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Progress in shakedown analysis with applications to composites

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Abstract This paper reports on recent developments in the field of direct methods, in particular shakedown analysis (SA) from theoretical and numerical points of view. Emphasis is placed on problems connected with the failure of fibre-reinforced periodic composites under variable loads where special attention is paid to the problem of interface debonding between fibre and matrix materials. The approach is based on a local SA in a representative volume element of the composite and the use of averaging techniques to study the influence of each component (matrix, fibre and interface) on the macroscopic response of such composite. For numerical applications, the interior-point difference-of-convex-functions algorithm (IPDCA) is proposed as an efficient method for solving large-scale problems.

Keywords Shakedown analysis · Averaging techniques · Non-linear optimisation · Interfacial damage

1 Introduction

Guaranteeing the integrity of structures or structural elements as a necessary condition for safe functioning has always been one of the most important tasks for mechanical and civil engineers. Consequently, the development of practical methods to predict whether under specific service conditions failure would occur or not was necessary, even in times when computers and modern numerical methods did not yet exist. One of the most powerful and successfully applied branches of engineering was limit analysis (LA), which stemmed from the combination of classical plasticity in the form of Tresca or von Mises yield conditions and the assumption of validity of the normality rule for plastic flow, the satisfaction of equilibrium conditions (in the case of static approaches), specific kinematical conditions related to the type of structure being considered (beam, plate, etc.) and guesses at the collapse mechanisms (in the case of kinematical approaches). From a methodological point of view it is important that in this context plasticity conditions did not complicate but helped to simplify the specific problems considered. Whilst LA aims at monotonous loading processes, shakedown analysis (SA) considers mechanical systems under variable, nondeterministic loads within deterministic bounds [1,2]. Obviously, this problem is more complicated than the problem of LA, and analytical solutions are more difficult to find, although the theoretical foundations of SA had been established before those of LA. (SA and LA were developed independently; today we know that LA can be considered as a particular case of SA. The reader interested in the fundamentals of LA and SA and their historical development is referred to [3–8]).

What is characteristic of LA and SA is the fact that information, regardless of whether or not failure occurs, is obtained directly, without solving an evolutionary problem. Therefore, LA and SA are often called direct methods in contrast to step-by-step methods. It is important to understand that the theoretical core of SA and LA is the same: (1) active failure mechanisms are dissipative and irreversible; (2) locally sustainable dissipation is bounded; (3) in a safe state, no dissipation occurs in a structure.

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Classical plasticity as the framework within which the fundamental theorems of SA and LA were formulated satisfies these conditions. But it is interesting to note that other material phenomena leading to failure fit within the same concept: it has been shown that different classes of material hardening, material damage and, under certain restrictive conditions, crack propagation can be addressed within the same theoretical framework, and many papers have been devoted to the extension of the classical theorems to these types of material phenomena (see [9] and references therein).

A rather new area of application with a strong practical impact is the study of structured materials, basically composites with at least one ductile phase. These materials can be considered as mechanical structures by themselves and therefore treated with the same methods as mechanical structures taken in the usual sense. The present paper, a continuation of earlier work by the authors [10, 11], focuses on the contribution of dissipative effects in the interface between matrix materials (here considered as metal type) and reinforcing fibres. Experimental and numerical observations show that the global behaviour of fibre-reinforced composites depends on the local interfacial damage. Following Carrère et al. [12] two domains can be evidenced as a function of the applied stress. Below the critical stress, interfaces remain undamaged, and above, interfaces are totally debonding. In this paper, the process of debonding is modelled within the framework of interface damage mechanics, where the displacement discontinuities during the progressive decohesion are related to constitutive equations extended by an anisotropic damage model [13, 14].

The presented work is based on a two-scale approach. On the macroscopic scale, the global response of the composite is analysed. On the microscopic scale, the influence of each component (matrix, fibre and interface) on the behaviour of the composite is investigated. The methods used are an averaging technique, in particular the homogenisation technique [15], combined with SA [1], applied to a representative volume element on the micro-level. This allows us, under the assumption of periodicity of the composite, to link the results of the SA on the micro-level to the overall material properties on the macro-level.

2 Definitions and general assumptions

A periodic medium can be defined by a unit cell (representative volume element) and three vectors (in 3D) of a translation invariance. Although the unit cell is not uniquely defined, the effective behaviour of the composite computed from different unit cells that generate the same microstructure should coincide (since the physical effective properties of the composite are well defined independently of the choice of the unit cell) [16]. The choice of the unit cell is often motivated by the differences in geometrical symmetries, which can be used to simplify the numerical solution of the local problem. We consider the problem of ductile (metal-type) matrix materials reinforced by straight fibres. In the case of transverse loading, this justifies the hypothesis of a plane state of strains. Thus, the problem is reduced to a 2D case with periodicity in the fibre section plane. (Note that this restriction is not essential for the theory developed in the sequel). The macroscopic behaviour of this heterogeneous material is observed on scale $\mathbf{x}$ and the mesoscopic behaviour on scale $\mathbf{y}$. A significant characteristic of fibre-reinforced composites is the limited strength of the interface between the fibres and the matrix. The micromechanical representative element of total surface Ω presented herein includes the fibre phase, the matrix phase occupying the domains Ω_+ and Ω_-, respectively, and the cohesive zone represented by the band-shaped domain Ω_δ of width $\delta << 1 (\Omega = \Omega_- \cup \Omega_+ \cup \Omega_\delta)$, as illustrated in Fig. 1, defined by

$$\Omega_\delta = \left[\mathbf{y} = \mathbf{y}_0 + \boldsymbol{\epsilon}\mathbf{n} \mid \forall \mathbf{y}_0 \in \Gamma, -\frac{\delta}{2} \leq \boldsymbol{\epsilon} \leq \frac{\delta}{2} \right]. \tag{1}$$

The strain field corresponding to the displacement field is derived as

$$\boldsymbol{\varepsilon} = \nabla_s \underline{\mathbf{u}} + ([\![\mathbf{u}]\!] \otimes_s \mathbf{n}\delta_R), \ \forall \mathbf{y} \in \Omega, \tag{2}$$

where $\underline{\mathbf{u}}$ is the continuous displacement field and $[\![\mathbf{u}]\!]$ the displacement jump vector, i.e. $[\![\mathbf{u}]\!] = \mathbf{u}^+ - \mathbf{u}^-$. Here, $\mathbf{u}^+$ and $\mathbf{u}^-$ are the displacement vectors at the interior and the exterior borders of the interface zone, $\mathbf{n}$ is the unit vector and δ_R is the Dirac delta function defined by

$$\delta_R = \begin{cases} 0 & \text{if } \mathbf{y} \in \Omega, \\ \frac{1}{\delta} & \text{if } \mathbf{y} \in \Omega_\delta, \\ 0 & \text{if } \mathbf{y} \in \Omega_+. \end{cases} \tag{3}$$

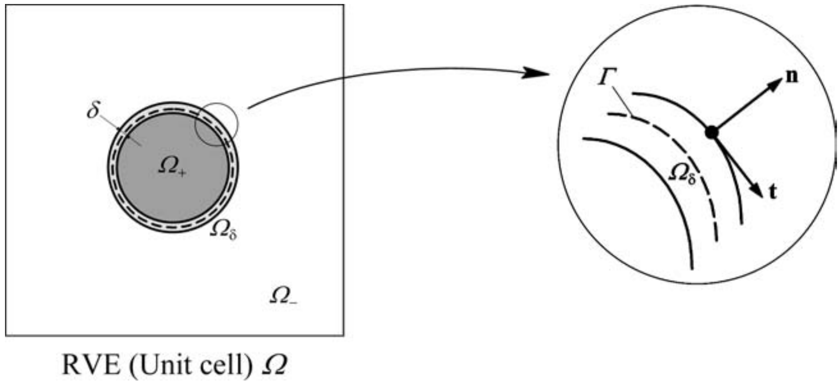

Fig. 1 Discontinuity interface Γ

Restricting our considerations to geometrically linear theory, the total strains $\boldsymbol{\varepsilon}(\mathbf{y})$ can be split into purely elastic $\boldsymbol{\varepsilon}^{\mathrm{e}}$ and purely inelastic $\boldsymbol{\varepsilon}^{\mathrm{in}}$ parts, respectively:

$$\boldsymbol{\varepsilon}(\mathbf{y}) = \boldsymbol{\varepsilon}^{\mathrm{e}}(\mathbf{y}) + \boldsymbol{\varepsilon}^{\mathrm{in}}(\mathbf{y}), \quad \forall \mathbf{y} \in \Omega. \tag{4}$$

Consequently, the total strain field $\boldsymbol{\varepsilon}$ can be decomposed independently into a continuous strain field $\underline{\boldsymbol{\varepsilon}}$ and a regularised strain field $\boldsymbol{\varepsilon}_\delta$ within the interface in the following way:

$$\boldsymbol{\varepsilon}(\mathbf{y}) = \underline{\boldsymbol{\varepsilon}}(\mathbf{y}) + \boldsymbol{\varepsilon}_\delta(\mathbf{y}), \quad \forall \mathbf{y} \in \Omega, \tag{5}$$

with

$$\underline{\boldsymbol{\varepsilon}}(\mathbf{y}) = \underline{\boldsymbol{\varepsilon}}^{\mathrm{e}}(\mathbf{y}) + \underline{\boldsymbol{\varepsilon}}^{\mathrm{in}}(\mathbf{y}), \quad \forall \mathbf{y} \in \Omega_- \cup \Omega_+, \tag{6}$$

$$\boldsymbol{\varepsilon}_\delta(\mathbf{y}) = \boldsymbol{\varepsilon}_\delta^{\mathrm{e}}(\mathbf{y}) + \boldsymbol{\varepsilon}_\delta^{\mathrm{in}}(\mathbf{y}), \quad \forall \mathbf{y} \in \Omega_\delta. \tag{7}$$

For the considered unit cell Ω, we adopt the usual homogenisation assumption for the local displacement field $\mathbf{u}$ at position $\mathbf{y}$

$$\mathbf{u} = \mathbf{E} \cdot \mathbf{y} + \mathbf{u}^{\mathrm{per}}, \tag{8}$$

where $\mathbf{E}$ is the macroscopic strain tensor and $\mathbf{u}^{\mathrm{per}}$ is a displacement field satisfying the periodicity conditions. Then, the Hill relationship holds [17]:

$$\boldsymbol{\Sigma} : \mathbf{E} =< \boldsymbol{\sigma} : \boldsymbol{\varepsilon} >= \frac{1}{|\Omega|} \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \, \mathrm{d}\Omega, \tag{9}$$

where

$$\Sigma(\mathbf{x}) =< \boldsymbol{\sigma}(\mathbf{y}) >= \frac{1}{|\Omega|} \int_\Omega \boldsymbol{\sigma}(\mathbf{y}) \mathrm{d}\Omega \tag{10}$$

and

$$\mathbf{E}(\mathbf{x}) =< \boldsymbol{\varepsilon}(\mathbf{y}) >= \frac{1}{|\Omega|} \int_{\partial\Omega} (\mathbf{u} \otimes_s \mathbf{n}) \mathrm{d}S. \tag{11}$$

Here, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are mesoscopic stresses and strains that also satisfy the periodicity condition. Within the unit cell, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ fulfil compatibility and equilibrium conditions, respectively.

For the plastic part of the matrix and/or fibre behaviour, we assume the validity of the generalised normality rule for plastic flow in subdifferential form, such that

$$\dot{\underline{\boldsymbol{\varepsilon}}}^{\mathrm{in}} \in \delta\varphi(\boldsymbol{\sigma}), \tag{12}$$

where $\delta\varphi(\boldsymbol{\sigma})$ denotes the subgradient of the plastic potential $\varphi(\boldsymbol{\sigma})$, which is the indicator function of a convex domain $\mathrm{F}(\mathbf{y})$ of all plastically admissible stress states. Here and in the sequel, superposed dots denote the rate of the considered quantity. Then we have

$$\boldsymbol{\sigma}(\mathbf{y}) \in \mathrm{F}(\mathbf{y}), \quad \forall \mathbf{y} \in \Omega_- \cup \Omega_+, \tag{13}$$

where $\mathrm{F}(\mathbf{y})$ is defined by means of a yield function $f(\boldsymbol{\sigma}, \mathbf{y})$,

$$\mathrm{F}(\mathbf{y}) = \left[\boldsymbol{\sigma} \mid f(\boldsymbol{\sigma}, \mathbf{y}) \leq 0, \ \forall \mathbf{y} \in \Omega_- \cup \Omega_+\right]. \tag{14}$$

The convexity of $f(\boldsymbol{\sigma}, \mathbf{y})$ and the validity of the generalised normality rule can be expressed by the maximum plastic work inequality

$$(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{(\mathrm{s})}) : \dot{\underline{\boldsymbol{\varepsilon}}}^{\mathrm{in}} > 0, \quad \forall \boldsymbol{\sigma}^{(\mathrm{s})}(\mathbf{y}) \in \bar{\mathrm{F}}(\mathbf{y}), \tag{15}$$

where $\boldsymbol{\sigma}^{(\mathrm{s})}$ is any safe state of stresses defined by

$$\bar{\mathrm{F}}(\mathbf{y}) = \left[\boldsymbol{\sigma}^{(\mathrm{s})} \mid f(\boldsymbol{\sigma}^{(\mathrm{s})}, \mathbf{y}) < 0, \ \forall \mathbf{y} \in \Omega_- \cup \Omega_+\right]. \tag{16}$$

3 Interfacial constitutive equations

The mechanical response of the interface is described by the relation between the normal and tangential relative displacements $[\![u_n]\!]$ and $[\![u_t]\!]$ and their respective normal and tangential tractions τ_n and τ_t. The relative displacement across the interface $[\![\mathbf{u}]\!]$ is decomposed into an elastic part $[\![\mathbf{u}^\mathrm{e}]\!]$ and an inelastic part $[\![\mathbf{u}^\mathrm{in}]\!]$

$$[\![\mathbf{u}]\!] = [\![\mathbf{u}^\mathrm{e}]\!] + [\![\mathbf{u}^\mathrm{in}]\!] \tag{17}$$

At each point of the interface, we define

$$u_\mathrm{n} = \mathbf{n} \cdot [\![\mathbf{u}]\!], u_\mathrm{t} = \mathbf{t} \cdot [\![\mathbf{u}]\!] \tag{18}$$

and

$$\tau_\mathrm{n} = \mathbf{n} \cdot \boldsymbol{\tau}, \tau_\mathrm{t} = \mathbf{t} \cdot \boldsymbol{\tau}, \tag{19}$$

where $\mathbf{n}$ and $\mathbf{t}$ form a right-hand coordinate system. The crack initiation and crack propagation between fibre and matrix is simulated by using a strain-based anisotropic damage model in the sense of Ju [18]. To this end, we separately introduce a normal damage variable d_n and a tangential damage variable d_t. To describe the interface constitutive equations, we introduce a thermodynamical potential

$$\Psi = \frac{1}{2}[\![\mathbf{u}^\mathrm{e}]\!]^\mathrm{T} \cdot \tilde{\mathbf{Q}}_\delta \cdot [\![\mathbf{u}^\mathrm{e}]\!] \tag{20}$$

with

$$[\![\mathbf{u}^\mathrm{e}]\!] = [\![u^\mathrm{e}_\mathrm{n}]\!]\mathbf{n} + [\![u^\mathrm{e}_\mathrm{t}]\!]\mathbf{t} \text{ and } \tilde{\mathbf{Q}}_\delta = (\mathbf{I} - \mathbf{d})\mathbf{Q}_\delta = (1 - d_\mathrm{n})Q_{\delta n}\mathbf{n} \otimes_\mathrm{s} \mathbf{n} + (1 - d_\mathrm{t})Q_{\delta t}\mathbf{t} \otimes_\mathrm{s} \mathbf{t}, \tag{21}$$

where $\mathbf{Q}_\delta$ represents the interface stiffness, i.e. $\mathbf{Q}_\delta = \mathbf{n} \cdot \mathbf{C} \cdot \mathbf{n}/\delta$. Here, $\mathbf{C}$ denotes the fourth-order elasticity tensor. This leads us to define, respectively, the traction forces as functions of elastic displacement discontinuities and damage variables and the thermodynamic forces associated with damage variables (see e.g. [13]):

$$\boldsymbol{\tau} = \frac{\partial \Psi}{\partial [\![\mathbf{u}^\mathrm{e}]\!]} = \tilde{\mathbf{Q}}_\delta \cdot [\![\mathbf{u}^\mathrm{e}]\!] \text{ and } \mathbf{Y} = -\frac{\partial \Psi}{\partial \mathbf{d}} = \frac{1}{2}[\![\mathbf{u}^\mathrm{e}]\!]^\mathrm{T} \cdot \mathbf{Q}_\delta \cdot [\![\mathbf{u}^\mathrm{e}]\!]. \tag{22}$$

In the sequel a superposed tilde indicates quantities related to the damaged state of the cohesive zone. It is assumed that both debonding and damage processes are irreversible with no cross-healing effect. Then we impose that the second principle of thermodynamics is satisfied in the restricted form of the Clausius–Duhem inequality

$$\boldsymbol{\tau} \cdot [\![\dot{\mathbf{u}}^\mathrm{in}]\!] \geq 0 \text{ and } \mathbf{Y} \cdot \dot{\mathbf{d}} \geq 0 \tag{23}$$

or, in index notation,

$$\tau_n[\![\dot{u}_n^{in}]\!] + \tau_t[\![\dot{u}_t^{in}]\!] \geq 0 \text{ and } \frac{1}{2}Q_{\delta n}[\![u_n^e]\!]^2\dot{d}_n + \frac{1}{2}Q_{\delta_t}[\![u_t^e]\!]^2\dot{d}_t \geq 0, \tag{24}$$

where the two inequalities are independent.

The inelastic parts of the displacement jump are expressed by the normality rule

$$[\![\dot{u}_n^{in}]\!] = \dot{\lambda}\frac{\partial g}{\partial \tau_n},\ [\![\dot{u}_t^{in}]\!] = \dot{\lambda}\frac{\partial g}{\partial \tau_t}, \tag{25}$$

where ($\dot{\lambda}$) is the plastic multiplier and g the inelastic gap potential defined by

$$g(\boldsymbol{\tau}, \mathbf{d}, \mathbf{y}) = \left[\left(\frac{\tau_n}{1-d_n}\right)^2 + \frac{1}{\beta^2}\left(\frac{\tau_t}{1-d_t}\right)^2\right]^{1/2} - \sigma_{\max} \leq 0, \forall \mathbf{y} \in \Omega_\delta, \tag{26}$$

which is a positive convex and differentiable function. The maximum plastic work inequality is valid, i.e.

$$(\boldsymbol{\tau} - \boldsymbol{\tau}^{(s)}) \cdot [\![\dot{\mathbf{u}}^{in}]\!] > 0, \quad \forall \boldsymbol{\tau}^{(s)}(\mathbf{y}) \in \bar{\mathrm{G}}(\mathbf{y}), \tag{27}$$

where

$$\bar{\mathrm{G}}(\mathbf{y}) = \left[\boldsymbol{\tau}^{(s)} \mid g(\boldsymbol{\tau}^{(s)}, \mathbf{d}, \mathbf{y}) < 0, \quad \forall \mathbf{y} \in \Omega_\delta\right]. \tag{28}$$

Here, $\sigma_{\max}$ is a measure of the bond strength of the cohesive zone and β is the interface shear-to-normal strength ratio.

As an example of a simple explicit form of (24), we cite the equations for the damage variables proposed by Bazant and Prat [19]:

$$d_n = 1 - \exp\left[-\left(\frac{\kappa_n}{a_n}\right)^n\right], d_t = 1 - \exp\left[-\left(\frac{\kappa_t}{a_t}\right)^t\right], \tag{29}$$

where the constants (n, a_n) and (t, a_t) are material parameters to be determined from experiment. The history parameters κ_n and κ_t are the damage thresholds at the current time t. They memorise the maximum values attained by $[\![\mathbf{u}_n]\!]$ and $[\![\mathbf{u}_t]\!]$ in the following way:

$$\kappa_n = \max\left[\kappa_{n_0}, \max_{t\in[0,T]}[\![u_n]\!]\right]; \kappa_t = \max\left[\kappa_{t_0}, \max_{t\in[0,T]}[\![u_t]\!]\right], \tag{30}$$

where κ_{n_0} and κ_{t_0} are the initial damage thresholds before any loading is applied.

4 Formulation of the shakedown problem

Adopting these definitions and hypotheses, the statical shakedown theorems can be stated as follows: if there exist a real number $\alpha > 1$, a time-independent field of periodic residual stresses $\mathring{\boldsymbol{\sigma}}^{(r)}$, a time-independent vector of traction forces $\mathring{\boldsymbol{\tau}}^{(r)}$ at the interface and an admissible domain P of macroscopic states of stress $\boldsymbol{\Sigma}$:

$$P = \left[\Sigma \mid \exists\boldsymbol{\sigma}^{(s)}, \exists\boldsymbol{\tau}^{(s)},\ \boldsymbol{\sigma}^{(s)}(\mathbf{y}) \in \bar{\mathrm{F}}(\mathbf{y}), \boldsymbol{\tau}^{(s)}(\mathbf{y}) \in \bar{\mathrm{G}}(\mathbf{y})\right], \tag{31}$$

then the fibre-reinforced composite material shakes down to the given domain of loading. Safe states of stresses $\boldsymbol{\sigma}^{(s)}$ and traction forces $\boldsymbol{\tau}^{(s)}$ are defined, respectively, by

$$\boldsymbol{\sigma}^{(s)} = \alpha\boldsymbol{\sigma}^{(c)} + \mathring{\boldsymbol{\sigma}}^{(r)}, \boldsymbol{\tau}^{(s)} = \alpha\boldsymbol{\tau}^{(c)} + \mathring{\boldsymbol{\tau}}^{(r)} \tag{32}$$

such that

$$\bar{\mathrm{F}}(\mathbf{y}) = \{\boldsymbol{\sigma}^{(s)} \,|\, f(\boldsymbol{\sigma}^{(s)}, \mathbf{y}) < 0, \forall \mathbf{y} \in \Omega_- \cup \Omega_+\} \tag{33}$$

and

$$\bar{\mathrm{G}}(\mathbf{y}) = \left[\boldsymbol{\tau}^{(\mathrm{s})} \,\middle|\, g(\boldsymbol{\tau}^{(\mathrm{s})}, \mathbf{d}, \mathbf{y}) < 0, \forall \mathbf{y} \in \Omega_\delta \right]. \tag{34}$$

Here, $\boldsymbol{\sigma}^{(\mathrm{c})}$ and $\boldsymbol{\tau}^{(\mathrm{c})}$ are, respectively, the stress field and the traction forces in a purely elastic representative volume element (RVE) under the same boundary conditions as for the RVE of the original problem such that the following relations hold:

$$\mathrm{Div}\,\boldsymbol{\sigma}^{(\mathrm{c})} = 0 \qquad \text{in } \Omega, \tag{35}$$

$$[\![\boldsymbol{\sigma}^{(\mathrm{c})}]\!] \cdot \mathbf{n}_\Gamma = 0 \qquad \text{on } \Gamma, \tag{36}$$

$$\mathbf{u}^{(\mathrm{c})} - \mathbf{E} \cdot \mathbf{y} \text{ periodic} \qquad \text{on } \partial\Omega, \tag{37}$$

$$\boldsymbol{\sigma}^{(\mathrm{c})} \cdot \mathbf{n} \text{ antiperiodic} \qquad \text{on } \partial\Omega, \tag{38}$$

$$\boldsymbol{\sigma}^{(\mathrm{c})} = \mathbf{C} : (\boldsymbol{\varepsilon}(\mathbf{u}^{\mathrm{per}}) + \mathbf{E}) \qquad \text{in } \Omega. \tag{39}$$

However, the field of the residual stresses $\mathring{\boldsymbol{\sigma}}^{(\mathrm{r})}$ satisfies

$$\mathrm{Div}\,\mathring{\boldsymbol{\sigma}}^{(\mathrm{r})} = \mathbf{0} \qquad \text{in } \Omega, \tag{40}$$

$$[\![\mathring{\boldsymbol{\sigma}}^{(\mathrm{r})}]\!] \cdot \mathbf{n}_\Gamma = 0 \qquad \text{on } \Gamma, \tag{41}$$

$$\mathring{\boldsymbol{\sigma}}^{(\mathrm{r})} \cdot \mathbf{n} \text{ antiperiodic} \qquad \text{on } \partial\Omega, \tag{42}$$

$$< \mathring{\boldsymbol{\sigma}}^{(\mathrm{r})} >= 0 \qquad \text{in } \Omega \tag{43}$$

and additionally

$$< \boldsymbol{\sigma}^{(\mathrm{s})}(\mathbf{y}) >= \boldsymbol{\Sigma}(\mathbf{x}), \tag{44}$$

where $\mathbf{n}_\Gamma$ is the normal unit vector at any point of Γ. For the proof, which follows due to the specific structure of the potentials (20) and (26) and the corresponding evolution laws of the classical argument, we refer the reader to [11].

5 Relative displacement bounding

As in the presented approach the damage $\mathbf{d}$ is a function of damage threshold $\boldsymbol{\kappa}$ (29), bounds for damage can be given in this special case by bounding the relative displacement $[\![\mathbf{u}]\!]$ which can be estimated from the extremal values of $[\![\mathbf{u}^{(\mathrm{c})}]\!]$ and $[\![\mathbf{u}^{\mathrm{R}}]\!]$

$$\mathrm{Max}[\![\mathbf{u}]\!] \leq \mathrm{Max}]\!]\mathbf{u}^{(\mathrm{c})}]\!] + \mathrm{Max}]\!]\mathbf{u}^{\mathrm{R}}]\!]. \tag{45}$$

Bounds to $[\![\mathbf{u}^{(\mathrm{c})}]\!]$ are obtained in a straightforward manner due to linearity in terms of the load multipliers. To evaluate bounds to $[\![\mathbf{u}^{\mathrm{R}}]\!]$, we use the reciprocal relation of elasticity extended to the case of noncompatible strains, which are identified here with the plastic strains

$$\int_{\partial\Omega} \hat{\mathbf{P}} \cdot \mathbf{u}\, \mathrm{d}S = \int_\Omega \hat{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}\, \mathrm{d}\Omega, \tag{46}$$

where $\hat{\boldsymbol{\sigma}}(\mathbf{y}, \mathbf{y}_0)$ stands for the purely elastic stress field in the RVE$^{(\mathrm{c})}$ loaded by the concentrated force ($\hat{\mathbf{P}}$) at point $\mathbf{y}_0$. Using relation (46), we deduce then

$$\int_\Gamma \hat{\mathbf{P}} \cdot [\![\mathbf{u}]\!]\, \mathrm{d}S = \int_{\Omega\delta} \hat{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}_\delta\, \mathrm{d}\Omega, \tag{47}$$

or in detail

$$\int_\Gamma \hat{\mathbf{P}} \cdot [\![\mathbf{u}^{(\mathrm{c})}]\!]\, \mathrm{d}S + \int_\Gamma \hat{\mathbf{P}} \cdot [\![\mathbf{u}^{\mathrm{R}}]\!]\, \mathrm{d}S = \int_{\Omega\delta} \hat{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}_\delta^{(\mathrm{c})}\, \mathrm{d}\Omega + \int_{\Omega\delta} \hat{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}_\delta^{(\mathrm{r})}\, \mathrm{d}\Omega + \int_{\Omega\delta} \hat{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}_\delta^{\mathrm{in}}\, \mathrm{d}\Omega. \tag{48}$$

Using a Betti–Maxwell relation,

$$\int_{\Omega\delta} \hat{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}_{\delta}^{(\mathrm{r})} \,\mathrm{d}\Omega = \int_{\Omega\delta} \boldsymbol{\sigma}_{\delta}^{(\mathrm{r})} : \hat{\boldsymbol{\varepsilon}} \,\mathrm{d}\Omega = 0, \tag{49}$$

we obtain

$$\int_{\Gamma} \hat{\mathbf{P}} \cdot [\![\mathbf{u}^{(\mathrm{c})}]\!] \mathrm{d}S = \int_{\Omega\delta} \hat{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}_{\delta}^{(\mathrm{c})} \,\mathrm{d}\Omega \tag{50}$$

and

$$\int_{\Gamma} \hat{\mathbf{P}} \cdot [\![\mathbf{u}^{\mathrm{R}}]\!] \mathrm{d}S = \int_{\Omega\delta} \hat{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}_{\delta}^{\mathrm{in}} \mathrm{d}\Omega. \tag{51}$$

If the yield surface contains the origin of the coordinate system and the material is initially virgin, then there exist two constants ζ and ξ such that

$$\zeta ||\dot{\boldsymbol{\varepsilon}}_{\delta}^{\mathrm{in}}|| \leq \boldsymbol{\sigma}_{\delta} : \dot{\boldsymbol{\varepsilon}}_{\delta}^{\mathrm{in}} \leq \xi ||\dot{\boldsymbol{\varepsilon}}_{\delta}^{\mathrm{in}}||, \tag{52}$$

where

$$\zeta = \inf \frac{\boldsymbol{\sigma}_{\delta} : \dot{\boldsymbol{\varepsilon}}_{\delta}^{\mathrm{in}}}{(\dot{\boldsymbol{\varepsilon}}_{\delta}^{\mathrm{in}} : \dot{\boldsymbol{\varepsilon}}_{\delta}^{\mathrm{in}})^{1/2}}, \quad \xi = \sup \frac{\boldsymbol{\sigma}_{\delta} : \dot{\boldsymbol{\varepsilon}}_{\delta}^{\mathrm{in}}}{(\dot{\boldsymbol{\varepsilon}}_{\delta}^{\mathrm{in}} : \dot{\boldsymbol{\varepsilon}}_{\delta}^{\mathrm{in}})^{1/2}}, \tag{53}$$

and

$$||\dot{\boldsymbol{\varepsilon}}_{\delta}^{\mathrm{in}}|| = (\dot{\boldsymbol{\varepsilon}}_{\delta}^{\mathrm{in}} : \dot{\boldsymbol{\varepsilon}}_{\delta}^{\mathrm{in}})^{1/2} = \left([\![\dot{u}_{\mathrm{n}}^{\mathrm{in}}]\!]^2 + [\![\dot{u}_{\mathrm{t}}^{\mathrm{in}}]\!]^2 \right)^{1/2}. \tag{54}$$

For initially virgin material, according to Dorosz [20], we have

$$||\boldsymbol{\varepsilon}_{\delta}^{\mathrm{in}}|| \leq \int_0^{\mathrm{t}} ||\dot{\boldsymbol{\varepsilon}}_{\delta}^{\mathrm{in}}|| \mathrm{d}t. \tag{55}$$

This allows us to bound the relative residual displacement by

$$\begin{aligned} \left| \int_{\Gamma} \hat{\mathbf{P}} \cdot [\![\mathbf{u}^{\mathrm{R}}]\!] \mathrm{d}S \right| &= \left| \int_{\Omega\delta} \hat{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}_{\delta}^{\mathrm{in}} \mathrm{d}\Omega \right| \\ &\leq \int_{\Omega\delta} ||\hat{\boldsymbol{\sigma}}|| \, ||\boldsymbol{\varepsilon}_{\delta}^{\mathrm{in}}|| \mathrm{d}\Omega \\ &\leq \int_{\Omega\delta} ||\hat{\boldsymbol{\sigma}}|| \frac{1}{\zeta} \left(\int_0^{\mathrm{t}} \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}_{\delta}^{\mathrm{in}} \mathrm{d}t \right) \mathrm{d}\Omega. \end{aligned} \tag{56}$$

Thus inequality (55) leads us to estimate the maximum relative residual displacement as

$$\left| [\![\mathbf{u}^{\mathrm{R}}]\!] \right| \leq \max_{\Omega} \left(\frac{||\hat{\boldsymbol{\sigma}}(\mathbf{y}, \mathbf{y}_0)||}{\zeta} \right) \frac{\alpha}{\alpha - 1} \int_{\Omega\delta} \int_0^{\mathrm{t}} \mathring{\boldsymbol{\sigma}}^{(\mathrm{r})}{}_{\delta} : \mathbf{C}^{-1} : \mathring{\boldsymbol{\sigma}}^{(\mathrm{r})}{}_{\delta} \mathrm{d}\Omega \mathrm{d}t. \tag{57}$$

The proposed shakedown condition for the determination of the macroscopic admissible domain P against failure due to interfacial damage or unlimited accumulation of inelastic deformations can then be expressed by the following optimisation problem:

Find

$$\alpha_{\mathrm{SD}} = \max_{\mathring{\boldsymbol{\tau}}^{(\mathrm{r})}, \mathring{\boldsymbol{\sigma}}^{(\mathrm{r})}, \mathbf{d}} \alpha, \tag{58}$$

subject to (40)–(43) and

$$\boldsymbol{\tau}^{(\mathrm{s})}(\mathbf{y}) \in \bar{\mathrm{G}}(\mathbf{y}), \quad \forall \mathbf{y} \in \Omega_{\delta} \tag{59}$$

$$\boldsymbol{\sigma}^{(\mathrm{s})}(\mathbf{y}) \in \bar{\mathrm{F}}(\mathbf{y}), \quad \forall \mathbf{y} \in \Omega_{+} \cup \Omega_{-}, \tag{60}$$

$$< \boldsymbol{\sigma}^{(\mathrm{s})} > = \boldsymbol{\Sigma}, \tag{61}$$

such that

$$\boldsymbol{\sigma}^{(s)} = \alpha\boldsymbol{\sigma}^{(c)} + \overset{\circ}{\boldsymbol{\sigma}}{}^{(r)}, \quad \boldsymbol{\tau}^{(s)} = \alpha\boldsymbol{\tau}^{(c)} + \overset{\circ}{\boldsymbol{\tau}}{}^{(r)}. \tag{62}$$

Condition (59) is considered as a failure criterion of the interface and assures that there is not complete debonding between matrix and fibre, and condition (60) assures that the safe state of stresses is plastically admissible. The problem to be solved is a convex optimisation problem with linear and nonlinear constraints. Due to the usual techniques of discretisation in engineering, the number of unknowns and subsidiary conditions is in general very large and not to be solved efficiently by classical optimisation algorithms.

6 A new optimisation algorithm

To overcome the problem of prohibitive time consumption for numerical calculations, a new, specially problem-adapted optimisation algorithm has been developed (IPDCA; interior-point difference-of-convex-functions algorithm). It is based on the interior-point method combined with the method of difference of convex functions (DC). The algorithm can be formulated as follows [21,22]:

Find a Karush-Kuhn-Tucker (KKT) point of the following nonlinear programming problem:

$$(\mathcal{P}_1)\begin{cases}\min f(x),\\ Ax - b = 0,\\ c(x) \geq 0,\\ x \in \mathbb{R}^n,\end{cases} \tag{63}$$

where f and $c : (\mathbb{R})^n \to (\mathbb{R})^{m_I}$ are two twice-continuous differentiable functions and c is supposed to be concave. $A \in (\mathbb{R})^{m_E \times n}$ is a surjective matrix and $b \in (\mathbb{R})^{m_E}$ a vector. We assume that we have a constraints qualification in the sense presented in (58)–(61).

The first step in the chosen interior-point approach is to add slack variables to each of the inequality constraints in $(\mathcal{P}_I)$ and to add linear constraints to handle free variable x. The problem $(\mathcal{P}_I)$ is then transformed into

$$(\mathcal{P}_m)\begin{cases}\min f(x),\\ Ax - b = 0,\\ c(x) - w = 0,\\ x - y + z = 0,\\ w \geq 0,\, y \geq 0,\, z \geq 0.\end{cases} \tag{64}$$

The second step is to consider the problem with the barrier objective function

$$(\mathcal{P}_m)\begin{cases}\min \bar{f}_\mu(w, x, y, z),\\ Ax - b = 0,\\ c(x) - w = 0,\\ x - y + z = 0,\\ w > 0,\, y > 0,\, z > 0,\end{cases} \tag{65}$$

with

$$\bar{f}_\mu(w, x, y, z) = f(x) - \mu\sum_{i=1}^{m_1}\log(w_i) - \mu\sum_{j=1}^{n}\log(y_j) - \mu\sum_{j=1}^{n}\log(z_j). \tag{66}$$

IPDCA requires a DC decomposition of $f = g - h$, where g and h are two convex functions. By linearising the concave component of the objective function, IPDCA solves approximately the problem:

$$(\mathcal{DC}_k)\begin{cases}\min \bar{g}_k(x),\\ Ax - b = 0,\\ c(x) - w = 0,\\ x - y + z = 0,\\ w \geq 0,\, y \geq 0,\, z \geq 0,\end{cases} \tag{67}$$

where $\bar{g}_k(x) = g(x) - \nabla^T h(x_k)x$.

The problem is then transformed into a sequence of problems with the logarithmic barrier function

$$(\mathcal{D}C_\mu)\begin{cases}\min g_\mu(w,x,y,z)-\nabla^{\mathrm{T}}h(x_k)x,\\ Ax-b=0,\\ c(x)-w=0,\\ x-y+z=0,\end{cases} \tag{68}$$

with

$$g_\mu(w,x,y,z)=g(x)-\mu\sum_{i=1}^{m_1}\log(w_i)-\mu\sum_{j=1}^{n}\log(y_j)-\mu\sum_{j=1}^{n}\log(z_j), \tag{69}$$
$$h_\mu(w,x,y,z)=h(x).$$

7 Numerical example

To illustrate the presented method, we consider a square unit cell assuming perfect bonding between fibre and metal matrix. The unit cell is discretized in space by 3D finite elements using the software package Ansys [23]. Geometry and the adopted mesh are shown in Fig. 3a. Here, SOLID45 is used, where each element is defined by eight nodes having 3 degrees of freedom at each node. For symmetry reasons, only a quarter of the unit cell is considered where the geometry and the adopted mesh are shown in Fig. 2. Note that the progressive decohesion is not taken into account in this example.

The adopted mechanical characteristics for the matrix and the fibres are given in Table 1.

To satisfy the boundary and symmetry conditions, the unit cell is subjected to biaxial uniform macroscopic strains at the edges. The following loading cases are studied:

(1) The strains E_{11} and E_{22} increase proportionally, which corresponds to the LA problem:

$$E_{11}=\mu E_{11}^0 \text{ and } E_{22}=\mu E_{22}^0 \text{ with } 0\le\mu\le\mu^+.$$

(2) The displacements E_{11} and E_{22} vary independently, which corresponds to the SA problem:

$$E_{11}=\mu_1 E_{11}^0 \text{ and } E_{22}=\mu_2 E_{22}^0 \text{ with } 0\le\mu_1\le\mu_1^+ \text{ and } 0\le\mu_2\le\mu_2^+.$$

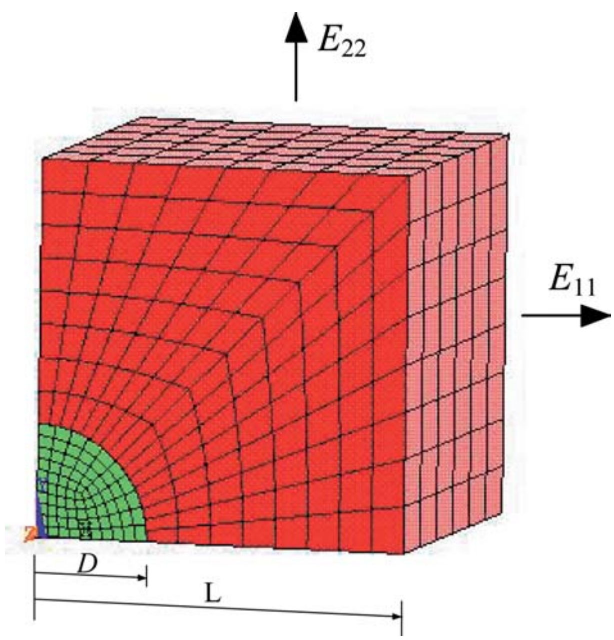

Fig. 2 Geometry and discretisation of unit cell

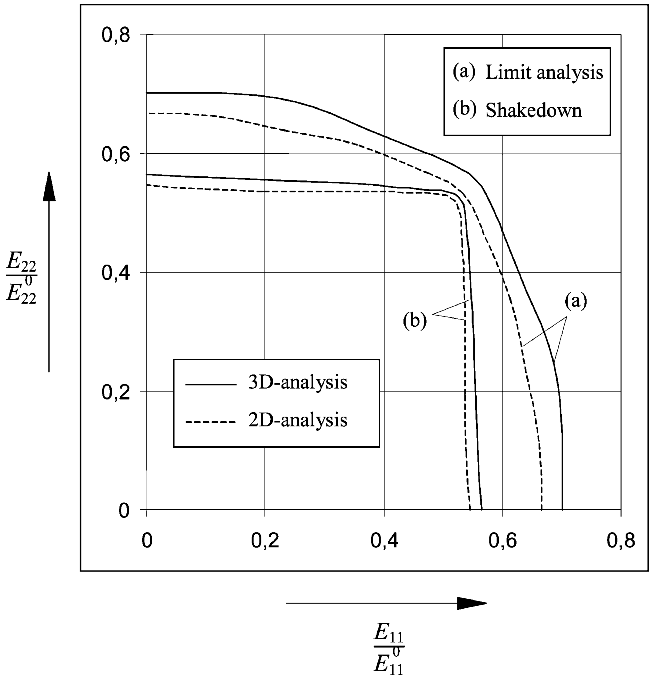

Fig. 3 Admissible domains of macroscopic strains

Table 1 Mechanical characteristics of fibre-reinforced composite

	Matrix	Fibre
Young's modulus E (GPa)	210	2.1
Poisson's ratio ν	0.3	0.2
Yield stress σ_Y (MPa)	280	140

It should be noted that fictitious mechanical characteristics have been chosen for matrix and fibres in order to point out the interaction between fibre and matrix. Figure 3 presents the admissible domains in the space of macroscopic strains (E_{11}, E_{22}), where the results obtained by Schwabe [24] using 2D triangular isoparametric elements with six nodes are taken for comparison. These bounds are obtained for the ratio of diameter D of the fibre and length L of the unit cell $D/L = 0.3$. The influence of patterns of periodicity, the size of the unit cell and the type of loading on the macroscopic domains can be found in Schwabe [24] by considering 2D analysis.

8 Conclusions

The presented formulation allows us to apply the lower-bound direct method more realistically to predict the long-time behaviour of periodic composite materials. In this paper, we address in particular the mechanisms of unlimited accumulation of the inelastic deformation in the matrix and decohesion between fibre and matrix. The numerical method is based on a finite-element analysis and mathematical programming, where a new algorithm is introduced in order to reduce computing time drastically. We observe that this algorithm is about 100 times faster than a standard optimisation code like Lancelot [25]. In future, frictional sliding and progressive decohesion between matrix and fibres will have to be included in the numerical scheme.

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References

1. Melan, E.: Theorie statisch unbestimmter Systeme aus ideal-plastischem Baustoff, vol. 145, pp. 195–218. Sitber Akad Wiss, Wien, Abt. IIa 145 (1936)
2. Koiter, W.T.: General theorems for elastic–plastic solids. In: Sneddon, I.N., Hill, R. (eds.) Progress in Solid Mechanics, pp. 165–221. North-Holland, Amsterdam (1960)
3. Gokhfeld, D.A., Cherniavsky, O.F.: Limit Analysis of Structures at Thermal Cycling. Sijthoff and Noordhoff, Leyden (1980)
4. Sawczuk, A., Binachi, G.: Plasticity today. Modelling, methods and applications. Elsevier, London (1985)
5. König, J.A.: Shakedown of Elastic–Plastic Structures. Elsevier, Amsterdam (1987)
6. Kleiber, M., König, J.A.: Inelastic Solids and Structures (A. Sawczuk memorial volume). Pineridge, Swansea (1990)
7. Save, M., Massonet, C., De Saxcé, G.: Plastic Limit Analysis of Plates, Shells and Disks. Elsevier/North-Holland, Amsterdam (1997)
8. Weichert, D., Maier, G.: Inelastic behaviour of structures under variable repeated loads: direct analysis methods. CISM Courses and Lectures no. 432. Springer, Berlin Heidelberg New York (2002)
9. Maier, G., Pastor, J., Ponter, A.R.S., Weichert, D.: Direct methods of limit and shakedown analysis. In: De Borst, R., Mang, H.A. (eds.) Numerical and Computational Methods, Chap. 12, Vol. 3. In: Milne, I., Ritchie, R.O., Karihaloo, B. (eds.) Comprehensive Structural Integrity. Elsevier–Pergamon, Amsterdam (2003)
10. Weichert, D., Hachemi, A., Schwabe, F.: Application of shakedown analysis to the plastic design of composites. Arch Appl Mech **69**, 623–633 (1999)
11. Hachemi, A., Weichert, D.: On the problem of interfacial damage in fibre-reinforced composites under variable loads. Mech Res Comm **32**, 15–23 (2005)
12. Carrère, N.; Kruch, S.; Vassel, A.; Chaboche, J.-L.; Damage mechanisms in unidirectional SiC/Ti composites under transverse creep loading: experiments and modeling. Int J Damage Mech **11**, 41–63 (2002)
13. Döbert, C., Mahnken, R., Stein, E.: Numerical simulation of interface debonding with a combined damage/friction constitutive model. Comput Mech **25**, 456–467 (2000)
14. Lissenden, C.J., Herakovich, C.T.: Numerical modelling of damage development and viscoplasticity in metal matrix composites. Comput Methods Appl Mech Eng **126**, 289–303 (1995)
15. Suquet, P.: Plasticité et homogénéisation. Ph.D. Thesis, Université Pierre et Marie Curie, Paris 6 (1982)
16. Michel, J.C., Moulinec, H., Suquet, P.: Effective properties of composite materials with periodic microstructure: a computational approach. Comput Methods Appl Mech Eng **172**, 109–143 (1999)
17. Hill, R.: Elastic properties of reinforced solids: some theoretical principles. J Mech Phys Solids **11**, 357–372 (1963)
18. Ju, J.W.: On energy-based coupled elastoplastic damage theories: constitutive modelling and computational aspects. Int J Solids Struct **25**, 803–833 (1989)
19. Bazant, P.Z, Prat, P.C.: Microplane model for brittle plastic material. J Eng Mech **114**, 1672–1702 (1988)
20. Dorosz, S.: An upper bound to maximum residual deflections of elastic-plastic structures at shakedown. Bull Acad Polon Sci Ser Sci Tech **24**, 167–174 (1976)
21. Akoa, F., Le Thi Hoai, An, Pham Dinh, Tao: An interior point algorithm with DC regularisation for nonconvex quadratic programming. In: Le Thi Hoai, An, Pham Dinh, Tao (eds.) Modelling, Computation and Optimization in Information Systems and Management Sciences, pp. 87–96. Hermes, London (2004)
22. Hachemi, A., Le Thi Hoai, An, Mouhtamid, S., Pham Dinh, Tao: Large-scale nonlinear programming and lower bound direct method in engineering applications. In: Le Thi Hoai, An; Pham Dinh, Tao (eds.) Modelling, Computation and Optimization in Information Systems and Management Sciences, pp. 299–310. Hermes Science, London (2004)
23. ANSYS Release 8.0. ANSYS Inc., Southpointe 275 Technology Drive, Canonsburg, PA 15317
24. Schwabe, F.: Einspieluntersuchungen von Verbundwerkstoffen mit periodischer Mikrostruktur. Mechanik Berichte, 2, IAM, Rheinisch-Westfälische Technische Hochschule, Aachen (2000)
25. Conn, A.R., Gould, N.I.M., Toint, Ph.L.: LANCELOT: a fortran package for large-scale nonlinear optimization (Release A). Springer, Berlin Heidelberg New York (1992)