Arch Appl Mech (2007) 77: 353–362
DOI 10.1007/s00419-006-0072-2

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Forced nonlinear oscillator with nonsymmetric dry friction

Received: 25 May 2006 / Accepted: 28 July 2006 / Published online: 18 October 2006

Abstract In this paper we discuss an approximately steady motion of an oscillator as a single whole with a constant “on the average” velocity. For that purpose we analyze the position and stability of some special points of the phase portrait. In the presence of internal excitation and nonsymmetric Coulomb dry friction, a motion of the oscillator with a constant “on the average” velocity is possible. The algebraic equation for this constant velocity is found. For different parameters of the model there exist at most three regimes of motion with a constant velocity, but only one or two of them are stable. The theoretical results obtained can be used for the design of worm-like moving robots.

Keywords Nonsymmetric dry friction · Oscillation · Locomotion

1 Introduction

A series of fundamental papers are devoted to the analysis of nonlinear oscillations of mechanical systems [1,2,5,10–14]. Some asymptotic methods for the solution of equations of nonlinear oscillations are presented in well-known monographs [1,2,4,5]. Nonlinear systems including discontinuous functions such as dry friction are especially considered in a series of publications [3,6,7,9,16,17,19,20]. In [19,20] the motion of two mass points connected by a linear spring is discussed. It is supposed that this linear oscillator is under the action of a small non-symmetric Coulomb dry friction force, i.e. the friction force is assumed to change in magnitude depending on the direction of motion. Excitation is carried out by the action of small internal harmonic forces. This oscillator is a mathematical model of a worm-like locomotion system. A limiting case of nonsymmetric friction, when motion is possible in one direction only, was proposed in [15] in connection with realistic computer animation of worms. A thorough discussion of such systems, where the point masses can also be equipped with massless steerable runners, described via knife-edge conditions, has been given in [18].

In this paper we consider the motion of a system of two equal mass points along a straight line under the action of a small non symmetric Coulomb dry friction force. The mass points are connected by a non linear spring with cubical non linearity. Excitation is carried due to the action of a small internal periodic force.

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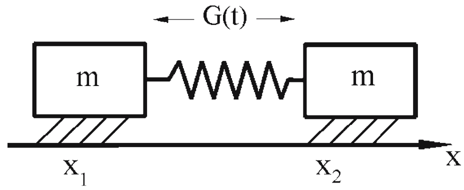

Fig. 1 Model of the system

2 Mathematical model

2.1 The equation of motion

The motion of a system of two material points (masses m) with coordinates x_1 and x_2, connected by a spring along an axis x is considered (Fig. 1).

The mass points are connected by a nonlinear spring with small cubic nonlinearity. The elastic force $f(x)$ in this case is

$$f(x) = -c(x - l_0) - \varepsilon m d (x - l_0)^3, \quad \varepsilon \ll 1. \tag{1}$$

Here $c > 0, d \geqslant 0$ and l_0 is the length of the spring in its undeformed state.

It is supposed that the mass points are under the action of a small nonsymmetric Coulomb dry frictional force $\varepsilon m F(\dot{x})$. The frictional force is taken to be different in magnitude depending on the direction of the motion of the body. The function $F(\dot{x})$ can be specified as follows:

$$F(\dot{x}) = \begin{cases} F_+, & \dot{x} > 0, \\ F_0, & \dot{x} = 0, \\ -F_-, & \dot{x} < 0. \end{cases} \tag{2}$$

Here $F_- \geqslant F_+ \geqslant 0$ are fixed, whereas F_0 may take any value in the interval $(-F_-, F_+)$.

Excitation is carried out due to the action of a small internal force

$$G(t) = \varepsilon m b \cos\psi, \quad \psi = \nu t. \tag{3}$$

From (1)–(3), designating $\omega^2 = c/m$ and replacing $x_2 \to x_2 - l_0$, we obtain a system of equations, retaining the original symbols:

$$\begin{aligned} \ddot{x}_1 + \omega^2(x_1 - x_2) &= -\varepsilon\left[F(\dot{x}_1) + d(x_1 - x_2)^3 + b\cos\psi\right], \\ \ddot{x}_2 + \omega^2(x_2 - x_1) &= -\varepsilon\left[F(\dot{x}_2) + d(x_2 - x_1)^3 - b\cos\psi\right]. \end{aligned} \tag{4}$$

2.2 A comment on F_0

The frictional force at rest, F_0, in the expression (2) changes in order to compensate the force $G(t)$ and the force of the spring in the state of rest ($\dot{x} = 0$). As long as the algebraic sum of these forces does not exceed the given maximum value of the frictional force of rest (F_- or F_+), the mass point stays in a static state. Such a situation can arise not only at the beginning, but also during the movement, as one or both mass points stay fixed over a certain time interval. This *stick-slip* motion under the action of dry friction is described in a series of publications [3,11,13,14,17]. Figure 2 shows charts for the type of movement described above, obtained by numerical integration of system (4) under the condition (2), where $\Delta = 10.0$, $\varepsilon = 0.01$, $\omega = 1.0$, $F_+ = 8.0$, $F_- = 9.0$, $b = 10.0$, $d = 10.0$.

The horizontal fragments on the curve $x_1(t)$ or $x_2(t)$ correspond to the time intervals where one or both mass points are fixed. In this paper we investigate the problem in the first-order approximation of the averaging method. It can be shown that the stick–slip effect should only be examined in the higher-order approximation, and so it is not considered in this paper.

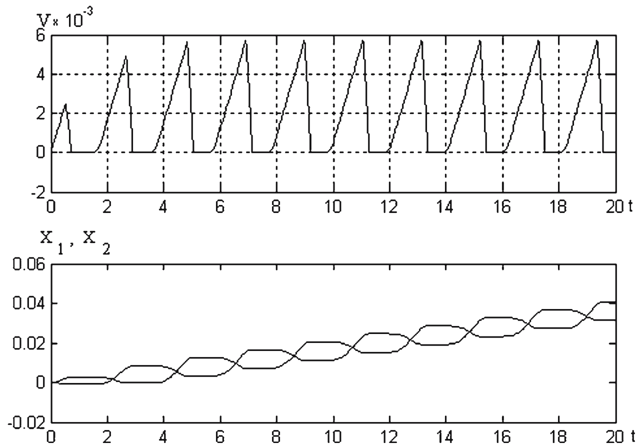

Fig. 2 Values of V, x_1 and x_2 versus time in the stick–slip case

2.3 Averaging procedure

To the system (4) we apply the procedure of averaging according to [5]. For this purpose we introduce new variables: the velocity of the center of mass V and the deviation of the mass points relative to the center of mass z:

$$V = (\dot{x}_1 + \dot{x}_2)\big/2, \quad z = (x_2 - x_1)\big/2. \tag{5}$$

We determine a value z as

$$z = a\cos(\Omega t + \vartheta), \quad \dot{z} = -a\Omega\sin(\Omega t + \vartheta), \tag{6}$$

where $\varphi = \Omega t + \vartheta$, $\Omega = \sqrt{2}\,\omega$.

Replacing V and z from expressions (5) and (6), system (4) can be written as

$$\begin{aligned}
\dot{V} &= -\frac{\varepsilon}{2}\left[F(V + a\Omega\sin\varphi) + F(V - a\Omega\sin\varphi)\right],\\
\dot{a} &= -\frac{\varepsilon}{2\Omega}\sin\varphi\left[F(V + a\Omega\sin\varphi) - F(V - a\Omega\sin\varphi) - 2da^3\cos^3\varphi + 2b\cos\psi\right],\\
\dot{\varphi} &= \Omega - \frac{\varepsilon}{2a\Omega}\cos\varphi\left[F(V + a\Omega\sin\varphi) - F(V - a\Omega\sin\varphi) - 2da^3\cos^3\varphi + 2b\cos\psi\right],\\
\dot{\psi} &= \nu.
\end{aligned} \tag{7}$$

where V and a are slow variables.

We investigate the system (7) in the vicinity of the main resonance $\nu = \Omega + \varepsilon\Delta$. For this purpose we introduce a new slow variable, $\xi = \psi - \varphi$, and exclude a fast variable, ψ; the result is the system (7) in the form

$$\begin{aligned}
\dot{V} &= -\frac{\varepsilon}{2}\left[F(V + a\Omega\sin\varphi) + F(V - a\Omega\sin\varphi)\right],\\
\dot{a} &= -\frac{\varepsilon}{2\Omega}\sin\varphi\left[F(V + a\Omega\sin\varphi) - F(V - a\Omega\sin\varphi) - 2da^3\cos^3\varphi + 2b\cos(\xi + \varphi)\right],\\
\dot{\xi} &= \varepsilon\left\{\frac{1}{2a\Omega}\cos\varphi\left[F(V + a\Omega\sin\varphi) - F(V - a\Omega\sin\varphi) - 2da^3\cos^3\varphi + 2b\cos(\xi + \varphi)\right] + \Delta\right\},\\
\dot{\varphi} &= \Omega - \frac{\varepsilon}{2a\Omega}\cos\varphi\left[F(V + a\Omega\sin\varphi) - F(V - a\Omega\sin\varphi) - 2da^3\cos^3\varphi + 2b\cos(\xi + \varphi)\right].
\end{aligned} \tag{8}$$

After averaging system (8) over a fast variable φ we obtain:

$$\dot V = \begin{cases} -\varepsilon\left(\frac{F_- + F_+}{\pi}\arcsin\frac{V}{a\,\Omega} - \frac{F_- - F_+}{2}\right), & \text{if } 0 \leqslant V < a\,\Omega,\\ -\varepsilon\,F_+, \quad \text{if } V \geqslant a\,\Omega, \end{cases}$$
$$\dot a = \begin{cases} -\frac{\varepsilon}{\Omega}\left(\frac{F_- + F_+}{\pi}\sqrt{1 - \frac{V^2}{a^2\,\Omega^2}} - \frac{b}{2}\sin\,\xi\right), & \text{if } 0 \leqslant V < a\,\Omega,\\ \varepsilon\,\frac{b}{2\,\Omega}\sin\,\xi, \quad \text{if } V \geqslant a\,\Omega, \end{cases} \tag{9}$$
$$\dot\xi = \varepsilon\left(\frac{b}{2\,a\,\Omega}\cos\,\xi + \Delta - \frac{3}{\Omega}\,d\,a^2\right).$$

2.4 The stationary regime

We are interested in an approximately steady motion as a single whole; therefore we seek the solution $\dot V = 0$ of the system (9):

$$V = a\,\Omega\cdot\sin\Phi, \quad \Phi = \frac{\pi}{2}\cdot\frac{F_- - F_+}{F_- + F_+}. \tag{10}$$

Since $V = \text{const}$, it follows from Eq. (10) that $a = \text{const}$ and further from system (9) that $\xi = \text{const}$. Then the second and the third equations of the system can be written as follows:

$$\frac{b}{2}\sin\,\xi = E\,\cos\Phi, \quad E = \frac{F_- + F_+}{\pi},$$
$$\frac{b}{2}\cos\,\xi = 3\,a^3 d - a\,\Delta\,\Omega. \tag{11}$$

Eliminating ξ from Eq. (11), we get the equation for stationary amplitude a:

$$a\left|3a^2 d - \Delta\,\Omega\right| = \sqrt{\frac{b^2}{4} - E^2\cos^2\Phi} \tag{12}$$

The necessary condition for the existence of a stationary regime is $b \geqslant 2\,E\,\cos\,\Phi$.

To investigate the stability of stationary amplitudes defined by Eq. (12), we consider the conditions for stability.

2.5 Conditions for stability

The variational equations for system (9) have the form:

$$\delta\,\dot V = -\varepsilon\frac{E}{a\,\Omega\,\cos\,\Phi}\cdot\delta\,V + \varepsilon\,\frac{E}{a}\,tg\,\Phi\cdot\delta\,a,$$
$$\delta\,\dot a = \varepsilon\,\frac{E}{a\,\Omega^2}\,tg\,\Phi\cdot\delta\,V - \varepsilon\,\frac{E}{a\,\Omega^2}\,tg\,\Phi\cdot\sin\,\Phi\cdot\delta\,a + \varepsilon\left(3\,\frac{d\,a^3}{\Omega} - \Delta\,a\right)\delta\,\xi, \tag{13}$$
$$\delta\,\dot\xi = -\,\varepsilon\left(9\frac{a\,d}{\Omega} - \frac{\Delta}{a}\right)\delta\,a - \varepsilon\,\frac{E}{a\,\Omega}\cos\,\Phi\cdot\delta\,\xi.$$

The characteristic polynomial $P(\lambda)$ for system (13) is

$$P\,(\lambda) = \lambda^3 + \lambda^2\cdot\,\varepsilon\,\frac{2\,E}{a\,\Omega\,\cos\,\Phi} + \lambda\cdot\varepsilon^2\left[\frac{E^2}{a^2\Omega^2}\left(1 + \sin^2\Phi\right) + 27\frac{a^4 d^2}{\Omega^2} + \Delta\left(\Delta - 12\frac{a^2 d}{\Omega}\right)\right]$$
$$+\,\varepsilon^3\frac{E}{\Omega\,\cos\,\Phi}\left[27\,\frac{a^3 d^2}{\Omega^2} + \frac{\Delta}{a}\left(\Delta - 12\frac{a^2 d}{\Omega}\right)\right]. \tag{14}$$

In order to locate all roots of the polynomial in the left half plane of the complex variable λ, the necessary and sufficient conditions are given by the Hurwitz criterion [8]. For the polynomial given by the formula (14) the Hurwitz criterion can be written in the form:

$$\begin{aligned}
&\frac{2E}{a\,\Omega\cos\Phi} > 0,\\
&\frac{E}{\Omega\cos\Phi}\left[\frac{2E^2}{a^3\Omega^2}\left(1+\sin^2\Phi\right)+27\frac{a^3d^2}{\Omega^2}+\frac{\Delta}{a}\left(\Delta-12\frac{a^2d}{\Omega}\right)\right] > 0,\\
&\frac{E}{\Omega\cos\Phi}\left[27\,\frac{a^3d^2}{\Omega^2}+\frac{\Delta}{a}\left(\Delta-12\frac{a^2d}{\Omega}\right)\right] > 0.
\end{aligned} \tag{15}$$

The conditions (15) can be reduced to a single condition:

$$27\,\frac{d^2}{\Omega^2}a^4-12\frac{d\,\Delta}{\Omega}\,a^2+\Delta^2 > 0. \tag{16}$$

3 Analysis of the model

Let us start with the investigation of the roots of Eq. (12) and with the analysis of their stability according to the condition (16).

3.1 The case $\Delta \leqslant 0$

In this case the stability condition (16) is satisfied. We write Eq. (12) for stationary amplitudes as $f(a)=0$, where $f(a)=3a^3d-a\,\Omega\,\Delta-L$. Here the value L is determined by the expression

$$L=\sqrt{\frac{b^2}{4}-E^2\cos^2\Phi}\geqslant 0. \tag{17}$$

Since $f(0)=-L$ is less than 0 and the expression for its derivative is

$$f'(a)=9a^2d-\Omega\,\Delta, \tag{18}$$

thus $f'(a) > 0$, and $f(a)$ is an increasing function. The equation $f(a)=0$ has a single positive root.

So, for $\Delta \leqslant 0$ there exists only one stable stationary amplitude a_1, located in the interval $0 < a_1 < a^*$, where $a^*=\min\left(\sqrt[3]{\frac{L}{3d}},\ -\frac{L}{\Omega\,\Delta}\right),\ \Delta \neq 0$.

The velocity corresponding to the stationary amplitude is calculated according to Eq. (10). Note that at $\Delta=0$, $a_1=\sqrt[3]{\frac{L}{3d}}$.

3.2 The case $\Delta > 0$

In this case the condition (16) leads to

$$0 < a < \frac{1}{3}\sqrt{\frac{\Omega\,\Delta}{d}},\quad a > \sqrt{\frac{\Omega\,\Delta}{3d}} \tag{19}$$

for stability, and

$$\frac{1}{3}\sqrt{\frac{\Omega\,\Delta}{d}} < a < \sqrt{\frac{\Omega\,\Delta}{3\,d}} \tag{20}$$

for instability.

First we consider the case where $0 < a < \sqrt{\frac{\Omega\,\Delta}{3\,d}}$. Then Eq. (12) for stationary amplitudes is given by $f(a) = 0$, where $f(a) = 3a^3 d - a\,\Omega\,\Delta + L$.

The expression for the first derivative has the form of Eq. (18), and for $\Delta > 0$ the function $f(a)$ has a minimum

$$a_{\min} = \frac{1}{3}\sqrt{\frac{\Omega\,\Delta}{d}}, \quad f(a_{\min}) = L - \frac{2}{9}\cdot\Omega\,\Delta\sqrt{\frac{\Omega\,\Delta}{d}}. \tag{21}$$

Since $f(0) = f\left(\sqrt{\frac{\Omega\,\Delta}{3d}}\right) = L > 0$, then if $f(a_{\min}) < 0$, the equation has two distinct positive roots; if $f(a_{\min}) > 0$, then there are no positive roots.

Considering Eqs. (17) and (21) we obtain the condition for the existence of two distinct roots as $b < M$, and the condition for the nonexistence of roots as $b > M$, thereby $M = 2\sqrt{\frac{4}{81}\cdot\frac{\Omega^3\Delta^3}{d} + E^2\cos^2\Phi}$.

There are two positive roots, which satisfy the conditions:

$$0 < a_1 < \frac{1}{3}\sqrt{\frac{\Omega\,\Delta}{d}}, \quad \frac{1}{3}\sqrt{\frac{\Omega\,\Delta}{d}} < a_2 < \sqrt{\frac{\Omega\,\Delta}{3d}}. \tag{22}$$

Comparing the inequalities (22) with the conditions of stability and instability (19) and (20), it follows that the stationary amplitude corresponding to the lesser root of $f(a) = 0$, a_1, is stable, and that the amplitude a_2, corresponding to the greater root, is unstable.

Now consider the case

$$a > \sqrt{\frac{\Omega\,\Delta}{3\,d}}. \tag{23}$$

In this case Eq. (12) for the stationary amplitudes is given by $f(a) = 0$, where $f(a) = 3a^3 d - a\,\Omega\,\Delta - L$. Given $f(0) = -L < 0$ and the derivative in Eq. (18), $f'(a) > 0$ if a satisfies condition (23). Thus $f(a)$ increases over the given interval, and the equation $f(a) = 0$ has a single positive root a_3.

Since a_3 satisfies the condition (23), it follows from Eq. (19) that the amplitude a_3 is stable.

3.3 Investigation results

Let us now summarize the results of the investigation.

For $b < 2\,E\,\cos\,\Phi$ there is no stationary mode.

For

$$\Delta < 0 \quad \text{and} \quad b > 2\,E\,\cos\,\Phi \tag{24}$$

there is only one stationary amplitude $a_1 : 0 < a_1 < a^*$, where $a^* = \min\left(\sqrt[3]{\frac{L}{3\,d}},\ -\frac{L}{\Omega\,\Delta}\right)$, $\Delta \neq 0$.

For $\Delta = 0$, there is $a_1 = \sqrt[3]{\frac{L}{3\,d}}$.

For $\Delta > 0$ and

$$2\,E\,\cos\,\Phi < b < 2\sqrt{\frac{4}{81}\cdot\frac{\Omega^3\Delta^3}{d} + E^2\cos^2\Phi} \tag{25}$$

there are three stationary amplitudes, a_1, a_2, and a_3, which satisfy the conditions

$$0 < a_1 < \frac{1}{3}\sqrt{\frac{\Omega\,\Delta}{d}}, \quad \frac{1}{3}\sqrt{\frac{\Omega\,\Delta}{d}} < a_2 < \sqrt{\frac{\Omega\,\Delta}{3d}}, \quad a_3 > \sqrt{\frac{\Omega\,\Delta}{3d}}.$$

Thereby the least amplitude a_1 and the greatest amplitude a_3 are stable, whereas the middle amplitude a_2 is unstable.

For $\Delta > 0$ and $b > 2\sqrt{\frac{4}{81}\cdot\frac{\Omega^3\Delta^3}{d} + E^2\cos^2\Phi}$ the only stable stationary amplitude is a_3:

$$a_3 > \sqrt{\frac{\Omega\,\Delta}{3d}}.$$

The velocity corresponding to the stationary amplitude is $V = a\,\Omega\cdot\sin\Phi$, $\Phi = \frac{\pi}{2}\cdot\frac{F_- - F_+}{F_- + F_+}$

4 Discussion of results and graphical illustrations

Figure 3 shows a chart for the velocity of the center of mass versus time, obtained by resolving of the exact equations of motion (4) with the following parameters: $\varepsilon = 0.01, \omega = 1.0, F_+ = 1.0, F_- = 2.0, b = 10.0, d = 10.0$. We retain the values of these parameters in the further calculations. For the case considered we let $\Delta = -10.0$ ($\varepsilon\,\Delta = -0.1$) and the equation for stationary amplitudes (12), obtained from the system of averaged equations (9), has only one stable solution, at $a = 0.28$. The corresponding value for the velocity of the center of mass V, calculated with Eq. (10) is $V = 0.20$. Figure 3 shows that after completion of the transitional process the exact velocity converges toward the value $V = 0.2$.

Figure 4 shows the dependence of the stationary amplitude a on the value of Δ, obtained from Eq. (12).

Since the frequency ν of the driving force is connected with Δ by formula $\nu = \Omega + \varepsilon\,\Delta$, the curve on the Fig. 4 is a resonance curve.

By solving condition (25) for Δ and combining it with condition (24) we obtain the following conditions: For

$$\Delta < \frac{3}{\Omega}\sqrt[3]{\frac{3}{4}d\left(\frac{b^2}{4} - E^2\cos^2\Phi\right)}, \tag{26}$$

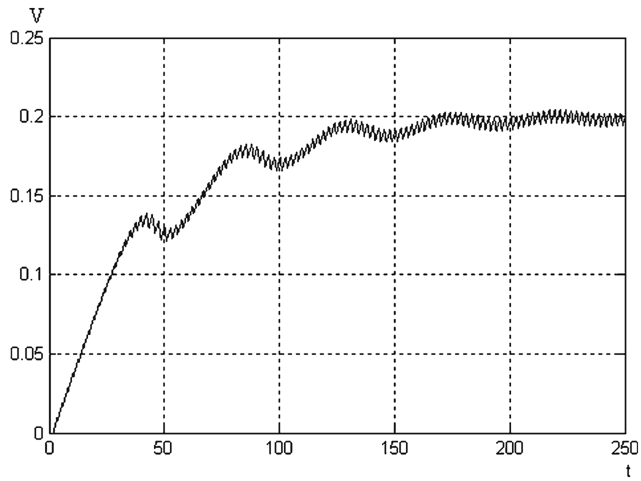

Fig. 3 Velocity versus time

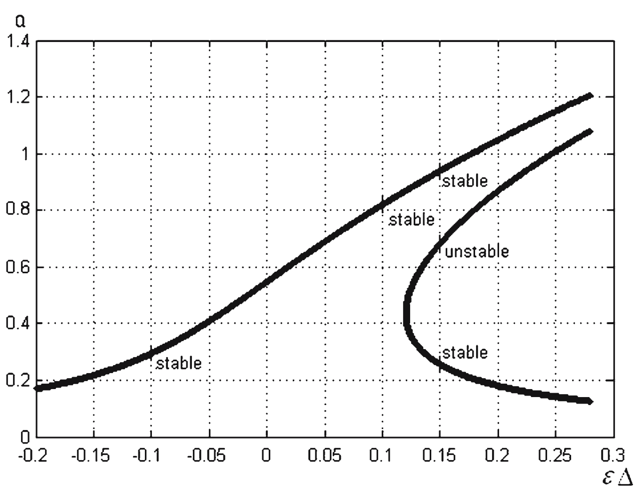

Fig. 4 Stationary amplitude a versus value $\varepsilon\,\Delta$

there is only one stable stationary amplitude a_1. For

$$\Delta > \frac{3}{\Omega}\sqrt[3]{\frac{3}{4}d\left(\frac{b^2}{4} - E^2\cos^2\Phi\right)}, \tag{27}$$

there are three stationary amplitudes a_1, a_2, and a_3 (in ascending order), of which only two are stable, the least and the greatest.

The value $\Delta = \frac{3}{\Omega}\sqrt[3]{\frac{3}{4}d\left(\frac{b^2}{4} - E^2\cos^2\Phi\right)}$ corresponds to the point of the vertical tangent to the resonance curve.

Let us consider some characteristic points on the resonance curve.

We take $\varepsilon\,\Delta = -0.1$. Then for the parameter values set as below there is only one stable stationary amplitude $a_1 = 0.28$. Condition (26) is satisfied, because this point is placed left of the vertical tangent. The next point considered corresponds to the positive value $\Delta = 10$ ($\varepsilon\,\Delta = 0.1$). Again, there is a single stable stationary amplitude $a_1 = 0.8$.

Now let us consider $\Delta = 15$. In this case condition (27) is satisfied, because the point $\varepsilon\,\Delta = 0.15$ is to the right of the vertical tangent. There are three stationary amplitudes, only two of which are stable.

Figures 5, 6 and 7 show charts of the stationary velocity V, amplitude a and phase ξ for various initial conditions.

For the curves in Fig. 5 the initial condition ($V_0 = 0.14$, $a_0 = 0.2$, $\xi_0 = 2.8$) is chosen so that the motion takes place with a stable stationary amplitude $a_1 = 0.27$. The corresponding value of stationary velocity of the center of mass is $V_1 = 0.18$.

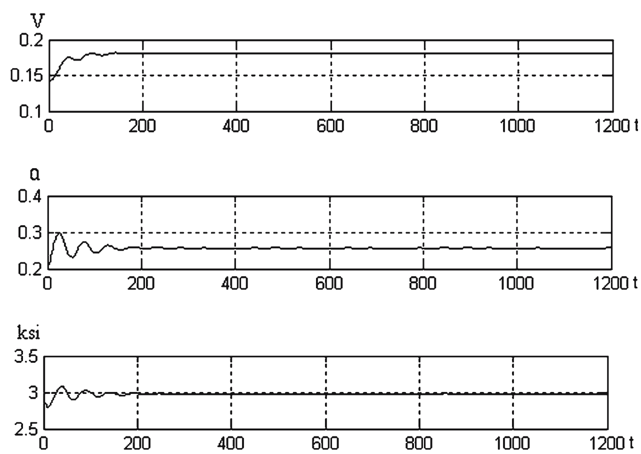

Fig. 5 Values of V, a and ξ versus time ($V_0 = 0.14$, $a_0 = 0.2$, $\xi_0 = 2.8$)

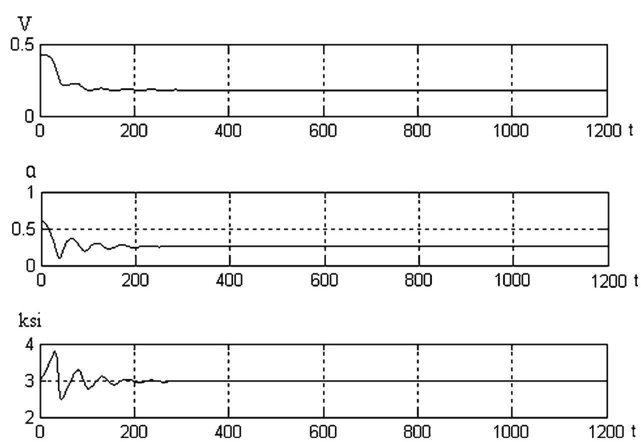

Fig. 6 Values of V, a and ξ versus time ($V_0 = 0.4$, $a_0 = 0.6$, $\xi_0 = 3.0$)

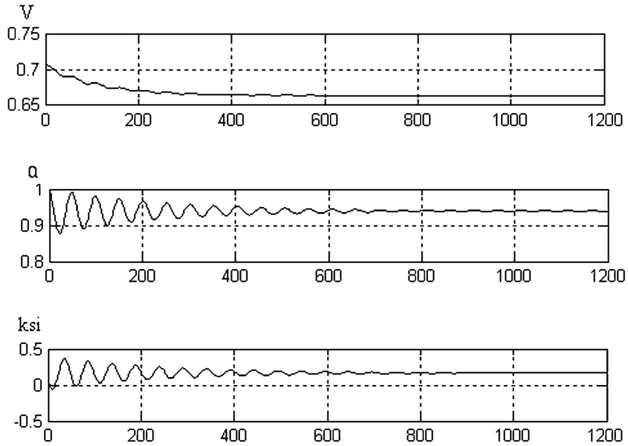

Fig. 7 Values of V, a and ξ versus time ($V_0 = 0.71$, $a_0 = 1.0$, $\xi_0 = 0$)

For the curves in Fig. 6 the initial condition ($V_0 = 0.4$, $a_0 = 0.6$, $\xi_0 = 3.0$) for the amplitude is chosen close to the unstable stationary amplitude $a_2 = 0.65$.

The charts reveal that the unstable stationary velocity $V_2 = 0.45$ corresponding to this amplitude drops toward the stable stationary velocity $V_1 = 0.18$.

For the curves on Fig. 7 the initial condition ($V_0 = 0.71$, $a_0 = 1.0$, $\xi_0 = 0$) is set up so that the motion occurs with the maximal stable stationary amplitude $a_3 = 0.95$. The value of the maximal stable stationary velocity of the center of mass is thereby $V_3 = 0.67$.

5 Conclusions

We have considered the oscillations of a nonlinear oscillator that consists of two mass points connected by a spring with cubic nonlinearity. The oscillator is exposed to the internal periodic force and to the nonsymmetrical force of dry friction.

The result of the analysis of the mathematical model allows us to make the following conclusions:

1. Using the averaging method an expression for the velocity of the system as a whole and an algebraic equation for the corresponding stationary amplitudes can be obtained.
2. There are at most three values for the stationary amplitude and accordingly for the velocity of the system as a single whole.
3. Depending on the values of the input parameters the conditions for stability and instability of the stationary amplitudes can be found. At most there exist two stable amplitudes.

Acknowledgments This project is supported by Deutsche Forschungsgemeinschaft (DFG, ZI 540/6-1). We thank Joachim Steigenberger for enlightening discussions on the dynamics of mechanical systems.

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