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The BEM for nonhomogeneous bodies

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Abstract The boundary element method (BEM) is developed for nonhomogeneous bodies. The static or steady-state response of such bodies leads to boundary value problems for partial differential equations (PDEs) of elliptic type with variable coefficients. The conventional BEM can be employed only if the fundamental solution of the governing equation is known or can be established. This is, however, out of question for differential equations with variable coefficients. The presented method uses simple, known fundamental solutions for homogeneous isotropic bodies to establish the integral equation. An additional field function is introduced, which is determined from a supplementary domain integral boundary equation. The latter is converted to a boundary integral by employing a domain meshless technique based on global approximation by radial basis function series. Then the solution is evaluated from its integral representation based on the known fundamental solution. The presented method maintains the pure boundary character of the BEM, since the discretization into elements and the integrations are limited only to the boundary. Without restricting its generality, the method is illustrated for problems described by second-order differential equations. Therefore, the employed fundamental solution is that of the Laplace equation. Several problems are studied. The obtained numerical results demonstrate the effectiveness and accuracy of the method. A significant advantage of the proposed method is that the same computer program is utilized to obtain numerical results regardless of the specific form of the governing differential operator.

Keywords Boundary element method · Nonhomogeneous bodies · Meshless · Partial differential equations · Analog equation

1 Introduction

The boundary element method (BEM) has emerged as a powerful alternative to the so-called domain methods, such as the finite difference method (FDM) and finite element method (FEM), particularly in cases where better accuracy is required or the domain methods are inefficient, for example, infinite domains. The most important feature of the BEM, however, is that it requires discretization of the boundary rather than the domain. Hence BEM computer codes are easier to use. This advantage is particularly important for design, since the process involves shape modifications and thus complete remeshing, which are difficult to carry out using FEM. To implement BEM for a given problem the integral representation of its solution is required, which can be derived if the fundamental solution of the governing differential equation is known or can be established. For many differential equations the fundamental solution is known and interesting engineering problems have been successfully solved using the BEM. Difficulties arise when the fundamental solution cannot be determined or it is too complicated and thus impractical to evaluate numerically. These difficulties become practically insurmountable when we come across problems pertaining to nonhomogeneous bodies where the coefficients

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of the differential equations are variable, whose fundamental solution, except for special problems [3], cannot be established. Therefore, effort has been given to simplify BEM formulations using simple fundamental solutions. The so-called domain boundary element methods (D/BEMs) belong to these formulations. Although these methods utilize simple fundamental solutions and maintain the boundary features of the BEM, they do require domain discretization, which spoils the pure boundary character. The dual reciprocity method (DRM) [9] appeared as the most promising method that overcomes these difficulties. Even this method is subject to a major limitation. Namely, for a given nonstandard differential operator, a dominant operator with known fundamental solution must be extracted, which is not always feasible, especially for differential equations with variable coefficients. The DRM is problem-dependent [1,10]. The method presented in this investigation is valid without any limitations. It is based on the concept of the analog equation [4,6–8], which converts the original problem to an equivalent one described by an equation having a simple known fundamental solution, for example, the general second-order equation is replaced by the Poisson equation. Actually, the proposed method introduces an additional unknown domain function, which represents the source of the substitute problem. This function is determined from a supplementary domain integral equation, which is converted to a boundary integral equation using a meshless technique based on global approximation by a radial basis function series. Thus, the pure boundary character of the method is maintained, since the discretization into elements and the integrations are limited only to the boundary. Once this source is established, the solution of the problem is obtained from the integral representation of the solution of the substitute problem, which is used as a mathematical formula. Without restricting the generality with regard to the degree of the partial differential equation (PDE), the method is illustrated for problems described by second-order PDEs. Several problems are studied. The numerical results obtained demonstrate the effectiveness and accuracy of the method.

2 Problem statement

The static or steady-state response of a nonhomogeneous body occupying the two-dimensional domain Ω in the xy-plane is governed by the boundary value problem

$$L(u) = g(\mathbf{x}) \quad \text{in } \mathbf{x} \in \Omega \tag{1}$$

$$\beta_1(\mathbf{x})u + \beta_2(\mathbf{x})q = \beta_3(\mathbf{x}) \quad \text{on } \mathbf{x} \in \Gamma, \tag{2}$$

where $u = u(\mathbf{x})$, $\mathbf{x}\{x, y\}$ is the unknown field function, $q = u_{,n}$ its normal derivative on Γ and

$$L(u) = A(\mathbf{x})\, u_{,xx} + 2B(\mathbf{x})\, u_{,xy} + C(\mathbf{x})\, u_{,yy} + D(\mathbf{x})\, u_{,x} + E(\mathbf{x})\, u_{,y} + F(\mathbf{x})\, u, \tag{3}$$

is the general second-order elliptic differential operator defined in Ω; $\Gamma = \cup_{i=0}^{i=K} \Gamma_i$ is the boundary, where $\Gamma_i \quad (i = 0, 1, 2, \ldots, K)$ are $K+1$ nonintersecting closed contours surrounded by the contour Γ_0 (see Fig. 1). Moreover, $\beta_i = \beta_i(\mathbf{x}),\quad i = 1, 2, 3$ are functions specified on Γ and $A(\mathbf{x}), B(\mathbf{x}), \ldots, F(\mathbf{x})$ position-dependent coefficients satisfying the ellipticity condition $B^2 - AC < 0$ at all points in $\Omega \cup \Gamma$.

Generally, conventional BEM cannot be applied for the problem (1)–(3), because it requires the establishment of the fundamental solution of the governing operator $L(u)$, which in general is not feasible. The BEM solution presented in this paper is based on the concept of the analog equation, which converts the original problem to an equivalent problem described by the Poisson equation with a fictitious source under the same boundary condition. This procedure is presented in the following section.

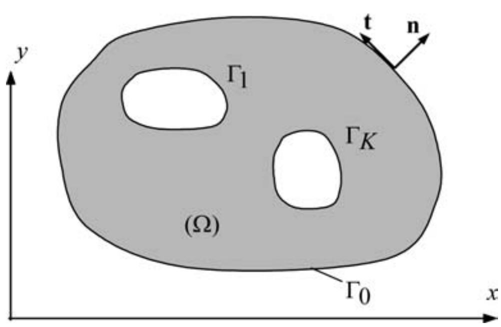


Fig. 1 Two-dimensional domain Ω occupied by the nonhomogeneous body

3 The solution procedure

Let $u = u(\mathbf{x})$ be the sought solution to the problem (1)–(2). This function is twice continuously differentiable in Ω. Thus, if the Laplace operator is applied to it, we have

$$\nabla^2 u = b(\mathbf{x}), \tag{4}$$

where $b(\mathbf{x})$ represents an unknown fictitious source.

Equation (4) indicates that the solution of Eq. (1) could be established by solving this equation under the boundary condition (2), if $b(\mathbf{x})$ is first established. This is accomplished following the procedure below.

We write the solution of Eq. (4) in integral form. Thus, we have [5]

$$\varepsilon u(\mathbf{x}) = \int_\Omega u^* b \, \mathrm{d}\Omega - \int_\Gamma (u^* q - q^* u) \mathrm{d}s \quad \mathbf{x} \in \Omega \cup \Gamma, \tag{5}$$

in which $u^* = \ell n r / 2\pi$ is the fundamental solution to Eq. (4) and $q^* = u,_n^*$ is its derivative normal to the boundary with $r = |\xi - \mathbf{x}| = [(\xi - x)^2 + (y - \eta)^2]^{1/2}$ being the distance between any two points $\mathbf{x}\{x, y\} \in \Omega \cup \Gamma$ and $\xi \in \Gamma$; ε is a constant which takes the values $\varepsilon = 1$ if $\mathbf{x} \in \Omega$ and $\varepsilon = \alpha/2\pi$ if $\mathbf{x} \in \Gamma$; α is the interior angle between the tangents of boundary at point $\mathbf{x}$. Note that it is $\varepsilon = 1/2$ for points where the boundary is smooth.

Equation (5), when applied to boundary points, yields the boundary integral Eq. [5]

$$\frac{1}{2} u(\mathbf{x}) = \int_\Omega u^* b \, \mathrm{d}\Omega - \int_\Gamma (u^* q - q^* u) \mathrm{d}s \quad \mathbf{x} \in \Gamma. \tag{6}$$

In the conventional BEM, the source $b(\mathbf{x})$ is known and Eq. (6) is combined with Eq. (2) to yield the unknown boundary quantities u and q. This, however, cannot be done here, because $b(\mathbf{x})$ is unknown. For this purpose, an additional integral equation is derived, which permits the establishment of the additional unknown field function $b(\mathbf{x})$. This equation results by applying the operator $L(\;)$ to Eq. (5) for points $\mathbf{x} \in \Omega$, $(\varepsilon = 1)$. Thus we have

$$\int_\Omega L\left(u^*\right) b \, \mathrm{d}\Omega - \int_\Gamma \left[L\left(u^*\right) q - L\left(q^*\right) u\right] \mathrm{d}s = g(\mathbf{x}), \quad \mathbf{x} \in \Omega \tag{7}$$

Equations (2), (6) and (7) can be combined and solved to yield all three unknown quantities, namely, u, q, b. Equations (6) and (7) are domain-boundary integral equations and could be solved using domain discretization to approximate the domain integrals. This, however, would spoil the advantages of the BEM over the domain methods. We can maintain the pure boundary character of the method by converting the domain integrals to boundary line integrals. This can be accomplished as follows.

We set

$$b = \sum_{j=1}^{M} \alpha_j f_j \tag{8}$$

where $f_j = f_j(r)$, $r = \left|\mathbf{x} - \mathbf{x}_j\right|$, is a set of radial basis approximating functions; $\mathbf{x}_j$ are collocation points in Ω and α_j are coefficients to be determined. Using Green's reciprocal identity [5] the domain integral in Eq. (6) becomes

$$\begin{aligned} \int_\Omega u^*(\mathbf{x}, \mathbf{y}) b(\mathbf{y}) \mathrm{d}\Omega_\mathbf{y} &= \sum_{j=1}^{M} a_j \int_\Omega u^*(\mathbf{x}, \mathbf{y}) f_j(\mathbf{y}, \mathbf{x}_j) \mathrm{d}\Omega_\mathbf{y} \\ &= \sum_{j=1}^{M} a_j \left\{ \frac{1}{2} \hat{u}_j(\mathbf{x}) + \int_\Gamma \left[u^*(\mathbf{x}, \xi) \hat{q}_j(\xi) - q^*(\mathbf{x}, \xi) \hat{u}_j(\xi)\right] \mathrm{d}s \right\}' \\ &\quad \mathbf{x}, \mathbf{y} \in \Omega, \;\; \xi \in \Gamma \end{aligned} \tag{9}$$

in which $\hat{u}_j(\mathbf{x})$ is a particular solution of the equation

$$\nabla^2 \hat{u}_j = f_j \quad j = 1, 2, \ldots, M \tag{10}$$

A particular solution of Eq. (10) can always be determined, if f_j is specified [5]. Hence, Eqs. (6) and (7) become

$$\frac{1}{2}u(\mathbf{x}) = \sum_{j=1}^{M} a_j \left[\frac{1}{2}\hat{u}_j(\mathbf{x}) + \int_\Gamma \left(u^*\hat{q}_j - q^*\hat{u}_j\right) \mathrm{d}s \right] - \int_\Gamma (u^* q - u q^*) \mathrm{d}s, \quad \mathbf{x} \in \Gamma \tag{11}$$

$$\sum_{j=1}^{M} a_j \left[L\left(\hat{u}_j(\mathbf{x})\right) + \int_\Gamma \left[L\left(u^*\right) \hat{q}_j - L(q^*)\hat{u}_j \right] \mathrm{d}s \right] - \int_\Gamma \left[L\left(u^*\right) q - L\left(q^*\right) u \right] \mathrm{d}s = g, \quad \mathbf{x} \in \Omega, \tag{12}$$

which can be combined with Eq. (2) and solved numerically to yield the boundary quantities u, q and the coefficients a_j. Then the solution of the problem at any point inside Ω will be evaluated from Eq. (5), which by virtue of Eq. (9) becomes

$$u(\mathbf{x}) = \sum_{j=1}^{M} a_j \left[\hat{u}_j(\mathbf{x}) + \int_\Gamma \left(u^*\hat{q}_j - q^*\hat{u}_j\right) \mathrm{d}s \right] - \int_\Gamma (u^* q - u q^*) \mathrm{d}s, \quad \mathbf{x} \in \Omega \tag{13}$$

4 Numerical implementation

The BEM with constant elements is used to approximate the boundary integrals in Eqs. (11) and (12). If N is the number of the boundary nodal points (see Fig. 2), then for node i Eq. (11) is written as

$$\frac{1}{2}u^i = \sum_{j=1}^{M} K_{ij} a^j + \sum_{k=1}^{N} \tilde{H}_{ik} u^k - \sum_{k=1}^{N} G_{ik} q^k, \quad i = 1, 2, \ldots, N \tag{14}$$

where

$$\tilde{H}_{ik} = \int_k q^*(r_{ik}) \mathrm{d}s \tag{15}$$

$$G_{ik} = \int_k u^*(r_{ik}) \mathrm{d}s \tag{16}$$

$$K_{ij} = \frac{1}{2}\hat{u}_j^i - \sum_{k=1}^{N} \tilde{H}_{ik} \hat{u}_j^k + \sum_{k=1}^{N} G_{ik} \hat{q}_j^k \tag{17}$$

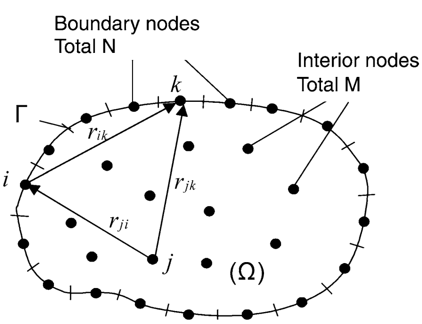


Fig. 2 Boundary discretization and domain nodal points

where $\int\limits_{k}$ indicates integration on element j and $\hat{u}_j^i = \hat{u}_j(r_{ji})$, $\hat{u}_j^k = \hat{u}_j(r_{jk})$.

Applying Eq. (14) to all boundary nodal points and using matrix notation yields

$$\mathbf{Hu} - \mathbf{Gq} + \mathbf{Ka} = \mathbf{0}, \tag{18}$$

where $\mathbf{a}$ is the vector of the M coefficients a_j; $\mathbf{u}$, $\mathbf{q}$ are the vectors of the N boundary nodal values of u and q, respectively, and

$$\mathbf{H} = \tilde{\mathbf{H}} - \frac{1}{2}\mathbf{I} \tag{19}$$

with $\mathbf{I}$ being the unit matrix.

Similarly, using the same boundary discretization for Eq. (12) and applying it to M nodal points inside Ω yields

$$\sum_{j=1}^{M} T_{ij} a^j + \sum_{k=1}^{N} S_{ik} u^k - \sum_{k=1}^{N} R_{ik} q^k = g^i, \quad i = 1, 2, \ldots, M, \tag{20}$$

where

$$S_{ik} = \int\limits_{k} L\left(q^*(r_{ik})\right) \mathrm{d}s \tag{21}$$

$$R_{ik} = \int\limits_{k} L\left(u^*(r_{ik})\right) \mathrm{d}s \tag{22}$$

$$T_{ij} = L\left(\hat{u}_j^i\right) - \sum_{k=1}^{N} S_{ik} \hat{u}_j^k + \sum_{k=1}^{N} R_{ik} \hat{q}_j^k, \tag{23}$$

or in matrix notation

$$\mathbf{Su} - \mathbf{Rq} + \mathbf{Ta} = \mathbf{g}, \tag{24}$$

where $\mathbf{g}$ is the vector of values of $g(\mathbf{x})$ at the M interior nodal points.

The boundary condition (2), when applied to the N boundary nodal points, yields

$$\boldsymbol{\beta}_1 \mathbf{u} + \boldsymbol{\beta}_2 \mathbf{q} = \boldsymbol{\beta}_3, \tag{25}$$

where $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2$ are $N \times N$ diagonal matrices and $\boldsymbol{\beta}_3$ $N \times 1$ vector including the values of β_i, $i = 1, 2, 3$ at the N boundary nodal points.

Equations (18), (24) and (25) constitute the following system of $2N + M$ unknowns

$$\mathbf{QX} = \mathbf{p}, \tag{26}$$

where

$$\mathbf{Q} = \begin{bmatrix} \mathbf{H} & -\mathbf{G} & \mathbf{K} \\ \boldsymbol{\beta}_1 & \boldsymbol{\beta}_2 & \mathbf{0} \\ \mathbf{S} & -\mathbf{R} & \mathbf{T} \end{bmatrix}, \quad \mathbf{X} = \begin{Bmatrix} \mathbf{u} \\ \mathbf{q} \\ \mathbf{a} \end{Bmatrix}, \quad \mathbf{p} = \begin{Bmatrix} \mathbf{0} \\ \boldsymbol{\beta}_3 \\ \mathbf{g} \end{Bmatrix}, \tag{27}$$

which can be solved to obtain the boundary quantities $\mathbf{u}$ and $\mathbf{q}$ as well as the coefficients $\mathbf{a}$.

The matrices $\mathbf{R}$, $\mathbf{S}$ and $\mathbf{T}$ are evaluated from the expressions

$$\mathbf{R} = \mathbf{AG}_{xx} + 2\mathbf{BG}_{xy} + \mathbf{CG}_{yy} + \mathbf{DG}_x + \mathbf{EG}_y + \mathbf{FG} \tag{28}$$

$$\mathbf{S} = \mathbf{A}\tilde{\mathbf{H}}_{xx} + 2\mathbf{B}\tilde{\mathbf{H}}_{xy} + \mathbf{C}\tilde{\mathbf{H}}_{yy} + \mathbf{D}\tilde{\mathbf{H}}_x + \mathbf{E}\tilde{\mathbf{H}}_y + \mathbf{F}\tilde{\mathbf{H}} \tag{29}$$

$$\mathbf{T} = \mathbf{A}\hat{\mathbf{U}}_{xx} + 2\mathbf{B}\hat{\mathbf{U}}_{xy} + \mathbf{C}\hat{\mathbf{U}}_{yy} + \mathbf{D}\hat{\mathbf{U}}_x + \mathbf{E}\hat{\mathbf{U}}_y + \mathbf{F}\hat{\mathbf{U}} - \mathbf{S}\hat{\mathbf{U}} + \mathbf{R}\hat{\mathbf{Q}}, \tag{30}$$

where $\mathbf{A}, \mathbf{B}, \ldots, \mathbf{F}$ are $M \times M$ diagonal matrices including the values of the coefficients of Eq. (1) at the M nodal points inside Ω. The matrices $\mathbf{G}, \mathbf{G}_x, \mathbf{G}_y, \ldots, \mathbf{G}_{yy}, \tilde{\mathbf{H}}, \tilde{\mathbf{H}}_x, \tilde{\mathbf{H}}_y, \ldots, \tilde{\mathbf{H}}_{yy}, \hat{\mathbf{U}}, \hat{\mathbf{U}}_x, \hat{\mathbf{U}}_y, \ldots, \hat{\mathbf{U}}_{yy}$ refer to the domain nodal points and result from direct differentiation of Eqs. (15), (16) and $\hat{\mathbf{U}} = \hat{\mathbf{U}}_j^k$, for example, $\left(G_{xy}\right)_{ik} = \int_k u,^*_{xy}(r_{ik})\mathrm{d}s$. Subsequently, the solution at any point $\mathbf{x} \in \Omega$ is evaluated from the discretized counterpart of Eq. (13), that is,

$$u(\mathbf{x}) = \sum_{j=1}^{M} K_{\mathbf{x}j} a^j + \sum_{k=1}^{N} \tilde{H}_{\mathbf{x}k} u^k - \sum_{k=1}^{N} G_{\mathbf{x}k} q^k, \tag{31}$$

where now

$$K_{\mathbf{x}j} = \hat{u}_j(\mathbf{x}) - \sum_{k=1}^{N} \tilde{H}_{xk} \hat{u}_j^k + \sum_{k=1}^{N} G_{xk} \hat{q}_j^k \tag{32}$$

because point $\mathbf{x}$ is inside Ω.

The above solution procedure is implemented adhering to the following steps:

1. Formulate $\mathbf{G}, \mathbf{H}, \mathbf{K}$ for the boundary points using Eqs. (15), (16), (17) and (19).
2. Formulate $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3$ for the boundary points.
3. Formulate $\mathbf{G}, \mathbf{G}_x, \mathbf{G}_y, \ldots, \mathbf{G}_{yy}, \tilde{\mathbf{H}}, \tilde{\mathbf{H}}_x, \tilde{\mathbf{H}}_y, \ldots, \tilde{\mathbf{H}}_{yy}, \hat{\mathbf{U}}, \hat{\mathbf{U}}_x, \hat{\mathbf{U}}_y, \ldots, \hat{\mathbf{U}}_{yy}$ for the interior points.
4. Formulate $\mathbf{R}$, $\mathbf{S}$ and $\mathbf{T}$ using Eqs. (28), (29) and (30)
5. Solve the simultaneous equation (26) to obtain the boundary nodal values $\mathbf{u}$, $\mathbf{q}$ and the coefficients $\mathbf{a}$.
6. For a given point $\mathbf{x} \in \Omega$ evaluate the row matrices $\mathbf{G}, \tilde{\mathbf{H}}, \mathbf{U}, \mathbf{G}_x, \tilde{\mathbf{H}}_x, \mathbf{U}_x, \ldots, \mathbf{G}_{yy}, \tilde{\mathbf{H}}_{yy}, \mathbf{U}_{yy}$ and compute the solution and its derivatives from

$$u,_{st}(\mathbf{x}) = \mathbf{K}_{st}\mathbf{a} + \tilde{\mathbf{H}}_{st}\mathbf{u} - \mathbf{G}_{st}\mathbf{q}, \quad s, t = 0, x, y$$

Note that the above notation defines $u,_{00}(\mathbf{x}) = u(\mathbf{x})$, $u,_{x0}(\mathbf{x}) = u,_x(\mathbf{x})$, $\mathbf{G}_{00} = \mathbf{G}$, $\mathbf{G}_{x0} = \mathbf{G}_x$ etc.

5 Examples

On the basis of the procedure presented in previous sections a FORTRAN code has been written for the solution of the boundary value problem (1)–(2). The employed radial basis functions f_j are the multiquadrics, which are defined as

$$f_j = \sqrt{r^2 + c^2}, \tag{33}$$

where c is the shape parameter and

$$r = \sqrt{(x - x_j)^2 + (y - y_j)^2} \quad (j = 1, 2, \ldots, M), \tag{34}$$

with $(x_j, y_j) \in \Omega$ being the collocation point inside. The particular solution of Eq. (10) for f_j given by Eq. (33) is obtained as

$$\hat{u}_j = -\frac{c^3}{3}\ell u(c\sqrt{r^2 + c^2} + c^2) + \frac{1}{9}(r^2 + 4c^2)\sqrt{r^2 + c^2}. \tag{35}$$

Certain example problems are presented which demonstrate the efficiency and accuracy of the proposed method.

Example 1

As a first example we consider a benchmark problem [2]. This problem is governed by the Poisson equation

$$\nabla^2 u = -\frac{10^6}{52} \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma \tag{36}$$

where $\Omega = \{(x, y) : -0.3 \leq x \leq 0.3, -0.2 \leq y \leq 0.2\}$. The exact value of u at the center is $u(0, 0) = 310.10$. The solution has been obtained using N constant boundary elements and M domain nodal points uniformly distributed on a rectangular domain. The obtained results for various values of N, M and c are shown in Table 1. The solution converges for $N = 140$.

Table 1 Dependence of the solution $u(0,0)$ on N, M and c in Example 1

N	$c = 1$		$c = 1.5$	$c = 3$
	$M = 25$	$M = 55$	$M = 25$	$M = 25$
20	312.71	312.69	312.71	312.67
40	310.65	310.64	310.65	310.64
80	310.19	310.19	310.19	310.19
140	310.10	310.10	310.10	310.10

Example 2

As a second example we obtain the solution of the following boundary value problem for the complete second-order equation

$$y^2 u_{xx} + 2xyu_{xy} + 2x^2 u_{yy} + xu_x - yu_y + u = 7x^2 + y^2 \quad \text{in } \Omega \tag{37}$$

$$u = x^2 + y^2 \quad \text{on } \Gamma \tag{38}$$

where Ω is the ellipse with semi-axes $a = 1.5$, $b = 1.0$.

The exact solution is

$$u = x^2 + y^2. \tag{39}$$

The obtained results for the solution and its derivatives are shown in Table 2. In all cases the error is less than 10^{-3}. It is worth noting that this method ensures great accuracy, not only for the solution, but also for its derivatives, a fact that is not guaranteed by other numerical methods, for example, FEM.

Example 3

As a third example we study the thermal distribution in a plane body having the irregular shape of Fig. 3. The thermal conductivity is taken to vary according to the law

$$k(x, y) = (2x + y + 2)^2 \tag{40}$$

The flux $q_n = -kT,_n$ hence the normal derivative $T,_n$, is prescribed along the sides AB and CD as $T,_n(x, 0) = -\frac{17+20x-15}{2(1+x)^2}$, $T,_n(0.5, y) = \frac{-45+46y+32y^2}{(3+y)^2}$, while the temperature on the remaining part of the boundary is $T_b(x, y) = \frac{6x^2-6y^2+20xy+30}{2x+y+2} + 100$.

The temperature $T(x, y)$ will be obtained as a solution of the following boundary value problem

$$T,_{xx} + T,_{yy} + (\ln k),_x T,_x + (\ln k),_y T,_y = 0, \quad \text{in } \Omega \tag{41}$$

Table 2 Comparison of analytical and numerical results for u and its derivative in Example 2. ($N = 100$, $M = 82$, $c = 1$). Upper value: computed; lower value: exact

x	y	u	u_x	u_y	u_{xx}	u_{yy}	u_{xy}
0.5000	0.0000	0.2496	0.9993	0.0000	1.9992	2.0014	0.0000
		0.2500	1.0000	0.0000	2.0000	2.0000	0.0000
0.4619	0.1275	0.2289	0.9225	0.2548	2.0065	2.0049	−0.0080
		0.2297	0.9239	0.2551	2.0000	2.0000	0.0000
0.3535	0.2357	0.1802	0.7049	0.4719	2.0037	2.0067	−0.0043
		0.1806	0.7071	0.4714	2.0000	2.0000	0.0000
0.1913	0.3079	0.1312	0.3828	0.6149	1.9970	2.0002	0.0012
		0.1314	0.3827	0.6159	2.0000	2.0000	0.0000
0.0000	0.3333	0.1104	0.0023	0.6646	2.0004	2.0041	0.0041
		0.1111	0.0000	0.6667	2.0000	2.0000	0.0000

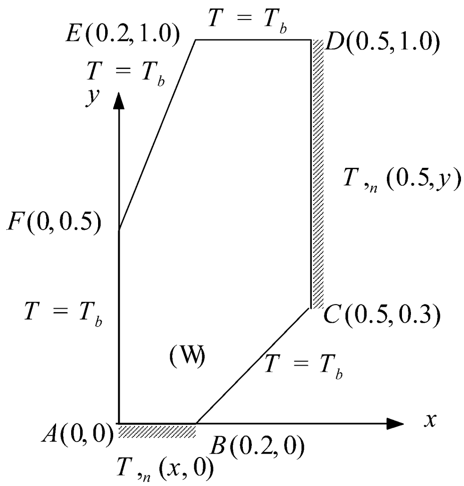


Fig. 3 Irregularly shaped plane body of Example 3

$$T_{,n} = T_{,n}\,(x, 0) \text{ on } AB,\ T_{,n} = T_{,n}\,(0.5, y) \quad \text{on } CD, \tag{42a}$$
$$T = T_b(x, y) \text{ on } BC,\ DE,\ EF \text{ and } FA \tag{42b}$$

The exact solution is

$$T(x, y) = \frac{6x^2 - 6y^2 + 20xy + 30}{2x + y + 2} + 100. \tag{43}$$

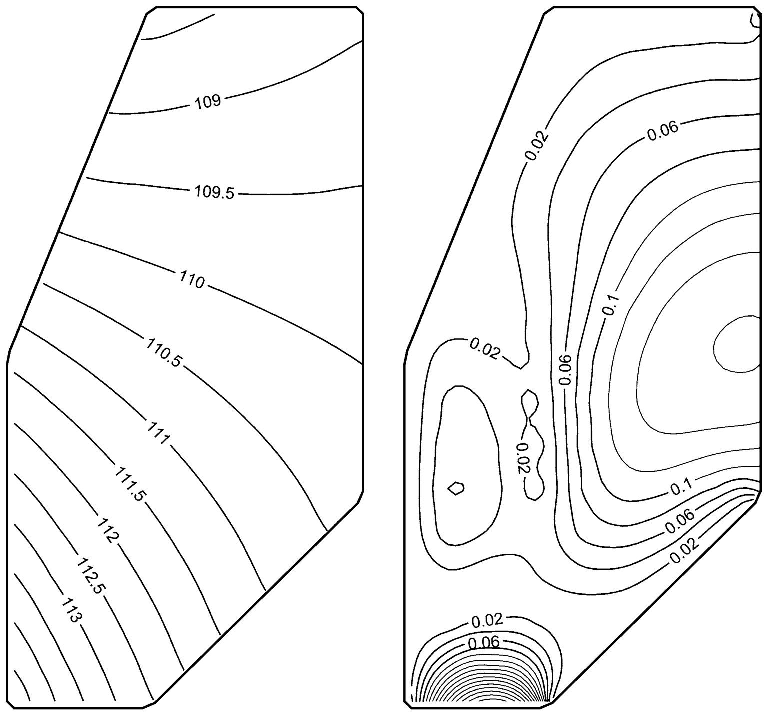
The obtained results for the temperature and the fluxes $q_x = -kT_{,x}$, $q_y = -kT_{,y}$ are given in Table 3 and compared with the exact ones. Moreover, the temperature distribution and its relative error are shown in Fig. 4.

6 Conclusions

From the presented analysis and the numerical examples the following main conclusions can be drawn.

Table 3 Comparison of analytical and numerical results for temperature and fluxes in Example 3. (N = 130, M = 82, c = 0.1). Upper value: computed; lower value: exact

x	y	T	q_x	q_y
	0.25	111.356	40.783	26.081
		111.364	40.500	25.750
	0.3125	111.138	37.437	27.979
		111.147	36.687	27.836
	0.375	110.914	33.577	30.035
		110.924	32.625	29.968
	0.4375	110.684	29.299	32.149
		110.694	28.312	32.148
	0.500	110.449	24.702	34.195
		110.458	23.750	34.375
0.25	0.5625	110.209	19.769	36.427
		110.217	18.937	36.648
	0.625	109.963	14.476	38.876
		109.970	13.875	38.968
	0.6875	109.712	9.020	41.236
		109.718	8.5625	41.336
	0.750	109.456	3.381	43.589
		109.462	3.000	43.750
	0.8125	109.196	−2.431	45.907
		109.200	−2.812	46.210
	0.875	108.932	−8.558	48.423
		108.935	−8.875	48.720



a Temperature T **b** Relative error of $T \times 10^3$

Fig. 4 Temperature and error distribution in Example 3

(1) As the method is limited to the boundary, it has all the advantages of the BEM, that is, the discretization and integration are performed only on the boundary.
(2) Simple, known fundamental solutions are employed. They depend only on the order of the differential equation and not on the specific differential operator which governs the problem under consideration.
(3) The computer program is the same and depends only on the order of the differential equation and not on the specific differential operator which governs the problem under consideration.
(4) The solution and its derivatives are computed at any point using the respective integral representation as mathematical formulas.
(5) Accurate numerical results are obtained using radial basis functions of multiquadric type with a relatively small number of domain collocation points.
(6) The concept of the analog equation in conjunction with radial basis functions approximation of the fictitious sources renders BEM a versatile computational method for solving difficult engineering problems for nonhomogeneous bodies.

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