + \omega_0^2\right)^2\right]$$

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