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Analytical solution for the electroelastic dynamics of a nonhomogeneous spherically isotropic piezoelectric hollow sphere

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Summary By introduction of a special dependent variable and separation of variables technique, the electroelastic dynamic problem of a nonhomogeneous, spherically isotropic hollow sphere is transformed to a Volterra integral equation of the second kind about a function of time. The equation can be solved by means of the interpolation method, and the solutions for displacements, stresses, electric displacements and electric potential are obtained. The present method is suitable for a piezoelectric hollow sphere with an arbitrary thickness subjected to arbitrary mechanical and electrical loads. Numerical results are presented at the end.

Keywords Piezoelectricity, Dynamics, Inhomogeneity, Transient solution, Hollow sphere

1 Introduction

Owing to the special coupling between the electric field and mechanical deformation, piezoelectric materials have been widely used in smart structures, active control and sensing devices. For homogeneous piezoelectric materials, the axisymmetric free and forced vibrations of piezoceramic hollow spheres have been studied in [1, 2]. The radial and three-dimensional free vibrations of piezoceramic hollow spheres have been investigated in [3–5]. The spherically symmetric free vibrations with the radial electric displacement vanishing on the boundary has been studied in [6]. The natural frequencies of a piezoceramic hollow sphere submerged in and filled with a compressible fluid were obtained in [7, 8]. The spherically symmetric steady-state response of a piezoceramic hollow sphere submerged in a compressible fluid was analyzed in [9], and that of a laminated spherical shell consisting of piezoelectric and elastic layers was solved in [10].

There are also many investigations that have been done for nonhomogeneous materials. Among them, the special case that the Young's modulus has a power-law dependence on the radial coordinate, while the linear thermal expansion coefficient and the Poisson's ratio are constants, has been considered by many scientists and engineers. For instance, general solutions have been obtained for a nonhomogeneous, orthotropic annular disk in plane stress subjected to uniform pressures at the internal and external surfaces, [11]. The pressured functionally graded material (FGM) in hollow cylinder and disk problems were investigated recently in [12]. The rotation problem of a nonhomogeneous, orthotropic composite cylinder was considered in [13]. The transient thermal stresses in a rotating, nonhomogeneous, cylindrically orthotropic composite tube and in a nonhomogeneous, spherically orthotropic, elastic medium with a spherical cavity were studied in [14] and [15], respectively. Exact solutions for a steady-state problem of the FGM, anisotropic cylinders subjected to thermal and mechanical loads have been obtained in [16]. The torsional oscillations of a finite, nonhomogeneous,

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piezoelectric, cylindrical shell were also investigated in [17]; have the analytical solution is only suitable for the class 622 crystals, but not for the class 6 mm crystals, that are usually met. In the above studies, the variation of the material density is mostly assumed to be the same as that of the Young's modulus, [13–17].

At present, most works on dynamic behavior of piezoelectric hollow spheres are concerned with the problems of free vibration and steady-state response. For the transient response of nonhomogeneous, piezoelectric hollow spheres, there are no related studies reported. In this paper, an analytical method is proposed to solve the spherically symmetric electroelastodynamic problem of a nonhomogeneous, piezoelectric, hollow sphere subjected to dynamic loads. Firstly, a new dependent variable is introduced to rewrite the governing equations, the mechanical boundary conditions as well as the initial conditions. Secondly, a special function is introduced to transform the inhomogeneous mechanical boundary conditions into the homogeneous ones. Thirdly, by virtue of the separation of variables technique, and utilizing the initial conditions as well as electrical boundary conditions, the second kind Volterra integral equation about a function with respect to time is derived, which can be solved by means of the interpolation method. The displacements, stresses, electric displacements and electric potential are obtained at the end. The present method is suitable for a nonhomogeneous, piezoelectric, hollow sphere with an arbitrary thickness subjected to arbitrary spherically symmetric mechanical and electrical loads. Numerical examples are considered and their discussion is presented.

2 Basic formulations

If a spherical coordinate system (r, θ, φ) with the origin identical to the center of the sphere is used, then for a spherically symmetric problem, we have $u_\theta = u_\varphi = 0$, $u_r = u_r(r, t)$ and $\Phi = \Phi(r, t)$, where $u_i (i = r, \theta, \varphi)$ and Φ are the components of the displacement and the electric potential, respectively. The strain–displacement relations are thus simplified as

$$\gamma_{rr} = \frac{\partial u_r}{\partial r}, \quad \gamma_{\theta\theta} = \frac{u_r}{r} \quad , \tag{1}$$

where $\gamma_{ii}(i = r, \theta)$ are the strain components. The constitutive relations of a spherically isotropic, radially polarized, piezoelectric medium are then read as, [18],

$$\begin{aligned} \sigma_{\theta\theta} &= (c_{11} + c_{12})\gamma_{\theta\theta} + c_{13}\gamma_{rr} + e_{31}\frac{\partial \Phi}{\partial r}, \\ \sigma_{rr} &= 2c_{13}\gamma_{\theta\theta} + c_{33}\gamma_{rr} + e_{33}\frac{\partial \Phi}{\partial r}, \\ D_r &= 2e_{31}\gamma_{\theta\theta} + e_{33}\gamma_{rr} - \varepsilon_{33}\frac{\partial \Phi}{\partial r} \quad , \end{aligned} \tag{2}$$

where c_{ij}, e_{ij} and ε_{33} are elastic, piezoelectric and dielectric constants. σ_{ij} and D_r are the components of stress and radial electric displacement, respectively. The equation of motion is

$$\frac{\partial \sigma_{rr}}{\partial r} + 2\frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \rho \frac{\partial^2 u_r}{\partial t^2} \quad , \tag{3}$$

where ρ is the mass density. In absence of free charge density, the charge equation of electrostatics is

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) = 0 \quad . \tag{4}$$

Let the inner and outer radii of the hollow sphere be a and b, respectively. Also, we assume that the inhomogeneity of the material is characterized by the following special law

$$\begin{aligned} &c_{11} = \xi^{2N}C_{11}, \quad c_{12} = \xi^{2N}C_{12}, \quad c_{13} = \xi^{2N}C_{13}, \quad c_{33} = \xi^{2N}C_{33}, \\ &e_{31} = \xi^{2N}E_{31}, \quad e_{33} = \xi^{2N}E_{33}, \quad \varepsilon_{33} = \xi^{2N}\Omega_{33}, \quad \rho = \xi^{2N}\rho_0, \quad \xi = \tfrac{r}{b} \ , \end{aligned} \tag{5}$$

where C_{ij}, E_{ij}, Ω_{33} and ρ_0 are known constants, and N can be an arbitrary real number. Substituting Eq. (5) into Eqs. (2) and (3), and rewriting Eqs. (1)–(4) in a nondimensional form, gives

$$\gamma_{rr} = \frac{\partial u}{\partial \xi}, \quad \gamma_{\theta\theta} = \frac{u}{\xi} \ , \tag{6}$$

$$\sigma_\theta = \xi^{2N}\left[(C_1 + C_2)\frac{u}{\xi} + C_3\frac{\partial u}{\partial \xi} + E_1\frac{\partial \phi}{\partial \xi}\right],$$

$$\sigma_r = \xi^{2N}\left[2C_3\frac{u}{\xi} + \frac{\partial u}{\partial \xi} + E_3\frac{\partial \phi}{\partial xi}\right], \tag{7}$$

$$D = \xi^{2N}\left[2E_1\frac{u}{\xi} + E_3\frac{\partial u}{\partial \xi} - \frac{\partial \phi}{\partial \xi}\right] \ ,$$

$$\frac{\partial \sigma_r}{\partial \xi} + 2\frac{\sigma_r - \sigma_\theta}{\xi} = \xi^{2N}\frac{\partial^2 u}{\partial \tau^2} \ , \tag{8}$$

$$\frac{1}{\xi^2}\frac{\partial}{\partial \xi}(\xi^2 D) = 0 \ , \tag{9}$$

where

$$C_i = \tfrac{C_{1i}}{C_{33}}\,(i = 1,2,3), \quad E_1 = \tfrac{E_{31}}{\sqrt{C_{33}\Omega_{33}}}, \quad E_3 = \tfrac{E_{33}}{\sqrt{C_{33}\Omega_{33}}},$$

$$\sigma_i = \tfrac{\sigma_{ii}}{C_{33}}\,(i = r,\theta), \quad \phi = \sqrt{\tfrac{\Omega_{33}}{C_{33}}}\tfrac{\Phi}{b}, \quad D = \tfrac{D_r}{\sqrt{C_{33}\Omega_{33}}}, \tag{10}$$

$$u = \tfrac{u_r}{b}, \quad \xi = \tfrac{r}{b}, \quad s = \tfrac{a}{b}, \quad \tau = \tfrac{c_v}{b}t, \quad c_v = \sqrt{\tfrac{C_{33}}{\rho_0}} \ .$$

The boundary conditions are

$$\sigma_r(s,\tau) = p_a(\tau), \quad \sigma_r(1,\tau) = p_b(\tau) . \tag{11}$$

$$\phi(s,\tau) = \phi_a(\tau), \quad \phi(1,\tau) = \phi_b(\tau) \ ,$$

where $p_a(\tau)$ and $p_b(\tau)$ are the prescribed dimensionless pressures acting on the internal and external surfaces, respectively, and $\phi_a(\tau)$ and $\phi_b(\tau)$ are the known dimensionless electric potential imposed on the internal and external surfaces, respectively.

The initial conditions are

$$\tau = 0 : u(\xi,0) = u_0(\xi), \quad \dot{u}(\xi,0) = v_0(\xi) \ , \tag{12}$$

where $u_0(\xi)$ and $v_0(\xi)$ are known functions, and a dot over a quantity denotes its partial derivative with respect to time.

3 Analytical solution

Firstly, the third equation in Eq. (7) is rewritten as

$$\frac{\partial \phi}{\partial \xi} = 2E_1\frac{u}{\xi} + E_3\frac{\partial u}{\partial \xi} - \frac{D}{\xi^{2N}} \ . \tag{13}$$

The solution of Eq. (9) is

$$D(\xi,\tau) = \frac{1}{\xi^2}\eta(\tau) \ , \tag{14}$$

where $\eta(\tau)$ is an undetermined function with respect to the dimensionless time τ. Substituting Eq. (13) into the first two equations in Eq. (7), and utilizing Eq. (14), gives

$$\begin{aligned}\sigma_\theta &= \xi^{2N}\left[(C_1^D + C_2^D)\frac{u}{\xi} + C_3^D\frac{\partial u}{\partial \xi}\right] - E_1\frac{\eta(\tau)}{\xi^2},\\ \sigma_r &= \xi^{2N}\left[2C_3^D\frac{u}{\xi} + C_0^D\frac{\partial u}{\partial \xi}\right] - E_3\frac{\eta(\tau)}{\xi^2}\quad ,\end{aligned} \tag{15}$$

where

$$C_1^D = C_1 + E_1^2,\quad C_2^D = C_2 + E_1^2,\quad C_3^D = C_3 + E_1E_3,\quad C_0^D = 1 + E_3^2\ . \tag{16}$$

Substituting Eq. (15) into Eq. (8), we obtain

$$\frac{\partial^2 u}{\partial \xi^2} + 2(N+1)\frac{1}{\xi}\frac{\partial u}{\partial \xi} - \frac{\mu_1^2}{\xi^2}u = \frac{1}{c_L^2}\frac{\partial^2 u}{\partial \tau^2} - 2\frac{E_1}{C_0^D}\frac{1}{\xi^{2N+3}}\eta(\tau)\ , \tag{17}$$

where

$$\mu_1 = \sqrt{2\frac{C_1^D + C_2^D - (2N+1)C_3^D}{C_0^D}},\quad c_L = \sqrt{C_0^D}\ . \tag{18}$$

Utilizing the second equation in Eq. (15), first of Eq. (11) can be rewritten as

$$\begin{aligned}\xi = s:&\quad C_0^D\tfrac{\partial u}{\partial \xi} + 2C_3^D\tfrac{u}{\xi} = s^{-2N}\left[p_a(\tau) + \tfrac{E_3}{s^2}\eta(\tau)\right],\\ \xi = 1:&\quad C_0^D\tfrac{\partial u}{\partial \xi} + 2C_3^D\tfrac{u}{\xi} = p_b(\tau) + E_3\eta(\tau)\ .\end{aligned} \tag{19}$$

Secondly, a new dependent variable $w(\xi,\tau)$ is introduced as

$$u(\xi,\tau) = \xi^{-\left(N+\frac{1}{2}\right)}w(\xi,\tau) \tag{20}$$

Then Eqs. (17), (19) and (12) become

$$\frac{\partial^2 w}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial w}{\partial \xi} - \frac{\mu^2}{\xi^2}w = \frac{1}{c_L^2}\frac{\partial^2 w}{\partial \tau^2} + X(\xi)\eta(\tau)\ , \tag{21}$$

$$\begin{aligned}\xi = s:&\quad \tfrac{\partial w}{\partial \xi} + h\tfrac{w}{\xi} = p_1(\tau),\\ \xi = 1:&\quad \tfrac{\partial w}{\partial \xi} + h\tfrac{w}{\xi} = p_2(\tau)\ ,\end{aligned} \tag{22}$$

$$w(\xi,0) = u_1(\xi),\quad \dot{w}(\xi,0) = v_1(\xi)\ , \tag{23}$$

where

$$\begin{aligned}&h = 2\tfrac{C_3^D}{C_0^D} - \left(N+\tfrac{1}{2}\right),\quad \mu = \sqrt{\mu_1^2 + \left(N+\tfrac{1}{2}\right)^2},\quad X(\xi) = -2\tfrac{E_1}{C_0^D}\xi^{-\left(N+\frac{5}{2}\right)},\\ &p_1(\tau) = \tfrac{s^{-N+\frac{1}{2}}}{C_0^D}\left[p_a(\tau) + \tfrac{E_3}{s^2}\eta(\tau)\right],\quad p_2(\tau) = \tfrac{1}{C_0^D}\left[p_b(\tau) + E_3\eta(\tau)\right],\\ &u_1(\xi) = \xi^{N+\frac{1}{2}}u_0(\xi),\quad v_1(\xi) = \xi^{N+\frac{1}{2}}v_0(\xi)\ .\end{aligned} \tag{24}$$

Thirdly, we transform the inhomogeneous boundary conditions into the homogeneous ones by taking

$$w(\xi,\tau) = w_1(\xi,\tau) + w_2(\xi,\tau)\ , \tag{25}$$

where $w_2(\xi,\tau)$ satisfies the inhomogeneous boundary conditions, which can be taken as

$$w_2(\xi,\tau) = f_1(\xi)p_a(\tau) + f_2(\xi)p_b(\tau) + f_3(\xi)\eta(\tau) \ , \tag{26}$$

where

$$\begin{aligned} &f_1(\xi) = s^{-N+\frac{1}{2}} \tfrac{A_2}{C_0^D}(\xi-1)^m, \quad f_2(\xi) = \tfrac{A_1}{C_0^D}(\xi-s)^m, \quad f_3(\xi) = E_3\left[\tfrac{1}{s^2} f_1(\xi) + f_2(\xi)\right], \\ &A_1 = \tfrac{1}{m(1-s)^{m-1} + h(1-s)^m}, \quad A_2 = \tfrac{1}{m(s-1)^{m-1} + h(s-1)^m/s} \ . \end{aligned} \tag{27}$$

Generally, we take $m = 2$ if the denominators of A_1 and A_2 are nonzero; otherwise, $m = 3$ or 4 etc. can be adopted. Substituting Eq. (25) into Eqs. (21)–(23), gives

$$\frac{\partial^2 w_1(\xi,\tau)}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial w_1(\xi,\tau)}{\partial\xi} - \frac{\mu^2}{\xi^2} w_1(\xi,\tau) = \frac{1}{c_L^2}\frac{\partial^2 w_1(\xi,\tau)}{\partial\tau^2} + g(\xi,\tau) \ , \tag{28}$$

$$\frac{\partial w_1(\xi,\tau)}{\partial\xi} + h\frac{w_1(\xi,\tau)}{\xi} = 0, \quad (\xi = s \text{ and } 1) \ , \tag{29}$$

$$w_1(\xi,0) = u_2(\xi) - f_3(\xi)\eta(0), \quad \dot{w}_1(\xi,0) = v_2(\xi) - f_3(\xi)\dot{\eta}(0) \ , \tag{30}$$

where

$$\begin{aligned} &g(\xi,\tau) = g_1(\xi,\tau) + g_2(\xi)\eta(\tau) + g_3(\xi)\ddot{\eta}(\tau), \\ &g_1(\xi,\tau) = f_4(\xi)p_a(\tau) + f_5(\xi)p_b(\tau) + \frac{1}{c_L^2}\left[f_1(\xi)\ddot{p}_a(\tau) + f_2(\xi)\ddot{p}_b(\tau)\right], \\ &g_2(\xi) = \tfrac{\mu^2}{\xi^2} f_3(\xi) - \tfrac{1}{\xi}\frac{\mathrm{d}f_3(\xi)}{\mathrm{d}\xi} - \frac{\mathrm{d}^2 f_3(\xi)}{\mathrm{d}\xi^2} + X(\xi), \quad g_3(\xi) = \tfrac{f_3(\xi)}{c_L^2}, \\ &f_4(\xi) = \tfrac{\mu^2}{\xi^2} f_1(\xi) - \tfrac{1}{\xi}\frac{\mathrm{d}f_1(\xi)}{\mathrm{d}\xi} - \frac{\mathrm{d}^2 f_1(\xi)}{\mathrm{d}\xi^2}, \quad \dot{\eta}(0) = \frac{\mathrm{d}\eta(\tau)}{\mathrm{d}\tau}\Big|_{\tau=0}, \\ &f_5(\xi) = \frac{\mu^2}{\xi^2} f_2(\xi) - \frac{1}{\xi}\frac{\mathrm{d}f_2(\xi)}{\mathrm{d}\xi} - \frac{\mathrm{d}^2 f_2(\xi)}{\mathrm{d}\xi^2} \end{aligned} \tag{31}$$

and

$$\begin{aligned} &u_2(\xi) = u_1(\xi) - f_1(\xi)p_a(0) - f_2(\xi)p_b(0), \\ &v_2(\xi) = v_1(\xi) - f_1(\xi)\dot{p}_a(0) - f_2(\xi)\dot{p}_b(0) \ . \end{aligned} \tag{32}$$

In Eq. (32), we denote $\dot{p}_a(0) = \frac{\mathrm{d}p_a(\tau)}{d\tau}\big|_{\tau=0}$ and $\dot{p}_b(0) = \frac{\mathrm{d}p_b(\tau)}{\mathrm{d}\tau}\big|_{\tau=0}$,

Then, by using the separation-of-variables technique, the solution of Eq. (28) can be assumed in the following form

$$w_1(\xi,\tau) = \sum_i R_i(\xi)F_i(\tau) \ , \tag{33}$$

where $F_i(\tau)$ are unknown functions of τ, and $R_i(\xi)$ is given by

$$R_i(\xi) = \mathrm{J}_\mu(k_i\xi)\mathrm{Y}(\mu,k_i,s) - \mathrm{Y}_\mu(k_i\xi)\mathrm{J}(\mu,k_i,s) \ , \tag{34}$$

in which $\mathrm{J}_\mu(k_i\xi)$ and $\mathrm{Y}_\mu(k_i\xi)$ are Bessel functions of the first and second kind, and of order μ, respectively. Quantities k_i, arranged in an ascending order, are a series of positive roots of the following eigenequation:

$$\mathrm{J}(\mu,k_i,s)\mathrm{Y}(\mu,k_i,1) - \mathrm{J}(\mu,k_i,1)\mathrm{Y}(\mu,k_i,s) = 0 \ , \tag{35}$$

where

$$\mathrm{J}(\mu,k_i,\xi) = \frac{\mathrm{d}J_\mu(k_i\xi)}{\mathrm{d}\xi} + h\frac{\mathrm{J}_\mu(k_i\xi)}{\xi}, \mathrm{Y}(\mu,k_i,\xi) = \frac{\mathrm{d}Y_\mu(k_i\xi)}{d\xi} + h\frac{Y_\mu(k_i\xi)}{\xi} \ . \tag{36}$$

It can be shown that $w_1(\xi,\tau)$ given in Eq. (33) satisfies the homogeneous boundary conditions as shown in Eq. (29). Substituting Eq. (33) into Eq. (28), gives

$$-c_L^2\sum_i k_i^2F_i(\tau)R_i(\xi)=\sum_i R_i(\xi)\frac{\mathrm{d}^2F_i(\tau)}{\mathrm{d}\tau^2}+c_L^2g(\xi,\tau)\ . \tag{37}$$

By virtue of the orthogonal property of Bessel functions, it is easy to verify the following equation

$$\int_s^1 \xi R_i(\xi)R_j(\xi)\mathrm{d}\xi=N_i\delta_{ij}\ , \tag{38}$$

where δ_{ij} is the Kronecker delta, and

$$N_i=\frac{1}{2k_i^2}\left\{\left[\frac{\mathrm{d}R_i(1)}{\mathrm{d}\xi}\right]^2-s^2\left[\frac{\mathrm{d}R_i(s)}{\mathrm{d}\xi}\right]^2+k_i^2\left[R_i^2(1)-s^2R_i^2(s)\right]-\mu^2\left[R_i^2(1)-R_i^2(s)\right]\right\}\ . \tag{39}$$

In Eq. (39), we denote $\frac{\mathrm{d}R_i(s)}{\mathrm{d}\xi}=\frac{\mathrm{d}R_i(\xi)}{\mathrm{d}\xi}\Big|_{\xi=s}$ and $\frac{\mathrm{d}R_i(1)}{\mathrm{d}\xi}=\frac{\mathrm{d}R_i(\xi)}{\mathrm{d}\xi}\Big|_{\xi=1}$. Utilizing Eq. (38), we can derive the following equation from Eq. (37)

$$\frac{\mathrm{d}^2F_i(\tau)}{\mathrm{d}\tau^2}+\omega_i^2F_i(\tau)=q_i(\tau)\ , \tag{40}$$

where

$$\begin{aligned}&q_i(\tau)=q_{1i}(\tau)+h_{1i}\eta(\tau)+h_{2i}\ddot{\eta}(\tau),\\&\omega_i=k_ic_L,\quad q_{1i}(\tau)=-\frac{c_L^2}{N_i}\int_s^1\xi g_1(\xi,\tau)R_i(\xi)\mathrm{d}\xi\,,\\&h_{1i}=-\frac{c_L^2}{N_i}\int_s^1\xi g_2(\xi)R_i(\xi)\mathrm{d}\xi,\quad h_{2i}=-\frac{c_L^2}{N_i}\int_s^1\xi g_3(\xi)R_i(\xi)\mathrm{d}\xi\ .\end{aligned} \tag{41}$$

The solution of Eq. (40) is

$$F_i(\tau)=B_{1i}\cos\omega_i\tau+\frac{B_{2i}}{\omega_i}\sin\omega_i\tau+\frac{1}{\omega_i}\int_0^\tau q_i(p)\sin\omega_i(\tau-p)\mathrm{d}p\ , \tag{42a}$$

where B_{1i} and B_{2i} are unknown constants. We can derive the following equation from Eq. (42a)

$$\dot{F}_i(\tau)=-\omega_iB_{1i}\sin\omega_i\tau+B_{2i}\cos\omega_i\tau+\int_0^\tau q_i(p)\cos\omega_i(\tau-p)\mathrm{d}p\ , \tag{42b}$$

According to Eq. (30) and Eq. (33) and utilizing Eq. (38), we obtain

$$F_i(0)=\ B_{1i}=I_{1i}+I_{2i}\eta(0),\quad \dot{F}_i(0)=B_{2i}=I_{3i}+I_{2i}\dot{\eta}(0)\ , \tag{43}$$

where

$$I_{1i}=\frac{1}{N_i}\int_s^1\xi u_2(\xi)R_i(\xi)\mathrm{d}\xi,\quad I_{2i}=\frac{-1}{N_i}\int_s^1\xi f_3(\xi)R_i(\xi)\mathrm{d}\xi,\quad I_{3i}=\frac{1}{N_i}\int_s^1\xi v_2(\xi)R_i(\xi)\mathrm{d}\xi\ . \tag{44}$$

Noticing that $q_i(\tau)$ in the Eq. (41) includes $\ddot{\eta}(\tau)$, we use the integration-by-parts formula to perform the integration of the term involving $\ddot{\eta}(p)$ in Eq. (42a) as

$$\int_0^\tau \ddot{\eta}(p)\sin\omega_i(\tau-p)\mathrm{d}p = -\dot{\eta}(0)\sin\omega_i\tau - \eta(0)\omega_i\cos\omega_i\tau + \omega_i\eta(\tau) - \omega_i^2\int_0^\tau \eta(p)\sin\omega_i(\tau-p)\mathrm{d}p \ . \tag{45}$$

Utilizing Eq. (45), Eq. (42a) can be rewritten as

$$F_i(\tau) = F_{1i}(\tau) + h_{2i}\eta(\tau) + \left(\frac{h_{1i}}{\omega_i} - h_{2i}\omega_i\right)\int_0^\tau \eta(p)\sin\omega_i(\tau-p)\mathrm{d}p \ , \tag{46}$$

where

$$F_{1i}(\tau) = B_{1i}\cos\omega_i\tau + \frac{B_{2i}}{\omega_i}\sin\omega_i\tau + \frac{1}{\omega_i}\int_0^\tau q_{1i}(p)\sin\omega_i(\tau-p)\mathrm{d}p - \frac{h_{2i}}{\omega_i}[\dot{\eta}(0)\sin\omega_i\tau + \eta(0)\omega_i\cos\omega_i\tau] \ . \tag{47}$$

In the following, we will determine $\eta(0)$, $\dot{\eta}(0)$ and $\eta(\tau)$ from the electric boundary conditions. Substituting Eq. (14) into Eq. (13), gives

$$\frac{\partial\phi}{\partial\xi} = 2E_1\frac{u}{\xi} + E_3\frac{\partial u}{\partial\xi} - \frac{\eta(\tau)}{\xi^{2(N+1)}} \ . \tag{48}$$

Substituting Eq. (25) into Eq. (20), utilizing Eqs. (26) and (33), we obtain

$$u(\xi,\tau) = \xi^{-\left(N+\frac{1}{2}\right)}\left[\sum_i R_i(\xi)F_i(\tau) + f_1(\xi)p_a(\tau) + f_2(\xi)p_b(\tau) + f_3(\xi)\eta(\tau)\right] \ . \tag{49}$$

Integrating Eq. (48) and utilizing Eq. (49), derives

$$\phi(\xi,\tau) = \phi_1(\xi)p_a(\tau) + \phi_2(\xi)p_b(\tau) + \phi_3(\xi)\eta(\tau) + \sum_i \phi_{4i}(\xi)F_i(\tau) + \phi_a(\tau) \ , \tag{50}$$

where

$$\begin{aligned}
\phi_1(\xi) &= 2E_1\int_s^\xi \xi^{-\left(N+\frac{3}{2}\right)}f_1(\xi)\mathrm{d}\xi + E_3\left[\xi^{-\left(N+\frac{1}{2}\right)}f_1(\xi) - s^{-\left(N+\frac{1}{2}\right)}f_1(s)\right],\\
\phi_2(\xi) &= 2E_1\int_s^\xi \xi^{-\left(N+\frac{3}{2}\right)}f_2(\xi)\mathrm{d}\xi + E_3\left[\xi^{-\left(N+\frac{1}{2}\right)}f_2(\xi) - s^{-\left(N+\frac{1}{2}\right)}f_2(s)\right],\\
\phi_3(\xi) &= 2E_1\int_s^\xi \xi^{-\left(N+\frac{3}{2}\right)}f_3(\xi)\mathrm{d}\xi + E_3\left[\xi^{-\left(N+\frac{1}{2}\right)}f_3(\xi) - s^{-\left(N+\frac{1}{2}\right)}f_3(s)\right] + f_6(\xi),\\
\phi_{4i}(\xi) &= 2E_1\int_s^\xi \xi^{-\left(N+\frac{3}{2}\right)}R_i(\xi)d\xi + E_3\left[\xi^{-\left(N+\frac{1}{2}\right)}R_i(\xi) - s^{-\left(N+\frac{1}{2}\right)}R_i(s)\right],\\
f_6(\xi) &= \begin{cases} \left[\xi^{-(2N+1)} - s^{-(2N+1)}\right]/(2N+1) & N \neq -0.5, \\ -\ln(\xi/s) & (N=-0.5) \ . \end{cases}
\end{aligned} \tag{51}$$

When $\xi = 1$, Eq. (50) gives

$$\phi_b(\tau) = \phi_1(1)p_a(\tau) + \phi_2(1)p_b(\tau) + \phi_3(1)\eta(\tau) + \sum_i \phi_{4i}(1)F_i(\tau) + \phi_a(\tau) \ . \tag{52a}$$

Then we have

$$\dot{\phi}_b(\tau) = \phi_1(1)\dot{p}_a(\tau) + \phi_2(1)\dot{p}_b(\tau) + \phi_3(1)\dot{\eta}(\tau) + \sum_i \phi_{4i}(1)\dot{F}_i(\tau) + \dot{\phi}_a(\tau) \ . \tag{52b}$$

If $\tau = 0$, we can determine $\eta(0)$ and $\dot{\eta}(0)$ from Eqs. (52) by virtue of Eq. (43):

$$\begin{aligned} \eta(0) &= \frac{\phi_b(0) - \phi_a(0) - \phi_1(1)p_a(0) - \phi_2(1)p_b(0) - \sum_i \phi_{4i}(1)I_{1i}}{\phi_3(1) + \sum_i \phi_{4i}(1)I_{2i},} \\ \dot{\eta}(0) &= \frac{\dot{\phi}_b(0) - \dot{\phi}_a(0) - \phi_1(1)\dot{p}_a(0) - \phi_2(1)\dot{p}_b(0) - \sum_i \phi_{4i}(1)I_{3i}}{\phi_3(1) + \sum_i \phi_{4i}(1)I_{2i}} \ . \end{aligned} \tag{53}$$

Substitute $\eta(0)$ and $\dot{\eta}(0)$ into Eqs. (43) and (47). Then B_{1i} and B_{2i} become known and $F_{1i}(\tau)$ is also a known function. Substituting Eq. (46) into Eq. (52a), derives

$$\psi(\tau) = L_1\eta(\tau) + \sum_i L_{2i} \int_0^\tau \eta(p) \sin\omega_i(\tau - p)\mathrm{d}p \ , \tag{54}$$

where

$$\psi(\tau) = \phi_b(\tau) - \phi_a(\tau) - \phi_1(1)p_a(\tau) - \phi_2(1)p_b(\tau) - \sum_i \phi_{4i}(1)F_{1i}(\tau),$$

$$L_1 = \phi_3(1) + \sum_i \phi_{4i}(1)h_{2i}, \quad L_{2i} = \phi_{4i}(1)\left(\frac{h_{1i}}{\omega_i} - h_{2i}\omega_i\right) \ . \tag{55}$$

From Eq. (54), we have

$$\dot{\psi}(\tau) = L_1\dot{\eta}(\tau) + \sum_i L_{2i}\omega_i \int_0^\tau \eta(p) \cos\omega_i(\tau - p)\mathrm{d}p \ , \tag{56}$$

It is noted that Eq. (54) is a Volterra integral equation of the second kind, [19], of which analytical solutions can be obtained only for certain cases. In the general case, numerical methods are usually adopted.

For $\eta(0)$ and $\dot{\eta}(0)$ have been obtained, we will construct recursive formula by making use of a cubic Hermite polynomial approximation of $\eta(\tau)$. Practically, accurate numerical results can be obtained efficiently by the following method. We first divide the time interval $[0, \tau_n]$ into n equal subintervals, with discrete time points $\tau_0 = 0, \quad \tau_1, \quad \tau_2, \quad \ldots \quad \tau_n$. Then the cubic Hermite polynomial at the interval $[\tau_{j-1}, \tau_j]$ is

$$\eta(\tau) = H_{0j}(\tau)\eta(\tau_{j-1}) + H_{1j}(\tau)\eta(\tau_j) + H_{2j}(\tau)\dot{\eta}(\tau_{j-1}) + H_{3j}(\tau)\dot{\eta}(\tau_j) \quad (j = 1, 2 \cdots n) \ . \tag{57}$$

where $\dot{\eta}(\tau_j)$ is the value $\dot{\eta}(\tau)$ at $\tau = \tau_j$, and

$$\begin{aligned} H_{0j}(\tau) &= \left(1 + 2\frac{\tau-\tau_{j-1}}{\tau_j-\tau_{j-1}}\right)\left(\frac{\tau-\tau_j}{\tau_j-\tau_{j-1}}\right)^2, \quad H_{1j}(\tau) = \left(1 + 2\frac{\tau_j-\tau}{\tau_j-\tau_{j-1}}\right)\left(\frac{\tau-\tau_{j-1}}{\tau_j-\tau_{j-1}}\right)^2, \\ H_{2j}(\tau) &= (\tau - \tau_{j-1})\left(\frac{\tau-\tau_j}{\tau_j-\tau_{j-1}}\right)^2, \quad H_{3j}(\tau) = (\tau - \tau_j)\left(\frac{\tau-\tau_{j-1}}{\tau_j-\tau_{j-1}}\right)^2, \ (j = 1, 2 \ldots n) \ . \end{aligned} \tag{58}$$

Substituting Eq. (57) into Eqs. (54) and (56), gives

$$\psi(\tau_j) = L_1\eta(\tau_j) + \sum_i L_{2i}\sum_{k=1}^{j}[\lambda_{0ijk}\eta(\tau_{k-1}) + \lambda_{1ijk}\eta(\tau_k) + \lambda_{2ijk}\dot{\eta}(\tau_{k-1}) + \lambda_{3ijk}\dot{\eta}(\tau_k)],$$
$$\dot{\psi}(\tau_j) = L_1\dot{\eta}(\tau_j) + \sum_i L_{2i}\omega_i\sum_{k=1}^{j}[\mu_{0ijk}\eta(\tau_{k-1}) + \mu_{1ijk}\eta(\tau_k) + \mu_{2ijk}\dot{\eta}(\tau_{k-1}) + \mu_{3ijk}\dot{\eta}(\tau_k)] \quad , \tag{59}$$

where

$$\lambda_{lijk} = \int_{\tau_{k-1}}^{\tau_k} H_{lk}(p)\sin\omega_i(\tau_j - p)\mathrm{d}p, \quad \mu_{lijk} = \int_{\tau_{k-1}}^{\tau_k} H_{lk}(p)\cos\omega_i(\tau_j - p)\mathrm{d}p,$$
$$(l = 0, 1, 2, 3;\ k = 1, 2 \ldots j;\ j = 1, 2 \ldots n) \quad . \tag{60}$$

Then we can derive the following formula from Eq. (59)

$$\eta(\tau_j) = \frac{b_{1j}k_{22j} - b_{2j}k_{12j}}{k_{11j}k_{22j} - k_{12j}k_{21j}}, \quad \dot{\eta}(\tau_j) = \frac{b_{2j}k_{11j} - b_{1j}k_{21j}}{k_{11j}k_{22j} - k_{12j}k_{21j}}, \quad (j = 1, 2 \ldots n) \quad , \tag{61}$$

where

$$k_{11j} = L_1 + \sum_i L_{2i}\lambda_{1ijj}, \quad k_{12j} = \sum_i L_{2i}\lambda_{3ijj},$$
$$k_{21j} = \sum_i L_{2i}\omega_i\mu_{1ijj}, \quad k_{22j} = L_1 + \sum_i L_{2i}\omega_i\mu_{3ijj},$$
$$b_{1j} = \psi(\tau_j) - \sum_i L_{2i}\sum_{k=1}^{j-1}[\lambda_{0ijk}\eta(\tau_{k-1}) + \lambda_{1ijk}\eta(\tau_k) + \lambda_{2ijk}\dot{\eta}(\tau_{k-1}) + \lambda_{3ijk}\dot{\eta}(\tau_k)]$$
$$\quad - \sum_i L_{2i}[\lambda_{0ijj}\eta(\tau_{j-1}) + \lambda_{2ijj}\dot{\eta}(\tau_{j-1})], \tag{62}$$
$$b_{2j} = \dot{\psi}(\tau_j) - \sum_i L_{2i}\omega_i\sum_{k=1}^{j-1}[\mu_{0ijk}\eta(\tau_{k-1}) + \mu_{1ijk}\eta(\tau_k) + \mu_{2ijk}\dot{\eta}(\tau_{k-1}) + \mu_{3ijk}\dot{\eta}(\tau_k)]$$
$$\quad - \sum_i L_{2i}\omega_i[\mu_{0ijj}\eta(\tau_{j-1}) + \mu_{2ijj}\dot{\eta}(\tau_{j-1})] \quad .$$

It should be pointed out that if $j = 1$, the second terms in b_{1j} and b_{2j} in Eq. (62) are zero. In Eq. (48), we have obtained $\eta(0)$ and $\dot{\eta}(0)$, based on which we can determine $\eta(\tau_j)$ and $\dot{\eta}(\tau_j), \quad (j = 1, 2 \ldots n)$ step by step by virtue of Eq. (61). After $\eta(\tau)$ is obtained, $u(\xi, \tau)$ and $\phi(\xi, \tau)$ also can be determined.

4 Numerical results and discussions

Example 1
The dynamic response of a nonhomogeneous, piezoelectric, hollow sphere subjected to a constant pressure suddenly applied on the internal surface is considered. The material constants are

$C_{11} = C_{22} = 139.0$ GPa, $C_{12} = 77.8$ GPa, $C_{13} = 74.3$ GPa, $C_{33} = 115.0$ GPa, $E_{31} = -5.2\mathrm{C/m}^2$, $E_{33} = 15.1\mathrm{C/m}^2$, $\Omega_{33} = 5.62 \times 10^{-9}\mathrm{C}^2/(\mathrm{Nm}^2)$, [2].

The boundary conditions are

$$p_a(\tau) = -\sigma_0 H(\tau), \quad p_b(\tau) = 0.0,$$
$$\phi_a(\tau) = 0.0, \quad \phi_b(\tau) = 0.0 \quad , \tag{63}$$

where σ_0 is a prescribed dimensionless constant pressure, and $H(\tau)$ is the Heaviside function. In the following, we take $\sigma_0 = 1.0$, $s = 0.5$, $m = 2$, and $\tau_n = \tau_{200} = 5$. We consider the first 40 terms in the series in Eq. (33) for numerical calculations.

Figure 1 shows the response of σ_r at $\xi = 0.75$ (the middle surface) in the hollow sphere for $N = -1$, $N = 0$ and $N = 1$. From the curves, we can see that the peak values of compressive stress increase quickly with the increase of N, while the peak values of tensile stress vary slightly with N.

Figures 2 and 3 depict the response of σ_θ at $\xi = 0.5$ (the internal surface) and $\xi = 1.0$ (the external surface) in the sphere for different values of N. From the curves, we find that, at the internal surface, the peak values of tensile stress decrease with the increase of N, while at the external surface, it is just the contrary. We know that the circumferential stress has the maximum value at the internal surface for a homogeneous, isotropic, hollow sphere subjected to a uniform pressure at the internal surface. Form the above studies, we can conclude that nonhomogeneous materials can be used to decrease the circumferential stress at the internal surface of the hollow sphere subjected to internal pressure. Actually, it is a very efficient way to make full use of the materials.

Figures 4 and 5 illustrate the distributions of dimensionless electric potential ϕ at different times for $N = -1$ and $N = 1$. Comparing Fig. 4 with Fig. 5, we find that the distributions of ϕ are different for $N = -1$ and $N = 1$. Note that the calculated electric potentials are zero both at the internal and external surfaces, which agrees with the prescribed electric boundary conditions. The correctness of the numerical results is thus clarified in this respect.

Example 2

The dynamic response of a nonhomogeneous, piezoelectric, hollow sphere subjected to a constant electric potential suddenly imposed on the external surface is considered here. The

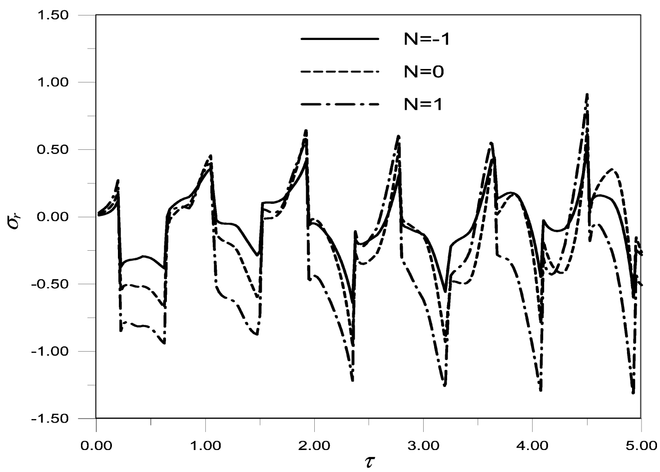

Fig. 1. Histories of dynamic stresses σ_r at the middle surface varying with N(example 1)

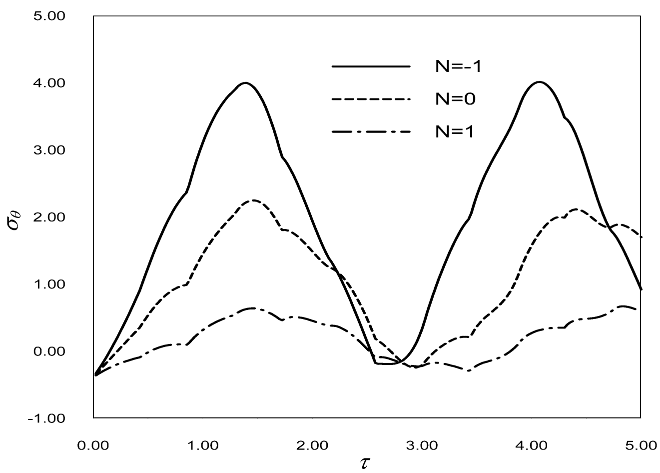

Fig. 2. Histories of dynamic stresses σ_θ at the inner surface varying with N (example 1)

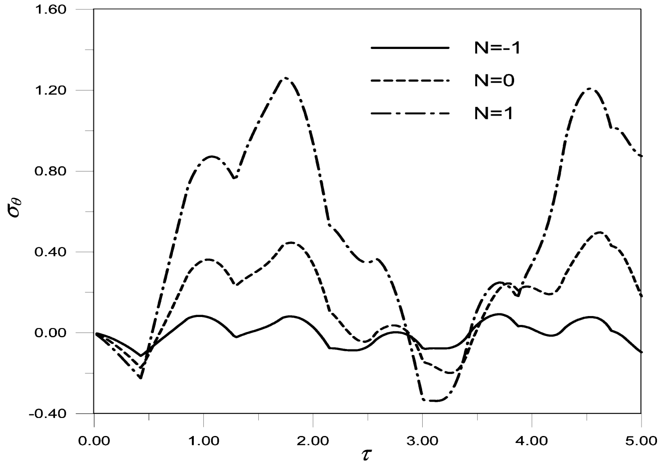

Fig. 3. Histories of dynamic stresses σ_θ at the outer surface varying with N (example 1)

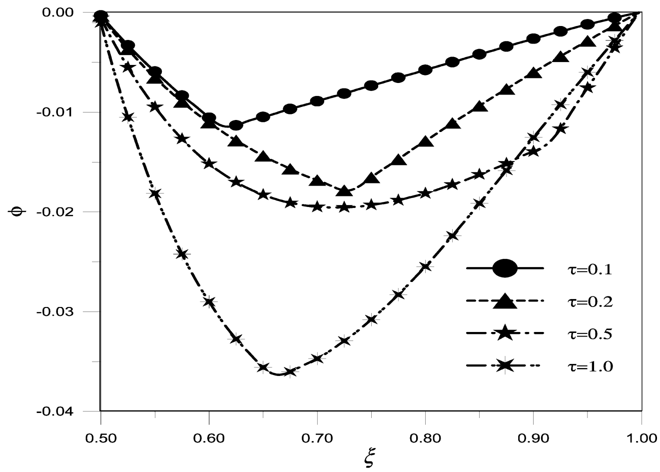

Fig. 4. Distribution of electric potential ϕ for $N = -1$ (example 1)

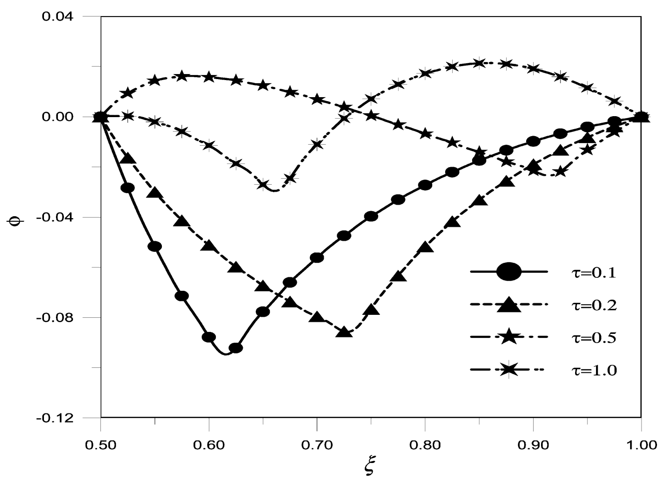

Fig. 5. Distribution of electric potential ϕ for $N = 1$ (example 1)

material constants are the same as those adopted in Example 1. The boundary conditions become

$$\begin{aligned} &p_a(\tau) = 0.0, \quad p_b(\tau) = 0.0, \\ &\phi_a(\tau) = 0.0, \quad \phi_b(\tau) = \phi_0 H(\tau) \ , \end{aligned} \tag{64}$$

where ϕ_0 is the prescribed constant dimensionless electric potential. For numerical calculations, the same parameters as that in Example 1 are employed, except that $\phi_0 = 1.0$ is used instead of $\sigma_0 = 1.0$.

Figure 6 shows the response of σ_r at $\xi = 0.75$ (the middle surface) in the hollow sphere for $N = -1$, $N = 0$ and $N = 1$. From the curves, we can see that the peak values of dynamic radial stress decrease with N.

Figures 7 and 8 depict the response of σ_θ at $\xi = 0.5$ (the internal surface) and $\xi = 1.0$ (the external surface) in the sphere for different values of N. From the curves, we find that, at the

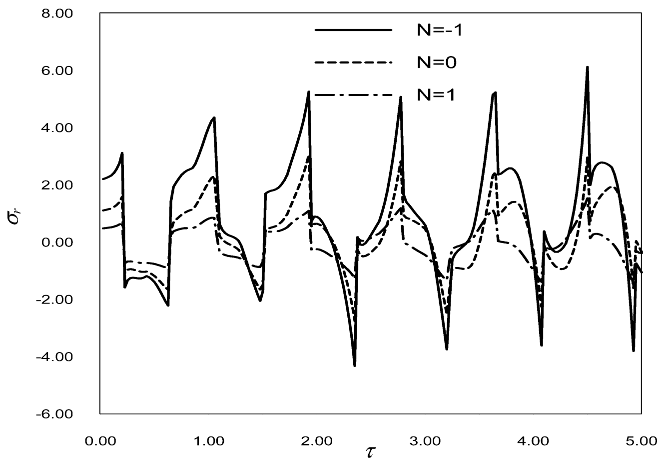

Fig. 6. Histories of dynamic stresses σ_r at the middle surface varying with N (example 2)

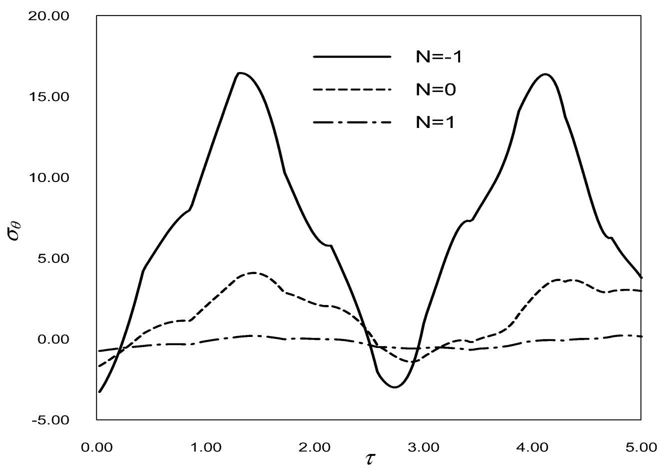

Fig. 7. Histories of dynamic stresses σ_θ at the inner surface varying with N (example 2)

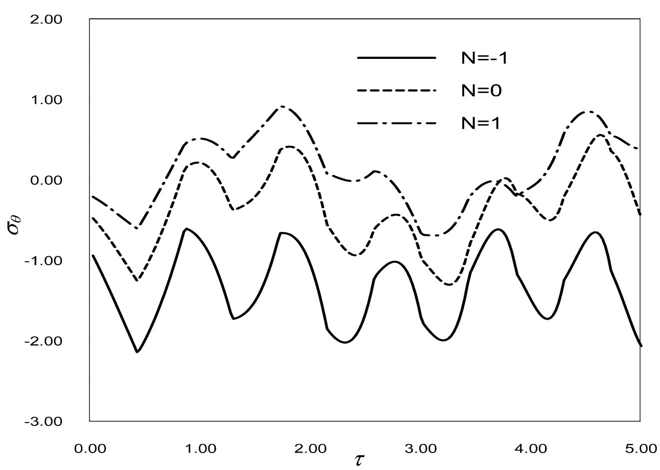

Fig. 8. Histories of dynamic stresses σ_θ at the outer surface varying with N (example 2)

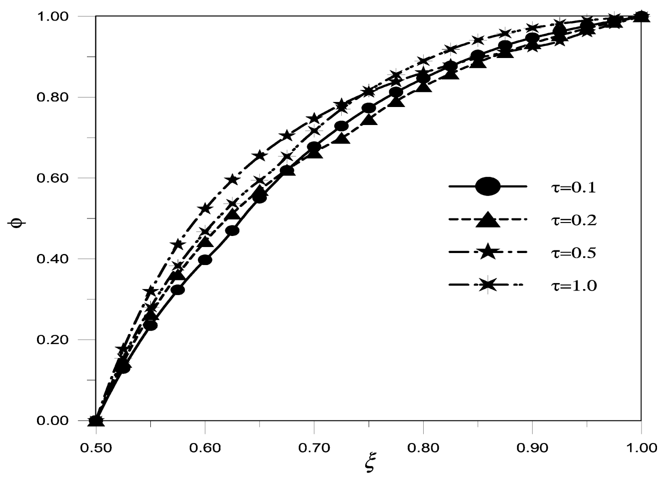

Fig. 9. Distribution of electric potential ϕ for $N = -1$ (example 2)

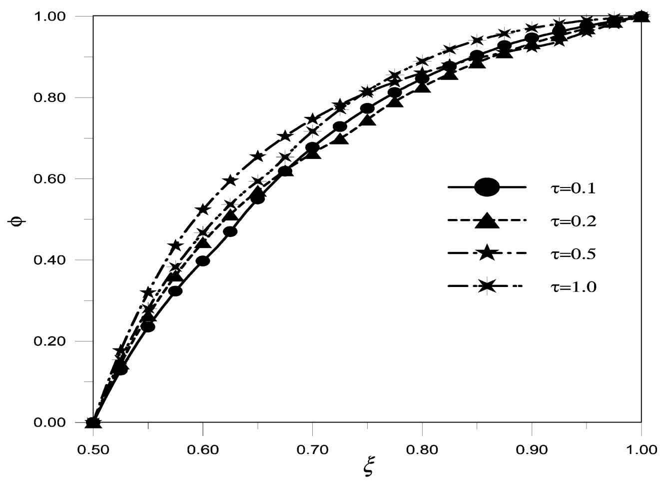

Fig. 10. Distribution of electric potential ϕ for $N = 1$ (example 2)

internal surface, the peak values of the tensile stress decrease quickly, while at the external surface, the peak values of the compressive stress decrease with N. We also notice that, at the external surface, the dynamic circumferential stress is compressive when $N = -1$, while it becomes tensile for $N = 0$ and $N = 1$. The peak values of tensile stress for $N = 1$ are larger than those for $N = 0$.

Figures 9 and 10 illustrate the distributions of dimensionless electric potential ϕ at different times for $N = -1$ and $N = 1$, respectively. Comparing Fig. 9 with Fig. 10, we also find that the distributions of ϕ are different for $N = -1$ and $N = 1$. Again, the calculated electric potentials are zero both at the internal and external surfaces, which satisfy the prescribed electric boundary conditions. The correctness of the numerical results is further clarified in this respect.

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Comments

(1) In terms of numerical accuracy for different number of terms considered in the series in Eq. (33), we find that the results vary very slightly between 30 terms and 40 terms. Therefore, we take for all computations the first 40 terms in the series.

(2) If the electric boundary conditions in Eq. (11b) are expressed by the electric displacement, only one boundary condition will be involved. That is because, if the electric displacement is prescribed on one surface, then the distribution of the electric displacement can be determined immediately from Eq. (14). In this case, from the beginning to Eq. (47), the displacement and stress solution can be determined and the procedure of solving integral equation can be avoided. The expression for electric potential can be written as Eq. (50). But if we want to determine $\phi(\xi, \tau)$ completely, one boundary condition related to ϕ must

be known. That is, either $\phi_a(\tau)$ or $\phi_b(\tau)$ should be prescribed. The relationship between $\phi_a(\tau)$ and $\phi_b(\tau)$ is given in Eq. (52a).

(3) If $H_{lk}(\tau)$ $(l = 0, 1, 2, 3)$ are polynomials of τ, the integration in Eq. (60) can be obtained explicitly, which can improve the computing accuracy. Using cubic Hermite polynomial to approximate $\eta(\tau)$, accurate results can be obtained efficiently, and it is also very stable for long time calculations. Based on many numerical tests, we find that the relative error is less than 10^{-6} for the time step $\Delta\tau \leq 0.1$. In order to obtain highly accurate results, we adopt $\Delta\tau = 0.025$ in the paper.

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