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COMMENT

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A survey of equations of motion in terms of inertial quasi-velocities for serial manipulators

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Abstract In this work we compare equations of motion using the so-called inertial quasi-velocities. As a result of these velocities we obtain two first-order decoupled equations of motion instead of one second-order differential equation of motion. The methods presented here, solve in a way, the problem of nonlinear dynamic decoupling. The first and the second method result from diagonalized Lagrangian robot dynamics (Jain and Rodriguez, IEEE Trans Robot Autom 11:571–584, 1995) and are known as normalized and unnormalized quasi-velocities. The third method described by Junkins and Schaub (J Astronaut Sci 45:279–295, 1997) offers eigenfactor quasi-coordinate velocities formulation for multibody dynamics. As a consequence of using transformation given by Loduha and Ravani (Trans ASME J Appl Mech 62:216–222, 1995) we obtain decoupled equations of motion in terms of modified generalized velocity components. Here we limit all these methods to serial manipulators. The novelty of this paper consists in physical interpretation of the quasi-velocities and discussion concerning equations of motion, the kinetic energy shaping, relationship between each of them and properties useful for simulation and control purposes. Also forward dynamics algorithms and their computational complexity in terms of new velocities are given. Simulation results illustrate the theoretical investigations. We conclude that all methods offer interesting possibilities for dynamic simulation and future control investigations.

Keywords Robot dynamics · Serial manipulators · Decoupled equation of motion · Quasi-velocities

Nomenclature

$\mathcal{N}$	number of joints and number of degrees of freedom
$\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}} \in R^{\mathcal{N}}$	vectors of generalized positions, velocities and accelerations, respectively
$\mathbf{M}(\boldsymbol{\theta}) \in R^{\mathcal{N}\times\mathcal{N}}$	system mass matrix in classical equations of motion
$\mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in R^{\mathcal{N}}$	vector of Coriolis and centrifugal forces in classical equations of motion
$\mathbf{G}(\boldsymbol{\theta}) \in R^{\mathcal{N}}$	vector of gravitational forces in classical equations of motion
$\mathbf{Q} \in R^{\mathcal{N}}$	vector of generalized forces in classical equations of motion
$\mathbf{M}_{\mathrm{p}}(\boldsymbol{\theta}) \in R^{\mathcal{N}\times\mathcal{N}}$	system mass matrix in Poincare's equations of motion
$\mathbf{C}_{\mathrm{p}}(\boldsymbol{\theta}, \mathbf{p}) \in R^{\mathcal{N}}$	vector of Coriolis and centrifugal forces in Poincare's equations of motion

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$\mathbf{G}_{\mathrm{p}}(\boldsymbol{\theta}) \in R^{\mathcal{N}}$	vector of gravitational forces in Poincare's equations of motion
$\mathbf{p} \in R^{\mathcal{N}}$	vector of quasi-velocities in Poincare's equations of motion
$\mathbf{Q}_{\mathrm{p}} \in R^{\mathcal{N}}$	vector of generalized forces in Poincare's equations of motion
$\mathbf{v}_{\mathrm{q}}, \dot{\mathbf{v}}_{\mathrm{q}} \in R^{\mathcal{N}}$	vector of kinematical quasi-velocities and its time derivative
$\boldsymbol{\mathcal{A}}(\boldsymbol{\theta}) \in R^{\mathcal{N}\times\mathcal{N}}$	dimensional configuration-dependent transformation matrix between quasi-velocities and joint velocities
$\mathbf{B}_0 \in R^{\mathcal{N}\times\mathcal{N}}$	dimensional configuration-dependent transformation matrix between joint velocities and quasi-velocities
$\mathbf{b}_0 \in R^{\mathcal{N}}$	additional configuration-dependent vector arising from matrix $\mathbf{B}_0$
$\dot{\mathbf{q}} \in R^{\mathcal{N}}$	vector of time derivatives of quasi-coordinates
$\mathbf{dq} \in R^{\mathcal{N}}$	vector of time differentials of quasi-coordinates
$\mathbf{h}^*(\boldsymbol{\theta}, \mathbf{v}_{\mathrm{q}}, t) \in R^{\mathcal{N}}$	vector which represents the sum of applied forces, gyroscopic terms, centrifugal forces and Coriolis effects
$\mathbf{F}_r \in R^{\mathcal{N}}$	generalized active forces vector
$\mathbf{F}_r^* \in R^{\mathcal{N}}$	generalized inertia forces vector
$\boldsymbol{\nu} \in R^{\mathcal{N}}$	vector of normalized quasi-velocities
$\boldsymbol{\xi} \in R^{\mathcal{N}}$	vector of unnormalized quasi-velocities
$\mathbf{C}(\boldsymbol{\theta}, \boldsymbol{\nu}) \in R^{\mathcal{N}}$	vector of Coriolis and centrifugal forces in diagonalized normalized equations of motion
$\mathbf{C}(\boldsymbol{\theta}, \boldsymbol{\xi}) \in R^{\mathcal{N}}$	vector of Coriolis and centrifugal forces in diagonalized unnormalized equations of motion
$\mathbf{G}_{\nu}(\boldsymbol{\theta}) = \mathbf{m}^{-1}(\boldsymbol{\theta})\mathbf{G}(\boldsymbol{\theta}) \in R^{\mathcal{N}}$	vector of gravitational forces in diagonalized normalized equations of motion
$\mathbf{D} = \mathbf{HPH}^{\mathrm{T}} \in R^{\mathcal{N}\times\mathcal{N}}$	articulated inertia about joint axes matrix
$\mathbf{H} \in R^{\mathcal{N}\times 6\mathcal{N}}$	projection operator for all joint axes
$\mathbf{P} \in R^{6\mathcal{N}\times 6\mathcal{N}}$	articulated inertia matrix
$\mathbf{G}_{\xi}(\boldsymbol{\theta}) = \mathbf{D}^{1/2}\mathbf{m}^{-1}(\boldsymbol{\theta})\mathbf{G}(\boldsymbol{\theta}) \in R^{\mathcal{N}}$	vector of gravitational forces in diagonalized unnormalized equations of motion
$\mathbf{m}(\boldsymbol{\theta}) \in R^{\mathcal{N}\times\mathcal{N}}$	spatial operator – "square root" of mass matrix $\mathbf{M}(\boldsymbol{\theta})$, namely $\mathbf{M}(\boldsymbol{\theta}) = \mathbf{m}(\boldsymbol{\theta})\mathbf{m}^{\mathrm{T}}(\boldsymbol{\theta})$ which is expressed as $\mathbf{m}(\boldsymbol{\theta}) = [\mathbf{I} + \mathbf{H}\boldsymbol{\phi}\mathbf{K}]\mathbf{D}^{1/2}$
$\boldsymbol{\phi} \in R^{6\mathcal{N}\times 6\mathcal{N}}$	rigid manipulator force transformation matrix
$\mathbf{K} \in R^{6\mathcal{N}\times\mathcal{N}}$	shifted Kalman gain matrix
$\dot{\mathbf{m}}(\boldsymbol{\theta}) \in R^{\mathcal{N}\times\mathcal{N}}$	time derivative of factor $\mathbf{m}(\boldsymbol{\theta})$
$\boldsymbol{\epsilon} \in R^{\mathcal{N}}$	vector of normalized quasi-moments
$\boldsymbol{\kappa} \in R^{\mathcal{N}}$	vector of unnormalized quasi-moments
$\mathcal{O}_k$	origin of the frame attached to the k-th link
$\theta_k, \dot{\theta}_k, \ddot{\theta}_k$ kth	generalized position, velocity and acceleration, respectively
$\mathbf{l}_{k,j} \in R^3$	vector from origin $\mathcal{O}_k$ to origin $\mathcal{O}_j$
$\boldsymbol{\phi}_{k,k-1} \in R^{6\times 6}$	operator which transforms rigid quantities from the kth to the $(k-1)$th joint defined as:

$$\boldsymbol{\phi}_{k,k-1} = \begin{bmatrix} \mathbf{I} & \tilde{\mathbf{l}}_{k,k-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

and $\tilde{\mathbf{l}}$ is a skew symmetric matrix

$$\tilde{\mathbf{l}} = \begin{bmatrix} 0 & -l_z & l_y \\ l_z & 0 & -l_x \\ -l_y & l_x & 0 \end{bmatrix}$$

where l_x, l_y, l_z are elements x, y and z of vector $\mathbf{l}$, $\mathbf{0}$ and $\mathbf{I}$ denote zero and unit matrices of appropriate dimensions

$^{k+1}\mathbf{R}_k \in R^{3\times3}$ direction cosine matrix between $\mathcal{O}_k$ and $\mathcal{O}_{k+1}$ according to the modified Denavit–Hartenberg notation [14]

$\boldsymbol{\psi}_{k,k-1} \in R^{6\times6}$ operator which transforms articulated quantities from the kth to the $(k-1)$th joint

α_k angle for the kth joint in the modified Denavit–Hartenberg notation [14]

$\mathbf{A}_k \in R^{6\times6}$ spatial rotation matrix defined as:

$$\mathbf{A}_k = \begin{bmatrix} ^{k+1}\mathbf{R}_k & \mathbf{0} \\ \mathbf{0} & ^{k+1}\mathbf{R}_k \end{bmatrix}$$

$(.)^{\mathrm{T}}$ transpose operation

$\mathbf{h}_k^{\mathrm{T}} \in R^3$ axis of rotation or axis of translation for the kth joint

$\mathbf{H}_k^{\mathrm{T}} \in R^6$ joint map matrix for the kth joint:

$$\mathbf{H}_k^{\mathrm{T}} = \left[\mathbf{h}_k,\ \mathbf{0}^{\mathrm{T}}\right]^{\mathrm{T}} \text{ rotational joint}$$
$$\mathbf{H}_k^{\mathrm{T}} = \left[\mathbf{0}^{\mathrm{T}},\ \mathbf{h}_k\right]^{\mathrm{T}} \text{ translational joint}$$

m_k mass of the kth link

$\mathbf{p}_k \in R^3$ vector from $\mathcal{O}_k$ to the kth link's center mass

$\mathcal{I}_k \in R^{3\times3}$ inertia tensor of kth link with respect to $\mathcal{O}_k$

$\mathbf{M}_k \in R^{6\times6}$ spatial inertia matrix of the kth link expressed in the coordinate $\mathcal{O}_k$ defined as:

$$\mathbf{M}_k = \begin{bmatrix} \mathcal{I}_k & \mathbf{m}_k\tilde{\mathbf{p}}_k \\ -\mathbf{m}_k\tilde{\mathbf{p}}_k & \mathbf{m}_k\mathbf{I} \end{bmatrix}$$

$\mathbf{V}_k \in R^6$ spatial velocity vector of the kth body frame, $\mathbf{V}_k = \begin{bmatrix} \boldsymbol{\omega}_k \\ \mathbf{v}_k \end{bmatrix}$

$\boldsymbol{\omega}_k \in R^3$ angular velocity of the kth body

$\mathbf{v}_k \in R^3$ linear velocity of the kth body

$\mathbf{n}_k \in \mathbf{R}^6$ spatial bias acceleration vector for the kth link: for rotational joint or translational joint defined as [36,37]

$$\mathbf{n}_k = \begin{bmatrix} ^{k+1}\mathbf{R}_k(\boldsymbol{\omega}_k \times \mathbf{h}_k\dot{\theta}_k) \\ ^{k+1}\mathbf{R}_k(\mathbf{v}_k \times \mathbf{h}_k\dot{\theta}_k) \end{bmatrix} \tag{1}$$

and for translational joint [36,37]

$$\mathbf{n}_k = \begin{bmatrix} \mathbf{0} \\ ^{k+1}\mathbf{R}_k(\boldsymbol{\omega}_k \times \mathbf{h}_k\dot{\theta}_k) \end{bmatrix} \tag{2}$$

$\mathbf{G}_{ak} \in R^6$ Kalman gain vector for the kth joint

$\mathbf{P}_k \in R^{6\times6}$ spatial articulated inertia matrix for the kth joint

D_k articulated inertia about the kth joint axis

ν_k kth normalized quasi-velocity

$\dot{\nu}_k$ kth normalized quasi-acceleration

ξ_k kth unnormalized quasi-velocity

$\dot{\xi}_k$ kth unnormalized quasi-acceleration

ϵ_k kth normalized quasi-moment

$C_{\nu k}$ Coriolis term for the kth joint in normalized diagonalized equations of motion

$G_{\nu k}$ gravity term for the kth joint in normalized diagonalized equations of motion

κ_k kth unnormalized quasi-moment

$C_{\xi k}$ Coriolis term for the kth joint in unnormalized diagonalized equations of motion

$G_{\xi k}$ gravity term for the kth joint in unnormalized diagonalized equations of motion

$\mathbf{b}_k \in R^6$ spatial bias forces vector calculated as (both for rotational and translational joints) [36,37]:

$$\mathbf{b}_k = \begin{bmatrix} \boldsymbol{\omega}_k \times \mathcal{I}_k\boldsymbol{\omega}_k + m_k\mathbf{p}_k \times [\boldsymbol{\omega}_k \times \mathbf{v}_k] \\ m_k[\boldsymbol{\omega}_k \times \mathbf{v}_k + \boldsymbol{\omega}_k \times (\boldsymbol{\omega}_k \times \mathbf{p}_k)] \end{bmatrix}$$

$\mathbf{G}_k \in R^3$	three-dimensional gravitational forces vector
$\mathbf{b}_{gk} = (-\mathbf{p}_k \times \mathbf{G}_k, -\mathbf{G}_k)^{\mathrm{T}} \in R^6$	spatial gravitational forces vector
τ_k	generalized force acting at the kth joint
$\boldsymbol{\tau} \in R^{\mathcal{N}}$	generalized force acting at the manipulator
$\boldsymbol{\eta} \in R^{\mathcal{N}}$	eigenfactor quasi-coordinate velocity (EQV) vector
$\dot{\boldsymbol{\eta}} \in R^{\mathcal{N}}$	eigenfactor quasi-coordinate acceleration vector
$\mathbf{SC}_e \in R^{\mathcal{N}\times\mathcal{N}}$	EQV rate transformation matrix
$\mathbf{D}_{\mathrm{e}} = \mathrm{diag}(\lambda_i) \in R^{\mathcal{N}\times\mathcal{N}}$	diagonal matrix containing eigenvalues of the mass matrix $\mathbf{M}(\boldsymbol{\theta})$
λ_i	ith eigenvalue of the mass matrix $\mathbf{M}(\boldsymbol{\theta})$
$\dot{\lambda}_i$	time derivative of the eigenvalue λ_i
$\mathbf{E} = \mathbf{C}_{\mathrm{e}}^{\mathrm{T}} \in R^{\mathcal{N}\times\mathcal{N}}$	orthogonal real matrix of eigenvectors of $\mathbf{M}(\boldsymbol{\theta})$
$\boldsymbol{\Omega}_{\mathrm{e}} \in R^{\mathcal{N}\times\mathcal{N}}$	skew symmetric matrix in which each element Ω_{ij} represents a generalized eigenvector axis angular velocity
$\mathbf{M}_\theta = \partial\mathbf{M}/\partial\boldsymbol{\theta}$	partial derivative with respect to the configuration vector $\boldsymbol{\theta}$
$\mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \boldsymbol{\eta}) \in R^{\mathcal{N}}$	vector of Coriolis and centrifugal forces in EQV formulation
$\boldsymbol{\varepsilon} \in R^{\mathcal{N}}$	vector of quasi-moments in EQV formulation
$\mathbf{u} \in R^{\mathcal{N}}$	generalized velocity component (GVC) vector
$\dot{\mathbf{u}} \in R^{\mathcal{N}}$	time derivative of the GVC vector
$\boldsymbol{\Upsilon} \in R^{\mathcal{N}\times\mathcal{N}}$	rate transformation matrix in GVC formulation which depends on mass matrix of the system and kinematical parameters
$\dot{\boldsymbol{\Upsilon}} \in R^{\mathcal{N}\times\mathcal{N}}$	time derivative of the matrix $\boldsymbol{\Upsilon}$
$\mathbf{N} \in R^{\mathcal{N}\times\mathcal{N}}$	diagonal mass matrix in GVC formulation
$\mathbf{C}\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \mathbf{u}\right) \in R^{\mathcal{N}}$	vector of Coriolis and centrifugal forces in GVC equations of motion
$\boldsymbol{\pi} \in R^{\mathcal{N}}$	vector of quasi-moments in GVC formulation
$\mathbf{J}_k \in R^{3\times\mathcal{N}}$	partial derivative of the kth body mass center position with respect to the inertial reference frame
$\boldsymbol{\Omega}_k \in R^{3\times\mathcal{N}}$	partial derivative of the kth body angular velocity with respect to the time derivative of the generalized coordinate vector
$\mathbf{W}_k = \tilde{\boldsymbol{\omega}}_k \in R^{3\times 3}$	angular velocity matrix associated with the kth body written in terms of the kth body
$\mathbf{f}_k \in R^3$ where $\mathbf{f}_k = \mathbf{f}_{gk} + \mathbf{f}_{ek}$	resultant active force acting at the mass center of the kth body, where $\mathbf{f}_{gk}$ and $\mathbf{f}_{ek}$ denote gravitational and external forces, respectively
$\boldsymbol{\tau}_{\mathrm{R}k} \in R^3$	where $\boldsymbol{\tau}_{\mathrm{R}k} = \boldsymbol{\tau}_{gk} + \boldsymbol{\tau}_{ek}$ resultant moment of the kth body, where $\boldsymbol{\tau}_{gk}$ and $\boldsymbol{\tau}_{ek}$ denote gravitational and external moments, respectively

1 Introduction

In robot dynamics there are two basic problems. First of them, an inverse dynamics problem, relies on finding the joint moments or forces from applied manipulator's trajectory (namely positions, joint velocities and accelerations). This problem is very important for fast manipulators because knowledge of dynamical model is required in control (for example feed-forward control). Second one, i.e. a forward dynamics problem, is to determine the joint accelerations which result from a set of applied joint moments or forces. The solution of this problem is useful for simulation purposes.

Manipulators as multirigid body systems are systems of interconnected bodies and their motion depends on behavior of all of them. Their dynamics is described by second-order nonlinear differential equations. In simulation dynamic equations require, in the most general case, taking the inverse of a time-varying, configuration variable-dependent mass matrix. This problem exists for various multibody mechanical systems, among others for serial manipulators. In this work we assume that manipulators are holonomic and unconstrained systems.

Dynamical equations of motion which describe the behavior of manipulators play an important role in problems of their design and control. The most often used classical description of motion is based on Euler–Lagrange or Newton–Euler principle [14,56,57]. There exists also other techniques (e.g. generalized speeds), which lead to the second-order differential equations [29]. However, our goal is to make dynamics of the manipulator easier for simulation and control purposes. Moreover, for the purpose of control the mass matrix should be diagonal. Due to this, many authors introduced different approaches in description of equations of motion.

The problem of simplification of dynamical equations of motion was considered in Refs. [23,33,58]. Factorization of the inertia matrix (which is always positive definite) of a $\mathcal{N}$-link robot manipulator as multiplication of a matrix and its transposition, and definition of a canonical transformation lead to robot dynamic equations which are particularly simple. For instance, Koditschek [33] used the coordinate transformation to solve this problem. Next, Gu and Loh [23] proposed a canonical transformation bases on a well-known Hamilton's equation. In their work they discussed possibility of using this transformation for simplification of the robot dynamics formulation and modeling. Afterward Spong [58] using tools from Hamiltonian mechanics and Riemannian geometry has given necessary and sufficient conditions for the existence of factorization of the inertia mass matrix $\mathbf{M}(\boldsymbol{\theta})$ of the $\mathcal{N}$-link manipulator which may lead to globally diagonalized dynamics. The necessary and sufficient condition of this factorization is vanishing of the Riemannian symbols of the first kind. These symbols are computable from Christoffel symbols. In this case we obtain two first-order differential equations of motion. The Lagrange equations are equivalent to Hamiltonian's equations and contain a set of the conjugate momentum and joint velocities. Another solution of the dynamical systems modeling is to introduce two first-order decoupling equations of motion instead of one differential second-order equation. As a result of the diagonalization process we obtain the so-called quasi-variables. Due to the fact that globally diagonalized dynamics rarely exists in practice for multibody systems Jain and Rodriguez [26] proposed the diagonalization in velocity space.

The dynamical description of motion can be simplified if we decompose the mass matrix. Saha [48,49] employed the inverse Gaussian elimination to achieve this aim. His algorithm is suitable both for the inverse and forward dynamics of the system and allows to avoid inversion of the generalized inertia matrix.

Sometimes differential equations of motion can be obtained easier if we introduce different velocities not the time derivatives of the generalized positions. In classical mechanics we describe the mechanical system using the generalized coordinates with their time derivatives or the generalized coordinates with quasi-velocities [13]. A set of the angular velocities together with Euler angles can be as an illustrative example. A goal of introducing the quasi-velocities is simplification of dynamic equations of motion of the system [6,8,16]. They serve also as more suitable quantities as generalized velocities for modeling of dynamics of mechanical systems [3,18,39]. Quasi-velocities, which are meaningful physical quantities, are understood as the time derivative of some quasi-coordinates [18,25,39]. Quasi-coordinates do not lead to obtaining information about the system trajectories but are very useful in rigid body dynamics. They are not meaningful in physical sense but only in terms of infinitesimal motion.

The most important difference between quasi-velocities $\mathbf{v}_{\mathrm{q}}$ and generalized velocities $\dot{\boldsymbol{\theta}}$ is expressed in the statement that the first are not integrable components of velocities [4,5], i.e.:

$$\mathbf{v}_{\mathrm{q}} = \mathbf{B}_0\dot{\boldsymbol{\theta}} + \mathbf{b}_0 \tag{3}$$

where components $\mathbf{B}_0$ and $\mathbf{b}_0$ are not integrable. Meaning of quasi-velocities one can explain in a manner given e.g. in Ref. [16]. Assuming vector $\dot{\mathbf{q}}$ as a vector of time derivatives of quasi-coordinates vector $\mathbf{q}$ one can obtain:

$$\mathbf{v}_{\mathrm{q}} = \dot{\mathbf{q}}, \quad \mathrm{d}\mathbf{q} = \mathbf{B}\mathrm{d}\boldsymbol{\theta}. \tag{4}$$

The quantities contained in vector $\mathrm{d}\mathbf{q}$ represent differentials of quasi-coordinates because they unlike true coordinates are, in general, not integrable.

Kwatny and Blankenship have stated [39] that it is possible to reduce Poincare's equations to the form:

$$\mathbf{M}_{\mathrm{p}}(\boldsymbol{\theta})\dot{\mathbf{p}} + \mathbf{C}_{\mathrm{p}}(\boldsymbol{\theta},\mathbf{p})\mathbf{p} + \mathbf{G}_{\mathrm{p}}(\boldsymbol{\theta}) = \mathbf{Q}_{\mathrm{p}} \tag{5}$$

where

$$\mathbf{C}_{\mathrm{p}}(\boldsymbol{\theta},\mathbf{p}) = \left[\frac{\partial \mathbf{M}_{\mathrm{p}}(\boldsymbol{\theta})\mathbf{p}}{\partial \boldsymbol{\theta}}\mathbf{Z}(\boldsymbol{\theta})\right] - \frac{1}{2}\left[\frac{\partial \mathbf{M}_{\mathrm{p}}(\boldsymbol{\theta})\mathbf{p}}{\partial \boldsymbol{\theta}}\mathbf{Z}(\boldsymbol{\theta})\right]^{\mathrm{T}} - \left[\sum_{j=1}^{\mathcal{N}} p_j \mathbf{X}_j^{\mathrm{T}}\right]\mathbf{Z}^{-\mathrm{T}}(\boldsymbol{\theta})\mathbf{M}_{\mathrm{p}}^{\mathrm{T}}(\boldsymbol{\theta}) \tag{6}$$

$$\mathbf{G}_{\mathrm{p}}(\boldsymbol{\theta}) = \mathbf{Z}^{\mathrm{T}}(\boldsymbol{\theta})\frac{\partial \mathcal{P}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^{T}}, \quad \mathbf{Q}_{\mathrm{p}} = \mathbf{Z}^{\mathrm{T}}(\boldsymbol{\theta})\mathbf{Q} \tag{7}$$

$\mathcal{P}(\boldsymbol{\theta})$ denotes the potential energy function and p_j is the j-th element of the vector $\mathbf{p}$. In comparison with [39] the term $\mathbf{C}_\mathrm{p}(\boldsymbol{\theta}, \mathbf{p})$ was corrected here. The generalized velocity vector $\dot{\boldsymbol{\theta}}$ is related to the quasi-velocity vector $\mathbf{p}$ as follows [39]:

$$\dot{\boldsymbol{\theta}} = \mathbf{Z}(\boldsymbol{\theta})\mathbf{p} \tag{8}$$

where the matrix $\mathbf{Z}(\boldsymbol{\theta})$ is a velocity transformation matrix. The dynamical equation (5) together with the Eq. (8) provide a closed set of equations. Because it is assumed [39] that the matrix $\mathbf{Z}$ is invertible i.e. there exists $\mathbf{Z}^{-1} = \mathbf{Y}$ then also

$$\mathbf{p} = \mathbf{Y}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}. \tag{9}$$

The quantity $\mathbf{X}_j = \left[[z_j, z_1][z_j, z_2]\ldots[z_j, z_\mathcal{N}]\right]$ present in Eq. (6) consists of commutators or Lie brackets $[z_i, z_j]$ which are columns of the matrix $\mathbf{Z} = [z_1\ z_2\ \ldots z_\mathcal{N}]$. Additionally, it was shown [39] that Poincare's equations can be derived from Lagrange's equations. It is always possible to express the Lagrangian, $L(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$, in terms of $\boldsymbol{\theta}$ and $\mathbf{p}$ because $L(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \tilde{L}(\boldsymbol{\theta}, \mathbf{p}) = \tilde{L}(\boldsymbol{\theta}, \mathbf{Y}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}})$. Therefore, Lagrange's equations can be written in the following form:

$$\frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial L}{\partial \dot{\boldsymbol{\theta}}} - \frac{\partial L}{\partial \boldsymbol{\theta}} = \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \tilde{L}}{\partial \mathbf{p}}\mathbf{Y}(\boldsymbol{\theta})\right) - \frac{\partial \tilde{L}}{\partial \mathbf{p}}\frac{\partial \mathbf{Y}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}}{\partial \boldsymbol{\theta}} - \frac{\partial \tilde{L}}{\partial \boldsymbol{\theta}} = \mathbf{Q}^\mathrm{T}. \tag{10}$$

From these equations one can derive formally Poincare's equations

$$\frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial \tilde{L}}{\partial \mathbf{p}} - \sum_{j=1}^{\mathcal{N}} p_j \frac{\partial \tilde{L}}{\partial \mathbf{p}}\mathbf{Y}\mathbf{X}_j - \frac{\partial \tilde{L}}{\partial \boldsymbol{\theta}}\mathbf{Z} = \mathbf{Q}^\mathrm{T}\mathbf{Z}. \tag{11}$$

This important result shows that Poincare's equations results from Lagrange's equations. Thus, assuming various kinds of quasi-velocities one can obtain, in fact, different equations in terms of quasi-velocities which correspond to Poincare's equations.

Quasi-velocities were considered also in [4,6,8] and understood as kinematical parameters [5,7] because they are kinematically dependent on generalized coordinates. Thus quasi-velocities in classical mechanics literature represent mainly strictly kinematical relationships between them and the generalized velocities. The transformation matrix depends on generalized coordinates.

In a robotics literature one can find also description based on generalized speeds (which are in fact a kind of quasi-velocities) [29]. These speeds depend on generalized coordinates and serve as new velocity variables. Their introduction together with generalized coordinates enables to bring equations of motion into particularly simple form. We can state that quasi-velocities are related to generalized velocities by means of some transformation matrix containing kinematical quantities.

There exist also methods of diagonalization of the mass matrix using the so-called inertial quasi-velocities. These quasi-velocities were described by Jain and Rodriguez [26], Junkins and Schaub [27,28,53] and Loduha and Ravani [40]. They, similarly as classical quasi-velocities are kinematically dependent on generalized velocities and contain kinematical parameters. Consider a matrix which transforms vector of the generalized velocities into vector of inertial quasi-velocities. The main difference consists in fact that the transformation matrix contains both kinematical (positions) and dynamical parameters of the system (masses, inertias). Because of that these velocities can be understood as inertial quasi-velocities to distinguish them from well-known kinematical quasi-velocities. As a result we obtain, instead of one second-order differential equation of motion, two first-order differential equations: the dynamical differential equation of motion and kinematical one. Notice, however that the dependence is not purely kinematical because of the transformation matrix which contains kinematic and inertial quantities. The normalized and unnormalized quasi-velocities introduced by Jain and Rodriguez [26] are based on spatial operator algebra notation described in Refs. [46,47]. Loduha and Ravani [40] used Kane's equations to propose method for decoupling of the second-order differential equations of motion. Junkins and Schaub method [27,28,52,53] takes advantage of spectral decomposition for obtaining the decoupled system mass matrix in equations of motion.

What are the advantages of using the inertial quasi-velocities? Firstly, diagonalization implies that at the same time instant each equation is decoupled (as time passes they are becoming coupled) from all other joint equations. Such equations can be used to design decoupling or noninteracting controllers. Secondly, by making use of quasi-velocities one can obtain natural and elegant splitting between momentum differential equation and kinematic differential equation. This splitting enables consideration of two first-order differential equations instead of one second-order equation. Thirdly, one can realize decomposition of the mass matrix to gain further insight into the manipulator dynamics. Fourthly, mass matrix is diagonal which simplifies its inversion (or no inversion is needed if we deal with the unit matrix). Fifthly, if the equations of motion are analyzed directly to determine the nature of the nonlinear behavior, we make our work easier. Besides that, new quasi-velocity vector may be orthogonal to the new Coriolis term which also simplifies control schemes (the Coriolis term does no work). Finally, in some cases design of a globally exponentially stabilizing controller is possible.

In this work we consider the classical equations of motion for robot manipulators and the following inertial quasi-velocities:

1. the normalized quasi-velocities (NQV) [26],
2. the unnormalized quasi-velocities (UQV) [26],
3. the eigenfactor quasi-coordinate velocities (EQV) [27,53],
4. the generalized velocity components (GVC) [40].

All these quasi-velocities are understood as some abstract velocities containing joint velocities, kinematical and dynamical parameters of the manipulator. This is another problem as "generalized speeds" defined by Kane and Levinson [29] because those speeds are dependent only on generalized coordinates, their time derivatives and time. Here-considered methods can be divided into two groups. The first of them represents methods which decompose mass matrix into two matrices (NQV and EQV). The second group consists of methods in which mass matrix is divided into three matrices (UQV and GVC). In this case the physical units are the same as using the generalized velocities because the inertial quantities are presented as coefficients.

Jain and Rodriguez [26] proposed robot dynamic algorithms in terms of quasi-velocities which are recursive in nature (recursions are described by using vector-matrix notation). On the contrary Junkins and Schaub (EQV formulation) [27,53] use the spectral decomposition of the system mass matrix. Loduha and Ravani solution 4 [40] is based on appropriate selection of a rate transformation matrix, which serves as decoupling of the dynamical equations of motion. This method is related to modified Kane's equations.

The aim of this work is to consider various equations of motion in terms of the inertial quasi-velocities as well as to give a comparison between all of them. Besides that one can point out the differences between the above-mentioned propositions of diagonalization. It will be shown how to use them to solve two fundamental problems in robot dynamics (inverse and forward) for serial manipulators. A tutorial of complexity is another important issue considered in this paper which to our best knowledge has not been resolved so far in detail.

The rest of the paper is organized as follows. Equations of motion of rigid body systems are discussed in the Sect. 2. Section 3 contains description of dynamical equations of motion in terms of various quasi-velocities. In Sect. 4 a comparison between inertial quasi-velocities is given. In Sect. 5 the forward dynamics algorithms are presented. Next Section is devoted computational complexity of the related algorithms. Simulation results are presented in Sect. 7. The last Section contains concluding remarks.

2 Equations of motion of rigid body systems

The rigid body dynamics is a part of theoretical mechanics. A set of rigid bodies connected in a chain we can consider as a multibody system. Classical approach to the dynamics of the multibody systems were described in many references. A review of known methods can be found e.g. in [45,54,64].

At the start of our survey we should refer to publications based on theoretical mechanics. Because the literature is rich then we have to limit only to some selected references. In the paper [30] Kharlamov considered equations of motion for a system of rigid bodies. The obtained equations were related to a general case and as was shown, in some cases, were integrable. Periodical solution of Poincaré problem for Hamiltonian systems was presented by Elfimov [19]. Mechanical system are sometimes described using Hamilton equations. Such equations for a system with constraints were derived in [34]. However, omitting the constraints the same equations can be used for the holonomic case. Savchenko et al. [50] presented equations of motion for a system of rigid bodies with a kinetic energy accumulator. Moreover, some properties of the system were shown in the paper. Similar problem of the kinetic energy accumulator was developed next in Ref. [51]. An interesting case of a system of connected bodies was considered by Bolgrabskaya and Savchenko [9]. The authors have shown both dynamic equations and their solutions. The increasing demand of high performance manipulators in industrial applications inspires a new methodology. Some works of specialists in theoretical mechanics has been devoted to manipulator motion and control. The approach consists of two steps. In the first step differential manipulator dynamic equations are derived. In the second step they are used to its motion control. The problem of an optimal manipulator motion control was discussed by Gubin [24]. Several papers concerning manipulator modeling and control were written by Vypov and Elfimov [60–62]. In Ref. [60] the authors presented dynamical equations of a tree-like manipulator. Based on a linearizing method the appropriate solution of manipulator equations was given. An analysis of motion of the manipulator was extended in [61]. However, in that work rotors (one rotor between two links) were taken into account. The manipulator model was, as previously, linearized. In Ref. [62] for a given manipulator model a kinematic as well as a dynamic measure of controllability were proposed. In the paper [31] Kharlamov summarized the state and pointed out development perspectives of classical tasks of rigid body dynamics based on the theoretical mechanics literature survey (the work contained 90 references).

In order to explain the problem of derivation of equations of motion also other scientists works should be mentioned. Multibody dynamics was considered in many books and journals. For example Wittenburg [64] presented not only body kinematics and dynamics but also classical problems both of single rigid body and multibody systems. Later, in 1990, the state of multibody system dynamics was described in the Multibody Systems Handbook [54]. The book contains formalisms which allow one to generate automatically the equations of motion. Moreover, the software for multibody systems was compared. The roots, the state-of-the-art and perspectives of multibody system dynamics were presented in the paper written by Schiehlen in 1997 [55]. The author divided the existing formalisms into two groups, namely the numerical and the symbolical ones. The numerical equations are generated for each time instant whereas the symbolical can be obtained by various formulas. The numerical approach was given e.g. by Vibet [59] and the symbolical approach e.g. by Cui et al. [15]. After that, actual problems and directions in the future development of rigid body dynamics were discussed in [35]. The paper was written based on materials of the Round Table 'Rigid Body Dynamics: Past, Present, Future' which was held in Donetsk in 1999.

In other papers equations of motion for multibody systems were also considered. Equations of motion in terms of quasi-variables are well known in the analytical mechanics. Papastavridis [43] presented derivation of various forms of the transitivity equations concerning the rigid body dynamics. The transitivity equations enabled transition from Lagrange's equations to the Eulerian equations. Moreover, the equations can be expressed using quasi-variables. A direct vectorial derivation of the Boltzmann–Hamel equations of motion of discrete mechanical system was presented in [44]. In those equations quasi-velocities were used too. An overview of the fundamental principles of analytical mechanics both for holonomic and nonholonomic case was given by Papastavridis in 1998 [45]. An interesting comparison between analytical techniques has been made by Baruh [3]. In the paper similarities and differences between the D'Alembert Principle and the Lagrangian Mechanics were discussed. As was shown in [42] equations of motion of multibody mechanisms can be obtained based on canonical momenta. Using the proposed formalism Hamilton's equations were established. However, the Hamilton's equations were different than the equations of motion expressed in terms of quasi-velocities considered in our paper.

3 Dynamical equations of motion in classical form and in terms of quasi-velocities

Consider the manipulator as an unconstrained system. Equations of motion in terms of generalized coordinates and their time derivatives [10,12,14,56,57] are as follows:

$$\mathbf{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \mathbf{G}(\boldsymbol{\theta}) = \boldsymbol{\tau} \tag{12}$$

where meaning of the symbols is given in *Nomenclature*. Equation (12) is coupled and nonlinear.

In classical mechanics quasi-velocities depend only on kinematical quantities. Three of described quasi-velocities, presented in Subsects. 3.1–3.3, belong to this group. Equations of motion arising from using such quasi-velocities do not lead to diagonal mass matrix.

One can use different kind of velocities, namely the inertial quasi-velocities, which diagonalize the mass matrix of the system. They differ from the classical quasi-velocities because contain both kinematical and dynamical quantities.

3.1 Dynamical equations of motion using kinematical quasi-velocities

Mechanical systems are sometimes described by dynamical equations of motion and kinematical relationship using quasi-velocities [13]. One can find many references referring to this kind of velocities both for unconstrained and constrained systems [2–8,18,39]. The usefulness of quasi-velocities and methods that employ them consists in elegance of formulation and unification of the form of the Lagrange equations. Very often they are more natural than generalized velocities. In some cases the obtained equations can be simpler than Lagrange equations. It is worth of notice that three of four formulations considered in this paper, namely NQV, UQV and EQV are based on Lagrange equations. If derivation of equations which contain the generalized coordinates and their time derivatives is very laborious and complicated, it is better to introduce linear combinations of generalized velocities. A review of the role of quasi-coordinates and quasi-velocities in this matter contains e.g. Ref. [18]. Discussion concerning various approaches of formulations containing quasi-velocities, their advantages and disadvantages can be found in [3].

In order to compare, in further part of this section and in Sect. 3, various quasi-velocities recall firstly the equations in terms of quasi-velocities. Consider an open chain link manipulator described in terms of the classical quasi-velocities [2,4,6,39]:

$$\mathbf{M}_{\mathrm{v}}(\boldsymbol{\theta})\dot{\mathbf{v}}_{\mathrm{q}} = \mathbf{h}^{*}(\boldsymbol{\theta}, \mathbf{v}_{\mathrm{q}}, t) \tag{13}$$

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\mathcal{A}}(\boldsymbol{\theta})\mathbf{v}_{\mathrm{q}} \tag{14}$$

where $\mathbf{v}_{\mathrm{q}}$, $\dot{\mathbf{v}}_{\mathrm{q}}$ denote the $(\mathcal{N} \times 1)$ vector of quasi-velocities and its time derivative, respectively, the expression $\mathbf{h}^{*}(\boldsymbol{\theta}, \mathbf{v}_{\mathrm{q}}, t) = \mathbf{Q}_{\mathrm{v}} - \mathbf{C}_{\mathrm{v}}(\boldsymbol{\theta}, \mathbf{v}_{\mathrm{q}}) - \mathbf{G}_{\mathrm{v}}(\boldsymbol{\theta})$ denotes $(\mathcal{N} \times 1)$ vector which represents the sum of applied forces, gyroscopic terms, centrifugal forces, Coriolis and gravitational effects, $\boldsymbol{\mathcal{A}}(\boldsymbol{\theta})$ is $(\mathcal{N} \times \mathcal{N})$ dimensional configuration dependent transformation matrix and t denotes time variable. Notice, however that Eqs. (13) and (14) can lead to simpler form of motion description using the generalized coordinates and generalized velocities, but the manipulator mass matrix $\mathbf{M}_{\mathrm{v}}(\boldsymbol{\theta})$ is, in general, not diagonal.

Observation 1 Comparing Eqs. (13) and (14) with Eqs. (5) and (8) one can notice that $\mathbf{M}_{\mathrm{p}}(\boldsymbol{\theta}) = \mathbf{M}_{\mathrm{v}}(\boldsymbol{\theta})$, $\mathbf{C}_{\mathrm{p}}(\boldsymbol{\theta}, \mathbf{p})\mathbf{p} = \mathbf{C}_{\mathrm{v}}(\boldsymbol{\theta}, \mathbf{v}_{\mathrm{q}})$, $\mathbf{G}_{\mathrm{p}}(\boldsymbol{\theta}) = \mathbf{G}_{v}(\boldsymbol{\theta})$, $\mathbf{Q}_{\mathrm{p}} = \mathbf{Q}_{\mathrm{v}}$, $\mathbf{p} = \mathbf{v}_{\mathrm{q}}$ and $\mathbf{Z}(\boldsymbol{\theta}) = \boldsymbol{\mathcal{A}}(\boldsymbol{\theta})$.

In a robotic literature some particular meaning have two approaches using quasi-velocities: Kane's equations and a formulation containing the spatial velocity vector. They are considered in the next two subsections.

3.2 Dynamical equations of motion in terms of generalized speeds

This formulation is known from works of Kane and Levinson (e.g. [29]) and is called Kane's equations. Its importance results from the fact that equations of motion in terms of GVC [40] uses Kane's description. Generalized speeds $\mathbf{u}_r$ are quantities which are associated with the motion of a system, rather than only with its configuration. They are functions of generalized coordinates and their time derivatives. Generalized speeds were also considered by authors of references [6,40]. What is an advantage of using equations of motion in this form? The Newton–Euler or the Lagrange's formulation are based on complicated mathematical description. A result equation containing only joint positions, velocities and accelerations is obtained. Kane's equations of motion are expressed sometimes in terms of other velocities as the generalized velocities. The advantage of this formulation relies on fact that the manipulator equations of motion can be presented in a particularly simple form. However, instead of the quasi-coordinates the generalized coordinates are employed. The obtained equations are expressed in terms of generalized speeds together with the generalized coordinates. The derivation of Kane's equations is based on the D'Alembert's principle.

Here we recall briefly this formulation assuming notation from [40]. According to D'Alembert's principle the sum of generalized active forces $\mathbf{F}_r$ and generalized inertia forces $\mathbf{F}_r^{*}$ equals zero. Therefore one can write:

$$\mathbf{F}_r + \mathbf{F}_r^{*} = \mathbf{0} \tag{15}$$

$$\dot{\boldsymbol{\theta}} = \boldsymbol{\mathcal{A}}(\boldsymbol{\theta})\mathbf{u}_r \tag{16}$$

where $\mathbf{u}_r$ denotes the $(\mathcal{N} \times 1)$ vector of generalized speeds, $\boldsymbol{\mathcal{A}}(\boldsymbol{\theta})$–the $(\mathcal{N} \times \mathcal{N})$ dimensional rate transformation matrix. Vectors $\mathbf{F}_r$ and $\mathbf{F}_r^*$ are described by the following equations:

$$\mathbf{F}_r = \sum_{k=1}^{\mathcal{N}} \left(\mathbf{V}_k^{\mathrm{T}} \mathbf{f}_k + \boldsymbol{\Gamma}_k^{\mathrm{T}} \boldsymbol{\tau}_k\right) \tag{17}$$

$$\mathbf{F}_r^* = -\sum_{k=1}^{\mathcal{N}} \left(m_k \mathbf{V}_k^{\mathrm{T}} \mathbf{a}_k + \boldsymbol{\Gamma}_k^{\mathrm{T}} \dot{\mathbf{H}}_k\right) \tag{18}$$

where $\mathbf{V}_k$ and $\boldsymbol{\Gamma}_k$ denote the $(3 \times \mathcal{N})$ partial linear velocity and partial angular matrices, respectively:

$$\mathbf{V}_k = \frac{\partial \mathbf{v}_k}{\partial \mathbf{u}_r^{\mathrm{T}}}, \quad \boldsymbol{\Gamma}_k = \frac{\partial \boldsymbol{\omega}_k}{\partial \mathbf{u}_k^{\mathrm{T}}}. \tag{19}$$

Other symbols denote: $\partial \mathbf{v}_k$, $\partial \boldsymbol{\omega}_k$ the partial linear velocity and the partial angular velocity, respectively, $\mathbf{f}_k$ is the resultant active force acting at the mass center of the kth body, $\boldsymbol{\tau}_k$ is the resultant moment, m_k is kth mass, $\mathbf{a}_k$ is (3×1) the mass center acceleration of the kth body and $\dot{\mathbf{H}}_k$ is the time rate of change of angular momentum of the kth body with respect to the inertial reference frame (this frame is fixed in the base of the manipulator).

In Eq. (15) generalized speeds are involved in quantities $\mathbf{V}_k$ and $\boldsymbol{\Gamma}_k$. Its solution leads to the form expressed by Eq. (13).

As can be noticed introduction of generalized speeds does not diagonalize the manipulator mass matrix. For simulation purposes this method needs inversion of the mass matrix, dynamical equations of motion are still second-order and coupled in the highest derivative term.

Observation 2 Notice that comparison of Eqs. (15) and (16) with Eqs. (5) and (8) leads to the following: $\mathbf{M}_{\mathrm{p}}(\boldsymbol{\theta})$, $\mathbf{C}_{\mathrm{p}}(\boldsymbol{\theta}, \mathbf{p})\mathbf{p}$ and $\mathbf{G}_{\mathrm{p}}(\boldsymbol{\theta})$ are contained in $\mathbf{F}_r^*$ whereas $\mathbf{Q}_{\mathrm{p}}$ is related to $\mathbf{F}_r$, $\mathbf{p} = \mathbf{u}_r$ and $\mathbf{Z}(\boldsymbol{\theta}) = \boldsymbol{\mathcal{A}}(\boldsymbol{\theta})$.

3.3 Dynamical equations of motion using twist vector

Formulations NQV and UQV use operators coming from the spatial algebra. Jain and Rodriguez method [26] leads to decomposition the manipulator mass matrix into three matrices. Similar decomposition (based also on the spatial velocity) was reported by Saha [48] and developed in reference [49]. The author proposed simple rules of the Gaussian elimination in order to decompose the mass matrix. As a result the inverse and the forward dynamics problems can be solved too. Saha defined the six-dimensional vector of twist which is equivalent to the spatial velocity vector $\mathbf{V}(k) = [\boldsymbol{\omega}_k^{\mathrm{T}} \;\; \mathbf{v}_k^{\mathrm{T}}]^{\mathrm{T}}$ for the kth link of manipulator [46,47]. For the manipulator consisting of $\mathcal{N}$ links composed by $\mathcal{N}$ revolute pairs, the $6\mathcal{N}$-dimensional generalized twist $\mathbf{t}_{\mathrm{V}}$ is defined as: $\mathbf{t}_{\mathrm{V}} \equiv [\mathbf{V}_1^{\mathrm{T}}, \ldots, \mathbf{V}_{\mathcal{N}}^{\mathrm{T}}]^{\mathrm{T}}$.

Based on the twist vector Saha introduced the following equations:

$$\mathbf{I}_{\mathrm{G}} \ddot{\boldsymbol{\theta}} = \hat{\boldsymbol{\tau}}, \quad \mathbf{I}_{\mathrm{G}} = \mathbf{A}_{\mathrm{G}}^{\mathrm{T}} \mathbf{M}_{\mathrm{G}} \mathbf{A}_{\mathrm{G}} \tag{20}$$

$$\mathbf{t}_{\mathrm{V}} = \mathbf{A}_{\mathrm{G}} \dot{\boldsymbol{\theta}} \tag{21}$$

where $\mathbf{I}_{\mathrm{G}}$ is the $(\mathcal{N} \times \mathcal{N})$ generalized inertia matrix (GIM), $\mathbf{t}_{\mathrm{V}}$ is the $(6\mathcal{N} \times 1)$ dimensional generalized twist vector, $\hat{\boldsymbol{\tau}} = \boldsymbol{\tau} - \mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) - \mathbf{G}(\boldsymbol{\theta})$, $\mathbf{A}_{\mathrm{G}}$ is the $(6\mathcal{N} \times \mathcal{N})$ natural orthogonal complement matrix and $\mathbf{M}_{\mathrm{G}}$ is the $(6\mathcal{N} \times 6\mathcal{N})$ generalized mass matrix defined as:

$$\mathbf{M}_{\mathrm{G}} \equiv \mathrm{diag}[\mathbf{M}_{\mathrm{G}1}, \ldots, \mathbf{M}_{\mathrm{G}\mathcal{N}}]. \tag{22}$$

Symbol $\mathbf{M}_{\mathrm{G}k}$ denotes extended mass of the kth link with respect to its mass center, which refers to the spatial inertia matrix (meaning of the symbols is explained in *Nomenclature*):

$$\mathbf{M}_{\mathrm{G}k} = \begin{bmatrix} \boldsymbol{\mathcal{I}}_k & \mathbf{0} \\ \mathbf{0} & m_k \mathbf{I} \end{bmatrix}. \tag{23}$$

Comparing Eqs. (20) and (12) one can see that the Coriolis and the gravitational terms are shifted into the matrix $\mathbf{I}_G$. On the contrary Eq. (21) shows only relationship between vectors $\mathbf{t}_V$ and $\dot{\boldsymbol{\theta}}$. Therefore $\mathbf{t}_V$ can be regarded as quasi-velocity. However, Eq. (20) is not expressed in terms of the quasi-velocity vector as was given by Eq. (13).

In spite of the diagonalized Lagrangian robot dynamics described in reference [26], which uses also the spatial velocity vector for obtaining the normalized and unnormalized quasi-velocities, the resultant equations of motion are different. Saha [48] concentrated on building the generalized inertia matrix (GIM), whereas Jain and Rodriguez [26] proposed the inertial quasi-velocities in order to diagonalize the mass matrix which appears in Eq. (12). The next difference relies on the fact that the twist vector contains only kinematical variables. On the contrary the inertial quasi-velocities depend also on dynamical quantities.

Decomposition of the GIM leads to the generalized acceleration vector $\ddot{\boldsymbol{\theta}}$. In Eq. (20) matrix $\mathbf{I}_G$ consists of the matrix $\mathbf{A}_G$ which depends on kinematical parameters of the manipulator and matrix $\mathbf{M}_G$ which contains its dynamical parameters (masses, inertias). As a consequence the matrix $\mathbf{A}_G$ is not diagonal. Therefore the problem of manipulator mass matrix inversion can be solved for simulation purposes the first-order decoupled equations of motion cannot be obtained.

Observation 3 Comparison of Equations. (20) and (21) with Eqs. (5) and (8) leads to conclusion that instead of the matrix $\mathbf{M}_p(\boldsymbol{\theta})$ the matrix $\mathbf{I}_G$ is used. The matrices $\mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$ and $\mathbf{G}(\boldsymbol{\theta})$ and the vector $\boldsymbol{\tau}$ are the same as in the classical equations of motion. Besides the twist vector $\mathbf{t}_V$ cannot be understood in the same way as $\mathbf{p}$ because it represents a spatial quantity. The transformation matrix $\mathbf{A}_G$ is related to the matrix $\mathbf{Z}^{-1}(\boldsymbol{\theta})$.

3.4 Dynamical equations of motion in terms of normalized and unnormalized quasi-velocities

The authors of reference [26] have proposed, instead of transformation in configuration space, a diagonalizing transformation in the velocity space. They have presented a diagonalized equations of motion, which are called diagonalized Lagrangian robot dynamics for two cases: normalized and unnormalized.

For serial manipulators decoupled equations of motion in terms of *normalized quasi-velocities* are as follows:

$$\dot{\boldsymbol{\nu}} + \mathbf{C}(\boldsymbol{\theta}, \boldsymbol{\nu}) + \mathbf{G}_\nu(\boldsymbol{\theta}) = \boldsymbol{\epsilon} \tag{24}$$

$$\boldsymbol{\nu} = \mathbf{m}^{\mathrm{T}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}. \tag{25}$$

The first equation represents dynamical equation of motion and the second the kinematical relationship between joint velocities vector $\dot{\boldsymbol{\theta}}$ and normalized quasi-velocity vector. Normalized quasi-velocities (total joint rates) $\boldsymbol{\nu}$ are related to the joint-angle velocities $\dot{\boldsymbol{\theta}}$ by the configuration-dependent linear transformation operator $\mathbf{m}^{\mathrm{T}}(\boldsymbol{\theta})$. It is invertible and depends also on elements of the mass matrix $\mathbf{M}(\boldsymbol{\theta})$ which is presented in the manipulator equations of motion. The transformation matrix $\mathbf{m}^{\mathrm{T}}(\boldsymbol{\theta})$ is an upper triangular matrix. Linear transformation can be expressed further by means of spatial algebra operators which are calculated in a recursive way (compare for details work [47]).

From a physical point of view variables $\boldsymbol{\nu}$ are time-derivatives of quasi-coordinates known in analytical dynamics. Quasi-coordinates as functions of generalized coordinates do not exist physically because mathematical relationship between quasi-coordinates and generalized coordinates are not integrable. Because of that configuration variables $\boldsymbol{\theta}$ are kept together with quasi-velocities. Time derivatives of normalized quasi-velocities depend on joint-angle velocities and joint-angle accelerations.

Between vectors and matrices presented in Eq. (12) and quasi-variables the following relationships are true [26]:

$$\mathbf{M}(\boldsymbol{\theta}) = \mathbf{m}(\boldsymbol{\theta})\mathbf{m}^{\mathrm{T}}(\boldsymbol{\theta}), \tag{26}$$

$$\boldsymbol{\epsilon} = \mathbf{m}^{-1}(\boldsymbol{\theta})\boldsymbol{\tau}, \tag{27}$$

$$\mathbf{C}(\boldsymbol{\theta}, \boldsymbol{\nu}) = \mathbf{m}^{-1}(\boldsymbol{\theta})\mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) - \dot{\mathbf{m}}^{\mathrm{T}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}. \tag{28}$$

Notice, that kinetic energy of the manipulator is expressed as:

$$\mathcal{K}(\boldsymbol{\theta}, \boldsymbol{\nu}) = \tfrac{1}{2}\dot{\boldsymbol{\theta}}^{\mathrm{T}}\mathbf{m}(\boldsymbol{\theta})\mathbf{m}^{\mathrm{T}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} = \tfrac{1}{2}\boldsymbol{\nu}^{\mathrm{T}}\boldsymbol{\nu}. \tag{29}$$

Therefore using vector $\boldsymbol{\nu}$ an alternative but simpler as in classical case expression for the kinetic energy can be obtained.

Calculating the time derivative of $\boldsymbol{\nu}$ given by Eq. (25) i.e. $\dot{\boldsymbol{\nu}} = \dot{\mathbf{m}}^{\mathrm{T}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} + \mathbf{m}^{\mathrm{T}}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}}$ we can transform Eq. (12) as follows:

$$\mathbf{m}(\boldsymbol{\theta})\mathbf{m}^{\mathrm{T}}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \mathbf{G}(\boldsymbol{\theta}) = \boldsymbol{\tau} \tag{30}$$

$$\mathbf{m}^{\mathrm{T}}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \mathbf{m}^{-1}(\boldsymbol{\theta})\mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) + \mathbf{m}^{-1}(\boldsymbol{\theta})\mathbf{G}(\boldsymbol{\theta}) = \mathbf{m}^{-1}(\boldsymbol{\theta})\boldsymbol{\tau} \tag{31}$$

$$\dot{\mathbf{m}}^{\mathrm{T}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} + \mathbf{m}^{\mathrm{T}}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \mathbf{m}^{-1}(\boldsymbol{\theta})\mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) - \dot{\mathbf{m}}^{\mathrm{T}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} + \mathbf{m}^{-1}(\boldsymbol{\theta})\mathbf{G}(\boldsymbol{\theta}) = \mathbf{m}^{-1}(\boldsymbol{\theta})\boldsymbol{\tau}. \tag{32}$$

Comparing terms in the last equation with notation in Eqs. (26)–(28) one can obtain dynamical equation of motion in the form of Eq. (24) which natural splitting between momentum differential equations and kinematic differential equations (24) and (25), respectively.

Observation 4 Equations (24) and (25) can be compared with Eqs. (5) and (9) and one can get $\mathbf{M}_{\mathrm{p}}(\boldsymbol{\theta}) = \mathbf{I}$ (the identity matrix), $\mathbf{C}_{\mathrm{p}}(\boldsymbol{\theta}, \mathbf{p})\mathbf{p} = \mathbf{C}(\boldsymbol{\theta}, \boldsymbol{\nu})$, $\mathbf{G}_{\mathrm{p}}(\boldsymbol{\theta}) = \mathbf{G}_{\nu}(\boldsymbol{\theta})$, $\mathbf{Q}_{\mathrm{p}} = \boldsymbol{\epsilon}$, $\mathbf{p} = \boldsymbol{\nu}$ and $\mathbf{Y}(\boldsymbol{\theta}) = \mathbf{m}^{\mathrm{T}}(\boldsymbol{\theta})$.
Unnormalized quasi-velocities are related to the normalized quasi-velocities but the transformation in this case does not depend on mass elements expressed by articulated inertia about joint axes $\mathbf{D}(\boldsymbol{\theta})$, i.e. [26]:

$$\boldsymbol{\xi} = \mathbf{D}^{-1/2}\boldsymbol{\nu} = \mathbf{D}^{-1/2}\mathbf{m}^{\mathrm{T}}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}. \tag{33}$$

Here the manipulator mass matrix is subdivided into three matrices: $\mathbf{M}(\boldsymbol{\theta}) = \mathbf{LDL}^{\mathrm{T}}$, where $\mathbf{L} \in R^{\mathcal{N}\times\mathcal{N}}$ is a lower-triangular matrix which depends on spatial operators defined in *Nomenclature*, i.e. $\mathbf{L} = [\mathbf{I} + \mathbf{H}\boldsymbol{\phi}\mathbf{K}]$. From Eqs. (29) and (33) it arises that $\mathbf{D}$ is a diagonal matrix. The articulated body inertia about joint axes follows from the fact that a manipulator consists of connected links (as a whole it is not a rigid body). Each element of matrix $\mathbf{D}$ represents inertia of all links closer to the tip of the manipulator. Matrix $\mathbf{M}(\boldsymbol{\theta})$ is a result of decomposition of the articulated inertia of the entire system of $\mathcal{N}$ bodies. In terms of UQV dynamical equations of motion have the following form:

$$\mathbf{D}\dot{\boldsymbol{\xi}} + \mathbf{C}(\boldsymbol{\theta}, \boldsymbol{\xi}) + \mathbf{G}_{\xi}(\boldsymbol{\theta}) = \boldsymbol{\kappa} \tag{34}$$

and the velocity transformation equation in the form of Eq. (33).

Slightly different results are received using UQV formulation. Notice that the kinetic energy in terms of UQV represents a sum which consists of independent components:

$$\mathcal{K}(\boldsymbol{\theta}, \boldsymbol{\xi}) = \frac{1}{2}\boldsymbol{\xi}^{\mathrm{T}}\mathbf{D}\boldsymbol{\xi} \tag{35}$$

Each component is associated with the kth quasi-velocity $\left(\mathcal{K}(\boldsymbol{\theta}, \boldsymbol{\xi}) = 1/2\sum_{k=1}^{\mathcal{N}} D_k\xi_k^2\right)$.

Relationships between normalized and unnormalized quantities are as follows [26]:

$$\boldsymbol{\kappa} = \mathbf{D}^{1/2}\boldsymbol{\epsilon}, \tag{36}$$

$$\mathbf{C}(\boldsymbol{\theta}, \boldsymbol{\xi}) = \mathbf{D}^{1/2}\mathbf{C}(\boldsymbol{\theta}, \boldsymbol{\nu}) - \mathbf{D}\frac{\mathrm{d}\mathbf{D}^{-1/2}}{\mathrm{d}t}\boldsymbol{\nu}. \tag{37}$$

Time derivative of $\boldsymbol{\xi}$ given by Eq. (33) is $\dot{\boldsymbol{\xi}} = \dot{\mathbf{D}}^{-1/2}\boldsymbol{\nu} + \mathbf{D}^{-1/2}\dot{\boldsymbol{\nu}}$ and therefore one can write after calculation following from (24) that:

$$\mathbf{D}^{1/2}\dot{\boldsymbol{\nu}} + \mathbf{D}^{1/2}\mathbf{C}(\boldsymbol{\theta}, \boldsymbol{\nu}) + \mathbf{D}^{1/2}\mathbf{G}_{\nu}(\boldsymbol{\theta}) = \mathbf{D}^{1/2}\boldsymbol{\epsilon}. \tag{38}$$

Denoting by $\mathbf{G}_{\xi}(\boldsymbol{\theta}) = \mathbf{D}^{1/2}\mathbf{G}_{\nu}(\boldsymbol{\theta})$ Eq. (38) leads directly to Eq. (34). Natural splitting between momentum differential equations and kinematic differential equations in this case can be obtained using Eqs. (34) and (33). The transformation matrix $\mathbf{D}^{-1/2}\mathbf{m}^{\mathrm{T}}(\boldsymbol{\theta})$ is here also upper triangular matrix.

Observation 5 Let us compare Eqs. (34) and (33) with Eqs. (5) and (9). Using the UQV we have $\mathbf{M}_{\mathrm{p}}(\boldsymbol{\theta}) = \mathbf{D}$ (a diagonal matrix), $\mathbf{C}_{\mathrm{p}}(\boldsymbol{\theta}, \mathbf{p})\mathbf{p} = \mathbf{C}(\boldsymbol{\theta}, \boldsymbol{\xi})$, $\mathbf{G}_{\mathrm{p}}(\boldsymbol{\theta}) = \mathbf{G}_{\xi}(\boldsymbol{\theta})$, $\mathbf{Q}_{\mathrm{p}} = \boldsymbol{\kappa}$, $\mathbf{p} = \boldsymbol{\xi}$ and $\mathbf{Y}(\boldsymbol{\theta}) = \mathbf{D}^{-1/2}\mathbf{m}^{\mathrm{T}}(\boldsymbol{\theta})$.

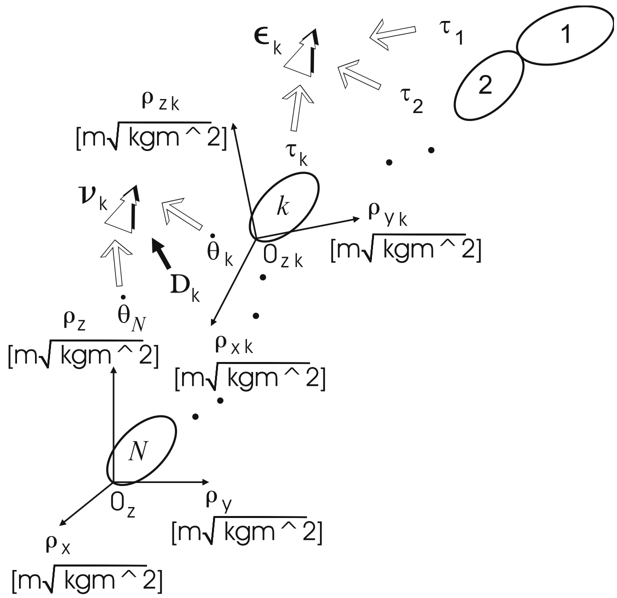

Fig. 1 Interpretation of quasi-variables for normalised quasi velocities (NQV) formulation

3.4.1 Remarks on physical interpretation of normalized quasi-variables

In Fig. 1 one can see quasi-variables for the kth joint. One can notice that components of $\boldsymbol{\nu}$ vector are time derivatives of certain quasi-velocities which are not generalized coordinates. Analytical form of these quasi-coordinates is not known. Quasi-velocities are functions of kinematical and dynamical parameters. Note that generalized coordinates in Eq. (24) are present (instead of using quasi-coordinates) together with new quasi-velocities. Besides that, normalized quasi-velocities depend on linear and angular joint velocities.

Notice that every component $D_k^{-1/2}\nu_k = \dot{\theta}_k + \delta_k$ (here D_k and ν_k denote the kth element of diagonal matrix $\mathbf{D}$ and the kth element of vector ν, respectively) is a sum of the kth joint velocity $\dot{\theta}_k$ and additional term $\delta_k = \sum_{i=k+1}^{\mathcal{N}} c_{ki}\dot{\theta}_i$ (links and joints are numbered in increasing order from the tip to the base of a manipulator). The last component reflects influence of all links which are located closer to the base starting from the $(k+1)$th joint. Each component of the additional term equals to a relative velocity crossing a joint multiplied by a weight coefficient. It depends on link mass parameters and kinematical parameters. In addition it is unitless and changes with time during motion of the links. In order to obtain ν_k quantity we normalize it by $D_k^{-1/2}$ coefficient. The element D_k called an articulated joint inertia contains both kinematical and dynamical parameters of the manipulator. These parameters are transformed from one link to next starting from its tip. Because of that D_k represents masses and inertias which are shifted recursively from the tip to the base. Therefore D_k reflects all links starting from the farthest distant link from the base to the kth link (this part of the manipulator is called a reduced part – compare reference [26]). At the same time one can notice that elements of matrix $\mathbf{m}(\boldsymbol{\theta})$ depend on inertial parameters of reduced part of the manipulator and their joint coordinates. Summarizing observe that ν_k component represents a resultant velocity which depends on kinematical and dynamical parameters of this part of the manipulator which is closer to the base (starting from the kth joint) and, additionally, on dynamical parameters of the reduced part of the manipulator in an instantaneous local coordinate frame. Quasi-moment ϵ_k depends only on the reduced part of the manipulator. As a consequence ν_k depends on reduced manipulator inertia and all generalized velocities between the base and the kth joint.

3.4.2 Remarks on physical interpretation of unnormalized quasi-variables

Figure 2 shows quasi-variables for the kth joint of the manipulator. One can notice that components of vector $\boldsymbol{\xi}$ are time derivatives of certain quasi-velocities which are not generalized coordinates. These quasi-velocities are functions of kinematical and dynamical parameters. Note, also that unnormalized quasi-velocities are similar

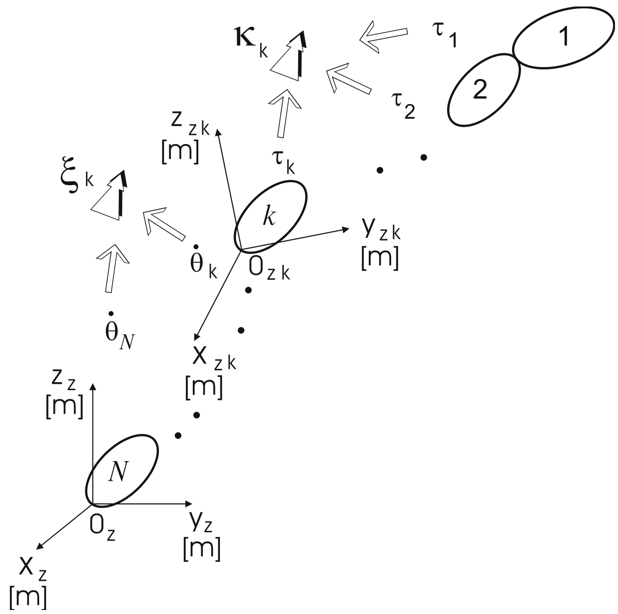

Fig. 2 Interpretation of quasi-variables for unnormalised quasi velocities (UQV) formulation

as normalized except the $\mathcal{N}$th joint (compare Figs. 1 and 2). UQV are calculated assuming the same physical units as generalized velocities whereas units of NQV contain additionally square root of the manipulator inertia.

Similarly, as previously we conclude that every component $\xi_k = \dot{\theta}_k + \delta_k$. One can easy notice that variables ξ_k have units of joint velocities. Quasi-moment κ_k depends only on the reduced part of the manipulator however it is not normalized by $D_k^{-1/2}$. Because of that quasi-moments have physical units of joint moments.

3.5 Eigenstructure quasi-velocity formulation for manipulators

The proposition of EQV given in [27] describes a very general class of constrained and unconstrained dynamical systems. This method is based on Lagrangian formulation. There exists then a similarity to the earlier presented methods introduced by Jain and Rodriguez. But the philosophy of Junkins and Schaub formulation is quite different. Diagonalization is not obtained in a recursive form. One has to calculate eigenvalues and eigenvectors for the whole system, and next, using mathematical relationships one has to obtain first-order quasi-velocity Lagrange's equations of motion. The method is based on decomposition of a mass matrix into the eigenvector matrix, its transposition and the eigenvalue matrix. By introducing eigenfactor quasi-coordinate velocities one can notice that in fact mass matrix is divided only into two matrices, similar as using normalized quasi-velocities. The new formulation induces however numerical problems with increasing number of degrees of freedom (dof). These problems were pointed out and solved in [28].

Because the second-order differential equations of motion are nontrivial to solve, Junkins and Schaub [27, 53] proposed the spectral decomposition of the mass matrix. As a result they obtained one first order dynamical differential equation and one first-order kinematical differential equation of motion which can be written in the following form:

$$\dot{\boldsymbol{\eta}} + \mathbf{S}^{-1}(\boldsymbol{\Omega}\mathbf{S} + \dot{\mathbf{S}})\boldsymbol{\eta} - \mathbf{S}^{-1}\mathbf{C}_{\mathrm{e}}\left(\frac{1}{2}\dot{\boldsymbol{\theta}}^{\mathrm{T}}\mathbf{M}_{\theta}\dot{\boldsymbol{\theta}}\right) = \mathbf{S}^{-1}\mathbf{C}_{\mathrm{e}}(\mathbf{Q} - \mathbf{G}(\boldsymbol{\theta})) \tag{39}$$

$$\dot{\boldsymbol{\theta}} = \mathbf{C}_{\mathrm{e}}^{\mathrm{T}}\mathbf{S}^{-1}\boldsymbol{\eta}, \tag{40}$$

with relationships used for spectral decomposition:

$$\begin{aligned}
&\mathbf{M} = \mathbf{C}_{\mathrm{e}}^{\mathrm{T}}\mathbf{D}_{\mathrm{e}}\mathbf{C}_{\mathrm{e}}, \quad \mathbf{C}_{\mathrm{e}}\mathbf{C}_{\mathrm{e}}^{\mathrm{T}} = \mathbf{I}, \quad \mathbf{C}_{\mathrm{e}} = \mathbf{E}^{\mathrm{T}},\\
&\mathbf{S} = \sqrt{\mathbf{D}_{\mathrm{e}}} = \mathrm{diag}(+\sqrt{\lambda_i}), \quad \mathbf{D}_{\mathrm{e}} = \mathrm{diag}(\lambda_i), \quad \mathbf{D}_{\mathrm{e}} = \mathbf{S}^{\mathrm{T}}\mathbf{S},\\
&\dot{\mathbf{S}} = \frac{1}{2}\boldsymbol{\Gamma}\mathbf{S}^{-1}, \quad \Gamma = \mathrm{diag}(\mu_{ii}), \quad \dot{\lambda}_i = \mu_{ii},
\end{aligned} \tag{41}$$

where $\dot{\lambda}_i$ represents time derivative of the ith eigenvalue, $\mathbf{E}$ is the orthogonal real eigenvector matrix, $\mathbf{D}_e$ is the eigenvalues matrix, and μ_{ii} are terms dependent on eigenvector derivatives of matrix $\mathbf{M}(\boldsymbol{\theta})$.

In this formulation the mass matrix $\mathbf{M}$ is decomposed into three matrices and depends on eigenvalues hidden in matrix $\mathbf{D}_e$ and eigenvectors hidden in matrix $\mathbf{C}_e$. The mass matrix factorization leads to:

$$\mathbf{M} = \mathbf{C}_e^T \mathbf{S}^T \mathbf{S} \mathbf{C}_e. \tag{42}$$

In more compact form these equations can be written as:

$$\dot{\boldsymbol{\eta}} + \mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \boldsymbol{\eta}) = \boldsymbol{\varepsilon} \tag{43}$$

$$\dot{\boldsymbol{\theta}} = \mathbf{W}\boldsymbol{\eta} \tag{44}$$

where $\mathbf{W} = \mathbf{C}_e^T \mathbf{S}^{-1}$ and

$$\mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \boldsymbol{\eta}) = \mathbf{S}^{-1}\left(\boldsymbol{\Omega}_e \mathbf{S} + \dot{\mathbf{S}}\right)\boldsymbol{\eta} - \mathbf{S}^{-1}\mathbf{C}_e\left(\frac{1}{2}\dot{\boldsymbol{\theta}}^T \frac{\partial \mathbf{M}}{\partial \boldsymbol{\theta}}\dot{\boldsymbol{\theta}}\right), \tag{45}$$

$$\boldsymbol{\varepsilon} = \mathbf{S}^{-1}\mathbf{C}_e(\mathbf{Q} - \mathbf{G}(\boldsymbol{\theta})). \tag{46}$$

$\mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \boldsymbol{\eta})$ and $\boldsymbol{\varepsilon}$ denote vector of a new Coriolis forces and vector of quasi-moments, respectively. The eigen-factor quasi-coordinate velocity vector is expressed as follows:

$$\boldsymbol{\eta} = \mathbf{W}^{-1}\dot{\boldsymbol{\theta}} = \mathbf{S}\mathbf{C}_e\dot{\boldsymbol{\theta}}. \tag{47}$$

In work [53] Eq. (39) is called the unconstrained Boltzmann–Hamel equation with the EQV vector $\boldsymbol{\eta}$. The kinetic energy is described here in the following simpler form [compare Eqs. (42] and (47)):

$$\mathcal{K}(\boldsymbol{\theta}, \boldsymbol{\eta}) = \frac{1}{2}\dot{\boldsymbol{\theta}}^T \mathbf{C}_e^T \mathbf{S}^T \mathbf{S} \mathbf{C}_e \dot{\boldsymbol{\theta}} = \frac{1}{2}\boldsymbol{\eta}^T \boldsymbol{\eta}. \tag{48}$$

New formulation described above replaces the second-order differential Lagrange's equations of motion (12) by two first differential equations, namely Eqs. (39) and (40). Notice that Eq. (40) contains matrix $\mathbf{C}_e$ and square root of the matrix $\mathbf{D}_e$. Finally, one can get decomposition of mass matrix $\mathbf{M}$ into two matrices $(\mathbf{S}\mathbf{C}_e)^T$ and $\mathbf{S}\mathbf{C}_e$. The transformation matrix $\mathbf{C}_e^T \mathbf{S}^{-1}$ is here a fully populated matrix in contrast to two earlier considered transformation matrices (notice that the eigenvector matrix $\mathbf{C}_e$ is, in general, fully populated).

An equivalence between new description and Lagrange's equations of motion for manipulators can be proven as follows. Recalling Eq. (12) one can calculate time derivative of Eq. (42), i.e. $\dot{\mathbf{M}} = \mathrm{d}/\mathrm{d}t[(\mathbf{S}\mathbf{C}_e)^T \mathbf{S}\mathbf{C}_e]$. Next from Eq. (40) arises that the time derivative of $\dot{\boldsymbol{\theta}}$ is $\ddot{\boldsymbol{\theta}} = \mathrm{d}/\mathrm{d}t[(\mathbf{C}_e^T \mathbf{S}^{-1})\boldsymbol{\eta}]$. In order to obtain the term $\mathbf{M}\ddot{\boldsymbol{\theta}} + \dot{\mathbf{M}}\dot{\boldsymbol{\theta}}$ which depend on matrices $\mathbf{S}$ and $\mathbf{C}_e$ one has to multiply both sides of Eq. (12) by $\mathbf{S}^{-1}\mathbf{C}_e$ (after using $\dot{\mathbf{C}}_e = -\boldsymbol{\Omega}\mathbf{C}_e$ [27]) and recall that $\mathbf{S} = \mathbf{S}^T$, $\dot{\mathbf{S}}^T = \dot{\mathbf{S}}$) which leads to Eq. (39), i.e. the first-order dynamical equations. The velocity transformation equation remains as before.

Observation 6 Recall Eqs. (43) and (44) and compare them with Eqs. (5) and (8) which results in $\mathbf{M}_p(\boldsymbol{\theta}) = \mathbf{I}$ (the identity matrix), $\mathbf{C}_p(\boldsymbol{\theta}, \mathbf{p})\mathbf{p} = \mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \boldsymbol{\eta})$, $\mathbf{G}_p(\boldsymbol{\theta}) = \mathbf{S}^{-1}\mathbf{C}_e\mathbf{G}(\boldsymbol{\theta})$, $\mathbf{Q}_p = \mathbf{S}^{-1}\mathbf{C}_e\mathbf{Q}$, $\mathbf{p} = \boldsymbol{\eta}$ and $\mathbf{Z}(\boldsymbol{\theta}) = \mathbf{W}$.

3.5.1 Remarks on physical interpretation of EQV and quasi-moments

Quasi-variables for the kth joint are shown in Fig. 3. In Refs. [27,53] authors pointed out similarity of new equations to Euler's rotational equations of motion. The authors of Ref. [27] understood new quasi-velocity η_k as the projection of the velocity vector $\dot{\boldsymbol{\theta}}$ onto the ith eigenvector and scaled by the ith eigenvalue square root. Notice, that this is not a physical interpretation because spectral decomposition of mass matrix $\mathbf{M}$ does not lead here to physical variables. Eigenvalues and eigenvectors have a mathematical sense which is different as compared to normalized and unnormalized quasi-velocities (recall that the last quasi-velocities have clear interpretation).

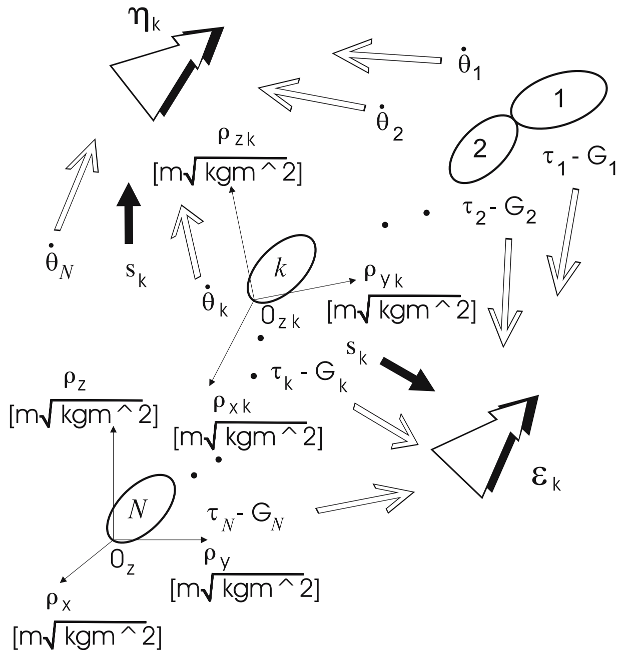

Fig. 3 Interpretation of quasi-variables for eigenfactor quasi-coordinate velocities (EQV) formulation

However, notice that every component η_k is a sum of all relative joint velocities $\dot{\theta}_k$ which arises from calculating eigenvalues and eigenvectors of the mass matrix $\mathbf{M}(\boldsymbol{\theta})$. These eigenvalues and eigenvectors reflect the influence of all links of manipulator and build weight coefficients in operator $\mathbf{SC}_\mathrm{e}$. These coefficients depend on link masses and on geometrical and kinematical parameters in real time instant and have physical units. They change during motion. If some elements in $\mathbf{SC}_\mathrm{e}$ matrix have big values, then the generalized velocity component η_k is strongly dependent on these joint velocities. One can say that the generalized velocity component η_k represents the resultant velocity in a local instantaneous frame of reference after including mass matrix elements. Similarly, quasi-moments ε_k reflects real moments at joints through operator $\mathbf{S}^{-1}\mathbf{C}_\mathrm{e}$ which involves eigenvalues and eigenvectors.

3.6 Decoupled equation of motion in terms of inertial generalized velocity components

The method described in [40] is related to dynamics of multibody systems. Loduha and Ravani used Kane's equations of motion instead of Lagrange's principle. Their proposition is called a congruency transformation. This means that for two given matrices $\mathbf{A}$ and $\mathbf{N}_\mathrm{A}$ there exists a transformation matrix $\boldsymbol{\Upsilon}$ which satisfies $\boldsymbol{\Upsilon}^\mathrm{T}\mathbf{A}\boldsymbol{\Upsilon} = \mathbf{N}_\mathrm{A}$. It is assumed that matrix $\mathbf{A} = -\mathbf{M}(\boldsymbol{\theta})$. Therefore, if two matrices $\mathbf{A}$ and $\mathbf{N}_\mathrm{A}$ are congruent, one can transform the first of them into the second using matrix $\boldsymbol{\Upsilon}$. The next problem relies on finding such transformation matrix which enables receiving the congruent matrix $\mathbf{N}_\mathrm{A}$. The matrix $\boldsymbol{\Upsilon}$ is composed [40] of m factors i.e. $\boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}_1\boldsymbol{\Upsilon}_2 \ldots \boldsymbol{\Upsilon}_m$, where m is the number of dof less one ($m = \mathcal{N} - 1$). One needs to calculate m factors because the resultant congruent matrix is upper diagonal with ones on the diagonal. Therefore for obtaining full transformation matrix it is sufficient to give only $\mathcal{N} - 1$ transformation components. It is assumed that the first matrix $\boldsymbol{\Upsilon}_1$ has only ones on the diagonal and nonzero elements in the first row (they are obtained by dividing elements of the first row of matrix $\mathbf{A}$ by element $-a_{11}$). The first congruency transformation gives matrix $\mathbf{A}^\mathrm{I} = \Upsilon_1^\mathrm{T}\mathbf{A}\boldsymbol{\Upsilon}_1$. The resulting matrix $\mathbf{A}^\mathrm{I}$ contains zero elements in row one and column one except the element a_{11} (other elements arise from earlier transformation). The second matrix $\boldsymbol{\Upsilon}_2$ is constructed in such a way, that one has to divide the second row of the matrix $\mathbf{A}^\mathrm{I}$ by element $-a_{22}^\mathrm{I}$ (this new matrix Υ_2 contains only ones on the diagonal and elements of the second row, whereas others are zeros). The next congruency matrix is equal $\mathbf{A}^\mathrm{II} = \boldsymbol{\Upsilon}_2^\mathrm{T}\mathbf{A}^\mathrm{I}\boldsymbol{\Upsilon}_2$. The sequence is repeated until a total of m transformations have been completed and the $\mathbf{A}$ matrix is converted into diagonal form. One additional requirement is that matrix $\boldsymbol{\Upsilon}$ has to be nonsingular. Here quasi-velocities are called generalized velocity components and understood as a linear combination of the first time derivatives of generalized coordinates.

Originally, the authors of [40] considered systems described by Kane's equations. Recall Kane's equation of motion in terms of the generalized velocity components [40] for $\mathcal{N}$ rigid bodies:

$$\boldsymbol{\Upsilon}^{\mathrm{T}}\mathbf{M}(\boldsymbol{\theta})\boldsymbol{\Upsilon}\dot{\mathbf{u}}+\sum_{k=1}^{\mathcal{N}}\Big[m_k\boldsymbol{\Upsilon}^{\mathrm{T}}\mathbf{J}_k^{\mathrm{T}}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{J}_k\boldsymbol{\Upsilon})\mathbf{u}+\boldsymbol{\Upsilon}^{\mathrm{T}}\Omega_k^{\mathrm{T}}\mathcal{I}_k\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\Omega}_k\boldsymbol{\Upsilon})\mathbf{u}+\boldsymbol{\Upsilon}^{\mathrm{T}}\boldsymbol{\Omega}_k^{\mathrm{T}}\mathbf{W}_k\mathcal{I}_k\boldsymbol{\Omega}_k \\ -\boldsymbol{\Upsilon}^{\mathrm{T}}\mathbf{J}_k^{\mathrm{T}}\mathbf{f}_k-\boldsymbol{\Upsilon}^{\mathrm{T}}\Omega_k^{\mathrm{T}}\boldsymbol{\tau}_k\Big]=\mathbf{0} \tag{49}$$

where

$$\mathbf{M}(\boldsymbol{\theta})=\sum_{k=1}^{\mathcal{N}}\left[m_k\mathbf{J}_k^{\mathrm{T}}\mathbf{J}_k+\boldsymbol{\Omega}_k^{\mathrm{T}}\mathcal{I}_k\boldsymbol{\Omega}_k\right]. \tag{50}$$

Transformation between joint velocities and generalized velocity components is defined as follows:

$$\dot{\boldsymbol{\theta}}=\boldsymbol{\Upsilon}\mathbf{u} \tag{51}$$

$$\mathbf{u}=\boldsymbol{\Upsilon}^{-1}\dot{\boldsymbol{\theta}} \tag{52}$$

where $\boldsymbol{\Upsilon}$ is a transformation matrix which depends on the mass matrix of the manipulator.

Equations (49) one can transform to another form using terms presented in Eq. (12). Calculating a time derivative of Eq. (51) one can get:

$$\ddot{\boldsymbol{\theta}}=\dot{\boldsymbol{\Upsilon}}\mathbf{u}+\boldsymbol{\Upsilon}\dot{\mathbf{u}}. \tag{53}$$

Inserting above equation into Eq. (12) gives

$$\mathbf{M}(\boldsymbol{\theta})(\dot{\boldsymbol{\Upsilon}}\mathbf{u}+\boldsymbol{\Upsilon}\dot{\mathbf{u}})+\mathbf{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})+\mathbf{G}(\boldsymbol{\theta})=\mathbf{Q} \tag{54}$$

$$\boldsymbol{\Upsilon}^{\mathrm{T}}\mathbf{M}(\boldsymbol{\theta})\boldsymbol{\Upsilon}\dot{\mathbf{u}}+\boldsymbol{\Upsilon}^{\mathrm{T}}\mathbf{M}(\boldsymbol{\theta})\dot{\boldsymbol{\Upsilon}}\mathbf{u}+\boldsymbol{\Upsilon}^{\mathrm{T}}\mathbf{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})+\boldsymbol{\Upsilon}^{\mathrm{T}}\mathbf{G}(\boldsymbol{\theta})=\boldsymbol{\Upsilon}^{\mathrm{T}}\mathbf{Q} \tag{55}$$

$$\boldsymbol{\Upsilon}^{\mathrm{T}}\mathbf{M}(\boldsymbol{\theta})\boldsymbol{\Upsilon}\dot{\mathbf{u}}+\boldsymbol{\Upsilon}^{\mathrm{T}}[\mathbf{M}(\boldsymbol{\theta})\dot{\boldsymbol{\Upsilon}}\mathbf{u}+\mathbf{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})]+\boldsymbol{\Upsilon}^{\mathrm{T}}\mathbf{G}(\boldsymbol{\theta})=\boldsymbol{\Upsilon}^{\mathrm{T}}\mathbf{Q}. \tag{56}$$

Equations of motion are then:

$$\mathbf{N}\dot{\mathbf{u}}+\mathbf{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}},\mathbf{u})=\boldsymbol{\pi} \tag{57}$$

$$\mathbf{N}=\boldsymbol{\Upsilon}^{\mathrm{T}}\mathbf{M}(\boldsymbol{\theta})\Upsilon \tag{58}$$

$$\mathbf{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}},\mathbf{u})=\boldsymbol{\Upsilon}^{\mathrm{T}}\left[\mathbf{M}(\boldsymbol{\theta})\dot{\Upsilon}\mathbf{u}+\mathbf{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\right] \tag{59}$$

$$\boldsymbol{\pi}=\boldsymbol{\Upsilon}^{\mathrm{T}}\left(\mathbf{Q}-\mathbf{G}(\boldsymbol{\theta})\right) \tag{60}$$

where $\mathbf{N}$ is diagonal matrix, $\mathbf{u},\dot{\mathbf{u}}$ are vector of generalized velocity components and its time derivative, $\mathbf{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}},\mathbf{u})$ is a new Coriolis force vector and $\boldsymbol{\pi}$ is a vector of quasi-forces. In this paper we will consider dynamic equations in the form of Eq. (57). Therefore serial manipulators can be described by Eqs. (49) and (50) using (52) in the form of (56) where:

$$\mathbf{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})=\sum_{k=1}^{\mathcal{N}}\left[\left(m_k\mathbf{J}_k^{\mathrm{T}}\dot{\mathbf{J}}_k+\boldsymbol{\Omega}_k^{\mathrm{T}}\mathcal{I}_k\dot{\boldsymbol{\Omega}}_k\right)\dot{\boldsymbol{\theta}}+\boldsymbol{\Omega}_k^{\mathrm{T}}\mathbf{W}_k\mathcal{I}_k\boldsymbol{\omega}_k\right] \tag{61}$$

$$\mathbf{G}(\boldsymbol{\theta})=-\sum_{k=1}^{\mathcal{N}}\left(\mathbf{J}_k^{\mathrm{T}}\mathbf{f}_{gk}+\boldsymbol{\Omega}_k^{\mathrm{T}}\boldsymbol{\tau}_{gk}\right) \tag{62}$$

$$\mathbf{Q}=\sum_{k=1}^{\mathcal{N}}\left(\mathbf{J}_k^{\mathrm{T}}\mathbf{f}_{ek}+\boldsymbol{\Omega}_k^{\mathrm{T}}\boldsymbol{\tau}_{ek}\right) \tag{63}$$

and $\mathbf{W}_k=[\boldsymbol{\omega}_k\times]$ denotes the cross product, i.e. $\tilde{\boldsymbol{\omega}}$ is a 3×3 skew symmetric matrix (see *Nomenclature*).

The kinetic energy is expressed (from Eqs. (58) and (51)) as follows:

$$\mathcal{K}(\boldsymbol{\theta}, \mathbf{u}) = \frac{1}{2}\mathbf{u}^{\mathrm{T}}\mathbf{M}(\boldsymbol{\theta})\mathbf{u} = \frac{1}{2}\mathbf{u}^{\mathrm{T}}\Upsilon^{\mathrm{T}}\mathbf{M}(\boldsymbol{\theta})\boldsymbol{\Upsilon}\mathbf{u} = \frac{1}{2}\mathbf{u}^{\mathrm{T}}\mathbf{N}\mathbf{u}. \tag{64}$$

The kinematic equation is described by Eq. (51). Natural splitting between momentum differential equations and kinematic differential equations in this case can be obtained using Eqs. (57) and (51). Inverse matrix $\boldsymbol{\Upsilon}^{-1}$ to the transformation matrix $\boldsymbol{\Upsilon}$ is similar to upper triangular matrix $\mathbf{D}^{-1/2}\mathbf{m}^{\mathrm{T}}(\boldsymbol{\theta})$ from Eq. (33) used by Jain and Rodriguez. Now we compare these both matrices. On the diagonal they have only unit elements which denotes that each kth inertial quasi-velocity contains the kth joint velocity. Both transformation matrices have nonzero elements behind diagonal at the same places. Besides that each kth inertial quasi-velocity depends on all joint velocities closer to the base of the manipulator. However, the methods of computing ξ_k and u_k are different. Every quasi-velocity ξ_k collects all velocities closer to the base of the manipulator and dynamical parameters (masses and inertias of links) closer to its tip (in this way we calculate weighting coefficients at each joint velocity closer to the base). Each quasi-velocity u_k can be obtained after manipulation of the mass matrix elements. At each step of the procedure described earlier in this subsection next quasi-velocity u_k is calculated. This procedure leads to the calculation of the matrix $\boldsymbol{\Upsilon}$ which gives the vector $\mathbf{u}$.

Observation 7 Comparing Eqs. (57) and (51) with Eqs. (5) and (8) one can observe that in this case $\mathbf{M}_{\mathrm{p}}(\boldsymbol{\theta}) = \mathbf{N}$ (a diagonal matrix), $\mathbf{C}_{\mathrm{p}}(\boldsymbol{\theta}, \mathbf{p})\mathbf{p} = \mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \mathbf{u})$, $\mathbf{G}_{\mathrm{p}}(\boldsymbol{\theta}) = \boldsymbol{\Upsilon}^{\mathrm{T}}\mathbf{G}(\boldsymbol{\theta})$, $\mathbf{Q}_{\mathrm{p}} = \boldsymbol{\Upsilon}^{\mathrm{T}}\mathbf{Q}$, $\mathbf{p} = \mathbf{u}$ and $\mathbf{Z}(\boldsymbol{\theta}) = \boldsymbol{\Upsilon}$.

3.6.1 Physical interpretation of GVC and quasi-moments

Quasi-variables of the kth joint are shown in Fig. 4 (in a sense of physical units they are similar to UQV shown in Fig. 2). Assuming as before inverse numbering (namely from the tip of the manipulator to its base) every component $u_k = \dot{\theta}_k + \delta_k$ is a sum of the kth relative joint velocity $\dot{\theta}_k$ and additional term $\delta_k = \sum_{i=k+1}^{\mathcal{N}} w_{ki}\dot{\theta}_i$. This term reflects the influence of all links towards the base from the kth link. Every term, in δ_k is equal to the relative joint velocity multiplied by a weighting coefficient. These coefficients depend on link masses and also on geometrical and kinematical parameters in actual time instant and have no physical units. They change during the motion of the manipulator. If elements δ_k arising from presence of other links as the kth have small values, then the generalized velocity component u_k is near to the relative joint velocity. On the other hand, if elements of the mass matrix have big values then term (coefficient) δ_k has also big value. It is equivalent to the fact that interactions from other links are big and have an essential contribution in component u_k. One can

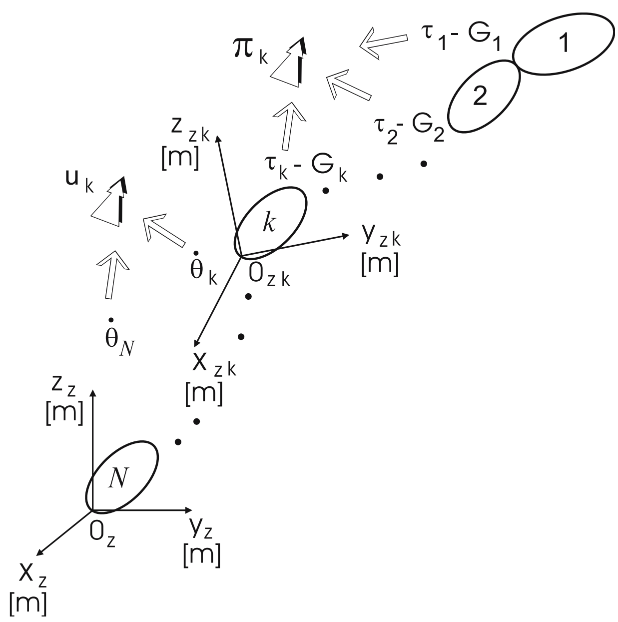

Fig. 4 Interpretation of quasi-variables for generalized velocity components (GVC) formulation

say that the generalized velocity component u_k represents the resultant velocity in a local instantaneous frame of reference.

Every quasi-moment π_k can be defined as:

$$\pi_k = \sum_{i=1}^{k-1} w_{ik}(\tau_i - G_i) + \tau_k - G_k. \tag{65}$$

Therefore π_k contains a sum of joint moments and gravitational forces closer to the tip of the manipulator as the kth joint plus forces acting at the kth joint. This sum is multiplied by weighting coefficients w_{ik}. Therefore all these forces of reduced manipulator enter into quasi-moment π_k.

3.7 Discussion concerning physical interpretation of inertial quasi-velocities

Consider again various inertial quasi-velocities and quasi-moments (compare Figs. 1 and 2). NQV and UQV are obtained recursively in a similar way. The main difference relies on introduction of the normalized factor D_k for each kth quasi-velocity. Either NQV or UQV represents an influence of all joint velocities during the motion of the manipulator. Because the transformation matrices $\mathbf{m}^{\mathrm{T}}(\boldsymbol{\theta})$ and $\mathbf{D}^{-1/2}\mathbf{m}^{\mathrm{T}}(\boldsymbol{\theta})$ are upper triangular, we obtain the relationship between ν_k (or ξ_k) and all joint velocities closer to the manipulator base. The quasi-moments are calculated earlier than the set of quasi-velocities (for NQV and UQV formulations, respectively). Each quasi-moment ϵ_k or κ_k transforms all joint moments closer to the manipulator tip (transformation matrices are, in both cases, lower triangular). Therefore, it represents the influence of all links closer to the tip during actual motion of the manipulator. It is also worth of noting that for NQV and UQV formulations, respectively, the quasi-moments do not contain gravitational forces acting at the manipulator.

For GVC formulation each quasi-velocity u_k which is obtained after adding a new link gives new interaction to the previous link mass [i.e. links from 1 to $(k-1)$-th (Fig. 4)]. The transformation matrix Υ^{-1} (upper triangular) is calculated from the known matrix $\boldsymbol{\Upsilon}$. It denotes that, similarly as for UQV formulation, one can compute quasi-moments π_k first and after that the inertial quasi-velocities u_k. These quasi-moments contain joint moments transformed from all links closer to the base. Also the transformation matrix $\boldsymbol{\Upsilon}^{\mathrm{T}}$ is lower triangular. However, on the contrary to UQV formulation, each quasi-moment π_k contains gravitational forces acting at links closer to the manipulator tip. Quasi-velocities u_k show the relationship between the kth quasi-velocity and all joint velocities closer to the base of the manipulator (similarly as ξ_k). The value of u_k arises from the motion which is a result of motion of all links closer to the base.

Now consider quasi-velocities η_k in EQV formulation. We have a similarity to quasi-velocities ν_k (NQV) because every η_k contains directly the inertia factor λ_i (the ith eigenvalue). However, the eigenvalues are calculated for the whole system (Fig. 3). Therefore η_k can be interpreted as a velocity which results from the motion of all coupled manipulator links. No recursion is given. The elements of the eigenvectors matrix represents the relationship between η_k and all joint velocities $\dot{\theta}_k$ which cause the actual motion of the manipulator. From the transformation matrix $\mathbf{SC}_{\mathrm{e}}$ (which is fully populated) arises that all joint velocities are presented in each quasi-velocity η_k (differently as in NQV, UQV and GVC formulations, respectively).

Compare now the quasi-moment vector $\boldsymbol{\varepsilon}$. Every quasi-moment ε_k depends on all joint moments of the manipulator. This differs ε_k from ϵ_k (in NQV formulation). Quasi-moments in EQV formulation (i.e. ε_k) contain gravitational forces (similarly as π_k in GVC formulation and differently as ϵ_k in NQV formulation). But, in this case, the physical units are different as for π_k and the same as for ϵ_k. We can see that the physical interpretation of EQV formulation is not so clear as for other formulations (NQV, UQV, GVC), because here the transformation matrix is fully populated. This denotes also that all joint velocities cause the motion of an artificial link (the mass which considers all links of the manipulator during its motion). Such interpretation is basically different as in other cases (i.e. for NQV, UQV and GVC formulations, respectively). However, we can assume that the mutual relationships between those quasi-variables and joint variables (joint velocities and joint moments) are hidden in a different way in transformation matrices.

3.8 Mathematical relationships between quasi-velocities

Now discuss the relationships existing between quasi-velocities. There are two groups of them: the first which contains the kinematical quasi-velocities and the second which concerns the inertial quasi-velocities.

Consider the first group. Generalized speeds can be practically understood as classical quasi-velocities given, e.g. in [13]. In Ref. [6] they are regarded as equivalent to independent speeds.

The twist vector introduced in reference [48] contains three angular velocities and three linear velocities for each link. For the entire system it is expressed as $\mathbf{Q}_V = [\boldsymbol{\omega}^T, \mathbf{v}^T]^T$ where $\boldsymbol{\omega}$ denotes vector of angular velocities and $\mathbf{v}$ vector of linear velocities, respectively. Therefore for robot manipulator composed of $\mathcal{N}$ links we have $6\mathcal{N}$ elements of vector of quasi-velocities (using stacked notation). Because of that these equations are directly not comparable with well-known quasi-velocities conception. It is visible from the kinematical quantities of the manipulator. However, transformation of the kinematical quasi-velocities vector $\mathbf{v}_q$ into the twist vector using matrices $\mathbf{A}_G$ and $\boldsymbol{\mathcal{A}}(\boldsymbol{\theta})$ exists. Mathematical relationship between both vectors arises from Eqs. (14) and (21) i.e.:

$$\mathbf{t}_V = \mathbf{A}_G \boldsymbol{\mathcal{A}}(\boldsymbol{\theta})\mathbf{v}_q. \tag{66}$$

In order to use the generalized speeds one has to replace vector $\mathbf{u}_r$ by vector $\mathbf{v}_g$. In this case the matrix $\mathcal{A}(\boldsymbol{\theta})$ has different elements as previous one. Notice, however that the natural orthogonal complement matrix $\mathbf{A}_G$ is expressed in the same physical units as parameters describing manipulator links. This fact distinguishes matrix $\mathbf{A}_G$ from matrix $\boldsymbol{\mathcal{A}}(\boldsymbol{\theta})$.

Next we compare inertial quasi-velocities. All of them contain dynamical quantities. We consider here two classes. In the first of them (NQV and EQV) transformation matrices evidently have inertia units. Besides that inertial quantities can be given as coefficients in the transformation matrix. Therefore both the normalized quasi-velocities and the eigenfactor quasi-coordinate velocities are expressed in some inertial-geometrical reference frames located at joints of the manipulator. Relationship between them follows from Eqs. (25) and (40):

$$\boldsymbol{\nu} = \mathbf{m}^T(\boldsymbol{\theta})\mathbf{C}_e^T\mathbf{S}^{-1}\boldsymbol{\eta}. \tag{67}$$

The matrix $\mathbf{m}(\boldsymbol{\theta})$ is obtained recursively and is upper triangular whereas the matrices $\mathbf{C}_e$ and $\mathbf{S}$ arise from spectral decomposition of mass matrix of the whole manipulator. Matrix $\mathbf{C}_e$ is fully populated and $\mathbf{S}$ is diagonal one. Physical units of NQV and EQV are still the same, i.e. [rad $\mathrm{m}\sqrt{\mathrm{kg}}/\mathrm{s}$].

The second class includes the UQV and the GVC. The relationship between both vectors is according to Eqs. (33) and (51):

$$\boldsymbol{\xi} = \mathbf{D}^{-1/2}\mathbf{m}^T(\boldsymbol{\theta})\boldsymbol{\Upsilon}\boldsymbol{u}. \tag{68}$$

The resultant transformation matrix is unitless and elements of vectors $\boldsymbol{\xi}$ and $\mathbf{u}$ have the same physical units as the generalized velocities i.e. [rad/s]. However, matrices $\mathbf{D}^{-1/2}$ (diagonal matrix) and $\mathbf{m}^T(\boldsymbol{\theta})$ which contain inertial quantities give after their multiplication a matrix with unitless elements. The resultant transformation matrix is upper triangular. Besides that, matrices of UQV formulation are obtained recursively whereas matrix $\boldsymbol{\Upsilon}$ arises from realization of the procedure given in [40]. This last matrix is also upper diagonal. Both UQV and GVC are expressed in some geometrical reference frames located at joints of the manipulator. But these quasi-velocities depend also on inertial quantities given as coefficients in the transformation matrices. This denotes that the reference frame can be understood as a geometrical one. But the inertial quantities arising from all coupled links causes the inertial motion of each kth artificial link (i.e. such link which is concerned as the kth inertial quasi-velocity instead of the kth joint velocity). These inertial quantities (quasi-velocities and quasi-moments) essentially depend on the motion of other links of the manipulator. Considering the motion in terms of joint velocities every link moves with own generalized velocity expressed in geometrical frame.

At the end one can write mixed relationships between the inertial quasi-velocities. NQV and UQV relationship arises from Eq. (33). Matrix $\mathbf{D}^{-1/2}$ causes that dependence on inertial quantities is not explicit. Both vectors $\boldsymbol{\xi}$ and $\boldsymbol{\nu}$ are calculated in a similar recursive way. Other relationships arise from Eqs. (25), (33), (47), and (51):

$$\boldsymbol{\nu} = \mathbf{m}^T(\boldsymbol{\theta})\Upsilon\mathbf{u} \tag{69}$$

$$\boldsymbol{\eta} = \mathbf{S}\mathbf{C}_e(\mathbf{m}^T(\boldsymbol{\theta}))^{-1}\mathbf{D}^{1/2}\boldsymbol{\xi} \tag{70}$$

$$\boldsymbol{\eta} = \mathbf{S}\mathbf{C}_e\boldsymbol{\Upsilon}\mathbf{u}. \tag{71}$$

Inertial quantities in explicit form are given in vectors $\boldsymbol{\nu}$ and $\boldsymbol{\eta}$ using matrices $\mathbf{m}^T(\boldsymbol{\theta})$, $\mathbf{D}^{1/2}$ and $\mathbf{S}$. The transformation matrix in Eq. (69) is upper diagonal and matrices in Eqs. (70) and (71) are fully populated.

4 Comparison of various quasi-velocities

Now we compare various kind of quasi-velocities according to some predefined criteria. These are simulation and some control requirements. Another important fact is a possibility of shaping of the potential energy and kinetic energy, and their relationships to the generalized velocities. The appropriate use of the quasi-velocities enables to take advantage of properties arising from equations of motion with diagonal mass matrix.

4.1 Properties useful for simulation purposes

As it was mentioned before for simulation purposes particularly important is the forward dynamics algorithm which leads to the acceleration vector $\ddot{\boldsymbol{\theta}}$. Choosing classical quasi-velocities, calculation of the generalized accelerations is not always easier because the mass matrix in dynamical equations of motion is fully populated. However, using for instance Kane's equations improves the computational efficiency of the dynamics algorithms [29].

Using the twist vector [48] it is possible to avoid inversion of the manipulator mass matrix because the generalized inertia matrix I_{G} is decomposed into three matrices. An advantage in this case is obvious. However, mass matrix is not diagonal. Additionally, the reverse Gaussian elimination using for calculation of joint accelerations vector $\ddot{\boldsymbol{\theta}}$ has recursive form. Such approach enables a deeper insight into manipulator dynamics.

On the contrary all methods based on the inertial quasi-velocities lead to diagonal manipulator mass matrix. In case of NQV and EQV one can obtain unit mass matrix and no inversion is needed to calculate quasi-acceleration. In case of UQV and GVC dynamical equations of motion we have to inverse the diagonal mass matrix. Diagonalized Lagrangian dynamics [26] which provides NQV and UQV formulations is based on recursive algorithms. EQV formulation [27,53] use spectral decomposition and this property is realized only numerically. The GVC approach [40] provides the diagonal matrix but does not use algorithms in similar form as diagonalized Lagrangian dynamics. It is rather closer to EQV formulation. One can say that useful properties for NQV and UQV formulations arise from their recursive nature whereas for EQV calculations are carried out only numerically. The method which leads to GVC formulation, i.e. the manner of calculation of transformation matrix $\boldsymbol{\Upsilon}$ result in a recursive process.

4.2 Properties useful for control purposes

The possibility of control of each variable independently and the orthogonality of the Coriolis term and the velocity vector are very important issues. The first feature is practically equivalent to equations of motion which have a diagonal mass matrix. The second feature denotes that the Coriolis term does no work and the controls can be simplified.

Dynamical equations of motion in terms of classical quasi-velocities, Kane's equations or equations using the twist vector are the second-order differential equations. On the contrary all considered here equations containing the inertial quasi-velocities are decoupled what denotes that the manipulator mass matrix is diagonal or the identity matrix. Besides that the inertial quasi-variables can be controlled separately.

In classical equations of motion (12) the Coriolis term is not orthogonal to the generalized velocity vector. However, this property is true only for some of decoupled equations of motion. The remarks given below concern equations expressed in terms of inertial quasi-velocities considered in this work.

- A point worth noting is that the authors of [26] have proved the orthogonality of Coriolis term $\mathbf{C}(\boldsymbol{\theta}, \boldsymbol{\nu})$ to generalized velocities ν, i.e. $\boldsymbol{\nu}^{\mathrm{T}}\mathbf{C}(\boldsymbol{\theta}, \boldsymbol{\nu}) = 0$. This condition means that Coriolis force does no work. There exists similar property in the equations of motion for a single rigid body which rotates with angular velocity vector $\boldsymbol{\omega}$, i.e. $\boldsymbol{\omega}^{\mathrm{T}}[\boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega}] = 0$, where $\mathbf{J}$ denotes the rotational inertia tensor. For standard equation of motion (12) the Coriolis forces term $\mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$ does work and the orthogonality condition is not true, i.e. $\dot{\boldsymbol{\theta}}^{\mathrm{T}}\mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \neq 0$. Natural splitting between momentum differential equations and kinematic differential equations can be obtained using Eqs. (24) and (25). The transformation matrix $\mathbf{m}^{\mathrm{T}}(\boldsymbol{\theta})$ is a triangular matrix.
- In Eq. (34) matrix D is diagonal. The Coriolis forces term $\mathbf{C}(\boldsymbol{\theta}, \boldsymbol{\xi})$ does work and the orthogonality condition is not true, i.e. $\boldsymbol{\xi}^{\mathrm{T}}\mathbf{C}(\boldsymbol{\theta}, \boldsymbol{\xi}) \neq 0$. One can notice that decomposition of the mass matrix $\mathbf{M}(\boldsymbol{\theta})$ into two or three matrices makes a difference between equations of motion and gives different properties. Natural splitting between momentum differential equations and kinematic differential equations in this case one can obtain using Eqs. (34) and (33). The transformation matrix $\mathbf{D}^{-1/2}\mathbf{m}^{\mathrm{T}}(\boldsymbol{\theta})$ is here a triangular matrix.

- New formulation described above replaces the second-order differential Lagrange's equations of motion (12) by two first differential equations (39) and (40). Notice, however, that Eq. (40) contains matrix $\mathbf{C}$ and a part of matrix $\mathbf{D}_e$. Because of that one can consider decomposition of mass matrix $\mathbf{M}$ into two matrices $(\mathbf{SC}_e)^T$ and $\mathbf{SC}_e$. The transformation matrix $\mathbf{C}_e^T\mathbf{S}^{-1}$ is here a fully populated matrix in contrast to two earlier considered transformation matrices (notice that the eigenvector matrix $\mathbf{C}_e$ is in general fully populated). Schaub and Junkins proved also [53] that this Coriolis term does no work, i.e. $\boldsymbol{\eta}^T\mathbf{C}\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \boldsymbol{\eta}\right) = 0$.
- In Eq. (57) the Coriolis forces term $\mathbf{C}\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \mathbf{u}\right)$ does work because the orthogonality condition is not true i.e. $\mathbf{u}^T\mathbf{C}\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \mathbf{u}\right) \neq 0$. Natural splitting between momentum differential equations and kinematic differential equations in this case can be obtained using Eqs. (57) and (51). The transformation matrix $\boldsymbol{\Upsilon}$ is similar to a triangular matrix of Jain and Rodriguez.

4.3 Gravitational forces, Coriolis terms and kinetic energy shaping

An important question is an influence of the gravitational and Coriolis terms, respectively, and the kinetic energy shaping in the decoupled equations of motion.

Recall for example the gravitational term in NQV formulation given in Eqs. (24) and (25). If the vector $\boldsymbol{\nu}$ is controlled one can shape the gravitational term $\mathbf{G}_\nu(\boldsymbol{\theta})$ because vector $\boldsymbol{\nu}$ contains matrix $\mathbf{m}(\boldsymbol{\theta})$. Performing transformation operation into joint space [i.e. multiplying Eq. (24) by $\mathbf{m}(\boldsymbol{\theta})$] we obtain the same gravitational term $\mathbf{G}(\boldsymbol{\theta})$ as in Eq. (12). Therefore we cannot shape the gravitational term in joint space of the manipulator using NQV formulation. Similar conclusions one can obtain for other formulations [using UQV Eqs. (34) and (33), EQV Eqs. (39) and (47), and GVC Eqs. (57)–(60) and (52)].

Next consider new Coriolis terms. Recall Eqs. (28), (37), (45), (59) for NQV, UQV, EQV and GVC formulations, respectively. Firstly, it can be seen that all Coriolis terms are more complicated than classical one, i.e. $\mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$ in Eq. (12). It is a result of diagonalization of manipulator mass matrix process. Calculations arising from the mass matrix decomposition are shifted into Coriolis terms (also into gravitational terms). Some advantages give the Coriolis terms in the normalized quasi-velocity and the eigenfactor quasi-coordinate velocity formulation, because these terms do not do any mechanical work. Notice, that the appropriate dynamical equations of motion contain the unit mass matrix. This denotes that the new quasi-velocity vector contains operator $\mathbf{m}^T(\boldsymbol{\theta})$ arising from decomposition of the mass matrix $\mathbf{M}(\boldsymbol{\theta})$. In other two formulations (UQV and GVC) the new mass matrix is diagonal and their elements have physical units the same as the matrix $\mathbf{M}(\boldsymbol{\theta})$. In that case the quasi-velocity vector contains inertial parameters given as coefficients. The Coriolis terms reflects some part of generalized moments vector $\boldsymbol{\tau}$ which arise from the diagonalization process. After performing transformation back to the joint space of the manipulator (for all quasi-velocity formulations) we do not obtain only the Coriolis term $\mathbf{C}\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\right)$ but additionally some terms arising from the decomposition of the manipulator mass matrix.

Next we consider some issues on the kinetic energy in terms of different inertial quasi-velocities vectors. For classical formulation the kinetic energy is expressed as follows:

$$\mathcal{K}\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\right) = \frac{1}{2}\dot{\boldsymbol{\theta}}^T\mathbf{M}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}. \tag{72}$$

In the simplest control law, e.g $\boldsymbol{\tau} = -\mathbf{c}_D\dot{\boldsymbol{\theta}}$ (where $\mathbf{c}_D$ is a positive definite matrix of coefficients) it is not possible to change directly the kinetic energy. However, one can shape it in a way by the generalized velocity vector.

Comparing variables in Eqs. (72) and (29) it can be observed that in classical control laws, which use vector of joint velocities $\dot{\boldsymbol{\theta}}$ (e.g. PD control in joint space, inverse dynamics control in joint space or velocity control), one can change the motion arising from control of vector $\dot{\boldsymbol{\theta}}$ but the kinetic energy cannot be controlled directly. In the contrary if one uses vector $\boldsymbol{\nu}$, kinetic energy can be controlled. It is obvious that here this energy is shaped in different way as using classical equations of motion (12). It is because the inertial quasi-velocity vector $\boldsymbol{\nu}$ contains additionally (besides the kinematical quantities) the dynamical parameters of the manipulator. It realizes directly kinetic energy control using, e.g. the following control law $\boldsymbol{\epsilon} = -\mathbf{c}_D\boldsymbol{\nu}$ because the energy consists only of the NQV vector $\boldsymbol{\nu}$ (and its transposition multiplied by $1/2$). A similar result can be obtained if one writes the kinetic energy of the system in terms of EQV vector [see Eqs. (42) and (47)].

Different results are received for NQV and GVC formulation, respectively. At a first glance both expressions (35) and (64) resemble Eq. (72). However, recall the appropriate quasi-velocities from Eqs. (33) and (52). They depend on inertial elements of the mass matrix. We conclude that by controlling vectors $\boldsymbol{\xi}$ or $\mathbf{u}$, it is possible to shape the kinetic energy in different way as using only the generalized velocity vector $\dot{\boldsymbol{\theta}}$. It is because these new velocities take into consideration the dynamical parameters of all links of the manipulator whereas vector $\dot{\boldsymbol{\theta}}$ represents only kinematical changes.

4.4 Inertial quasi-velocities and generalized velocities

It is also of interest to examine differences between the generalized velocities and the inertial quasi-velocities. Therefore recall Eqs. (25), (33), (47) and (52). All of them depend on vector $\dot{\boldsymbol{\theta}}$ and on rate transformation matrices which contain kinematical and dynamical quantities of the manipulator. Therefore, the above-mentioned equations represent not only purely kinematic relationships between new velocities and the generalized velocities but in fact some kinematic-inertial ones. Using NQV, UQV, EQV or GVC formulations we observe actual position of the manipulator and additionally the influence of its links at each time instant. Vectors $\boldsymbol{\xi}$ and $\mathbf{u}$ have the same as vector $\dot{\boldsymbol{\theta}}$ physical units i.e. [rad/s] whereas vectors $\boldsymbol{\nu}$ and $\boldsymbol{\eta}$ different, namely [rad $\mathrm{m}\sqrt{\mathrm{kg}}/\mathrm{s}$]. It arises from the fact that in the two latter cases the inertial quantities are given in an explicit form.

In classical equations of motion one uses the pair of vector $\{\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\}$. Using the inertial quasi-velocities we have the following pairs of vectors: $\{\boldsymbol{\theta}, \boldsymbol{\nu}\}$, $\{\boldsymbol{\theta}, \boldsymbol{\xi}\}$, $\{\boldsymbol{\theta}, \boldsymbol{\eta}\}$ and $\{\boldsymbol{\theta}, \mathbf{u}\}$. All these pairs contain the generalized coordinates vector. Some authors [26,27,53] compared inertial quasi-velocities with the angular velocities (where oriention is described by Euler angles).

4.5 Critical remarks concerning the inertial quasi-velocities

Now we return to the question of using the generalized coordinates for the purpose of manipulator position control. From the above considerations we know that the dynamical equations of motion in terms of the inertial quasi-velocities contain that vector. It is a well-understood description which involves the generalized coordinates vector for position determination. In some cases one can find integrals of inertial quasi-velocities and obtain some curvilinear trajectories. They are quite different to joint positions and depend also on dynamical quantities. However, quasi-coordinates are sometimes understood as time integrals of quasi-velocities [18,25]. They establish, e.g. the internal state space model and are used for feedback control [25]. On the other hand the generalized coordinates used together with the inertial rates lead to decoupled equations of motion expressed in two different spaces (the first is concerned with joint position whereas the second with new velocity). However, the presence of the joint position vector $\boldsymbol{\theta}$ enables the transformation from the quasi-space to the joint space of the manipulator.

The second problem produce the dynamical parameters in the inertial quasi-velocities. Their presence requires to know the full model of the manipulator in order to control it. Introducing the inertial quasi-velocities while the model is not known precise determined seems questionable.

5 Forward dynamics algorithms using inertial quasi-velocities

In this section we present the implementation of the forward dynamics algorithms in terms of inertial quasi-velocities i.e. in NQV, UQV, EQV and GVC formulations, respectively. The forward dynamics problem can be defined as follows. We assume an arbitrary manipulator with both rotational and prismatic joints. At the tip of the manipulator no forces or torques are acting. The forward dynamics algorithm calculates generalized accelerations at the joints, assuming that the joint forces or moments, the initial conditions for joint positions and velocities are known. Algorithms presented here allow the reader to become acquainted with the solutions of this problem.

5.1 Diagonalized Lagrangian dynamics

A forward dynamics problem in terms of quasi-velocities consists of two recursions. The first of them starts from the base of the manipulator towards its tip and the second is in opposite direction. Both recursions have in general matrix-vector form.

Firstly, consider normalized quasi-velocities case. In compact form the algorithm for normalized quasi-velocities is as follows (different operations for unnormalized quasi-velocities are given in brackets). Operations are realized for all joints and at each time instant.

1. Assume initial conditions for the joint position and velocities $\theta_{0k}, \dot{\boldsymbol{\theta}}_{0k}$, also the joint moments τ_k, and dynamical and kinematical parameters of the manipulator.
2. Compute the joint trajectories $\boldsymbol{\theta}_k$.
3. Build spatial operators $\mathbf{A}_k$, $\boldsymbol{\phi}_k$, $\mathbf{M}_k$ and $\mathbf{H}_k$ and compute operators $\mathbf{P}_k$, D_k, $\mathbf{G}_{ak}$ and $\boldsymbol{\psi}_k$.
4. Compute the normalized quasi-velocities ν_k (the unnormalized quasi-velocities ξ_k) and the joint velocities $\dot{\boldsymbol{\theta}}_k$.
5. Compute the normalized quasi-moments ϵ_k (unnormalized quasi-moments κ_k) based on known joint moments τ_k.
6. Compute the spatial bias forces $\mathbf{b}_k$, the gravitational forces $\mathbf{b}_{gk}$ and the normalized gravitational terms $G_{\nu k}$ (the normalized gravitational terms $G_{\xi k}$).
7. Use spatial operators to obtain the normalized Coriolis terms $C_{\nu k}$ (the unnormalized Coriolis terms $C_{\xi k}$).
8. Compute the normalized quasi-accelerations $\dot{\nu}_k$ (the unnormalized quasi-accelerations $\dot{\xi}_k$) using ϵ_k, $C_{\nu k}$ and $G_{\nu k}$ (κ_k, $C_{\xi k}$ and $G_{\xi k}$).

In explicit form we can write both algorithms as follows:

Input data:
Initial conditions: $\theta_{0k}, \dot{\theta}_{0k}$;
Dynamical parameters of manipulator: $\mathcal{I}_k$, m_k and $m_k\mathbf{p}_k$;
Kinematical parameters $\mathbf{h}_k$, $\mathbf{l}_k$ (or θ_k for translational joint), angle α_k;
Joint moments τ_k.
Output data:
Quasi–accelerations $\dot{\nu}_{ik}$, where $k = 1, \ldots, \mathcal{N}$; $i = 0, 1, 2, \ldots$.

1. Input operators $\mathbf{M}_k$ i $\mathbf{H}_k$
   Assign $i := 0$.
2. Compute θ_i for $i > 0$ as result of prediction and integration (for $i - 1$) $\dot{\theta}$ and known value θ. For $(i = 0)$ we have θ_{0k}.
3. Input operators $\mathbf{A}_k(\theta_i)$, $\boldsymbol{\phi}_{k,k-1}(\theta_i)$, for $k = 1, \ldots, \mathcal{N}$.

   Remark For rotational joints operator $\mathbf{A}_k$ is variable ($\boldsymbol{\phi}_{k,k-1}$ is constant), whereas for translational joints operator $\boldsymbol{\phi}_{k,k-1}$ is variable (and $\mathbf{A}_k$ is constant).
4. For the instant time ith compute:
   (a) initial conditions:

$$\mathbf{P}_0^+ = \mathbf{0}, \tag{73}$$

   (b) for $k = 1$ to $\mathcal{N}$:

$$\mathbf{P}_k = \boldsymbol{\phi}_{k,k-1}\mathbf{P}_{k-1}^+\boldsymbol{\phi}_{k,k-1}^{\mathrm{T}} + \mathbf{M}_k \tag{74}$$

$$D_k = \mathbf{H}_k\mathbf{P}_k\mathbf{H}_k^{\mathrm{T}} \tag{75}$$

$$\mathbf{G}_{ak} = \frac{\mathbf{P}_k\mathbf{H}_k^T}{D_k} \tag{76}$$

$$\mathbf{P}_k^+ = \mathbf{A}_k[\mathbf{I} - \mathbf{G}_{ak}\mathbf{H}_k]\mathbf{P}_k\mathbf{A}_k^{-1}. \tag{77}$$

   End loop.
5. Compute quasi-velocity ν_i for the ith time instant as result of prediction and integration of $\dot{\boldsymbol{\nu}}$ and known previous value $\boldsymbol{\nu}$.
   If $i = 0$ do:
   (a) initial conditions:

$$\mathbf{V}_{\mathcal{N}+1} = \mathbf{0}, \tag{78}$$

(b) for $k = \mathcal{N}$ to 1:

$$\mathbf{V}_k^+ = \boldsymbol{\phi}_{k+1,k}^{\mathrm{T}} \mathbf{V}_{k+1} \tag{79}$$

$$\nu_k = D_k^{1/2} \left[\dot{\boldsymbol{\theta}}_k + \mathbf{G}_{\mathrm{a}k}^{\mathrm{T}} \mathbf{A}_k^{-1} \mathbf{V}_k^+ \right] \tag{80}$$

$$\mathbf{V}_k = \mathbf{A}_k^{-1} \mathbf{V}_k^+ + \mathbf{H}_k^{\mathrm{T}} \dot{\theta}_k. \tag{81}$$

End loop.

6. For $i > 0$ time instant:
(a) initial conditions:

$$\mathbf{V}_{\mathcal{N}+1} = \mathbf{0},$$

(b) for $k = \mathcal{N}$ to 1:

$$\begin{aligned} \mathbf{V}_k^+ &= \boldsymbol{\phi}_{k+1,k}^{\mathrm{T}} \mathbf{V}_{k+1} \\ \dot{\theta}_k &= D_k^{-1/2} \nu_k - \mathbf{G}_{\mathrm{a}k}^{\mathrm{T}} \mathbf{A}_k^{-1} \mathbf{V}_k^+ \\ \mathbf{V}_k &= \mathbf{A}_k^{-1} \mathbf{V}_k^+ + \mathbf{H}_k^{\mathrm{T}} \dot{\theta}_k. \end{aligned} \tag{82}$$

End loop.

7. For the ith time instant do:
(a) initial conditions:

$$\mathbf{z}_0^+ = \mathbf{0}, \tag{83}$$

(b) for $k = 1$ to $\mathcal{N}$ compute:

$$\boldsymbol{\psi}_{k+1,k} = \boldsymbol{\phi}_{k+1,k} \mathbf{A}_k [\mathbf{I} - \mathbf{G}_{\mathrm{a}k} \mathbf{H}_k] \tag{84}$$

$$\mathbf{z}_k = \boldsymbol{\phi}_{k,k-1} \mathbf{z}_{k-1}^+ \tag{85}$$

$$\epsilon_k = D_k^{-1/2} [\tau_k - \mathbf{H}_k \mathbf{z}_k] \tag{86}$$

$$\mathbf{z}_k^+ = \mathbf{A}_k \left[\mathbf{z}_k + \mathbf{G}_{\mathrm{a}k} D_k^{1/2} \epsilon_k \right]. \tag{87}$$

End loop.

8. Calculate input operator $\tilde{\boldsymbol{\Omega}}_{\delta k}$ for rotational joint:

$$\tilde{\boldsymbol{\Omega}}_{\delta k} = \begin{bmatrix} \dot{\boldsymbol{\theta}}_k \tilde{\mathbf{h}}_k & \mathbf{0} \\ \mathbf{0} & \dot{\theta}_k \tilde{\mathbf{h}}_k \end{bmatrix}. \tag{88}$$

For translational joints operator $\tilde{\boldsymbol{\Omega}}_{\delta k}$ is zero matrix.

9. For the ith time instant compute $\mathbf{b}_k$ and $\mathbf{b}_{\mathrm{g}k}$.
10. For the ith time instant do:
(a) initial conditions:

$$\dot{\mathbf{P}}_0 = \mathbf{0}, \tag{89}$$

$$\mathbf{y}_0 = \mathbf{0}, \tag{90}$$

$$\mathbf{y}_{g0} = \mathbf{0}, \tag{91}$$

(b) for $k = 1$ to $\mathcal{N}$:

$$\mathbf{X}_k = \tilde{\boldsymbol{\Omega}}_{\delta k} \mathbf{P}_k \tag{92}$$

$$\dot{\mathbf{P}}_k = \boldsymbol{\psi}_{k,k-1} \dot{\mathbf{P}}_{k-1} \boldsymbol{\psi}_{k,k-1}^{\mathrm{T}} + \mathbf{X}_k + \mathbf{X}_k^{\mathrm{T}} \tag{93}$$

$$\mathbf{y}_k = \boldsymbol{\psi}_{k,k-1} \mathbf{y}_{k-1} + 2\mathbf{b}_k + \left[\mathbf{X}_k^{\mathrm{T}} - \mathbf{X}_k \right] \mathbf{V}_k + \boldsymbol{\psi}_{k,k-1} \dot{\mathbf{P}}_{k-1} \mathbf{V}_{k-1} - \dot{\mathbf{P}}_k \boldsymbol{\psi}_{k+1,k}^{\mathrm{T}} \mathbf{V}_{k+1} \tag{94}$$

$$\mathbf{y}_{\mathrm{g}k} = \boldsymbol{\psi}_{k,k-1} \mathbf{y}_{\mathrm{g}(k-1)} + 2\mathbf{b}_{\mathrm{g}k} + \left[\mathbf{X}_k^{\mathrm{T}} - \mathbf{X}_k \right] \mathbf{V}_k + \boldsymbol{\psi}_{k,k-1} \dot{\mathbf{P}}_{k-1} \mathbf{V}_{k-1} - \dot{\mathbf{P}}_k \boldsymbol{\psi}_{k+1,k}^{\mathrm{T}} \mathbf{V}_{k+1} \tag{95}$$

$$C_{\nu k} = \tfrac{1}{2} D_k^{-1/2} \mathbf{H}_k \mathbf{y}_k \tag{96}$$

$$G_{\nu k} = \tfrac{1}{2} D_k^{-1/2} \mathbf{H}_k \mathbf{y}_{\mathrm{g}k}. \tag{97}$$

End loop.

11. Compute quasi-accelerations for the ith time instant:

$$\dot{\nu}_k = \epsilon_k - C_{\nu k} - G_{\nu k}. \tag{98}$$

12. If $i\Delta t$ is smaller than assumed final time, then $i := i + 1$ and return to point 2.

Next observe the differences which are necessary for implementation of the unnormalized quasi-velocities.

Input data:
$\dot{\xi}_{ik}$;
where $k = 1, \ldots, \mathcal{N}$; $i = 0, 1, 2, \ldots$.

Steps 1 to 4 are the same as for the algorithm in terms of the normalized quasi-velocities.

5. Compute quasi-velocity ξ_i for the ith time instant as result of prediction and integration of $\dot{\xi}$ and known previous value ξ.
If $i = 0$ do:
   1. initial conditions:

$$\mathbf{V}_{\mathcal{N}+1} = \mathbf{0}, \tag{99}$$

   2. from $k = \mathcal{N}$ to 1 compute:

$$\mathbf{V}_k^+ = \boldsymbol{\phi}_{k+1,k}^{\mathrm{T}} \mathbf{V}_{k+1} \tag{100}$$

$$\xi_k = \dot{\boldsymbol{\theta}}_k + \mathbf{G}_{\mathrm{a}k}^{\mathrm{T}} \mathbf{A}_k^{-1} \mathbf{V}_k^+ \tag{101}$$

$$\mathbf{V}_k = \mathbf{A}_k^{-1} \mathbf{V}_k^+ + \mathbf{H}_k^{\mathrm{T}} \dot{\boldsymbol{\theta}}_k. \tag{102}$$

6. For $i > 0$ time instant do:
   1. initial conditions:

$$\mathbf{V}_{\mathcal{N}+1} = \mathbf{0},$$

   2. for $k = \mathcal{N}$ to 1 compute:

$$\begin{aligned} \mathbf{V}_k^+ &= \boldsymbol{\phi}_{k+1,k}^{\mathrm{T}} \mathbf{V}_{k+1} \\ \dot{\theta}_{ik} &= \xi_{ik} - \mathbf{G}_{\mathrm{a}k}^{\mathrm{T}} \mathbf{A}_k^{-1} \mathbf{V}_k^+ \\ \mathbf{V}_k &= \mathbf{A}_k^{-1} \mathbf{V}_k^+ + \mathbf{H}_k^{\mathrm{T}} \dot{\theta}_k. \end{aligned} \tag{103}$$

7. For the i time instant do:
   1. initial conditions:

$$\mathbf{z}_0^+ = \mathbf{0}, \tag{104}$$

   2. from $k = 1$ to $\mathcal{N}$ compute:

$$\boldsymbol{\psi}_{k+1,k} = \boldsymbol{\phi}_{k+1,k} \mathbf{A}_k \left[\mathbf{I} - \mathbf{G}_{\mathrm{a}k} \mathbf{H}_k\right] \tag{105}$$

$$\mathbf{z}_k = \boldsymbol{\phi}_{k,k-1} \mathbf{z}_{k-1}^+ \tag{106}$$

$$\kappa_k = \tau_k - \mathbf{H}_k \mathbf{z}_k \tag{107}$$

$$\mathbf{z}_k^+ = \mathbf{A}_k [\mathbf{z}_k + \mathbf{G}_{\mathrm{a}k} \kappa_k]. \tag{108}$$

Steps 8 and 9 are the same as for the algorithm in terms of the normalized quasi-velocities.

10. For the i time instant do:
   1. initial conditions:

$$\dot{\mathbf{P}}_0 = \mathbf{0} \tag{109}$$

$$\mathbf{y}_0 = \mathbf{0}, \tag{110}$$

2. from $k = 1$ to $\mathcal{N}$ compute:

$$\mathbf{X}_k = \tilde{\mathbf{\Omega}}_{\delta k}\mathbf{P}_k \tag{111}$$

$$\dot{\mathbf{P}}_k = \boldsymbol{\psi}_{k,k-1}\dot{\mathbf{P}}_{k-1}\boldsymbol{\psi}^{\mathrm{T}}_{k,k-1} + \mathbf{X}_k + \mathbf{X}^{\mathrm{T}}_k \tag{112}$$

$$\mathbf{y}_k = \boldsymbol{\psi}_{k,k-1}\mathbf{y}_{k-1} + \mathbf{b}_k - \mathbf{X}_k\mathbf{V}_k + \dot{\mathbf{P}}_k\mathbf{H}^{\mathrm{T}}_k\xi_k \tag{113}$$

$$C_{\xi k} = \mathbf{H}_k\mathbf{y}_k \tag{114}$$

$$G_{\xi k} = \mathbf{H}_k\mathbf{y}_{gk}. \tag{115}$$

11. Compute quasi-accelerations for the ith time instant:

$$\dot{\xi}_k = \frac{\left[\kappa_k - C_{\xi k} - G_{\xi k}\right]}{D_k}. \tag{116}$$

12. If $i\,\Delta t$ is smaller than assumed final time, then $i := i + 1$ and return to point 2.

5.2 EQV forward dynamics algorithm

In this subsection the forward dynamics algorithm using the EQV is presented.

1. Assume as input data the kinematical and dynamical parameters of the manipulator.
2. Calculate the rotation matrix ${}^{k+1}\mathbf{R}_k$.
3. Describe the kinetic energy $\mathcal{K}$ of the system and the potential energy $\mathcal{U}$.
4. From the kinetic energy expression calculate the mass matrix $\mathbf{M}(\boldsymbol{\theta})$.
5. Evaluate the gravity term $\mathbf{G} = \partial\mathcal{U}/\partial\boldsymbol{\theta}$.
6. Execute the spectral decomposition of the mass matrix $\mathbf{M}(\boldsymbol{\theta})$, i.e. calculate the eigenvalues matrix $\mathbf{D}_{\mathrm{e}} = \mathrm{diag}(\lambda_i)$ and the eigenvector matrix $\mathbf{E} = \mathbf{C}^{\mathrm{T}}_{\mathrm{e}} = [\mathbf{c}_{\mathrm{e}1}, \ldots, \mathbf{c}_{\mathrm{e}\mathcal{N}}]$.
7. Calculate the matrix $\mathbf{S} = \sqrt{\mathbf{D}_{\mathrm{e}}} = \mathrm{diag}\left(+\sqrt{\lambda_i}\right) = \mathrm{diag}(s_i)$.
8. Compute the eigenfactor quasi-velocity vector $\boldsymbol{\eta} = \mathbf{S}\mathbf{C}_{\mathrm{e}}\dot{\boldsymbol{\theta}}$.
9. From the mass matrix $\mathbf{M}(\boldsymbol{\theta})$ evaluate its time derivative $\dot{\mathbf{M}}$ and next the matrix $\boldsymbol{\mu} = \mathbf{C}_{\mathrm{e}}\dot{\mathbf{M}}\mathbf{C}^{\mathrm{T}}_{\mathrm{e}}$ (its elements are $\mu_{ij} = \mathbf{c}_{\mathrm{e}i}\dot{\mathbf{M}}\mathbf{c}_{\mathrm{e}j}$ and $\mu_{ii} = \dot{\lambda}_i$) and finally $\mathbf{\Omega}_{\mathrm{e}} = [\Omega_{\mathrm{e}ij}]$ expressed as:

$$\Omega_{\mathrm{e}ij}(t_1) = \frac{\mu_{ij}}{\lambda_j - \lambda_i} \quad \text{for } \left|\lambda_j - \lambda_i\right| \geq \delta \tag{117}$$

$$\Omega_{\mathrm{e}ij}(t_1) = \Omega_{\mathrm{e}ij}(t_0) + \dot{\Omega}_{\mathrm{e}ij}(t_0)\,(t_1 - t_0) \quad \text{for } \left|\lambda_j - \lambda_i\right| < \delta \tag{118}$$

$$\tilde{\mathbf{\Omega}}_{\mathrm{e}} = \begin{bmatrix} 0 & \Omega_{\mathrm{e}12} & \ldots & \Omega_{\mathrm{e}1\mathcal{N}} \\ -\Omega_{\mathrm{e}12} & 0 & \ldots & \Omega_{\mathrm{e}2\mathcal{N}} \\ \ldots & \ldots & \ldots & \ldots \\ -\Omega_{\mathrm{e}1\mathcal{N}} & -\Omega_{\mathrm{e}2\mathcal{N}} & \ldots & 0 \end{bmatrix}. \tag{119}$$

10. To obtain $\dot{\mathbf{S}} = 1/2\mathbf{\Gamma}\mathbf{S}^{-1}$ calculate the matrices $\mathbf{\Gamma} = \mathrm{diag}(\mu_{ii})$ and $\mathbf{S}^{-1}$.
11. Compute the Coriolis term and also the quasi-moments vector for the ith time instant according to Eqs. (45) and (46).
12. Calculate the quasi-accelerations for the ith time instant using:

$$\dot{\boldsymbol{\eta}} = \boldsymbol{\varepsilon} - \mathbf{C}\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \boldsymbol{\eta}\right). \tag{120}$$

5.3 GVC forward dynamics algorithm

Now the forward dynamics algorithm in terms of GVC is outlined.

1. Assume as input data the kinematical and dynamical parameters of the manipulator.
2. Calculate the rotation matrices ${}^{k+1}\mathbf{R}_k$.

3. Compute the partial derivatives of $\mathbf{r}_k$ body's mass center position with respect to the vector of generalized coordinates ($\mathbf{r}_k$ is the Euclidean position of body k's mass center with respect to the inertial reference frame)

$$\mathbf{J}_k = \frac{\partial \mathbf{r}_k}{\partial \dot{\boldsymbol{\theta}}^{\mathrm{T}}}, \quad \text{where } k = 1, 2, \ldots, \mathcal{N} \tag{121}$$

and the partial derivatives of body k's angular velocity with respect to the time derivative of the generalized coordinate vector

$$\boldsymbol{\Omega}_k = \frac{\partial \boldsymbol{\omega}_k}{\partial \dot{\boldsymbol{\theta}}^{\mathrm{T}}}, \quad \text{where } k = 1, 2, \ldots, \mathcal{N} \tag{122}$$

4. Calculate the mass matrix of the manipulator according to Eq. (50).
5. Compose the transformation matrix $\boldsymbol{\Upsilon}$ from the matrix $\mathbf{M}(\boldsymbol{\theta})$ where $\boldsymbol{\Upsilon} = \boldsymbol{\Upsilon}_1\boldsymbol{\Upsilon}_2\boldsymbol{\Upsilon}_3 \ldots \boldsymbol{\Upsilon}_m$, and the diagonal matrix $\mathbf{N} = \boldsymbol{\Upsilon}^{\mathrm{T}}\mathbf{M}(\boldsymbol{\theta})\boldsymbol{\Upsilon}$.
6. Compute the GVC vector from Eq. (52).
7. Calculate the Coriolis term according to Eq. (59) using Eqs. (51) and (61).
8. Transform joint moments τ_k and the gravitational term into quasi-moments described by Eqs. (60), (62) and (63).
9. Calculate the time derivatives of GVC vector according to:

$$\dot{\mathbf{u}} = \mathbf{N}^{-1}[\boldsymbol{\pi} - \mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \mathbf{u})]. \tag{123}$$

6 Computational complexity of the dynamical equations

In this section we consider scalar operations (multiplications, additions, subtractions and divisions) required to implement the algorithms presented in the previous sections. The calculation concerned arithmetic operations is based on several assumptions. Firstly, we assume that link-to-link coordinate orientation transformation is the modified Denarit–Hartenberg orientation matrix.

We have implemented the concept of customizing the dynamic equations to reduce the computational requirements [32]. In this convention the nonzero elements of a vector are denoted by subscript variables, and the zero and unit elements by 0 and 1, respectively. We regard the nonzero elements as variables and the zero elements as zeros. This procedure guarantees that every two mathematically equivalent expressions are denoted by the same variable name. The direction cosine matrix can be splitted up into two planar rotation matrices [11]. In that case every matrix has an invariant part with respect to planar rotations, one can see that we significantly save the number of operations. The problem of computational complexity of the forward dynamics concerned robot manipulator was considered by Featherstone [20,21] and by Featherstone and Orin in Ref. [22]. Besides that some earlier results one can find in Refs. [36–38].

6.1 Diagonalized Lagrangian robot dynamics

In Table 1 (which is supplement of table given in Ref. [38]) the resulting numbers of operations, with the above assumptions, for the normalized and unnormalized equations of motion are presented ($\mathcal{N} > 2$). In this table a row with "Subtotal" indicates the number of operations which are common for both equations of motion. Notice, that both algorithms are $O(\mathcal{N})$. The columns indicated by 1 give the number of operations for the general purpose manipulator and columns indicated by 2 the numbers for the the manipulators which have axes either parallel or perpendicular and have displacement along direction of x axis in the matrix $\boldsymbol{\phi}(k, k-1)$. In the first case total numbers of arithmetic operations is $1,019\mathcal{N} - 909$ and for the second case we obtain $811\mathcal{N} - 603$. In Table 1 we do not include the calculations required to evaluate the sines and cosines. In constructing this table we have taken into account the initial conditions for the various recursions in order to minimize the number of arithmetic operations. These conditions concern both the base of the manipulator and its tip. In many of recursions we can take into account the next link adjacent to the initial link which leads to $\mathcal{N} > 2$ as a condition for Table 1.

Table 1 Computational complexity of normalized (NQV) and unnormalized (UQV) quasi velocities algorithms

Recursions	Multiplications/divisions		Additions/subtractions	
	1	2(-)	1	2(-)
$\mathbf{P}$	$70\mathcal{N} - 108$	$34\mathcal{N} - 58$	$92\mathcal{N} - 105$	$53\mathcal{N} - 64$
$\mathbf{D}$	0	0	0	0
$\mathbf{G}$	$5\mathcal{N} - 1$	$5\mathcal{N} - 1$	0	0
$\mathbf{V}^+$	$6\mathcal{N} - 12$	$2\mathcal{N} - 4$	$6\mathcal{N} - 12$	$2\mathcal{N} - 4$
$\mathbf{V}$	$16\mathcal{N} - 29$	$11\mathcal{N} - 22$	$10\mathcal{N} - 17$	$5\mathcal{N} - 9$
$\mathbf{z}$	$6\mathcal{N} - 12$	$2\mathcal{N} - 4$	$6\mathcal{N} - 12$	$2\mathcal{N} - 4$
$\mathbf{b}$	$59\mathcal{N} - 43$	$55\mathcal{N} - 53$	$40\mathcal{N} - 27$	$38\mathcal{N} - 35$
$\mathbf{b}_g$	$6\mathcal{N}$	$6\mathcal{N}$	$3\mathcal{N}$	$3\mathcal{N}$
$\mathbf{X}$	$18\mathcal{N} - 12$	$18\mathcal{N} - 12$	0	0
$\dot{\mathbf{P}}$	$115\mathcal{N} - 6$	$79\mathcal{N} - 6$	$144\mathcal{N} - 6$	$113\mathcal{N} - 6$
Subtotal 1	$301\mathcal{N} - 223$	$212\mathcal{N} - 160$	$304\mathcal{N} - 179$	$219\mathcal{N} - 122$
$\dot{\boldsymbol{\theta}}$	$7\mathcal{N} - 10$	$7\mathcal{N} - 11$	$6\mathcal{N} - 10$	$6\mathcal{N} - 11$
$\boldsymbol{\epsilon}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N} - 1$	$\mathcal{N} - 1$
$\mathbf{z}^+$	$23\mathcal{N} - 4$	$14\mathcal{N} - 3$	$15\mathcal{N} - 8$	$10\mathcal{N} - 7$
$\mathbf{y}$	$158\mathcal{N} - 223$	$134\mathcal{N} - 211$	$159\mathcal{N} - 249$	$140\mathcal{N} - 217$
$\mathbf{y}_g$	$21\mathcal{N}$	$17\mathcal{N}$	$17\mathcal{N}$	$12\mathcal{N}$
$\mathbf{C}(\boldsymbol{\theta}, \boldsymbol{\nu})$	$2\mathcal{N} - 1$	$2\mathcal{N} - 1$	0	0
$\mathbf{G}_\nu(\boldsymbol{\theta})$	$2\mathcal{N} - 1$	$2\mathcal{N} - 1$	0	0
$\dot{\boldsymbol{\nu}}$	0	0	$2\mathcal{N}$	$\mathcal{N}$
Subtotal 2	$214\mathcal{N} - 239$	$177\mathcal{N} - 227$	$200\mathcal{N} - 268$	$171\mathcal{N} - 236$
Normalized equations of motion	$515\mathcal{N} - 462$	$389\mathcal{N} - 387$	$504\mathcal{N} - 447$	$390\mathcal{N} - 358$
$\dot{\boldsymbol{\theta}}$	$5\mathcal{N} - 8$	$5\mathcal{N} - 9$	$5\mathcal{N} - 8$	$5\mathcal{N} - 9$
$\boldsymbol{\kappa}$	0	0	$\mathcal{N} - 1$	$\mathcal{N} - 1$
$\mathbf{z}^+$	$21\mathcal{N} - 4$	$13\mathcal{N} - 3$	$13\mathcal{N} - 8$	$9\mathcal{N} - 7$
$\mathbf{y}$	$59\mathcal{N} - 83$	$47\mathcal{N} - 71$	$61\mathcal{N} - 89$	$50\mathcal{N} - 78$
$\mathbf{y}_g$	$21\mathcal{N}$	$17\mathcal{N}$	$17\mathcal{N}$	$12\mathcal{N}$
$\mathbf{C}(\boldsymbol{\theta}, \boldsymbol{\xi})$	0	0	0	0
$\mathbf{G}_\xi(\boldsymbol{\theta})$	0	0	0	0
$\dot{\boldsymbol{\xi}}$	$\mathcal{N}$	$\mathcal{N}$	$2\mathcal{N}$	$\mathcal{N}$
Subtotal 3	$107\mathcal{N} - 95$	$83\mathcal{N} - 83$	$99\mathcal{N} - 106$	$79\mathcal{N} - 95$
Unnormalized equations of motion	$408\mathcal{N} - 318$	$295\mathcal{N} - 243$	$403\mathcal{N} - 285$	$298\mathcal{N} - 217$

6.2 Eigenstructure quasi-velocity formulation for manipulators

Information about computational complexity concerning the EQV forward dynamics algorithm is given in Table 2. It is worth to notice that the computational complexity arising from this table contain only numbers of arithmetical operations under assumption that spectral decomposition was done earlier. Junkins and Schaub [28] used for the decomposition purpose Jacobi method. However the above-mentioned method is suitable

Table 2 Computational complexity of eigenfactor quasi-vector velocities (EQV) forward dynamics algorithm

Component	Multiplications/divisions	Additions/subtractions
${}^{k+1}\mathbf{R}_k$	$4\mathcal{N}$	0
$\mathbf{M}(\boldsymbol{\theta})$	$6\mathcal{N}^2 + 19\mathcal{N} + 8$	$2\mathcal{N}^3 + 3\mathcal{N}^2 + 10\mathcal{N}$
$\mathbf{G}(\boldsymbol{\theta})$	$\mathcal{N}^2 + \mathcal{N}$	$\mathcal{N}^2 - \mathcal{N}$
$\mathbf{S}$	$\mathcal{N}$	0
$\boldsymbol{\eta}$	$\mathcal{N}^2 + \mathcal{N}$	$\mathcal{N}^2 - \mathcal{N}$
$\boldsymbol{\mu}$	$2\mathcal{N}^3$	$2\mathcal{N}^3 - 2\mathcal{N}^2$
$\mathbf{col}\left(\dot{\boldsymbol{\theta}}^T \left(\partial\mathbf{M}/\partial\theta_k\right) \dot{\boldsymbol{\theta}}\right)$	$\mathcal{N}^3 + \mathcal{N}^2 + \mathcal{N}$	$\mathcal{N}^3 - \mathcal{N}$
$\boldsymbol{\Omega}$	$(1/2)\left(\mathcal{N}^2 - \mathcal{N}\right)$	$(1/2)\left(\mathcal{N}^2 - \mathcal{N}\right)$
$\dot{\mathbf{S}}$	$3\mathcal{N}$	0
$\mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \boldsymbol{\eta})$	$4\mathcal{N}^2 - \mathcal{N}$	$2\mathcal{N}^2 - \mathcal{N}$
$\boldsymbol{\varepsilon}$	$\mathcal{N}^2$	$\mathcal{N}^2 - \mathcal{N}$
$\dot{\boldsymbol{\eta}}$	0	$\mathcal{N}$
Subtotal	$3\mathcal{N}^3 + 14(1/2)\mathcal{N}^2 + 28(1/2)\mathcal{N} + 8$	$5\mathcal{N}^3 + 6(1/2)\mathcal{N}^2 + 5(1/2)\mathcal{N}$

Table 3 Computational complexity of generalized velocity components (GVC) forward dynamics algorithm

Component	Multiplications/divisions	Additions/subtractions
${}^{k+1}\mathbf{R}_k$	$4\mathcal{N}$	0
$\mathbf{M}(\boldsymbol{\theta})$	$6\mathcal{N}^2 + 19\mathcal{N} + 8$	$2\mathcal{N}^3 + 3\mathcal{N}^2 + 10\mathcal{N}$
$\boldsymbol{\Upsilon}$	$(1/2)\mathcal{N}^2 - 1(1/2)\mathcal{N} + 1$	$(1/2)\mathcal{N}^2 - 1(1/2)\mathcal{N} + 1$
$\mathbf{u}$	$(1/2)\mathcal{N}^2 - (1/2)\mathcal{N}$	$(1/2)\mathcal{N}^2 - (1/2)\mathcal{N}$
$\mathbf{C}\left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \mathbf{u}\right)$	$3\mathcal{N}^2 + 19\mathcal{N} + 24$	$4\mathcal{N}^2 + 4\mathcal{N} + 15$
$\boldsymbol{\pi}$	$4\mathcal{N}$	$\mathcal{N}^2 + 2\mathcal{N}$
$\dot{\mathbf{u}}$	$2\mathcal{N}$	$\mathcal{N}$
Subtotal	$10\mathcal{N}^2 + 46\mathcal{N} + 33$	$2\mathcal{N}^3 + 9\mathcal{N}^2 + 15\mathcal{N} + 16$

for not very large systems. According to Ref. [17] the Jacobi method is used for systems with maximal rank $\mathcal{N} < 10$. From this work we conclude that the additionally spectral decomposition of mass matrix gives about $20\mathcal{N}^3$ arithmetical operations [17]. It is an essential source of the computational complexity. Therefore the forward dynamics algorithm needs $28\mathcal{N}^3 + 21\mathcal{N}^2 + 34\mathcal{N} + 8$ arithmetical operations.

6.3 Decoupled equation of motion in terms of GVC

Computational complexity of the GVC forward dynamics algorithm is given in Table 3. The given number of arithmetical operations is "Subtotal" because we have to calculate operations needed to obtain matrices Υ^{-1} and $\mathbf{N}$. Additionally calculation of these matrices gives about $(1/6)\mathcal{N}^3$ arithmetical operations (Cholesky–Banachiewicz method) [17]. This algorithm has then $2(1/6)\mathcal{N}^3 + 19\mathcal{N}^2 + 61\mathcal{N} + 49$ arithmetical operations.

6.4 Comparison of different forward dynamics algorithms

It is also of interest to compare results of the above considerations with other forward dynamics algorithms. They are summarized in Table 4. In order to compare the results we calculate number of arithmetical operation for three cases $\mathcal{N} = 3$, $\mathcal{N} = 6$ and $\mathcal{N} = 12$ dof, respectively.

Table 4 Number of operations for different forward dynamics algorithms, $\mathcal{N} \geq 2$

Method	Computational complexity	Number of arithmetical operations		
		$\mathcal{N} = 3$	$\mathcal{N} = 6$	$\mathcal{N} = 12$
Articulated body [11]	$579\mathcal{N} - 526$	1, 211	2, 948	6, 422
First Walker and Orin [63]	$(1/3)\mathcal{N}^3 + 130(1/2)\mathcal{N}^2$ $+197(1/6)\mathcal{N} - 33$	1, 742	5, 920	21, 701
Second Walker and Orin [63]	$(1/3)\mathcal{N}^3 + 66(1/2)\mathcal{N}^2$ $+261(1/6)\mathcal{N} - 33$	1, 358	4, 000	13, 253
Third Walker and Orin [63]	$(1/3)\mathcal{N}^3 + 21(1/2)\mathcal{N}^2$ $+358(1/6)\mathcal{N} - 113$	1, 164	2, 882	7, 857
Fourth Walker and Orin [63]	$132(1/2)\mathcal{N}^2 + 207\mathcal{N} - 31$	1, 783	5, 981	21, 533
Brandl et al. [11]	$470\mathcal{N} - 420$	990	2, 400	5, 220
Numerical solution of the dynamics equations [36]	$(1/3)\mathcal{N}^3 + 20(1/2)\mathcal{N}^2$ $+341(1/6)\mathcal{N} - 395$	822	2, 462	7, 227
Kalman filtering and smoothing [11]	$477\mathcal{N} - 503$	928	2, 359	5, 221
McMillan and Orin [41]	$429\mathcal{N} - 507$	780	2, 067	4, 641
Saha [48]	$394\mathcal{N} - 696$	486	1, 668	4, 032
Canonical momenta [42]	$363\mathcal{N} - 475$	614	1, 703	3, 881
Normalized equations of motion (NQV)	$1,019\mathcal{N} - 909$	2, 148	5, 205	11, 319
Unnormalized equations of motion (UQV)	$811\mathcal{N} - 603$	1, 830	4, 263	9, 129
Junkins and Schaub (EQV)	$28\mathcal{N}^3 + 21\mathcal{N}^2 + 34\mathcal{N} + 8$	1, 055	7, 016	51, 824
Loduha and Ravani (GVC)	$2(1/6)\mathcal{N}^3 + 19\mathcal{N}^2$ $+61\mathcal{N} + 49$	462	1, 567	7, 261

For a 3 dof manipulator GVC algorithm has better computational complexity than other algorithms. Also EQV results are better than Walker and Orin forward algorithm or articulated body [63]. It is also better than NQV or UQV algorithm, but not effective than others. NQV and UQV have bigger computational complexity in this case.

For a 6 dof manipulator different results are obtained. EQV algorithm is the worst whereas NQV and UQV are better than the first and fourth Walker and Orin algorithms. An exception is algorithm in terms of GVC which still gives better result.

For a 12 dof manipulator GVC forward algorithm is not so effective as earlier. Normalized and unnormalized forward algorithms are only slightly better than the first, second and fourth Walker and Orin algorithms. The EQV forward algorithm is also the worst in this case. The numerical method used by Junkins and Schaub [28,52] is suitable however for systems containing under 10 d.o.f. [17].

From Table 4 we conclude that the EQV forward algorithm is suitable practically for 3 or 4 dof manipulators because it gives satisfactory computational complexity (such systems were considered in Refs. [27,28,53]). NQV and UQV algorithms have better computational complexity only than some classical forward algorithms 6 and 12 dof, but they are not effective as the others. GVC forward algorithm gives quite good results for systems with 3 and 6 dof. Afterward GVC loses its superiority. Therefore GVC, NQV and UQV are suitable for 6 or <6 dof manipulators.

In all algorithms given in Table 4 the number of operation required to perform integration in the forward dynamics problem is not included.

7 Simulation results

In order to illustrate a character of quasi-velocities we present simulation investigations for model of 3 dof 3-*D* manipulator direct drive arm (DDArm). The DDA robot is characterized by the following set of dynamic parameters [1]:

- link masses: $m_1 = 19.67$, $m_2 = 53.01$ and $m_3 = 67.13$ kg;
- link inertias: $J_{xx1} = 0.1825$, $J_{xx2} = 3.8384$, $J_{xx3} = 23.1568$, $J_{xy1} = J_{xy2} = J_{xy3} = 0$, $J_{xz1} = -0.0166$, $J_{xz2} = 0$, $J_{xz3} = 0.3145$, $J_{yy1} = 0.4560$, $J_{yy2} = 3.6062$, $J_{yy3} = 20.4472$, $J_{yz1} = 0$, $J_{yz2} = -0.0709$, $J_{yz3} = 1.2948$, $J_{zz1} = 0.3900$, $J_{zz2} = 0.6807$ and $J_{zz3} = 0.7418$ kgm^2;
- distance: axis of rotation – mass center: $p_{x1} = 0.0158$, $p_{y2} = -0.0643$, $p_{y3} = -0.0362$, $p_{z1} = 0.0166$, $p_{z2} = -0.1480$ and $p_{z3} = 0.5337$ m;
- length of link: $l_2 = 0.462$ m;
- angle $\alpha : \alpha_1 = \alpha_2 = -90°$ and $\alpha_3 = 0°$.

Kinematical scheme of DDArm manipulator is given in Fig. 5. For simulation a fifth-order polynomial in joint space was chosen to generate the desired trajectory.

Start points are (with index *i*): $\theta_{i1} = -7/6 * \pi$, $\theta_{i2} = 269.1/180 * \pi$, $\theta_{i3} = -5/9 * \pi$ rad, and final points (with index *f*) $\theta_{f1} = 2/9 * \pi$, $\theta_{f2} = 19.1/180 * \pi$, $\theta_{f3} = 5/6 * \pi$ rad, with time duration $t_f = 1.3$ s.

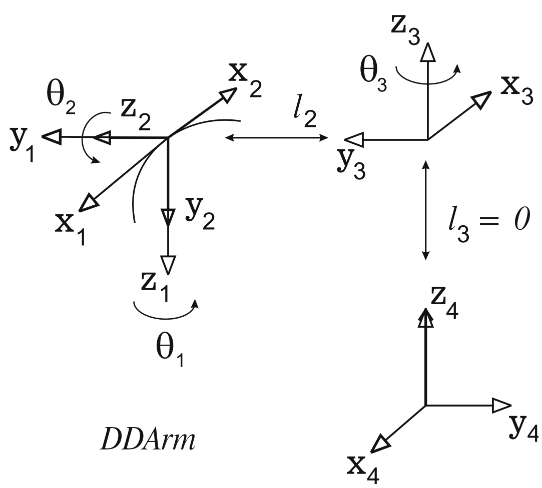

Fig. 5 Kinematical structure of the direct drive arm (DDArm) manipulator

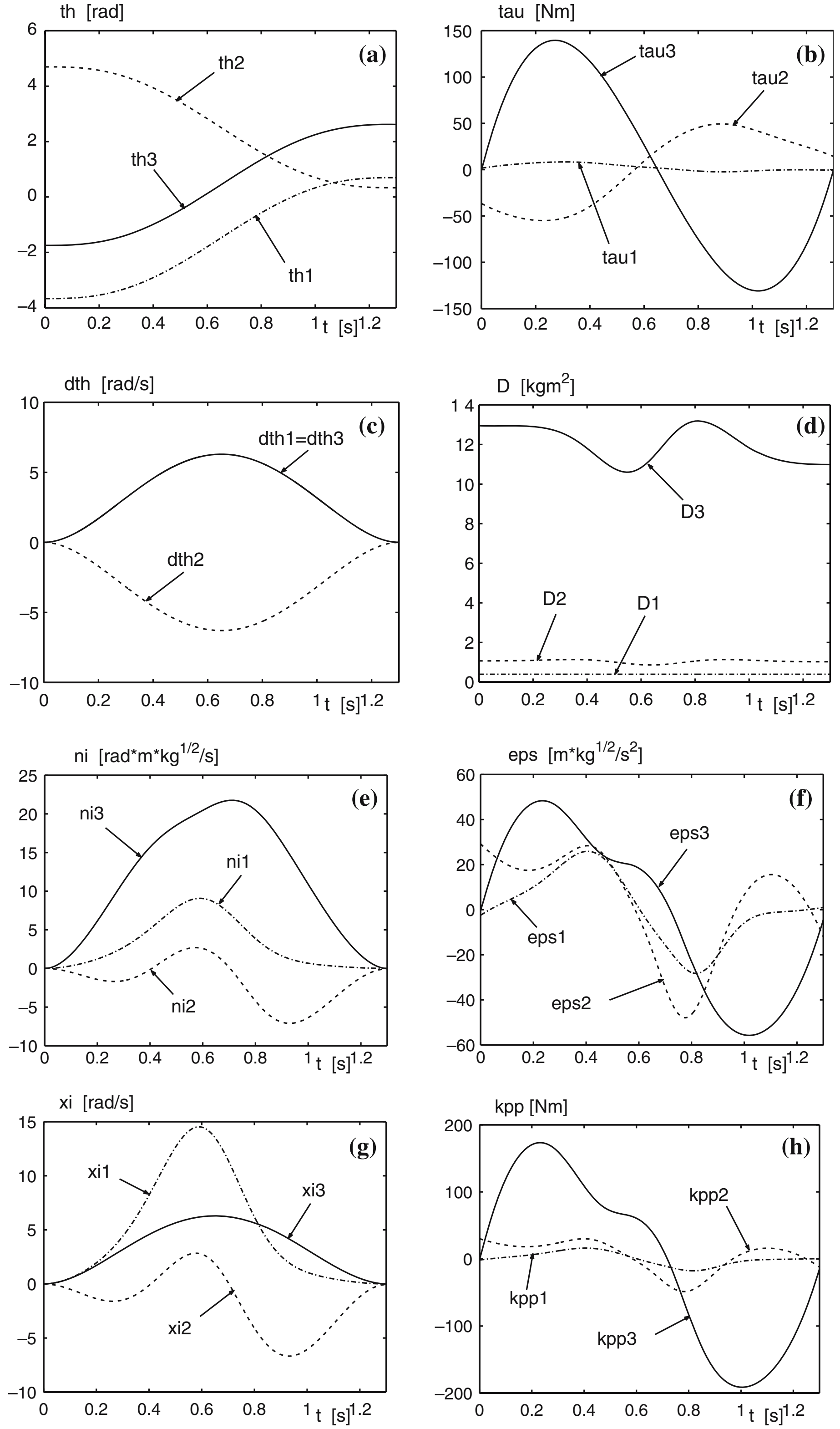

Fig. 6 Simulation results: **a** desired trajectory profiles in joint space, **b** desired joint moments τ_k(tau) for all joints (MATLAB/SIMULINK), **c** realized velocities in joint space, **d** articulated inertias $D_k(D)$ about the kth joint axis, **e** NQV $\nu_k(ni)$, **f** normalized quasi-moments ϵ_k(eps), **g** UQV ξ_k(xi), **h** unnormalized quasi-moments κ_k(kpp)

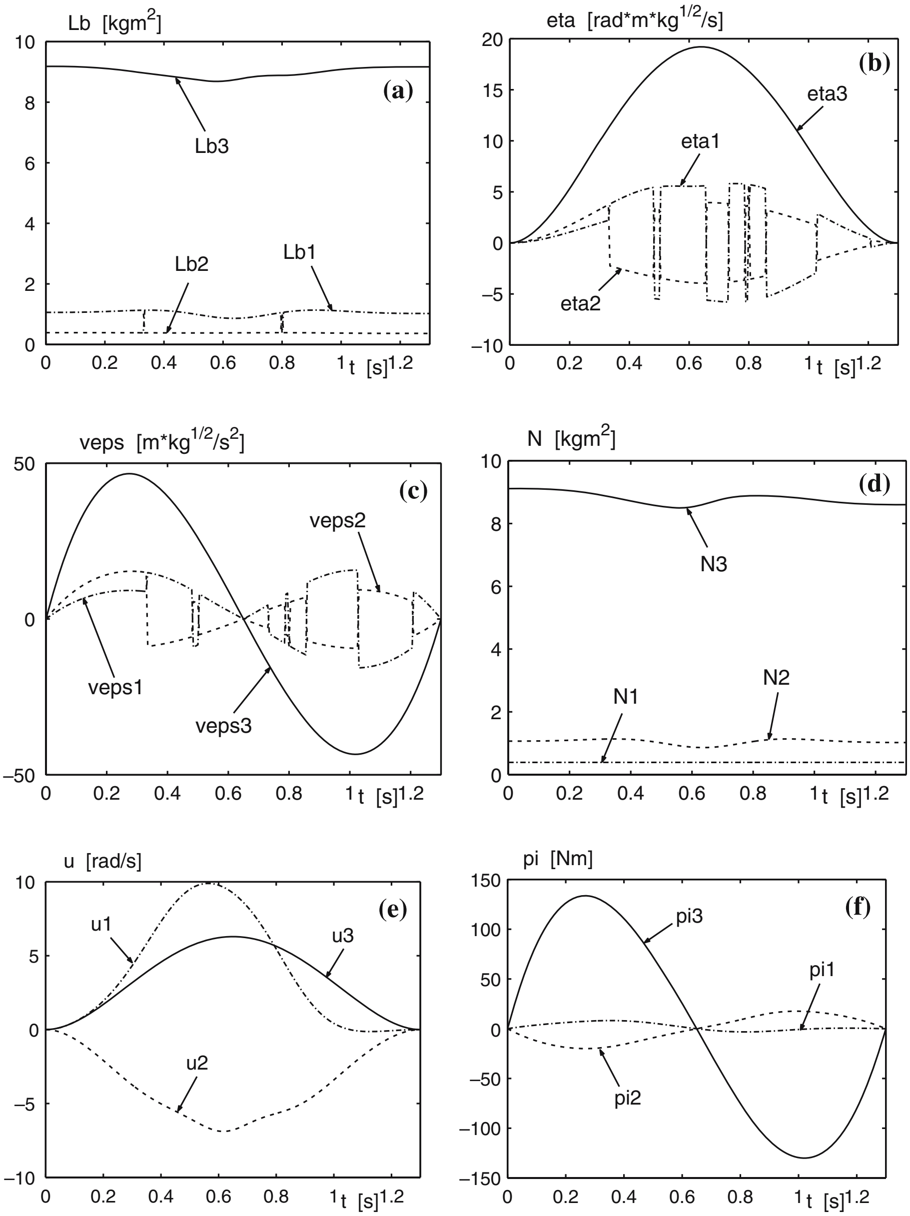

Fig. 7 Simulation results: **a** eigenvalues arising from spectral decomposition λ_k(Lb) for EQV formulation, **b** EQV eta$_k$(eta) for all joints, **c** quasi-moments for EQV formulation, **d** elements of diagonal matrix **N** for GVC formulation, **e** quasi-velocities $u_k(u)$ (GVC for all joints, **f** quasi-moments π_k(pi) for GVC formulation

Maximal peak velocity was $\dot{\boldsymbol{\theta}}_{k\max} = 6.29$ rad/s for each link, and maximal acceleration $\ddot{\boldsymbol{\theta}}_{k\max} = 14.91$ rad/s^2 (for $k = 1, 2, 3$). The assumed trajectories are similar as in [1].

For NQV and UQV cases the appropriate programs were coded in MATLAB with the fixed step size 0.005 s. For EQV and GVC cases we used MATLAB with SIMULINK and the same fixed step size.

Simulation results are given in Figs. 6 and 8. In Fig. 6a desired joint trajectory and desired velocity profile for three joints are shown. Figure 6b shows input moments for all joints of the manipulator (obtained using MATLAB with SIMULINK). Next Fig. 6c shows realized joint velocities. For the second and third joint they have the same values. In Fig. 6d one can see the obtained quantities D_k (for NQV and UQV cases they are identical). Quasi-velocities for NQV ν_k are compared in Fig. 6e. Unlike joint velocities they have different values in terms of new description of motion. Figure 6f gives quasi-moments ϵ_k for all joint. Notice, that quasi-velocities ν_k and quasi-moments ϵ_k have different physical units as joint velocities and joint moments, respectively. It is because they are realized in different inertial–geometrical space. Appropriate to UQV (unnormalized case)

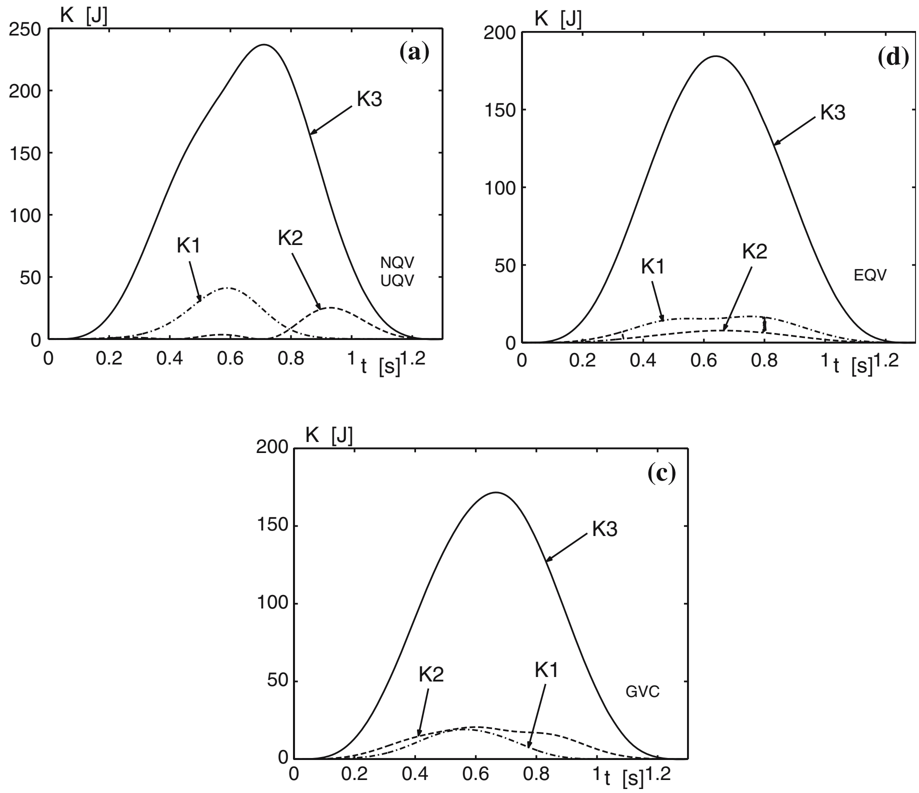

Fig. 8 Simulation results: **a** kinetic energy for all joints for NQV/UQV formulation, **b** kinetic energy for all joints for EQV formulation, **c** kinetic energy for all joints for GVC formulation

quantities are shown in Fig. 6g and h, respectively. Physical units of quasi-velocities and quasi-moments are the same as joint velocities and moments. Here the quantities represent influence of all links but in the same as classical geometrical space. In Fig. 7a one can see the eigenvalues λ_k arising from spectral decomposition of manipulator mass matrix in EQV (Junkins–Schaub method). These quantities reflect changes of link inertias for the kth link during the motion of the manipulator. The next two, Figs. 7b and c, compare EQV and quasi-moments for Junkins–Schaub method, respectively. They are expressed in the same inertial–geometrical space as NQV normalized quasi-velocities. However, results of velocities decoupling is here quite different for the first and the second joint. The last three (Figs. 7d, e, f) are concerned with GVC (Loduha–Ravani method). In Fig. 7d elements of diagonal matrix **N** are given. They reflect, from a physical point of view, changes of link inertias during the motion of the manipulator. Figure 7e compares new quasi-velocities. The first of them is distinctly decoupled from the third one. Figure 7f shows joint quasi-moments. Notice, that their final values are equal zero because gravitational forces are not transformed directly into the GVC space. In Fig. 8a, b and c the kinetic energy for all joints of manipulator was shown. Figure 8a presents results obtained for NQV and UQV, Fig. 8b for EQV and Fig. 8c for GVC. The most of energy is transferred by the third quasi-velocity $K3$ in each case. But for NQV/UQV in the first time interval more energy is transferred by the first quasi-velocity $K1$ than by the second ($K2$). At the end the situation changes. For EQV case $K1$ is dominant in comparison with $K2$ and for GVC $K2$ is dominant. The transferred kinetic energy is in relationship with the appropriate quasi-velocities, i.e. NQV/UQV, EQV and GVC. Hence values of $K1$, $K2$ and $K3$ are different for each case. They contained the part of energy which arises from couplings among links and in each kind of velocity the method is different.

8 Conclusions

In this work we have compared several equations of motion described in terms of various inertial quasi-velocities, namely Jain–Rodriguez NQV and UQV, Junkins–Schaub the EQV and Loduha–Ravani GVC. These formulations are different but all give a possibility of using them for manipulator behavior description and control. After application of these methods one can obtain, instead of the second-order differential equation of motion, two kind of equations: a first-order decoupled differential equation of motion and a first-order relationship between joint velocities and quasi-velocities. Mass matrix of manipulator has a diagonal form which simplify its inversion. Besides that in Jain and Rodriguez formulation deeper insights into manipulator dynamics was observed because recursions involve spatial operators.

Equations of motion in terms of various inertial quasi-velocities and remarks concerned the physical interpretation of these variables were presented. A comparison of these new formulations, differences between them and well-known quasi-velocities or the generalized velocities were reported. It describes properties useful for simulation and control purposes, including possibility of the kinetic energy shaping. The forward dynamics algorithms for robotic manipulator were proposed with their computational complexity comparison. It was shown that the inertial quasi-velocities are suited for control of the manipulator if its model is exactly known.

For the purpose of control we have to transform physical variables into quasi-velocity space and realize control in this space and at the end we have to transform quasi-moments into physical moments which are input signals to manipulator. The inertial quasi-velocities in NQV and EQV formulations have the same physical units but give different quasi-velocities what arises from their different transformation equations.

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