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Buckling of an elastically restrained multi-step non-uniform beam with multiple cracks

Q. S. Li

Summary A model of a massless rotational spring is adopted to describe the local flexibility induced by cracks in a beam. The governing differential equation for buckling of a multi-step non-uniform beam with spring supports, each step of which has an arbitrary number of cracks, is expressed in the terms of the bending moment. Linearly independent solutions of the governing equation are derived for five different types of non-uniform beams. The main advantage of the proposed method is that the eigenvalue equation of a multi-step non-uniform beam with any kind of two-end supports, any finite number of cracks and spring supports at intermediate points can be conveniently determined from a second-order determinant based on the fundamental solutions developed in this paper. The decrease in the determinant's order, as compared with previously developed procedures, leads to significant savings in the computational effort. Two numerical examples are given to illustrate the application of the proposed method and to study the effect of cracks on the critical buckling force. The accuracy of the proposed method is verified through numerical examples.

Keywords Buckling, Stability, Crack, Non-uniform beam

1 Introduction

Lightweight structures have been extensively used in civil, mechanical, aerospace engineering etc., and stability problems of such structures have become of increasing importance. Most structural parts are operated under fatigue loading conditions causing cracks in the over-stressed zones. Cracks that are found to exist in structural members lower the structural integrity and should be considered in the buckling analysis of structures.

The stability of columns with a single crack subjected to follower and vertical loads was studied in [1]. Paper [2] investigated the vibration and stability of cracked rotating blades. An analytical approach for vibration analysis of a cracked beam was developed in [3]. This approach leads to a system of $(4n+4)$ equations for establishing the eigenvalue equation in the case of n cracks within the beam. An improved analytical method for calculating natural frequencies of a beam with an arbitrary number of cracks was proposed in [4]. This procedure leads to a system of $(n+2)$ linear equations for determining the eigenvalue equation for a beam with n cracks. All studies mentioned above have been confined to uniform structural members with cracks.

However, non-uniform beams are widely used to achieve a better distribution of the strength and weight of structural members or machine parts and sometime to satisfy architectural and functional requirements. The vibration and stability of a non-uniform Timoshenko beam has been studied in [5], but only one crack in the beam was considered. There is still need to further the research on buckling of non-uniform beams with multiple cracks.

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In this paper, a new exact approach to the buckling analysis of multi-step non-uniform beams with an arbitrary number of cracks and elastic supports is proposed based on novel fundamental solutions. The main advantage of the proposed method is that the eigenvalue equation of such a beam can be conveniently determined from a system of two linear equations, i.e. by solving a second-order determinant. The decrease in the determinant's order, as compared with previous procedures (e.g., [3], [4]), leads to significant savings in the computational effort and costs associated with buckling analysis of cracked beams. Moreover, the procedure presented in [4] is confined to analysis of a clamped-free uniform beam with cracks, while the present method can be used to analyze a non-uniform beam with any kind of two-end support, any finite number of cracks and springs at intermediate points. The numerical examples show that the effects of cracks on the critical buckling force depend on the number, depth and location of the cracks, and prove that the proposed procedure is exact and efficient.

The finite element method (FEM) can be also used for buckling analysis of a structure with an arbitrary number of cracks. However, the FEM analysis will lead to determinants of high order. The availability of exact solutions should help in examining the accuracy of the approximated or numerical solutions. It should be also mentioned that the present analytical method and solutions can be easily implemented and could provide an insight into the physics of the problem.

2 Theory

A multi-step non-uniform beam with rotational and translational spring supports subjected to concentrated axial forces, as shown in Fig. 1, is considered in this paper for buckling analysis. The axial force N_i acting on the i-th step of the beam is given by

$$N_i = \sum_{k=0}^{i-1} a_k P \quad , \tag{1}$$

where $a_k P$ is an axial external load directly acting on the $(k+1)$-th step of the beam.

It is assumed that the number of cracks in the i-th step beam is n_i. A model of a massless rotational spring, [6], is adopted here to describe the local flexibility induced by cracks in the beam, as shown in Fig. 2. The n_i cracks are located at sections $x_{i1}, x_{i2}, \ldots, x_{in_i}$ such that $0 < x_{i1} < x_{i2} < \cdots < x_{in_i} < L_i$, where L_i is the length of the i-th step of the beam (Fig. 3), and

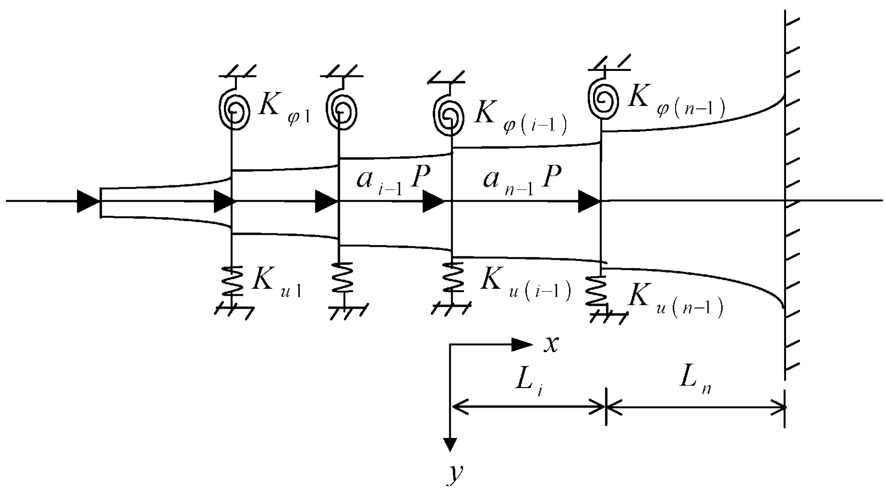

Fig. 1. A multi-step non-uniform beam with spring supports

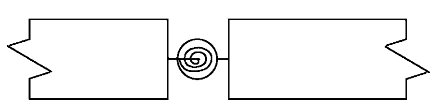

Fig. 2. A model of a massless rotational spring to represent local flexibility induced by a crack

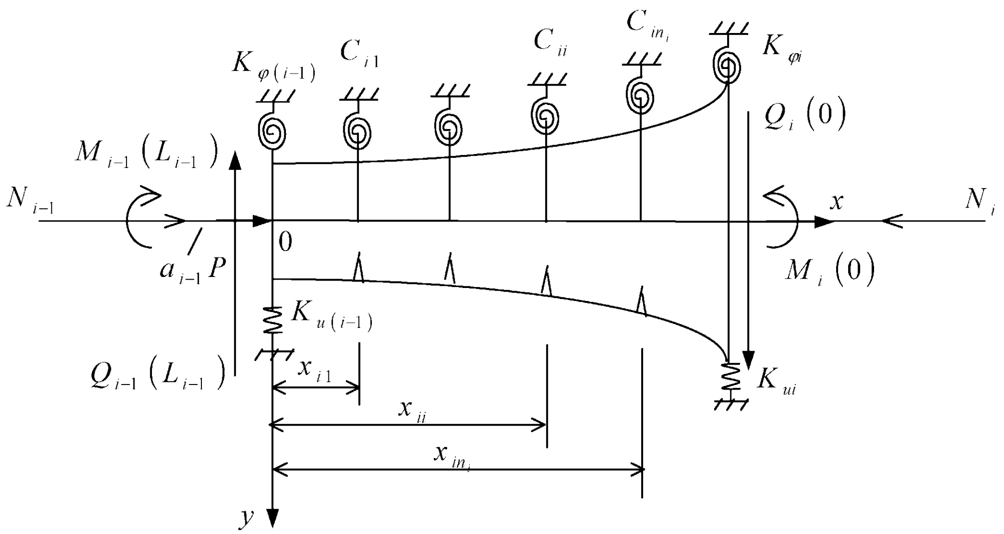

Fig. 3. The i-th beam step with n_i cracks

the origin of the coordinates is set at the left end of the i-th beam step ($i = 1, 2, \ldots, n$). It can be seen from Fig. 3 that the i-th beam step is divided into $(n_i + 1)$ segments by the n_i cracks. The symbols introduced in Fig. 3 are simplified representations for describing the local flexibility induced by the cracks. The effect of the ij-th crack is that the rotation at the ij-th section $(x = x_{ij})$ has a jump. It can be expressed as

$$\Delta\psi_{ij} = C_{ij} y_i''(x_{ij}) \ , \tag{2}$$

where $\Delta\psi_{ij}$ and $y_i(x_{ij})$ are the jump of the rotation and deflection at the ij-th section, respectively, the prime indicates differentiation with respect to the variable x, C_{ij} is the flexibility of the ij-th rotational spring representing the effect of the ij-th crack, which is a function of the crack depth and beam height. For a one-sided opening crack in a structural member with rectangular section, C_{ij} can be expressed as, [1, 6],

$$C_{ij} = 5.346 h_{ij} f(\xi_{ij}) \ , \tag{3}$$

where h_{ij} is the height of the cross-section at $x = x_{ij}$ and ξ_{ij} is the ratio

$$\xi_{ij} = \frac{a_{ij}}{h_{ij}} \ . \tag{4}$$

Here, a_{ij} is the depth of the ij-th crack. The function $f(\xi_{ij})$ is called the flexibility function and can be expressed as, [1, 6],

$$f(\xi_{ij}) = 1.862\,\xi_{ij}^2 - 3.95\,\xi_{ij}^3 + 16.375\,\xi_{ij}^4 - 37.226\,\xi_{ij}^5 + 76.81\,\xi_{ij}^6 - 126\,\xi_{ij}^7 + 172\,\xi_{ij}^8 - 143.97\,\xi_{ij}^9 + 66.56\,\xi_{ij}^{10} \ . \tag{5}$$

The case for two-sided cracks can be considered similarly, [4]. It was discussed in [1, 2, 6] that Eq. (5) is not only valid for vibration analysis of cracked structures, but also can be applied for the analysis of cracked beams under compression.

The difference between the buckling shape of a cracked beam and that of the corresponding uncracked beam is that the slope of the cracked beam at the section where a crack occurs has a jump. Thus, it is necessary to investigate the buckling of an uncracked beam first, which is the basis for buckling analysis of the cracked beam.

The differential equation for buckling of the i-th step of the uncracked beam can be written as, [7],

$$\frac{\mathrm{d}^2 M_i(x)}{\mathrm{d}x^2} + \frac{N_i}{K_i(x)} M_i(x) = 0, \quad x \in [0, L_i] \ , \tag{6}$$

where $M_i(x)$, N_i and $K_i(x)$ are the bending moment, axial force and flexural stiness of the i-th beam step at section x, respectively.

The general solution of Eq. (6) can be expressed in the form

$$M_i(x) = D_{i3}S_{i3}(x) + D_{i4}S_{i4}(x) \quad , \tag{7}$$

where S_{i3} and S_{i4} are the linearly independent special solutions, and D_{i3}, D_{i4} are integral constants of Eq. (6), respectively. It is obvious that the analytical solutions of Eq. (6) are dependent on the distribution of the flexural stiffness $K_i(x)$. Thus, the analytical solution of Eq. (6) may be obtained by means of a reasonable selection of the flexural stiffness distribution. As suggested in [8–10], the functions that can be used to approximate the variation of stiffness are algebraic polynomials, exponential functions, trigonometric series or their combinations. For the following five cases of $K_i x$), which describe many cases of ordinary structural members, solutions S_{i3} and S_{i4} are found in this paper. 525

Case 1:

$$K_i(x) = \left(\alpha_i e^{-\beta_i \frac{x}{L_i}} - c_i\right)^{-1} \quad . \tag{8}$$

Substituting Eq. (8) into Eq. (6) and setting

$$\eta = e^{\frac{\beta_i x}{2L_i}} \quad ,$$

one obtains the set of linearly independent solutions as follows

$$S_{i3}(x) = \mathrm{J}_{v_i}\left(a_i e^{\frac{\beta_i x}{2L_i}}\right),$$
$$S_{i4}(x) = \begin{cases} \mathrm{J}_{-v_i}\left(a_i e^{\frac{\beta_i x}{2L_i}}\right), & v_i \text{ is a non-integer,} \\ \mathrm{Y}_{v_i}\left(a_i e^{\frac{\beta_i x}{2L_i}}\right), & v_i \text{ is an integer} \quad , \end{cases} \tag{9}$$

where

$$a_i^2 = \frac{4\alpha_i N_i L_i^2}{\beta_i^2}, \qquad v_i = \frac{4c_i N_i L_i^2}{\beta_i^2} \quad . \tag{10}$$

Here, $\mathrm{J}_{v_i}(\cdot)$ and $\mathrm{Y}_{v_i}(\cdot)$ are the first and second kind Bessel functions of the order v_i , respectively.

Case 2:

$$K_i(x) = \alpha_i(1 + \beta_i x)^{b_i} \quad . \tag{11}$$

Letting

$$g = 1 + \beta_i x, \quad M_i = g^{\frac{1}{2}} z, \quad n_i^2 = \frac{N_i}{\alpha_i \beta_i^2} \quad , \tag{12}$$

then Eq. (6) becomes

$$\frac{\mathrm{d}^2 z}{\mathrm{d}g^2} + \frac{1}{g}\frac{\mathrm{d}z}{\mathrm{d}g} + \left[n_i^2 g^{-b_i} - \frac{1}{4g^2}\right] z = 0 \quad . \tag{13}$$

Setting

$$\eta = \frac{n_i}{q_i} g^{q_i} \quad , \tag{14}$$

then Eq. (13) becomes a Bessel equation of the ν_i-th order. For this case, there are two linearly independent solutions:

$$S_{i3}(x) = (1+\beta_i x)^{\frac{1}{2}} J_{\nu_i}\left[\frac{n_i}{q_i}(1+\beta_i x)^{\frac{1}{2\nu_i}}\right]$$

$$S_{i4}(x) = \begin{cases} (1+\beta_i x)^{\frac{1}{2}} J_{-\nu_i}\left[\frac{n_i}{q_i}(1+\beta_i x)^{\frac{1}{2\nu_i}}\right], & \nu_i \text{ is a non-integer,} \\ (1+\beta_i x)^{\frac{1}{2}} Y_{\nu_i}\left[\frac{n_i}{q_i}(1+\beta_i x)^{\frac{1}{2\nu_i}}\right], & \nu_i \text{ is an integer,} \end{cases} \quad (15)$$

$$q_i = \frac{2-b_i}{2}, \quad \nu_i = \frac{1}{2-b_i} \quad .$$

If $b_i = 2$, then $\nu_i = \infty$, and the solutions found above are not valid anymore. For this case, Eq. (13) becomes an Euler equation, and the solutions are found as

$$\left.\begin{aligned} S_{i3}(x) &= (1+\beta_i x)^{\frac{1}{2}+\sqrt{-B_i}}, \\ S_{i4}(x) &= (1+\beta_i x)^{\frac{1}{2}-\sqrt{-B_i}}, \end{aligned}\right\} \quad \text{for } B_i = \frac{N_i}{\alpha_i \beta_i^2} - \frac{1}{4} < 0 \quad , \quad (16)$$

or

$$\left.\begin{aligned} S_{i3}(x) &= (1+\beta_i x)^{\frac{1}{2}} \cos[\sqrt{B_i}\,\ln(1+\beta_i x)], \\ S_{i4}(x) &= (1+\beta_i x)^{\frac{1}{2}} \sin[\sqrt{B_i}\,\ln(1+\beta_i x)], \end{aligned}\right\} \quad \text{for } B_i = \frac{N_i}{\alpha_i \beta_i^2} - \frac{1}{4} > 0 \quad . \quad (17)$$

Case 3:

$$\frac{N_i}{K_i(x)} = \left(a_i x^2 + b_i x + c_i\right)^{-2} \quad . \quad (18)$$

Substituting Eq. (18) into Eq. (6) and letting

$$M(x) = \left(a_i x^2 + b_i x + c_i\right)^{\frac{1}{2}} \eta(\xi_i) \quad \text{with } \xi_i = \int \frac{dx}{a_i x^2 + b_i x + c_i} \quad , \quad (19)$$

one obtains

$$\frac{d^2\eta}{d\xi^2} + A_i \eta = 0 \quad , \quad (20)$$

where

$$A_i = 1 + a_i c_i - \frac{1}{4} b_i^2 \quad . \quad (21)$$

The linearly independent solutions are

$$\left.\begin{aligned} S_{i3}(x) &= (a_i x^2 + b_i x + c_i)^{\frac{1}{2}} \sin\sqrt{A_i}\; x, \\ S_{i4}(x) &= (a_i x^2 + b_i x + c_i)^{\frac{1}{2}} \cos\sqrt{A_i}\; x, \end{aligned}\right\} \quad \text{for } A_i > 0 \quad , \quad (22)$$

or

$$\left.\begin{aligned} S_{i3}(x) &= (a_i x^2 + b_i x + c_i)^{\frac{1}{2}} e^{\sqrt{-A_i}\, x}, \\ S_{i4}(x) &= (a_i x^2 + b_i x + c_i)^{\frac{1}{2}} e^{-\sqrt{-A_i}\, x}, \end{aligned}\right\} \quad \text{for } A_i < 0 \quad , \quad (23)$$

or

$$\left.\begin{aligned} S_{i3}(x) &= (a_i x^2 + b_i x + c_i)^{\frac{1}{2}}, \\ S_{i4}(x) &= (a_i x^2 + b_i x + c_i)^{\frac{1}{2}} \xi_i, \end{aligned}\right\} \quad \text{for } A_i = 0 \quad . \quad (24)$$

Case 4:

$$K_i(x) = \left(x^2 + b_i^2\right), \quad b_i > 0 \ . \tag{25}$$

The linearly independent solutions for this case are

$$\begin{aligned} S_{i3}(x) &= (x^2 + b_i)^{\frac{1}{2}} \sin \xi_i, \\ S_{i4}(x) &= (x^2 + b_i)^{\frac{1}{2}} \cos \xi_i \ , \end{aligned} \tag{26}$$

where

$$\xi_i = \left(\frac{N_i + b_i}{b_i}\right)^{\frac{1}{2}} \arctan \frac{x}{b_i^{\frac{1}{2}}} \ . \tag{27}$$

Case 5:

$$K_i(x) = K_i = \text{const} \tag{28}$$

For the case of a uniform beam considered, we have

$$\begin{aligned} S_{i3}(x) &= \sin k_i x, \\ S_{i4}(x) &= \cos k_i x, \\ k_i^2 &= \frac{N_i}{K_i} \ . \end{aligned} \tag{29}$$

Using Eq. (7) yields

$$Q_i(x) = D_{i3} S'_{i3}(x) + D_{i4} S'_{i4}(x) \ , \tag{30}$$

$$y_i(x) = D_{i3} g_i(x) + D_{i4} q_i(x) + D_{i2}(x) + D_{i1} \ , \tag{31}$$

$$\theta_i(x) = D_{i3} g'_i(x) + D_{i4} q'_i(x) + D_{i2} \ , \tag{32}$$

where $g(x)$ and $q(x)$ are obtained by integrating $S_{i3}(x)$ and $S_{i4}(x)$ twice, respectively, while D_{i1} and D_{i2} are integration constants.

In order to establish the eigenvalue equation for buckling, it is useful to construct a set of fundamental solutions, $\overline{S_{ij}}(\mathrm{x})(j = 1, 2, 3, 4)$, which would satisfy the following normalization condition at the origin of the co-ordinate system:

$$\begin{bmatrix} \bar{S}_{i1}(0) & \bar{S}'_{i1}(0) & \bar{S}''_{i1}(0) & \bar{S}'''_{i1}(0) \\ \bar{S}_{i2}(0) & \bar{S}'_{i2}(0) & \bar{S}''_{i2}(0) & \bar{S}'''_{i2}(0) \\ \bar{S}_{i3}(0) & \bar{S}'_{i3}(0) & \bar{S}''_{i3}(0) & \bar{S}'''_{i3}(0) \\ \bar{S}_{i4}(0) & \bar{S}'_{i4}(0) & \bar{S}''_{i4}(0) & \bar{S}'''_{i4}(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \ . \tag{33}$$

The set of fundamental solutions can be determined from

$$\begin{bmatrix} \bar{S}_{i1}(x) \\ \bar{S}_{i2}(x) \\ \bar{S}_{i3}(x) \\ \bar{S}_{i4}(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ g_i(0) & g'_i(0) & g''_i(0) & g'''_i(0) \\ q_i(0) & q'_i(0) & q''_i(0) & q'''_i(0) \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ x \\ g_i(x) \\ q_i(x) \end{bmatrix} , \tag{34}$$

where

$$g_i(0) = g_i(x)|_{x=0}, \quad q_i(0) = q_i(x)|_{x=0} \ . \tag{35}$$

The deflection of the i-th beam step can be expressed in terms of the fundamental solutions as

$$y_{ij}(x) = y_{i1}(x) + \sum_{k=1}^{j-1} C_{ik} y''_{ik}(x_{ik}) \bar{S}_{i2}(x - x_{ik}) H(x - x_{ik}), \quad (i = 1, 2, \ldots, n, \ \ j = 1, 2, \ldots, n_i + 1) \ , \tag{36}$$

where $H(\cdot)$ is the Heaviside function and

$$y_{i1}(x) = y_{i0}\bar{S}_{i1}(x) + \theta_{i0}\bar{S}_{i2}(x) - \frac{M_{i0}}{K_i(0)}\bar{S}_{i3}(x) - \frac{1}{K_i(0)}[Q_{i0} - \mu_i(0)M_{i0}]\bar{S}_{i4}(x) \ . \tag{37}$$

Here, $y_{i0}, \theta_{i0}, M_{i0}$ and Q_{i0} are the deflection, slope, bending moment and shear force of the i-th beam step at $x = 0$, respectively, while $\mu_i(x)$ is defined as

$$\mu_i(x) = \frac{K'(x)}{K(x)} \ . \tag{38}$$

Using the following relations:

$$\begin{aligned}
y_{i0} &= y_{i-1,n_{i-1}+1}(L_{i-1}), \\
\theta_{i0} &= \theta_{i-1,n_{i-1}+1}(L_{i-1}), \\
M_{i0} &= M_{i-1,n_{i-1}+1}(L_{i-1}) - K_{\varphi i}\theta_{i-1,n_{i-1}+1}(L_{i-1}), \\
Q_{i0} &= Q_{i-1,n_{i-1}+1}(L_{i-1}) + K_{u(i-1)}y_{i-1,n_{i-1}+1}(L_{i-1}) + a_i P\theta_{i-1,n_{i-1}+1}(L_{i-1}) \ ,
\end{aligned} \tag{39}$$

at the boundaries of two neighboring beam step results in the following recurrence formula:

$$\begin{aligned}
y_{ij}(x) = {} & y_{i-1,n_{i-1}+1}(L_{i-1})\bar{S}_{i1}(x) + \theta_{i-1,n_{i-1}+1}(L_{i-1})\bar{S}_{i2}(x) \\
& - \frac{1}{K_i(0)}\left[M_{i-1,n_{i-1}+1}(L_{i-1}) - K_{\varphi i}\theta_{i-1,n_{i-1}+1}(L_{i-1})\right]\bar{S}_{i3}(x) \\
& - \frac{1}{K_i(0)}\left\{Q_{i-1,n_{i-1}+1}(L_{i-1}) + K_{u(i-1)}y_{i-1,n_{i-1}+1}(L_{i-1}) + a_{i-1}P\theta_{i-1,n_{i-1}+1}(L_{i-1})\right. \\
& \qquad \left. - \mu_i(0)\left[M_{i-1,n_{i-1}+1}(L_{i-1}) - K_{\varphi(i-1)}\theta_{i-1,n_{i-1}+1}(L_{i-1})\right]\right\}\bar{S}_{i4}(x) \\
& + \sum_{k=1}^{j-1} C_{ik}y''_{ik}(x_{ik})\bar{S}_2(x - x_{ik})H(x - x_{ik}) \ ,
\end{aligned} \tag{40}$$

where $K_{\varphi i}$ and K_{ui} are the stiffnesses of the rotational spring and translational spring supports, respectively. Using $y_{11}(x)$ and the recurrence formula we can determine $y_{ij}(x)(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n_i + 1)$. The eigenvalue equation can be easily determined using the boundary conditions and the recurrence formula. In the following, there are given several typical examples.

Example 1
A fixed-free multi-step cracked beam with spring supports at $(n - 1)$ intermediate points (Figs. 1 and 3).

The boundary conditions for this case are

$$M_{10} = 0, \quad Q_{10} = a_0 P\theta_{10} \tag{41}$$

$$y_{n,n_n+1}(L_n) = 0, \quad \theta_{n,n_n+1}(L_n) = 0 \tag{42}$$

The deflections of the first segment in the first beam step can be obtained using Eqs. (41) and (37) as follows:

$$y_{11}(x) = y_{10}\bar{S}_{11}(x) + \theta_{10}\left[\bar{S}_{12}(x) - \frac{1}{K_1(0)} a_0 P \bar{S}_{14}(x)\right] \; . \tag{43}$$

Using $y_{11}(x)$ and Eq. (37) we can determine $y_{1j}(x)(j = 1, 2, \ldots, n_1 + 1)$, then using $y_{1,n_1+1}(x)$ and Eq. (40) results in $y_{n,n_n+1}(x)$. The eigenvalue equation can be obtained by Eq. (42) as

$$\begin{aligned} &y_{n-1,n_{n-1}+1}(L_{n-1})\bar{S}_{n1}(L_n) + \theta_{n-1,n_{n-1}+1}(L_{n-1})\bar{S}_{n2}(L_n) \\ &\quad - \frac{1}{K_n(0)}\left[M_{n-1,n_{n-1}+1}(L_{n-1}) - K_{\varphi,n-1}\theta_{n-1,n_{n-1}+1}(L_{n-1})\right]\bar{S}_{n3}(L_n) \\ &\quad - \frac{1}{K_n(0)}\{Q_{n-1,n_{n-1}+1}(L_{n-1}) + K_{u(n-1)}y_{n-1,n_{n-1}+1}(L_{n-1}) + a_{n-1}P\theta_{n-1,n_{n-1}+1}(L_{n-1}) \\ &\qquad\quad - \mu_n(0)\left[M_{n-1,n_{n-1}+1}(L_{n-1}) - K_{\varphi,n-1}\theta_{n-1,n_{n-1}+1}(L_{n-1})\right]\}\bar{S}_{n4}(L_n) \\ &\quad + \sum_{k=1}^{n_n} C_{nk}y''_{nk}(L_n)\bar{S}_{n2}(L - x_{nk}) = 0 \; , \end{aligned} \tag{44}$$

$$\begin{aligned} &y_{n-1,n_{n-1}+1}(L_{n-1})\bar{S}'_{n1}(L_n) + \theta_{n-1,n_{n-1}+1}(L_{n-1})\bar{S}'_{n2}(L_n) \\ &\quad - \frac{1}{K_n(0)}\left[M_{n-1,n_{n-1}+1}(L_{n-1}) - K_{\varphi,n-1}\theta_{n-1,n_{n-1}+1}(L_{n-1})\right]\bar{S}'_{n3}(L_n) \\ &\quad - \frac{1}{K_n(0)}\{Q_{n-1,n_{n-1}+1}(L_{n-1}) + K_{u(n-1)}y_{n-1,n_{n-1}+1}(L_{n-1}) + a_{n-1}P\theta_{n-1,n_{n-1}+1}(L_{n-1}) \\ &\qquad\quad - \mu_n(0)\left[M_{n-1,n_{n-1}+1}(L_{n-1}) - K_{\varphi,n-1}\theta_{n-1,n_{n-1}+1}(L_{n-1})\right]\}\bar{S}'_{n4}(L_n) \\ &\quad + \sum_{k=1}^{n_n} C_{nk}y''_{nk}(L_n)\bar{S}'_{n2}(L_n - x_{nk}) = 0 \; . \end{aligned} \tag{45}$$

It can be seen that Eqs. (44) and (45) contain only two unknown parameters, y_{10} and θ_{10}, which are included in $y_{11}(x)$ due to Eq. (43).

Setting equal to zero the determinant of the matrix of the coefficients at y_{10} and θ_{10} results in the eigenvalue equation.

Example 2

A hinged-hinged multi-step cracked beam with spring supports at $(n-1)$ intermediate points

The boundary conditions for this case can be written as

$$y_{10} = 0, \quad M_{10} = 0 \; , \tag{46}$$

$$y_{n,n_n+1}(L_n) = 0, \quad M_{n,n_n+1}(L_n) = 0, \quad \text{i.e., } y''_{n,n_n+1}(L_n) = 0 \; . \tag{47}$$

Using Eqs. (37) and (46), we have

$$y_{11}(x) = \theta_{10}\bar{S}_{12}(x) - \frac{1}{K_1(0)} Q_{10}\bar{S}_{14}(x) \; . \tag{48}$$

Then, using $y_{11}(x)$ and Eq. (36), one obtains $y_{1j}(x)(j = 2, \ldots, n_1 + 1)$. The deflection of the last beam step, $y_{n,n_n+1}(x)$, can be determined using $y_{1,n_1+1}(x)$ and the recurrence formula, Eq. (40). Differentiating $y_{n,n_n+1}(x)$ with respect to x twice results in $y''_{n,n_n+1}(x)$.

Since $y_{11}(x)$ contains only two unknown parameters, θ_{10} and Q_{10}, there are the same two unknown parameters in the expressions of $y_{n,n_n+1}(x)$ and $y''_{n,n_n+1}(x)$. Letting $x = L_n$ and setting $y_{n,n_n+1}(L_n)$ and $y''_{n,n_n+1}(x)$ equal to zero, one obtains two homogeneous algebraic equations including θ_{10} and Q_{10}. The eigenvalue equation can be established by setting equal to zero the determinant of the matrix of the coefficients at θ_{10} and Q_{10} in the homogeneous equations.

Example 3
A fixed-hinged multi-step cracked beam with spring supports at $(n-1)$ intermediate points

If the left end of the first beam step is hinged, then $y_{11}(x)$ has the same form as Eq. (48). The eigenvalue equation can be obtained by use of Eq. (42).

3
Numerical examples

Example 1
In order to illustrate the proposed method, buckling analysis of a uniform hinged-hinged beam on an elastic foundation shown in Fig. 4 is considered here. This problem is simplified as a beam with concentrated translational spring supports at five intermediate points shown in Fig. 5.

The structural parameters are $K_{ui} = L\beta/6$ and $K = EI$, with β-the module of the foundation, L-the length of the beam, $E = 2.8 \times 10^{10}\,\mathrm{N/m^2}$, $b = 0.45\,\mathrm{m}$, $h = 1.05\,\mathrm{m}$, resulting in $I = bh^3/12 = 4.3411 \times 10^{-2}\,\mathrm{m^4}$, $K = 1.2155 \times 10^9\,\mathrm{N \cdot m^2}$

The procedure for determining the critical buckling force of the beam shown in Fig. 5 is as follows:

(1) Determination of the fundamental solutions

The special solutions for the bending moment, $M(x)$, can be found from Eq. (29). Integrating $M(x)$ with respect to x twice yields the general solution for the deflection of the uncracked beam as

$$y(x) = D_1 \times 1 + D_2 x + D_3 \sin kx + D_4 \cos kx \quad , \tag{49}$$

where $1, x, \sin kx$ and $\cos kx$ are a set of four linearly independent solutions for the deflection, and k is given by

$$k = \frac{P}{K} \quad . \tag{50}$$

Using the set of four linearly independent solutions and Eq. (34) results in the fundamental solutions as

$$\bar{S}_1(x) = 1, \bar{S}_2(x) = x, \bar{S}_3(x) = k^{-2} - k^{-2}\cos kx, \bar{S}_4(x) = k^{-2}x - k^{-3}\sin kx \quad . \tag{51}$$

(2) Determination of the flexibility of the rotational springs representing the effects of cracks $(i = 1, 2)$, s. Fig. 5.

The locations of the cracks are given as, comp. Fig. 4,

$$x_{31} = \frac{7}{18}L, \quad x_{32} = \frac{8}{18}L, \quad x_{41} = \frac{10}{18}L, \quad x_{42} = \frac{11}{18}L \quad .$$

The dimensionless depths of the cracks are $\xi_{31} = \xi_{32} = \xi_{41} = \xi_{42} = 0.1$.

Using Eqs. (3) and (5) yields the flexibility of the rotational springs

$$C_{31} = C_{32} = C_{41} = C_{42} = 8.9819 \times 10^{-2} \quad .$$

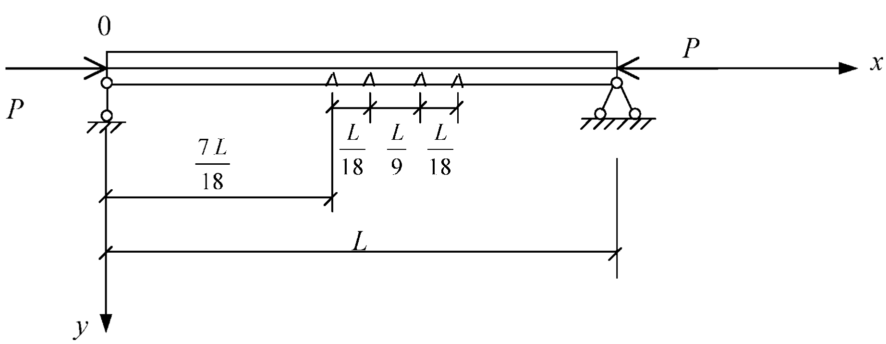

Fig. 4. A cracked uniform beam on an elastic foundation

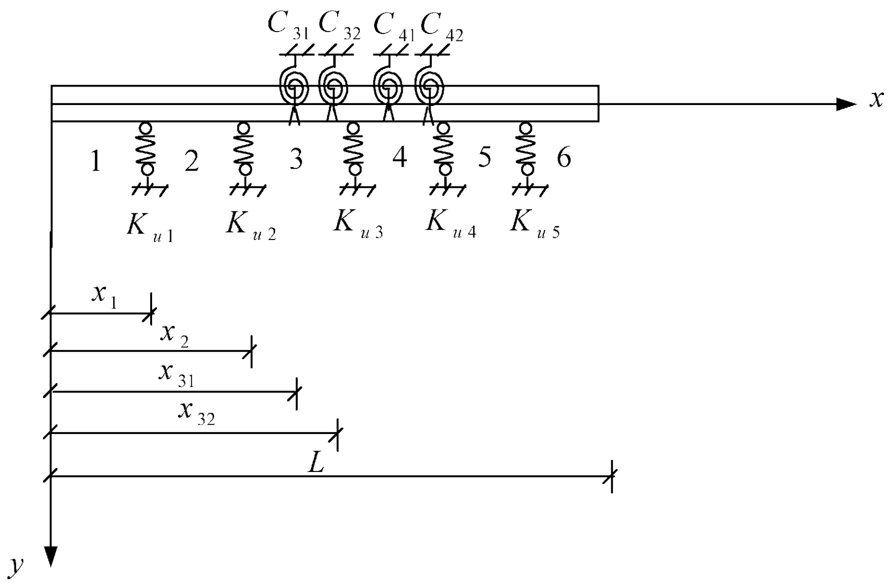

Fig. 5. A cracked uniform beam with translational spring supports at five intermediate points

(3) Determination of the deflection

$$y(x) = y_1(x) - \sum_{k=1}^{5} \frac{1}{K} K_{uk} y_k(x_k) \bar{S}_4(x - x_k) H(x - x_k) + \sum_{j=1}^{2} C_{3j} y_3''(x_{3j}) \bar{S}_2(x - x_{3j}) H(x - x_{3j})$$
$$+ \sum_{j=1}^{2} C_{4j} y_4(x_{4j}) \bar{S}_2(x - x_{4j}) H(x - x_{4j}) \quad , \tag{52}$$

where $y_k(x)$ is the deflection of the k-th segment, $x \in [x_{k-1}, x_k]$

Since the boundary conditions at the left end are $y_1(0) = 0$, $M_1(0) = 0$, the deflection of the first beam step can be found from Eq. (37) as

$$y_1(x) = \theta_{10} \bar{S}_2(x) - \frac{Q_{10}}{K} \bar{S}_4(x) \quad . \tag{53}$$

(4) Determination of the eigenvalue equation

The boundary conditions at the right end are

$$y(L) = 0, \quad y''(L) = 0 \quad . \tag{54}$$

By substitution of Eq. (52) into Eq. (54), a system of two linear homogeneous algebraic equations with respect to θ_{10} and Q_{10} is obtained

$$y_1(L) - \sum_{k=1}^{5} \frac{1}{K} K_{uk} y_k(x_k) \left[k^{-2}(L - x_k) - k^{-3} \sin k(L - x_k)\right]$$
$$+ \sum_{j=1}^{2} C_3 y_3''(x_{3j})(L - x_{3j}) + \sum_{j=1}^{2} C_{4j} y_4''(x_{4j})(L - x_{4j}) = 0 \quad , \tag{55}$$

$$y_1(L) - \sum_{k=1}^{5} \frac{1}{K} K_{uk} y_k''(x_k) k^{-1} \sin k(L - x_k) = 0 \quad . \tag{56}$$

The eigenvalue equation is determined by setting equal to zero the determinant of the matrix of the coefficients at θ_{10} and Q_{10} in Eqs. (55) and (56).

(5) Determination of the critical buckling force

Solving the eigenvalue equation results in the critical buckling force as

$$P_{cr}^{(1)} = \frac{17.3974K}{L^2} \quad \text{for } \beta = \frac{80K}{L^4} \quad .$$

The critical buckling force of the corresponding uncracked beam on an elastic foundation is found as

$$P_{cr} = \frac{17.8670K}{L^2} \quad .$$

If the supports from the elastic foundation are simplified as ten concentrated translational springs attached to the beam at intermediate points, and the constant of each spring is $K_{ui} = L\beta/12$, then it is found that

$$P_{cr}^{'} = \frac{17.8675K}{L^2}$$

Based on Tables 2–5 in Timoshenko and Gere, [11], we obtain

$$P_{cr}^{''} = \frac{17.9746K}{L^2} \quad .$$

The FEM with cubic interpolation functions was also adopted to analyze this problem for comparison purposes. The beam shown in Fig. 5 was divided into 80 beam elements in the FEM analysis. The critical buckling force has been determined at

$$P_{cr}^{'''} = \frac{17.8667K}{L^2}$$

It can be seen that P_{cr} is very close to $P_{cr}^{'}$, $P_{cr}^{''}$ and $P_{cr}^{'''}$, suggesting that a beam on elastic foundation can be simplified as a beam with translational spring supports at intermediate points for buckling analysis. The above comparison also demonstrates the present method does not lead to an unfavorable error transfer when the number of spans considered in the analysis is increased to eleven. For the considered cracks' depth it is found that $P_{cr}^{(1)}$ is by 2.63% less than P_{cr}.

If $\xi_{31} = \xi_{32} = \xi_{41} = \xi_{42} = 0.3$, then

$$P_{cr}^{(2)} = \frac{16.6843K}{L^2} \quad ,$$

and $P_{cr}^{(2)}$ is by 6.62% less than P_{cr}.

If $\xi_{31} = \xi_{32} = \xi_{41} = \xi_{42} = 0.4$, then

$$P_{cr}^{(3)} = \frac{15.9627K}{L^2} \quad .$$

and $P_{cr}^{(3)}$ is by 10.66% less than P_{cr}.

Using FEM one obtains, respectively,

$$\bar{P}_{cr}^{(1)} = \frac{17.3971K}{L^2}, \quad \bar{P}_{cr}^{(2)} = \frac{16.6841K}{L^2}, \quad \bar{P}_{cr}^{(3)} = \frac{15.9524K}{L^2} \quad .$$

It is evident that all the numerical results determined from FEM are in close agreement with those obtained by the present method, thus illustrating reliability of the proposed method.

If the four cracks would occur at the first and second beam steps and $x_{11} = L/18, x_{12} = 2L/18, x_{21} = 4L/18, x_{22} = 5L/18, \xi_{11} = \xi_{12} = \xi_{21} = \xi_{22} = 0.3$, then

$$P_{cr}^{(4)} = 17.0962K/L^2 \ ,$$

i.e. $P_{cr}^{(4)}$ is by 4.31% less than P_{cr}.

Example 2

A cantilever beam with translational spring supports and cracks is shown in Fig. 6. The structural parameters are as follows:
$L_1 = L_2 = L_3 = 40\,\text{m}, K_{u0} = 1.94 \times 10^3\,\text{Nm}^{-1}, K_{u1} = 1.56 \times 10^3\,\text{Nm}^{-1}, K_{u2} = 1.08 \times 10^3\,\text{Nm}^{-1}$,

$$K_i(x) = \alpha_i e^{\beta_i \frac{x}{L_i}}, \quad E = 2.78 \times 10^{10}\,\text{N/m}^2 \ ,$$

$$\alpha_i = \frac{E b_i h_i^3}{12} \ ,$$

$$\beta_1 = 0.49, \quad \beta_2 = 0.28, \quad \beta_3 = 0.22 \ ,$$

$$\begin{aligned}
b_1 &= 0.50\,\text{m}, \quad h_1 = 1.00\,\text{m}, \quad \alpha_1 = 1.1583 \times 10^9\,\text{Nm}^2, \\
b_2 &= 0.55\,\text{m}, \quad h_2 = 1.10\,\text{m}, \quad \alpha_2 = 1.6959 \times 10^9\,\text{Nm}^2, \\
b_3 &= 0.60\,\text{m}, \quad h_3 = 1.20\,\text{m}, \quad \alpha_3 = 2.4019 \times 10^9\,\text{Nm}^2,
\end{aligned}$$

$$x_{31} = 20\,\text{m}, \quad x_{32} = 30\,\text{m}, \quad x_{33} = 35\,\text{m}, \quad \xi_{31} = \xi_{32} = \xi_{33} = 0.1 \ .$$

The procedure for determining the critical buckling forces of the beam shown in Fig. 6 is as follows:

(1) Determination of the fundamental solutions

The special solutions for the bending moment, $M_i(x)$, can be found from Eq. (9). Integrating $M_i(x)$ twice with respect to x results in $y_i(x)$ as

$$y_i(x) = D_{i1} \times 1 + D_{i2}x + D_{i3}\mathrm{Y}_0\left(\lambda_i e^{-\frac{\beta x}{2L_i}}\right) + D_{i4}\mathrm{J}_0\left(\lambda_i e^{-\frac{\beta x}{2L_i}}\right) \ , \tag{57}$$

where

$$\lambda_1^2 = \frac{4(P)L^2}{\alpha_1 \beta_1^2}, \quad \lambda_2^2 = \frac{4(2.5P)L^2}{\alpha_2 \beta_2^2}, \quad \lambda_3^2 = \frac{4(4.5P)L^2}{\alpha_3 \beta_3^2} \quad \text{for } L = 40\,\text{m}.$$

Using the special solutions, 1, x, $\mathrm{Y}_0\left(\lambda_i e^{-\frac{\beta x}{2L_i}}\right), \mathrm{J}_0\left(\lambda_i e^{-\frac{\beta x}{2L_i}}\right)$ and Eq. (34), one obtains the fundamental solutions as

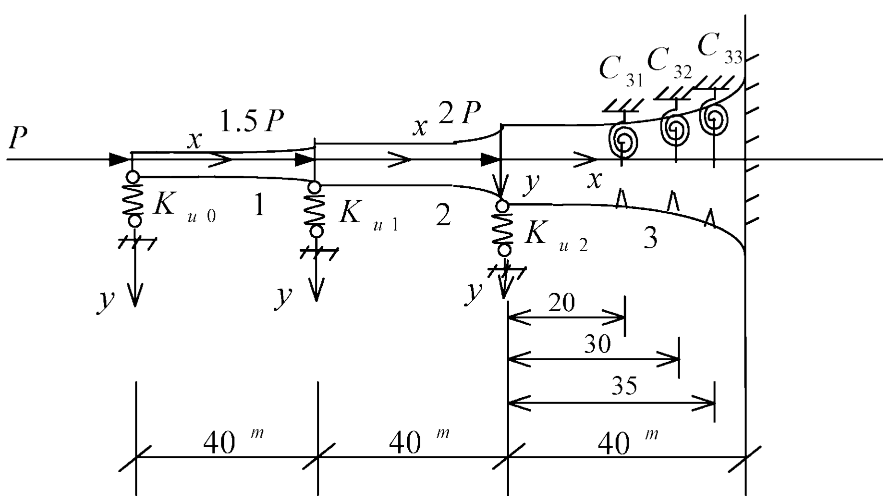

Fig. 6. A cracked cantilever beam with translational spring supports at three points

$$
\begin{aligned}
\bar{S}_{i1}(x) &= 1, \\
\bar{S}_{i2}(x) &= x, \\
\bar{S}_{i3}(x) &= A_{31} + A_{32}x + \frac{J_0''(\lambda_i)}{D} Y_0\left(\lambda_i e^{-\frac{\beta_i x}{2L_i}}\right) - \frac{Y_0''(\lambda_i)}{D} J_0\left(\lambda_i e^{-\frac{\beta_i x}{2L_i}}\right), \\
\bar{S}_{i4}(x) &= A_{41} + A_{42}x - \frac{J_0''(\lambda_i)}{D} Y_0\left(\lambda_i e^{-\frac{\beta_i x}{2L_i}}\right) - \frac{Y_0''(\lambda_i)}{D} J_0\left(\lambda_i e^{-\frac{\beta_i x}{2L_i}}\right) \;,
\end{aligned}
\tag{58}
$$

where

$$
\begin{aligned}
D &= Y_0''(\lambda_i)J_0'''(\lambda_i) - Y_0'''(\lambda_i)J_0''(\lambda_i), \\
A_{31} &= \frac{1}{D}\left[Y_0'''(\lambda_i)J_0(\lambda_i) - J_0'''(\lambda_i)Y_0(\lambda_i)\right], \\
A_{32} &= \frac{1}{D}\left[Y_0'''(\lambda_i)J_0'(\lambda_i) - J_0'''(\lambda_i)Y_0'(\lambda_i)\right], \\
A_{41} &= \frac{1}{D}\left[J_0''(\lambda_i)Y_0(\lambda_i) - Y_0''(\lambda_i)J_0(\lambda_i)\right], \\
A_{42} &= \frac{1}{D}\left[J_0''(\lambda_i)Y_0'(\lambda_i) - Y_0''(\lambda_i)J_0'(\lambda_i)\right] \;.
\end{aligned}
$$

(2) Determination of the flexibility of the rotational springs

The locations of the $(3i)$-th cracks are described by x_{3i}, and the flexibilities of the $(3i)$-th rotational springs are found from Eqs. (3) and (5) as

$$
\begin{aligned}
C_{31} &= 0.1064\,(h_{31} = 1.2444\,\text{m}), \\
C_{32} &= 0.1084\,(h_{32} = 1.2672\,\text{m}), \\
C_{33} &= 0.1095\,(h_{33} = 1.2804\,\text{m}) \;.
\end{aligned}
$$

(3) Determination of the deflection

The boundary conditions at the left end of the beam are

$$
M_{10} = 0, Q_{10} = K_{u0}y_{10} + P\theta_{10} \;. \tag{59}
$$

Using Eqs. (59) and (37) one obtains the deflection of the first beam step as

$$
y_1(x) = y_{10}\bar{S}_{111}(x) + \theta_{10}\bar{S}_{122}(x) \;, \tag{60}
$$

where

$$
\begin{aligned}
\bar{S}_{111}(x) &= \bar{S}_{11}(x) - \frac{K_{u0}}{\alpha_1}\bar{S}_{14}(x) \;, \\
\bar{S}_{122}(x) &= \bar{S}_{12}(x) - \frac{1}{\alpha_1}P\bar{S}_{14}(x) \;.
\end{aligned}
$$

Functions $y_2(x)$ and $y_3(x)$ can be determined using Eq. (40).

(4) Determination of the eigenvalue equation

Using the boundary conditions at the right end results in the eigenvalue equation.

(5) Determination of the critical buckling force

Solving the eigenvalue equation one obtains the critical buckling force as $P_{cr}^{(1)} = 1.0089 \times 10^6$ N. The critical buckling force of the corresponding uncracked beam is found as $P_{cr} = 1.0365 \times 10^6$ N. Thus, $P_{cr}^{(1)}$ is by 2.66% less than P_{cr}. If one chooses $\xi_{31} = \xi_{32} = \xi_{33} = 0.3$, then $P_{cr}^{(2)} = 0.9743 \times 10^6$ N and one obtains that $P_{cr}^{(2)}$ is by 6% less than P_{cr}. If $\xi_{31} = \xi_{32} = \xi_{33} = 0.4$, then one gets $P_{cr}^{(3)} = 0.9321 \times 10^6$ N, and observes that $P_{cr}^{(3)}$ is by 10.07% less than P_{cr}. If the three cracks occur at the first beam step only at $x_{11} = 20$ m, $x_{12} = 30$ m, $x_{13} = 35$ m, with $\xi_{11} = \xi_{12} = \xi_{13} = 0.3$, then the critical buckling force for this case is found as $P_{cr}^{(4)} = 1.0091 \times 10^6$ N.

This means that $P_{cr}^{(4)}$ is by 2.64% less than P_{cr}.

It can be seen from the above results that the effect of cracks on the critical buckling force depends on their number, depth and location.

4 Conclusions

In order to obtain analytical solutions of the governing differential equation for buckling of a non-uniform beam, the equation is expressed in terms of the bending moment. Linearly independent solutions are derived for five cases of the distribution of the flexural beam stiffness. A model of a massless rotational spring is adopted to describe the local flexibility induced by cracks. The deflections and the eigenvalue equation for buckling of an elastically restrained multi-step non-uniform beam with an arbitrary number of cracks are obtained based on the fundamental solutions developed in this paper. The main advantage of the proposed method is that the eigenvalue equation of a non-uniform beam with any kind of two-end supports, any number of cracks and spring supports at intermediate points can be conveniently determined from a second-order determinant. Due to the decrease in the determinant order, as compared with existing procedures (e.g., [3, 4]), the computational time required by the present method for solving the stability problem can be reduced significantly. The numerical examples show that the effect of cracks on the critical buckling force depends on the number, depth and location of cracks. They show that the proposed procedure is sufficiently exact and efficient.

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