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On a moving Griffith crack in anisotropic piezoelectric solids

A. K. Soh, J.-X. Liu, K. L. Lee, D.-N. Fang

Summary The generalized plane problem of a finite Griffith crack moving with constant velocity in an anisotropic piezoelectric material is investigated. The combined mechanical and electrical loads are applied at infinity. Based on the extended Stroh formalism, the closed-form expressions for the electroelastic fields are obtained in a concise way. Numerical results for PZT-4 piezoelectric ceramic are given graphically. The effects on the hoop stress of the velocity of the crack and the electrical to mechanical load ratios are analyzed. The propagation orientation of a moving crack is also predicted in terms of the criterion of the maximum tensile stress. When the crack speed vanishes, the results of the present paper are in good agreement with those given previously in the literature.

Keywords Piezoelectric material, Moving crack, Stroh formalism, Electroelastic field, Crack branching

1 Introduction

Piezoelectric materials produce an electric field when stressed, and deform when subjected to an electric field. Such intrinsic coupling has attracted wide industrial applications of piezoelectric materials in various technologies. Typical examples include electromechanical transducers, delay lines, medical instruments, denotation devices, sonar equipment, microelectronic components and the newly emerging smart (adaptive) structures. In general, some defects or cracks may be produced in piezoelectric materials in the course of their manufacturing. When they are subjected to mechanical and electrical loads, stress concentrations due to these defects can give rise to critical crack growth and subsequent mechanical failure. Therefore, it is of great importance to study the electro-elastic interaction and fracture behavior of piezoelectric materials.

The first to consider the crack problem in piezoelectric materials was Parton, [1]. In his work, the electric potential φ and the normal component D_n of electric displacements across the crack surface were assumed to be continuous, i.e. $\varphi^+ = \varphi^-, D_n^+ = D_n^-$. Later, in [2] another

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set of electric boundary conditions on the crack faces was proposed, namely $D_n^+ = D_n^- = 0$, and the crack in a piezoelectric medium was modeled by the method of distributed dislocations and electric dipoles. The conditions presented in [1] and [2], respectively, are called the *permeable* and *impermeable* electric boundary conditions. Indeed, since the pioneering works [1] and [2], a number of analytical researches concerning the fracture problems of piezoelectric materials have been carried out and great progress has been made. References [3–12] give some examples of the plane and generalized plane crack problems in piezoelectric solids. It is worthwhile noting that these theoretical investigations were focused on the stress and electric fields induced by *stationary* cracks. There were a few investigations that took into account the dynamic problems of cracks moving at a constant speed in piezoelectric materials. In [13] a moving anti-plane crack (mode-III crack) was first studied in a homogeneous piezoelectric material, and it was shown that the stress and electric displacement intensity factors are independent of the speed of the crack. Paper [14] analyzed the problem of a crack moving along an interface of two dissimilar piezoelectric half-planes. The authors concluded that the stress and electric displacement intensity factors are related to the crack speed, while the impermeable electric boundary condition was used. More recently, paper [15] investigated the behavior of a moving crack in a piezoelectric ceramic strip, and [16] examined a moving interfacial crack between piezoelectric and elastic layers. Both in [15] and [16] the permeable electric boundary condition was adopted. Based on the two kinds of electric boundary conditions above, paper [17] considered the same problem as in [14] and calculated the stresses disturbed by a moving interface crack. In [13–17], Yoffe's assumption was adopted of a crack with a fixed length propagating at a constant velocity in a purely elastic isotropic solid [18]. It appears that such an assumption may not be realistic since it demands that the crack opens at one end and closes at the other one at the same speed. However, it is important to note that the angular variation of the dynamic stress field near the moving crack tip is independent of the (fictitious) crack length. Freund, [19], has pointed out, that the near-tip stress field of the Yoffe moving crack is the same as that of the self-similar moving crack tip, i.e., when the opposite crack tip is moving at the same speed but in the opposite direction. This is why many researchers have extended the Yoffe's model to the study the moving crack problems in purely elastic orthotropic materials [20–30] and for piezoelectric ones [13–17].

All the above-mentioned studies on the moving crack problems in piezoelectric media are limited to the anti-plane deformation of transversely isotropic piezoelectric media. In this paper, we consider the generalized plane problem of a finite Griffith crack moving in a general anisotropic piezoelectric material. The main objective is to study the dynamic features of the hoop stress and crack branching. The impermeable electric boundary conditions along the crack surfaces, which are commonly assumed, see [2–8, 11–14 and 17], are implemented in the present study. The paper is organized as follows. Section 2 introduces the basic equations of piezoelectric media based on the quasi-electrostatic approximation and, hence, formulates the steady-state version of Stroh formalism for piezoelectricity. The solution to the moving crack problem is derived in Sect. 3. Numerical results are given for PZT-4 piezoelectric ceramic in Sect. 4, and conclusions are given in Sect. 5.

2
Basic equations and Stroh formalism

2.1
Basic equations

In a rectangular coordinate system $x_i (i = 1, 2, 3)$, the momentum balance equations and quasistatic Maxwell equation for quasi-electrostatic piezoelectricity are as follows, [31]:

$$\sigma_{ij,j} = \rho \frac{\partial^2 u_i}{\partial t^2}, \quad D_{i,i} = 0 \ , \tag{1}$$

where ρ is the density of the material, u_i, σ_{ij} and D_i are the elastic displacements, stresses, and electric displacements, respectively, and a subscript comma denotes partial differentiation with respect to one of the coordinates x_i. The constitutive relations are

$$\sigma_{ij} = c_{ijkl} u_{k,l} + e_{ijk} \varphi_{,k}, \quad D_i = e_{ikl} u_{k,l} - \varepsilon_{ik} \varphi_{,k} \ , \tag{2}$$

where φ is the electric potential, the electric fields E_i are related to φ as

$$E_i = -\varphi_{,i} \ ;$$

and c_{ijkl}, e_{kij} and ε_{ij} are the elastic stiffnesses, piezoelectric stress and dielectric constants, respectively.

For a two-dimensional problem, in which all the variables are independent of x_3, Eqs. (1) and (2) can be expressed in the following compact form:

$$\mathbf{t}_{1,1} + \mathbf{t}_{2,2} = \rho \mathbf{g} \ddot{\mathbf{U}} \ , \tag{3}$$

$$\mathbf{t}_1 = \mathbf{Q}\mathbf{U}_{,1} + \mathbf{R}\mathbf{U}_{,2}, \quad \mathbf{t}_2 = \mathbf{R}^{\mathrm{T}}\mathbf{U}_{,1} + \mathbf{T}\mathbf{U}_{,2} \ , \tag{4}$$

where

$$\mathbf{U} = [u_1, u_2, u_3, \varphi]^{\mathrm{T}}, \quad \mathbf{t}_\beta = \left[\sigma_{\beta 1}, \sigma_{\beta 2}, \sigma_{\beta 3}, D_\beta\right]^{\mathrm{T}} (\beta = 1, 2), \quad \mathbf{g} = \mathrm{diag}[1, 1, 1, 0] \ .$$

Matrices $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{T}$ are related to the material constants by

$$\mathbf{Q} = \begin{bmatrix} c_{i1k1} & e_{1i1} \\ e_{1k1}^{\mathrm{T}} & -\varepsilon_{11} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} c_{i1k2} & e_{1i2} \\ e_{2k1}^{\mathrm{T}} & -\varepsilon_{12} \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} c_{i2k2} & e_{2i2} \\ e_{2k2}^{\mathrm{T}} & -\varepsilon_{22} \end{bmatrix} \ . \tag{5}$$

Substituting Eq. (4) into Eq. (3) leads to

$$\mathbf{Q}\mathbf{U}_{,11} + \left(\mathbf{R} + \mathbf{R}^{\mathrm{T}}\right)\mathbf{U}_{,12} + \mathbf{T}\mathbf{U}_{,22} = \rho \mathbf{g} \frac{\partial^2 \mathbf{U}}{\partial t^2} \ . \tag{6}$$

2.2 Stroh formalism

The electroelastic fields arising from a two-dimensional steady-state motion of a piezoelectric material at constant speed v in the positive x_1-direction are as follows:

$$\mathbf{U} = \mathbf{U}(x_1 - vt, x_2), \quad \mathbf{t}_\beta = \mathbf{t}_\beta(x_1 - vt, x_2) \ . \tag{7}$$

Introducing the Galiean transformation

$$x = x_1 - vt, \quad y = x_2, \quad z = x_3 \ ,$$

one obtains $\partial/\partial x = \partial/\partial x_1, \partial/\partial t = -v\partial/\partial x$. Thus, Eq. (6) can be written as

$$\left(\mathbf{Q} - \rho v^2 \mathbf{g}\right)\mathbf{U}_{,xx} + \left(\mathbf{R} + \mathbf{R}^{\mathrm{T}}\right)\mathbf{U}_{,xy} + \mathbf{T}\mathbf{U}_{,yy} = \mathbf{0} \ , \tag{8}$$

which is the governing differential equation for the steady-state electroelastic fields. Note that the structure of Eq. (8) is identical to that of the static case when $(\mathbf{Q} - \rho v^2 \mathbf{g})$ is identified with $\mathbf{Q}$.

Adopting Stroh formalism, [32], for anisotropic elasticity, a general solution to Eq. (8) can be sought in the form

$$\mathbf{U} = \mathbf{a} f(\zeta), \zeta = x + \mu y \ , \tag{9}$$

where μ and $\mathbf{a}$ are a constant and a constant vector, respectively, which are to be determined; and $f(\zeta)$ is an arbitrary function of variable ζ subject to the requirement of twice-differentiability. Substitution of Eq. (9) into Eq. (8) results in

$$\left[\mathbf{Q} - \rho v^2 \mathbf{g} + \mu\left(\mathbf{R} + \mathbf{R}^{\mathrm{T}}\right) + \mu^2 \mathbf{T}\right]\mathbf{a} = \mathbf{0} \ , \tag{10}$$

which is a nonlinear eigenvalue problem. A nontrivial solution of $\mathbf{a}$ requires that the determinant of its coefficient matrix must be zero, i.e.,

$$\det\left[\mathbf{Q} - \rho v^2 \mathbf{g} + \mu\left(\mathbf{R} + \mathbf{R}^{\mathrm{T}}\right) + \mu^2 \mathbf{T}\right] = 0 \ . \tag{11}$$

This is a polynomial of degree eight for μ. The root μ of Eq. (11) depends not only on the material constants and its orientation, but also on the speed v of the steady-state motion. As discussed in [33] when studying the existence of surface waves in piezoelectric crystals, the static result that no root μ is real does not hold for arbitrary v except for subsonic motion, which will be considered next. If μ_α $(\alpha = 1, 2, 3, 4)$ are assumed to be the four distinct roots with positive imaginary parts, and $\mathbf{a}_\alpha$ are the associated eigenvectors, the general solution can then be expressed as

$$\mathbf{U} = 2\Re \sum_{\alpha=1}^{4} \mathbf{a}_\alpha f_\alpha(\zeta_\alpha) \quad , \tag{12}$$

where $\Re$ denotes the real part and $\zeta_\alpha = x + \mu_\alpha y$.

Substituting Eq. (12) into Eq. (4) and by using Eq. (3), the stress and electric displacement vectors can be expressed as

$$\mathbf{t}_1 = -\mathbf{\Phi}_{,y} + \rho v^2 \mathbf{g} \mathbf{U}_{,x}, \quad \mathbf{t}_2 = \mathbf{\Phi}_{,x} \quad , \tag{13}$$

in which

$$\mathbf{\Phi} = 2\Re \sum_{\alpha=1}^{4} \mathbf{b}_\alpha f_\alpha(\zeta_\alpha) \quad , \tag{14}$$

where $\mathbf{\Phi} = \{\phi_1, \phi_2, \phi_3, \phi_4\}^{\mathrm{T}}$ is called the generalized stress function vector, and $\mathbf{b}_\alpha$ can be determined from $\mathbf{a}_\alpha$ by the following relation:

$$\mathbf{b}_\alpha = \left(\mathbf{R}^{\mathrm{T}} + \mu_\alpha \mathbf{T}\right)\mathbf{a}_\alpha = -\left[\left(\mathbf{Q} - \rho v^2 \mathbf{g}\right)\mu_\alpha^{-1} + \mathbf{R}\right]\mathbf{a}_\alpha \quad . \tag{15}$$

Let's introduce two 4×4 matrices

$$\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4], \quad \mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4] \quad , \tag{16}$$

and a function vector

$$\mathbf{f}(\zeta_\alpha) = [f_1(\zeta_1), f_2(\zeta_2), f_3(\zeta_3), f_4(\zeta_4)]^{\mathrm{T}} \quad . \tag{17}$$

Then Eqs. (12) and (14) can be rewritten as

$$\mathbf{U} = 2\Re[\mathbf{A}\mathbf{f}(\zeta_\alpha)], \quad \mathbf{\Phi} = 2\Re[\mathbf{B}\mathbf{f}(\zeta_\alpha)] \quad . \tag{18}$$

Equations (13) and (18) together with the relations given by Eq. (15) are the main results of this section. In these expressions, the only unknown is the function vector $\mathbf{f}(\zeta_\alpha)$. The appropriate form of $\mathbf{f}(\zeta_\alpha)$ depends on the boundary conditions of the problems considered.

3
Solution to the moving crack problem

3.1
Full-field solution

Consider an infinite piezoelectric medium containing a moving Griffith crack of length $2a$ as shown in Fig. 1. The medium is subjected to remote uniform electro-mechanical loads given by $\mathbf{t}_2^\infty = [\sigma_{21}^\infty, \sigma_{22}^\infty, \sigma_{23}^\infty, D_2^\infty]^{\mathrm{T}}$. The crack surfaces are traction- and charge-free. Similar to the assumption made by Yoffe, [18], for pure elasticity, and used in [13–17] for anti-plane piezoelectricity and in [20–30] for orthotropic elasticity, in the present study the crack is also assumed to be propagating at a constant velocity v and without change in length along the positive x_1-axis. The coordinate system (x_1, x_2, x_3) is stationary, while the coordinate system (x, y, z) is attached to the crack. The solution to the problem considered can be obtained by superposing two solutions. The first one is the homogenous solution obtained by assuming that the crack is absent from the infinite piezoelectric medium subjected to uniform loads $\mathbf{t}_2^\infty$, and

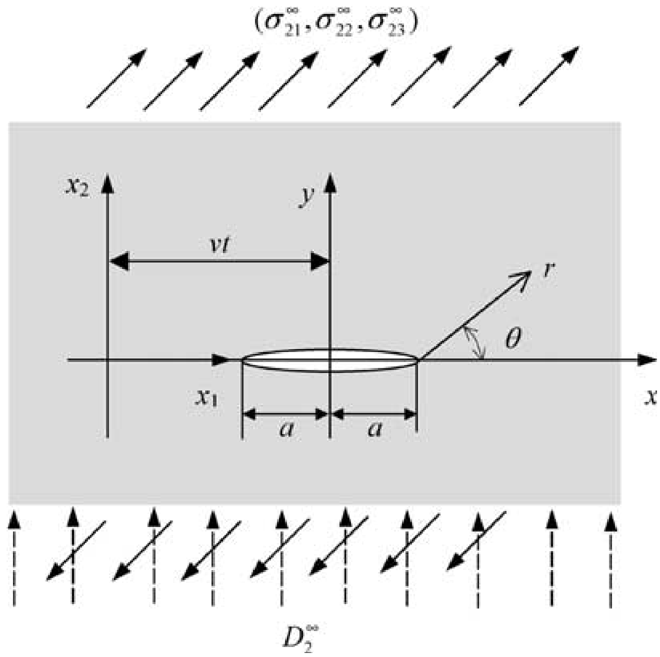

Fig. 1. Schematic representation of a moving crack in a piezoelectric solid subjected to far-field electromechanical loading

the second one is the disturbed solution induced by the crack surfaces which are subjected to $-\mathbf{t}_2^{\infty}$. The subscripts h and d are used to denote the physical variables related to the homogenous and disturbed solutions, respectively.

For the disturbed solution, the boundary conditions in terms of $\mathbf{\Phi}_d$ are as follows:

$$\begin{aligned} &\mathbf{\Phi}_d(x, y = 0^+) = \mathbf{\Phi}_d(x, y = 0^-), && |x| < \infty, \\ &\mathbf{\Phi}_d(x, y = 0^+) = \mathbf{\Phi}_d(x, y = 0^-) = -x\mathbf{t}_2^{\infty}, && |x| < a \ , \end{aligned} \tag{19}$$

$$\mathbf{\Phi}_d \to \mathbf{0}, |\zeta| \to \infty \ . \tag{20}$$

Based on the arguments presented in [5] for the static case of piezoelectricity and in [34] for anisotropic elasticity, the function $\mathbf{f}_d(\zeta_\alpha)$ corresponding to the disturbed solution, which satisfies the boundary conditions (19) and (20), is given by

$$\mathbf{f}_d(\zeta_\alpha) = \frac{1}{2}\left\langle\left\langle \sqrt{\zeta_\alpha^2 - a^2} - \zeta_\alpha \right\rangle\right\rangle \mathbf{B}^{-1}\mathbf{t}_2^{\infty} \ , \tag{21}$$

where $\langle\langle f_\alpha \rangle\rangle$ denotes a 4×4 matrix, for which the α-th diagonal element is f_α and the other elements are zero.

For the homogeneous solution, from the condition

$$\mathbf{\Phi}_{h,x} = 2\Re\left[\mathbf{B}\mathbf{f}_{h,x}(\zeta_\alpha)\right] = \mathbf{t}_2^{\infty} \ , \tag{22}$$

one obtains

$$\mathbf{f}_h(\zeta_\alpha) = \frac{1}{2}\langle\langle \zeta_\alpha \rangle\rangle \mathbf{B}^{-1}\mathbf{t}_2^{\infty} \ . \tag{23}$$

We note that the constant vector denoting the translation of a rigid body and the equipotential field have been omitted. Finally, superposition of the two solutions gives the total solution as follows

$$\mathbf{U} = \Re\left[\mathbf{A}\left\langle\left\langle \sqrt{\zeta_\alpha^2 - a^2} \right\rangle\right\rangle \mathbf{B}^{-1}\right]\mathbf{t}_2^{\infty}, \quad \mathbf{\Phi} = \Re\left[\mathbf{B}\left\langle\left\langle \sqrt{\zeta_\alpha^2 - a^2} \right\rangle\right\rangle \mathbf{B}^{-1}\right]\mathbf{t}_2^{\infty} \ . \tag{24}$$

which provides the full-field solution of the problem.

3.2 Electroelastic fields near the crack tip

By substituting Eq. (24) into Eq. (13), the stresses and electric displacements can be obtained as follows:

$$\mathbf{t}_1 = \Re\left[\rho v^2 \mathbf{g}\mathbf{A}\left\langle\left\langle \frac{\zeta_\alpha}{\sqrt{\zeta_\alpha^2 - a^2}} \right\rangle\right\rangle \mathbf{B}^{-1} - \mathbf{B}\left\langle\left\langle \frac{\mu_\alpha \zeta_\alpha}{\sqrt{\zeta_\alpha^2 - a^2}} \right\rangle\right\rangle \mathbf{B}^{-1}\right] \mathbf{t}_2^\infty \ , \tag{25}$$

$$\mathbf{t}_2 = \Re\left[\mathbf{B}\left\langle\left\langle \frac{\zeta_\alpha}{\sqrt{\zeta_\alpha^2 - a^2}} \right\rangle\right\rangle \mathbf{B}^{-1}\right] \mathbf{t}_2^\infty \ . \tag{26}$$

It is obvious that the distributions of the electroelastic fields near the crack tip are of great interest to us. By introducing a polar coordinate system (r, θ) with the origin at the crack right tip, as shown in Fig. 1, we have

$$\zeta_\alpha = a + r(\cos\theta + \mu_\alpha \sin\theta) \ . \tag{27}$$

When r is small compared to the half-length a of the crack, Eqs. (25) and (26) for the stresses and electric displacements can be rewritten as

$$\mathbf{t}_1 = \sqrt{\frac{a}{2r}}\Re\left[\rho v^2 \mathbf{g}\mathbf{A}\left\langle\left\langle \frac{1}{\Lambda_\alpha(\theta)} \right\rangle\right\rangle \mathbf{B}^{-1} - \mathbf{B}\left\langle\left\langle \frac{\mu_\alpha}{\Lambda_\alpha(\theta)} \right\rangle\right\rangle \mathbf{B}^{-1}\right] \mathbf{t}_2^\infty \ , \tag{28}$$

$$\mathbf{t}_2 = \sqrt{\frac{a}{2r}}\Re\left[\mathbf{B}\left\langle\left\langle \frac{1}{\Lambda_\alpha(\theta)} \right\rangle\right\rangle \mathbf{B}^{-1}\right] \mathbf{t}_2^\infty \ , \tag{29}$$

where

$$\Lambda_\alpha = \sqrt{\cos\theta + \mu_\alpha \sin\theta} \ . \tag{30}$$

4 Numerical results and discussion

In this section, the numerical calculations for the stresses near the moving crack tip are performed using expressions (28) and (29). The used material is PZT-4 piezoelectric ceramic that possesses the transversely isotropic property. The polarized direction of PZT-4 is perpendicular to the crack plane. The constitutive relations for such crack configuration and the explicit expressions for the eigenvalue, Eq. (10), are given in the **Appendix**. The material constants for PZT-4 piezoelectric ceramic are listed in Table 1.

In our calculations, a pure mechanical load, a pure electrical load and a combination of both loads are applied. The velocity of the crack is less than c, where

$$c = \sqrt{\frac{c_{66}}{\rho}}$$

is the lowest Bulk wave speed when the plane wave propagates along the x_1-direction in the transversely isotropic piezoelectric medium, [35]. Figure 2 shows the variation with the

Table 1. Material constants of PZT-4 piezoelectric ceramic

Elastic stiffnesses ($\times 10^{10}$ N/m^2)					Piezoelectric coefficients (C/m^2)			Dielectric constants ($\times 10^{-10}$ C/Vm)	
c_{11}	c_{12}	c_{13}	c_{33}	c_{44}	e_{31}	e_{33}	e_{15}	ε_{11}	ε_{33}
13.9	7.78	7.43	11.3	2.56	−6.98	13.84	13.44	60	54.7

position θ of the normalized hoop stress $\sigma_\theta\sqrt{2r/a}$ near the crack tip for different normalized crack speeds v/c when purely mechanical load is applied along the y-direction as the tensile stress $\sigma_{22}^\infty = 2\,\text{MPa}$. From Fig. 2, it can be seen that the maximum hoop stress σ_θ occurs at $\theta = 0°$ for low crack velocity. This means that the crack has a tendency to propagate along its original plane when the criterion of the maximum tensile stress is used. For high crack velocity, the hoop stress σ_θ is maximum for an angle $\theta \neq 0°$ at which the crack will deviate from its original plane. This phenomenon is called the crack branching. The angle corresponding to the crack branching is defined as the branch angle.

Variation of the normalized hoop stress $\sigma_\theta\sqrt{2r/a}$ with angular position θ for different crack speeds is shown in Fig. 3, where the pure negative electrical load, $D_2^\infty = -0.002\,\text{C/m}^2$ is applied. The "negative" means that the direction of applied electric load is opposite to the direction of poling. It can be observed that the crack tends to deviate from its straight line path under a purely electrical load for all the crack speeds, which is in agreement with the experimental observation in [36] and the theoretical investigations in [6] and [7] for the stationary crack in piezoelectric materials. On the other hand, Fig. 3 indicates that the higher the crack moving velocity, the larger the branch angle.

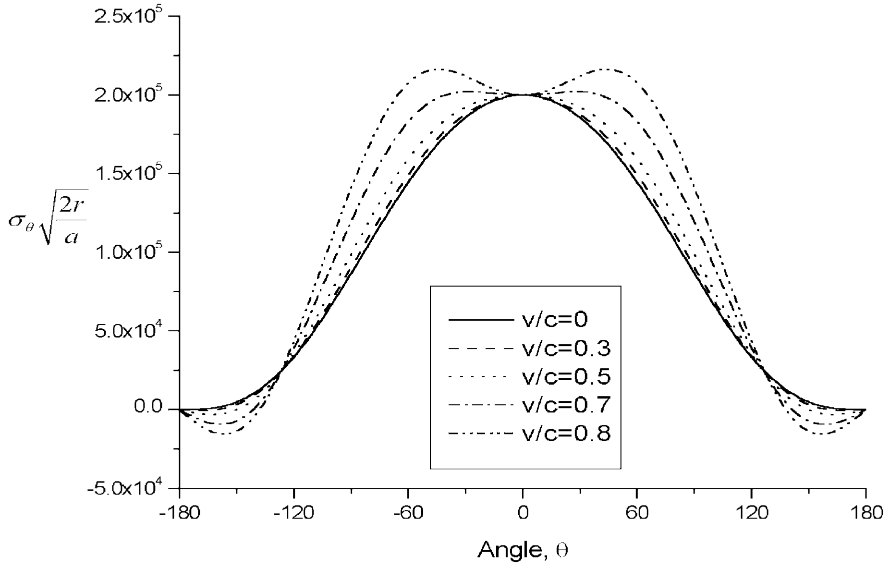

Fig. 2. Angular variation of the normalized hoop stress due to pure mechanical load ($\sigma_{22}^\infty = 2\,MPa$)

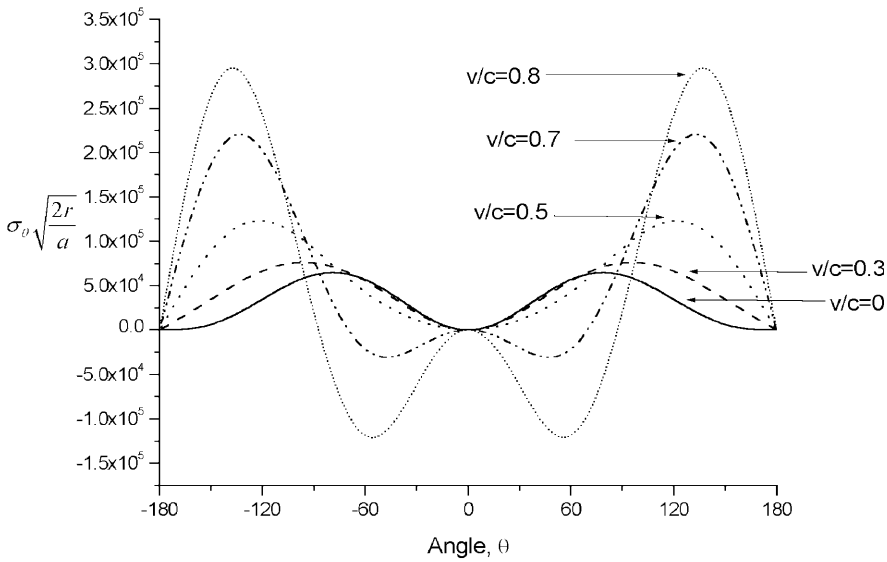

Fig. 3. Angular variation of the normalized hoop stress due to pure electrical load at $D_2^\infty = -0.002\,\text{C/m}^2$

For the case of the combined electromechanical loading ($\sigma_{22}^{\infty} = 2\,\text{MPa}, D_2^{\infty} = -0.002\,\text{C/m}^2$), the angular variation of the normalized hoop stress $\sigma_\theta\sqrt{2r/a}$ is illustrated in Fig. 4 for different crack velocities. In this case, the hoop stresses are tensile for all θ and the crack branching appears for each crack speed. When the crack velocities v are equal to 0, 0.3c and 0.7c, respectively, Fig. 5 present the angular variation of the normalized hoop stress $\sigma_\theta\sqrt{2r/a}$ for different electrical-to-mechanical-load ratios

$$\gamma = \frac{D_2^{\infty}}{\left(\frac{e_{33}}{c_{33}}\right)\sigma_{22}^{\infty}} \quad .$$

For lower crack velocity, a negative electric load can cause the crack branching, but a positive electric load can't. However, when the crack velocity is higher, a positive electric load can also give rise to the occurrence of crack branching. This implies that the influence of electric loads on crack propagation increases with the increase of the crack velocity.

5 Conclusions

The general solution for a moving crack in an anisotropic piezoelectric material has been presented under far-field electrical and mechanical loading. The numerical results reveal that the crack velocity, the direction of the electrical load and electrical-to-mechanical-load ratio have great influence on the stress distributions in the vicinity of the crack tip and the propagation orientation of the moving crack. Especially, when the combined electrical and mechanical loads are applied, the positive electric load makes the crack tend to deviate from its original path and propagate in an oblique direction for higher crack velocity, which is impossible for a stationary crack. Finally, it should be pointed out that the stress distributions obtained for the case in which the crack velocity vanishes are consistent with those given in [6, 7] for the stationary crack.

Appendix

The constitutive equations of transversely isotropic piezoelectric materials with the x_2-axis parallel to the poling direction in matrix notation are

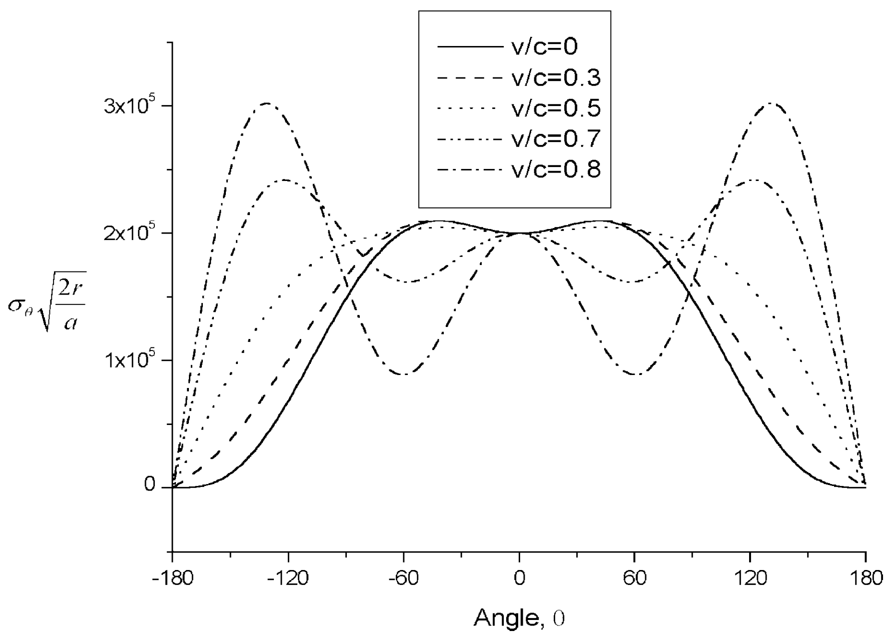

Fig. 4. Angular variation of the normalized hoop stress due to combined electromechanical load ($\sigma_{22}^{\infty} = 2\,\text{MPa}$, $D_2^{\infty} = -0.002\,\text{C/m}^2$)

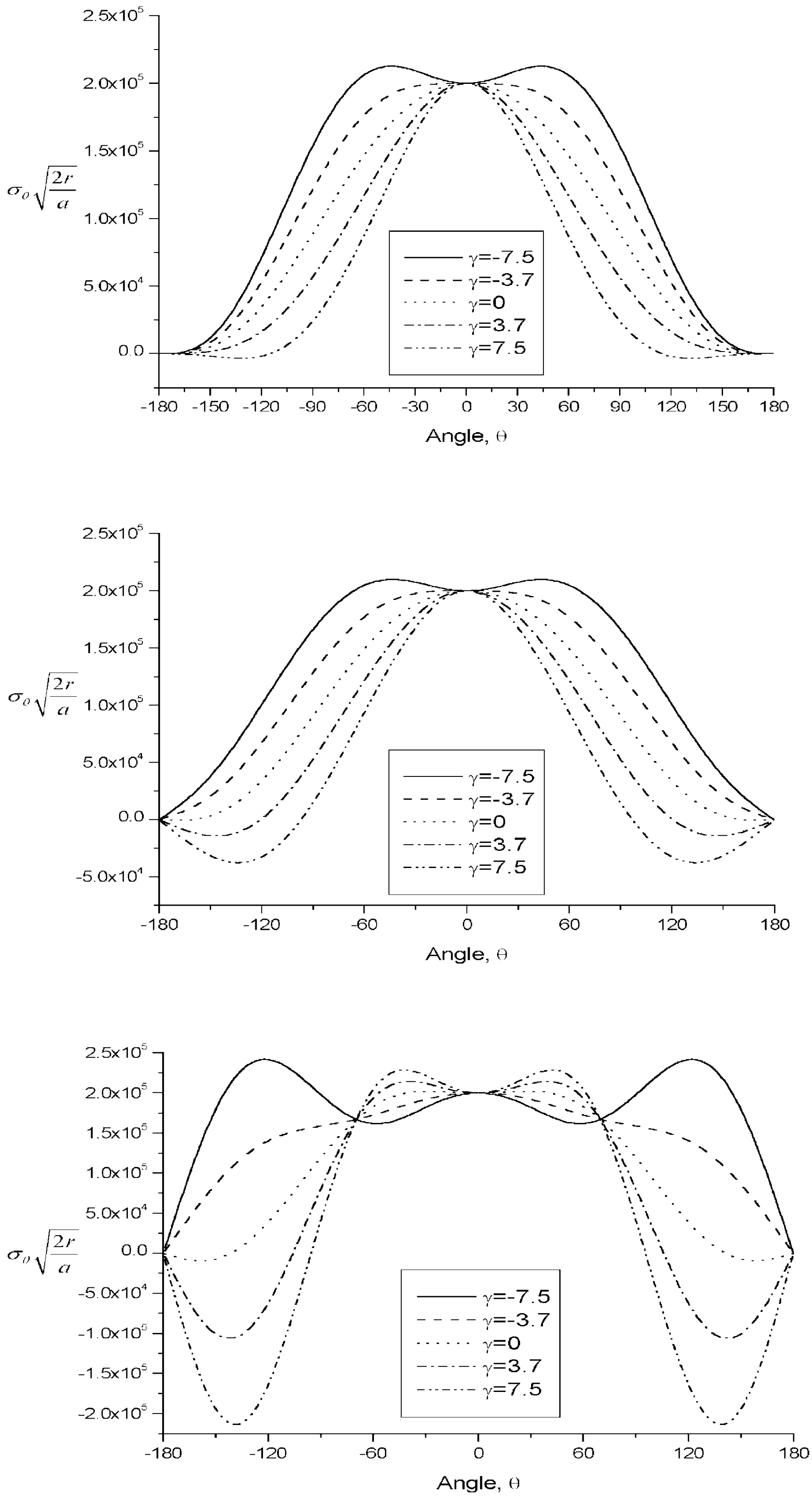

Fig. 5. Angular variation of the normalized stress for different electrical-to-mechanical-load ratios ($\sigma_{22}^{\infty} = 2\,\text{MPa}$) when the crack velocity v is (a) 0, (b) $0.3c$ and (c) $0.7c$

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{13}\\ \sigma_{12}\end{bmatrix}=\begin{bmatrix}c_{11}&c_{13}&c_{12}&0&0&0\\ c_{13}&c_{33}&c_{13}&0&0&0\\ c_{12}&c_{13}&c_{11}&0&0&0\\ 0&0&0&c_{44}&&\\ 0&0&0&&c_{66}&\\ 0&0&0&&&c_{44}\end{bmatrix}\begin{bmatrix}\gamma_{11}\\ \gamma_{22}\\ \gamma_{33}\\ 2\gamma_{23}\\ 2\gamma_{13}\\ 2\gamma_{12}\end{bmatrix}-\begin{bmatrix}0&e_{31}&0\\ 0&e_{33}&0\\ 0&e_{31}&0\\ 0&0&e_{15}\\ 0&0&0\\ e_{15}&0&0\end{bmatrix}\begin{bmatrix}E_1\\ E_2\\ E_3\end{bmatrix}, \qquad (\mathbf{A}.1a)$$

and

$$\begin{bmatrix}D_1\\ D_2\\ D_3\end{bmatrix}=\begin{bmatrix}0&0&0&0&0&e_{15}\\ e_{31}&e_{33}&e_{31}&0&0&0\\ 0&0&0&e_{15}&0&0\end{bmatrix}\begin{bmatrix}\gamma_{11}\\ \gamma_{22}\\ \gamma_{33}\\ 2\gamma_{23}\\ 2\gamma_{13}\\ 2\gamma_{12}\end{bmatrix}+\begin{bmatrix}\varepsilon_{11}&&\\ &\varepsilon_{33}&\\ &&\varepsilon_{11}\end{bmatrix}\begin{bmatrix}E_1\\ E_2\\ E_3\end{bmatrix}, \qquad (\mathbf{A}.1b)$$

where

$$c_{66}=\frac{c_{11}-c_{12}}{2} \quad . \qquad (\mathbf{A}.1c)$$

The matrices **Q**, **R** and **T** are

$$\mathbf{Q}=\begin{bmatrix}c_{11}&0&0&0\\ 0&c_{44}&0&e_{15}\\ 0&0&c_{66}&0\\ 0&e_{15}&0&-\varepsilon_{11}\end{bmatrix}, \quad \mathbf{R}=\begin{bmatrix}0&c_{13}&0&e_{31}\\ c_{44}&0&0&0\\ 0&0&0&0\\ e_{15}&0&0&0\end{bmatrix}, \quad \mathbf{T}=\begin{bmatrix}c_{44}&0&0&0\\ 0&c_{33}&0&e_{33}\\ 0&0&c_{44}&0\\ 0&e_{33}&0&-\varepsilon_{33}\end{bmatrix}. \qquad (\mathbf{A}.2)$$

The eigenproblem given by Eq. (10) becomes

$$\begin{bmatrix}c_{11}-\rho v^2+c_{44}\mu^2 & c_s\mu & 0 & e_s\mu\\ c_s\mu & c_{44}-\rho v^2+c_{33}\mu^2 & 0 & e_{15}+e_{33}\mu^2\\ 0 & 0 & c_{66}-\rho v^2+c_{44}\mu^2 & 0\\ e_s\mu & e_{15}+e_{33}\mu^2 & 0 & -\varepsilon_{11}-\varepsilon_{33}\mu^2\end{bmatrix}\begin{bmatrix}a_1\\ a_2\\ a_3\\ a_4\end{bmatrix}=0 \;, \qquad (\mathbf{A}.3)$$

where

$$c_s=c_{13}+c_{44}, \quad e_s=e_{15}+e_{31} \quad .$$

Equation (**A**.3) indicates that the anti-plane elastic deformation u_3 decouples from the piezo-electric fields (u_1, u_2, φ) denoted by (a_1, a_2, a_4). The equations for determination of the eigenvalue μ are

$$d_3\mu^6+d_2\mu^4+d_1\mu^2+d_0=0, \quad c_{66}-\rho v^2+c_{44}\mu^2=0 \;, \qquad (\mathbf{A}.4)$$

where

$$\begin{aligned}
d_3 &= c_{44}\left(c_{33}\varepsilon_{33}+e_{33}^2\right),\\
d_2 &= c_{33}\left(c_{44}\varepsilon_{11}+e_{15}^2\right)+\varepsilon_{33}\left[\left(c_{11}-\rho v^2\right)c_{33}+c_{44}\left(c_{44}-\rho v^2\right)-c_s^2\right]\\
&\quad +2e_{33}(c_{44}e_{15}-c_se_s)+\left(c_{11}-\rho v^2\right)e_{33}^2,\\
d_1 &= \left(c_{11}-\rho v^2\right)\left(c_{44}-\rho v^2\right)\varepsilon_{33}+c_{44}e_{15}^2+\varepsilon_{11}\left[\left(c_{11}-\rho v^2\right)c_{33}+c_{44}\left(c_{44}-\rho v^2\right)-c_s^2\right]\\
&\quad +2e_{15}\left[\left(c_{11}-\rho v^2\right)e_{33}-c_se_s\right]+\left(c_{44}-\rho v^2\right)e_s^2,\\
d_0 &= \left(c_{11}-\rho v^2\right)\left[\left(c_{44}-\rho v^2\right)\varepsilon_{11}+e_{15}^2\right] \;.
\end{aligned} \qquad (\mathbf{A}.5)$$

Similar to the static case, the roots of the first expression of Eq. (**A**.4) have the following form:

$$\mu_1 = i\xi, \quad \mu_2 = \eta + i\tau, \quad \mu_4 = -\eta + i\tau, \ (\xi, \eta, \tau) > 0 \ , \tag{A.6a}$$

while the root of the second expression of Eq. (**A**.4) is

$$\mu_3 = \sqrt{\frac{c_{66} - \rho v^2}{c_{44}}} \ . \tag{A.6b}$$

The corresponding eigenvectors are

$$\mathbf{a}_\alpha = \begin{bmatrix} A_{1\alpha} \\ A_{2\alpha} \\ 0 \\ A_{4\alpha} \end{bmatrix}, \quad (\alpha = 1, 2, 4), \quad \mathbf{a}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \ , \tag{A.7}$$

in which

$$\begin{aligned} A_{1\alpha} &= -\left(c_{44} - \rho v^2 + c_{33}\mu_\alpha^2\right)\left(\varepsilon_{11} + \varepsilon_{33}\mu_\alpha^2\right) - \left(e_{15} + e_{33}\mu_\alpha^2\right)^2, \\ A_{2\alpha} &= \left[c_s\left(\varepsilon_{11} + \varepsilon_{33}\mu_\alpha^2\right) - e_s\left(e_{15} + e_{33}\mu_\alpha^2\right)\right]\mu_\alpha, \\ A_{4\alpha} &= \left[c_s\left(e_{15} + e_{33}\mu_\alpha^2\right) - e_s\left(c_{44} - \rho v^2 + c_{33}\mu_\alpha^2\right)\right]\mu_\alpha \ . \end{aligned} \tag{A.8}$$

Once $\mathbf{a}_\alpha$ have been obtained, $\mathbf{b}_\alpha$ can be computed from Eq. (15).

References

1. **Parton, V.Z.:** Fracture mechanics of piezoelectric materials. Acta Astronautica 3 (1976) 671–683
2. **Deeg, W.F.:** The analysis of dislocation, crack, and inclusion problems in piezoelectric solids. Ph.D. Thesis, USA, Standford University 1980
3. **Sosa, H.:** On the fracture mechanics of piezoelectric solids. Int J Solids Struct 29 (1992) 2613–2622
4. **Pak, Y.E.:** Linear electro-elastic mechanics of piezoelectric materials. Int J Fracture 54 (1992) 79–100
5. **Suo, Z.; Kuo, C.M.; Barnett, D.M.; Willis, J.R.:** Fracture mechanics for piezoelectric ceramics. J Mech Phys Solids 40 (1992) 739–765
6. **Park, S.B.; Sun, C.T.:** Effect of electric fields on fracture of piezoelectric ceramics. Int J Fracture 70 (1995) 203–216
7. **Kumar, S.; Singh, R.N.:** Crack propagation in piezoelectric materials under combined mechanical and electrical loadings. Acta Mat 44 (1996) 173–200
8. **Pan, E.:** A BEM analysis of fracture mechanics in 2D anisotropic piezoelectric solids. Eng Anal Boundary Elements 23 (1999) 67–76
9. **Wang, T.C.; Han, X.L.:** Fracture mechanics of piezoelectric materials. Int J Fracture 98 (1999)15–35
10. **Shindo, Y.; Watanabe, K.; Narita, F.:** Electroelastic analysis of a piezoelectric ceramic strip with a central crack, Int J Eng Sci 38 (2000) 1–19
11. **Wang, B.L.; Noda, N.:** Mixed mode crack initiation in piezoelectric ceramic strip. Theo appl Fracture Mech 34 (2000) 35–47
12. **Wang, B.L.; Han, J.C.; Du, S.Y.:** New consideration for the fracture of piezoelectric materials under electromechanical loading. Mech Res Commun 27 (2000) 435–444
13. **Chen, Z.T.; Yu, S.W.:** Anti-plane Yoffe crack problem in piezoelectric materials. Int J Fracture 84 (1997) L41-L45
14. **Chen, Z.T.; Karihaloo, B.L.; Yu, S.W.:** A Griffith crack moving along the interface of two dissimilar piezoelectric materials. Int J Fracture 91 (1998) 197–203
15. **Kwon, J.H.; Lee, K.Y.; Kwon, S.M.:** Moving crack in a piezoelectric ceramic strip under anti-plane shear loading. Mech Res Commun 27 (2000) 327–332
16. **Kwon, J.H.; Lee, K.Y.:** Moving interfacial crack between piezoelectric ceramic and elastic layers. Eur J Mech 19 (2000) 979–987
17. **Li, X.F.; Fan, T.Y.; Wu, X.F.:** A moving Mode-III crack at the interface between two dissimilar piezoelectric materials. Int J Eng Sci 38 (2000) 1219–1234
18. **Yoffe, E.H.:** The moving Griffith crack. Phil Mag 42 (1951) 739–750
19. **Freund, L.B.:** Dynamic Fracture Mechanics. New York, Cambridge University Press 1990
20. **Kassir, M.K.; Tse, S.:** Moving Griffith crack in an orthotropic material. Int J Eng Sci 21 (1983) 315–325
21. **Arcisz, M.; Sih, G.C.:** Effect of orthotropy on crack propagation. Theor Appl Fracture Mech 1 (1984) 225–238

22. **Danyluk, H.T.; Singh, B.M.:** Closed form solutions for a finite length moving in an orthotropic layer of finite thickness. Lett Appl Eng Sci (22) (1984) 637–644
23. **Piva, A.:** Elastodynamic crack problems in an anisotropic medium through a complex variable approach. Quart Appl Math 44 (1986) 441–445
24. **Piva, A.; Viola, E.:** Crack propagating in an orthotropic medium. Eng Fract Mech 29 (1988) 535–548
25. **Viola, E.; Piva, A.; Radi, E.:** Crack propagation in an orthotropic medium under general loading. Eng Fract Mech 34 (1989) 1155–1174
26. **De, J.; Patra, B.:** Moving Griffith crack in an orthotropic strip. Int J Eng Sci 28 (1990) 809–819
27. **Lee, K.H.; Hawong, J.S.; Choi, S.H.:** Dynamic stress intensity factors K_{I}, K_{II} and dynamic crack propagation characteristics of orthotropic materials. Eng Fract Mech 53 (1996) 119–140
28. **Das, S.; Patra, B.:** Stress intensity factors for moving interfacial crack between boned dissimilar fixed orthotropic layers. Comput Struct 69 (1998) 59–472
29. **Das, S.; Patra, B.; Debnath, L.:** Stress intensity factors for a moving interfacial Griffith crack in composite media. Int J Eng Sci 37 (1999) 453–475
30. **Das, S.; Debnath, L.:** On a moving Griffith crack at the interface of two bonded dissimilar orthotropic half-planes. Z Angew Math Mech 81 (2001) 281–287
31. **Maugin, G.A.:** Continuum Mechanics of Electromagnetic Solids. Amsterdam, North-Holland 1988
32. **Stroh, A.N.:** Steady state problems in anisotropic elasticity. J Math Phys 41 (1962) 77–103
33. **Lothe, J.; Barnett, D.M.:** Integral formalism for surface waves in piezoelectric crystals. Existence considerations. J Appl Phys 47 (1976) 1799–1807
34. **Ting, T.C.T.:** Anisotropic elasticity: Theory and applications. Oxford, Oxford University Press 1996
35. **Daros, C.H.; Antes, H.:** On strong ellipticity conditions for piezoelectric materials of the crystal classes 6 mm and 622. Wave Motion 31 (2000) 237–253
36. **McHenry, K.D.; Koepke, B.G.:** Electric fields effects on subcritical crack growth in PZT. In: Brádt, R. C., Hasselman, D. P., Lange, F. F. (eds.) Fracture Mechanics of Ceramics, pp 337–352, 1983