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The Discovery of the Vector Representation of Moments and Angular Velocity

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Dedicated to the memory of CLIFFORD A. TRUESDELL (1919–2000)

Quas utinam qui legent scire possint quam inuitus susceperim scribendos, quo facilius caream stultitiae atque arrogantiae crimine, qui me mediis interposuerim Caesaris scriptis. Constat enim inter omnes nihil tam operose ab aliis esse perfectum, quod non horum elegantia commentariorum superetur.

Aulus Hirtius, on Caesar's *Commentaries*

History of science: the analysis of specific concepts in their historical origins and settings.

C. Truesdell [1980, p. 5]

Abstract

As a consequence of the new understanding of the general dynamics of rigid bodies induced by the researches of Euler and d'Alembert, between 1759 and 1826 several mathematicians discovered the vectorial properties of moments of vector quantities and angular velocity. Among those who investigated these matters are some of the leading mathematicians of the period: Euler, Lagrange, Laplace, Poinsot, Poisson and Cauchy. The detailed development of their results gave rise to the establishment of vector mechanics at the middle of nineteenth century. The present study attempts to identify and draw together the different threads that make up this story, which yields new insights into the relationship between mechanics and geometry.

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1. Introduction

While historical studies of the parallelogram law for velocities and forces abound, the history of the discovery of the vector representation for moments of vectors and angular velocity is still to be written. This does not mean, of course, that the subject has not been studied at all. It is possible to find a number of isolated statements or illuminating remarks in several books and articles, but they do not delineate the whole picture. As far as I know, nobody as ever attempted to write a complete account of the subject. What follows is an outline of such a work.

The introduction of vectors in mechanics falls naturally into two distinct periods, divided by an interval of about half a century. In fact, force and velocity belong to the elements of point mechanics, while moment of forces and angular velocity are mainly used to describe the motion of rigid bodies. Therefore, the vector representation of the latter kind of quantities must be traced in papers written after the publication of d'Alembert's *Recherches sur la Précession des Équinoxes* (1749) and Euler's *Découverte d'un nouveau Principe de Mécanique* (1752), and we should not expect to find anything important before 1750.

The reader will see that this is a consequential history: the stream of discoveries flows seamlessly from one author to the next, each of them adding something to the results obtained by his predecessors.

In this article I use the term "vector" quite freely. It would have been more appropriate to employ everywhere the locution "directed segment," for this is what these early authors really had in their minds, but its usage would have led to a cumbersome mode of expression. Of course, it was not until 1845 that the word "vector" was coined by Sir W. R. Hamilton, but this is the term that we should use if we had to translate the old formulations into modern mathematical language. The same must be said about my use of vector algebra, which is, strictly speaking, somewhat anachronistic for works of the beginning of the 19th century.

2. Frisi discovers the parallelogram of infinitesimal rotations (1759)

The first statement and proof that the infinitesimal rotations about concurrent axes can be composed according to the parallelogram law are due to the Italian mathematician Paolo Frisi (1728–1784). He gave at least three different accounts of his discovery [1759, 1783a, 1783b], but they differ only in details. Frisi is not usually credited with this result; his contribution to the principles of mechanics was discovered by R. Marcolongo [1905].

Frisi considers a rigid body with a fixed point *O* which rotates simultaneously about two different axes through *O*. Obviously, the displacement of any point is the vector sum of the displacements due to the two different rotations. Frisi decomposes them into their components along the plane passing through the axes and perpendicular to it. Hence he shows that this is the same decomposition that would be found as a result of a rotation about a third axis through *O*, making angles which stand in a particular proportion with the given axes:

> Si planum aliquod *ZHzh* binis viribus ita urgeatur ut earum una circa axem *Zz*, altera circa axem alterum *Hh*, seorsim rotari possit, & angulares rotationum duarum celeritates sint inter se ut *C:1*; binis viribus simul impressis planum omne rotabitur circa axem tertium *MTm*, jacentem in eodem plano, & qui a prioribus axibus *Zz*, *Hh* declinabit angulis *MTZ*, *MTH*, quorum sinus inter se erunt ut *1:C*. [Frisi, 1783, p. 133]

Implicitly, Frisi states that infinitesimal rotations may be represented geometrically by directed segments:

> In . . . quocumque corpore binos motus rotationis in motum unum componi, eadem prorsus ratione, qua duae vires duobus lateribus parallelogrammi alicujus expressae tertiam vim componunt, quae diagonali exprimitur [Frisi, 1759]

It is noteworthy that Frisi's proof does not require the introduction of a system of coordinates and makes use only of the parallelogram law; this a remarkable fact in a century dominated by purely analytic methods.

As we shall see in our analysis of Lagrange's work in Sect. 9, Frisi's discovery apparently went unnoticed and had no effect on the development of mechanics.

3. Further research on the composition of rotations

Tommaso Perelli (1704–1783), a minor Italian mathematician, deserves to be mentioned here for having studied the composition of infinitesimal rotations independently of Frisi.[1] We know very little about Perelli, who was a professor at the university of Pisa. While he is remembered as a mathematician by Montucla and Lalande, he was also an astronomer, a botanist, a hydraulic engineer and a Greek scholar. Unfortunately, it seems that his many activities prevented him from completing his works, for he

[1] His contribution to the theory of rotations was first noticed by Marcolongo [1905, pp. 9–12], to whose paper we refer.

published only two papers. We know of his work on the composition of rotations from Frisi's introduction to his treatise on mechanics:

> Clarissimus etiam Perellius, qui tunc temporis in Pisana Academia Astronomiae Professor erat, et quocun saepius de hisce omnibus simul omnes colloquebamur, demonstrationem aliam theorematis rotationum compositarum se invenisse dixerat ac plura ad rotationis motum spectantia eidem theorematis adjecisse. [Frisi, 1783a]

According to one of his biographers, Perelli wrote an account of his discovery, but he never published it. Probably he had studied only the infinitesimal rotations, for the composition of finite rotations was then considered a much more difficult subject.[2]

In passing, we remark that the components of the angular velocity vector with respect to a rectangular system of coordinates had been introduced explicitly by Euler in his paper "Du mouvement de rotation des corps solides autour d'un axe variable" [1758, § 28; = *L. Euleri Opera*, (2)8, p. 213].

4. Euler discovers the vector representation of moments of forces (1780)

The discovery that moments of forces are vectors was made by Leonhard Euler (1707–1783), and can be found in two papers presented to the Petersburg Academy in 1780 and published in consecutive pages of the same volume only in 1793.[3] This is not really surprising, since Euler is considered the creator of the general law of moment of momentum[4] and since moments of forces figure prominently in his many papers on the theory of elasticity and of rigid bodies.

In the first paper, entitled "De momentis virium respectu axis cuiuscunque inveniendis..." [1793a], Euler tries to obtain a formula for the moment of a force about an axis – a concept that up to then had been defined only by means of a geometric description – as the product of the intensity of the force by the length of the common perpendicular to the axis and to the line of action of the force. Thus Euler is led to solve a problem in analytic geometry, namely: to find the length of the common perpendicular to two assigned straight lines in rectangular Cartesian coordinates. In fact, more than half of the memoir is dedicated to this task, and Euler carefully separates the basic geometrical results from their applications to mechanics. Supposing that one of the straight lines passes through the origin of the coordinates, he obtains the following formula

$$m \sin \omega = (Gh - Hg)a + (Hf - Fh)b + (Fg - Gf)c$$

where m is the distance between the two lines, a, b, c are the coordinates of a point on the other line, F, G, H and f, g, h are the direction cosines respectively of the first and

[2] See the opinion of no less an authority than Euler [1776, p. 207; = *L. Euleri opera* (2)20, p. 98; see also Lagrange's *Oeuvres*, XI, p. 64, footnote]. The composition of finite rotations was deduced by Olinde Rodrigues after almost a century [Rodrigues, 1840].

[3] These dates were given by P. H. Fuss [1843, p. CI].

[4] For the history of the law of moment of momentum see Truesdell [1964], Galluzzi [1979], Maltese [1992; 1995; 1996] and Caparrini [1999].

the second line, and ω is the angle between them. This is a remarkable result in itself, which should be cited in the histories of analytic geometry.[5]

With this main geometric theorem stated, Euler can now turn his attention to the definition of moment. By a direct application of his previous result, Euler easily obtains the desired expression for the moment of a force about an axis. He sees that it can be expressed in the form

$$fP + gQ + hR,$$

where P, Q, R are respectively the moments about the axes Ox, Oy, Oz, and f, g, h are the cosines of the angles formed by the axis of moments with the coordinate axes. Obviously, this result is very similar to the theorem which says that the component of a force along a given line can be obtained by adding its projections on three orthogonal axes. In fact, it indicates that moments of forces can be represented by a directed segment and can be decomposed by means of the parallelogram law. Euler sees its meaning immediately:

> COROLLARIUM 2. Momenta igitur virium pro ternis axibus inter se normalibus eodem prorsus modo componi possunt, quo vires simplices componi solent. Si enim puncto a applicatae fuerint vires P, Q, R, secundum directiones af, ag, ah, ex iis componitur vis secundum directionem $az = fP + gQ + hR$, quae egregia harmonia maxima attentione digna est censenda, atque in universam Mechanicam hinc non contemnenda incrementa redundare possunt. [Euler, 1793a, § 35; = *L. Euleri Opera*, (2)9, p. 398]

Euler took up the same problem again in a second paper, entitled "Methodus facilis omnium virium momenta respectu axis cuius cunque determinandi" [1793b], which was probably composed immediately after the first. The scope is declared right at the beginning: To establish the formula of moments by using the first principles of statics. In fact, the greater part of the first memoir is pure geometry. This time Euler replaces the assigned force with a system of equivalent forces for which the calculation of the moment about the given axis is easier. While this second memoir does not contain anything new concerning the principles of mechanics, it shows how important this discovery was for Euler.

These two papers were written when Euler was already past his seventieth year and completely blind. They contain his last major contribution to the principles of mechanics, which is as important as any of the preceding ones. However, we note that he does not appear to have made any application of his discovery: perhaps the times were not ripe for a vectorial formulation of rotational dynamics.

As we will see in the following pages, what we may justly call *Euler's formula for moments* was cited and used by Prony [1800], Poinsot [1803], Pouillet de Lisle [1804], Poisson [1808; 1828; 1833], Lagrange [1811], Binet [1815; 1823], Bordoni [1822]. Thus it played an important part in the formulation of vector mechanics.

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[5] The same problem was solved by G. Monge [1801, §§ 12–13] and by A. L. Cauchy [1826a, *Préliminaires*, Problème VII; = *Oeuvres de Cauchy*, (2)5, p. 40]. Clearly, the formula for the distance between two straight lines is, from a modern point of view, a mixed product of vectors. In fact, it can be demonstrated easily by means of vector calculus; see, for example, the text book by Vygodskij [1971, § 165].

5. Laplace and the invariable plane (1798)

The mechanics of rigid bodies was not the only source of ideas that led to the geometrization of the theory of moments. Apparently unaware of Euler's results, a few years later Pierre-Simon Laplace (1749–1827) published two articles based essentially on the law of transformation of moments in passing from one system of coordinates to another. While Euler had considered moments of forces, Laplace studied moments of momenta.

The first paper, entitled "Mémoire sur la détermination d'un plan qui reste toujours parallèle a lui même, ..." [1798], has as its starting point the *principe de la conservation des aires*. It asserts:

> ... si l'on considère un système de corps dont les masses sont m, m', m'', ... etc., en mouvement dans l'espace, et si l'on suppose que ces corps ne soient soumis qu' à leurs action réciproques, dues à des attractions, à des répulsions ou à toute autre cause; la somme des aires planes décrites dans un instant infinitement petit par les rayons vecteurs de ces corps autour du centre de gravité du système, multipliées respectivement par leurs masses, et projettées sur un même plan, reste constante pendant tout le mouvement. [Poisson, 1808]

This is obviously a scalar formulation of the principle of conservation of moment of momentum. Laplace considers what we would now call the components of the total angular momentum and looks for their laws of transformation under a rotation of system of coordinates. By a straightforward algebraic calculation, he finds the expression of their projections in the new system. He obtains some rather complicated formulae, for he uses the Eulerian angles to express the rotation of the new system with respect to the old one.

We see that Laplace has within his hands the discovery of vectorial representation of moments. However, the use of an unsymmetrical notation obscures the geometrical meaning of his formulae; thus Laplace misses this important result.

By equating to zero two of the projections on the new coordinate plans (*i.e.*, two components of the total angular momentum), Laplace shows that it is possible to find a plane with two properties: its direction is fixed in space and the sum of the projections of the areas on it is a maximum. He calls it *plan invariable*. Of course, today we see that it is simply a plane orthogonal to the total angular momentum vector.

Shortly thereafter Laplace wrote a second paper on the same subject, whose title was simply "Sur la Mécanique" [1799]. It is only two pages long and there is not a single formula: evidently the reader is is expected to supply the calculations for himself. Here Laplace remarks that the invariable plane is orthogonal to the axis of moments, which he calls *axe du plus grand moment*.[6] We may conjecture that he had by then read Euler's papers, and that he wanted implicitly to comment on the connection between the two treatments.

The existence of the invariable plane was partly known by earlier writers. For example it was used by Lagrange in his famous memoir on the problem of the three bodies,[7] so that he was able to simplify his equations by putting two constants equal to zero. It is

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[6] This term was then widely used thoughout the first half of the 19th century.

[7] Lagrange [1772; = *Oeuvres*, VI, p. 253].

possible that the invariable plane had been used by some of the mathematicians who had studied the motion of a rigid body about a fixed point during the eighteenth century;[8] we shall see below (Sect. 8) that this result probably belongs to Poisson.

Laplace's discovery was then considered a new general theorem of mechanics and was included in the most important treatises of that period.[9] It led to a better understanding of the vectorial properties of moment of momentum.

6. Prony and the diffusion of Euler's formula (1800)

According to Poisson [1827, p. 357], by the time Euler's two papers were published, the situation caused by the revolution made it difficult for French mathematicians to have access to them. It seems that they became known in France through an influential textbook on mechanics written by Gaspard Clair François Marie Riche de Prony (1755–1839), the *Mécanique Philosophique ou Analyse Raisonnée des divers parties de la science de l'équilibre et du mouvement* [1800]. Here Prony writes:

> J'ai donné les formules nécessaires pour avoir les trois sommes des momens par rapport aux trois axes coordonnés, lorsqu'on connaît les angles que ces trois axes forment avec les directions des forces. Or, on déduit, de ces trois sommes, celles des momens par rapport à un axe de direction quelconque rencontrant les trois autres à leur point commun d'intesection, par une formule d'une simplicité et d'une élégance telle qu'on peut la regarder comme une des belles de la mécanique. [Prony, 1800, p. 110]

In a footnote to the text Prony observes:

> Ce théorème a été donné par Euler dans le tome VII des nouveaux *Actes de Saint-Pétersbourg*: Laplace, de son côté, l'avait déduit de ses belles recherches sur la position d'un plan qui reste toujours parallèle à lui même, dans le mouvement d'un système de corps agissant les unes sur les autres, *Journal de l'École polytechnique*, n.º 5. J'en ai donné une démonstration élémentaire dans le n.º 9 du même journal. [Prony, 1800, p. 110]

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[8] Considering the motion of a rigid body about a fixed point *O* under the action of forces which have no moment about *O* it is easy to demonstrate that the moment of momentum of the body remains constant throughout the motion, which means that it has an invariable plane. The first instance that I was able to find of the use of the invariable plane in the theory of rigid bodies is contained in the first edition of Poisson's *Traité de Mécanique* [1811; t. II, n. 388]. Of course, every student of mechanics knows the role that the invariable plane played in Poinsot's geometrical treatment of the theory of rigid bodies in his *Théorie nouvelle de la rotation des corps* [1834; 1851].

[9] Laplace's *Traité de mécanique céleste* [1799. Livre I, chap. IV, n. 21; = *Oeuvres*, I, pp. 63–69.] and *Exposition du Système du Monde* [VI ed.: 1836. Livre I, chap. V; = *Oeuvres*, VI, p. 199. Livre IV, chap. IV; = *Oeuvres*, VI, p. 218]. Poisson's *Traité de Mécanique*, first edition [1811: t. II, livre III, chap. VIII, § I, nn. 462–466, pp. 278–285] and second edition [1833; t. II, livre III, chap. IX, § III, nn. 560–563, pp. 463–475]. Lagrange's *Mécanique analitique*, second edition [1811; t. I, Seconde partie, sect. III, § II, nn. 10–11, pp. 265–69; = *Oeuvres*, XI, pp. 282–285].

The last phrase is puzzling, for in the memoir cited by Prony there is no hint of such a demonstration.[10]

Using Euler's formula for moments, Prony is immediately able to obtain Laplace's theory of the invariable plane. Let the system of forces be referred to any rectangular axes meeting in a point O, and let α, β, γ, be the angles formed with the coordinate axes by a given straight line d through O. Then the moment about d will will be a maximum if

$$\cos\alpha = \frac{P}{\sqrt{P^2+Q^2+R^2}}; \ \cos\beta = \frac{Q}{\sqrt{P^2+Q^2+R^2}}; \ \cos\gamma = \frac{R}{\sqrt{P^2+Q^2+R^2}};$$

where P, Q, R, are the moments of the forces about the coordinate axes. This line is Laplace's *axe du plus grand moment*.

Thus the theory of the invariable plane becomes at once a simple corollary of Euler's theorem. While the contents of Prony's book is entirely derivative, he has the merit of having clarified and made the first results in the geometric theory of moments generally known.

7. Poinsot creates the theory of couples (1803)

A general vectorial approach to statics was first obtained by Louis Poinsot (1777–1859). His theories are reasonably well known by historians, and thus we will not describe them in detail.

The first work published by Poinsot was the *Éléments de Statique* in 1803. The title is somewhat misleading since this is not really a didactic treatise: it is a systematic attempt to develop the foundations of statics deductively by means of a purely geometrical approach. It was clearly influenced by Monge's *Traité élémentaire de Statique* [1788], a modest little textbook with which it shares a purely geometrical approach to statics (very unusual in a period dominated by the analytic methods of Euler and Lagrange) and the subdivision of the subject. The *Éléments de Statique* at once became a classic of mechanics and it remained one of the best known works of mathematics throughout the nineteenth century.[11]

Poinsot's claim to fame in statics consists in having created the concept of *couple* of forces, which he accomplished in the *Éléments de Statique*. A couple is a system of two equal, parallel and oppositely directed forces; the measure of its dynamical effects is the product of the intensity of the forces by the distance between their lines of action. Poinsot was able to demonstrate that a couple can be turned and translated in its plane and that it can be transported to any plane parallel to its own without changing its effect. Several couples can be composed in a single resultant couple. Moreover, he showed that

[10] This fact has also been remarked on by I. Grattan-Guinness [1990, vol. 1, § 5.2.7, p. 297]. According to Grattan-Guinness, Prony gave a second proof of Euler's formula in a booklet, now very rare, entitled *Démonstration d'un théorème sur la composition des moments des forces* [1803]. Moreover, Prony used again the geometric theory of moments in his *Leçons de Mécanique analytique* [1810–15, vol. 1, p. 67].

[11] There were at least twelve editions of this book, the last being published in 1877.

if we represent a couple with a segment perpendicular to its plane we can compound two couples by means of the parallelogram law. Thus he was able to simplify and clarify at the same time the laws concerning the equilibrium of the moments: they were simply the algebraical expression of the geometrical law of composition of the couples. Poinsot therefore reduced all the laws of the statics of the rigid body to the well known rules for the composition of forces.

In 1806 a celebrated paper appeared, entitled "Sur la composition des moments et des aires" , in which Poinsot developed further his theory of couples. It is divided into three parts. The second part contains the theory of the central axis. In the third part Poinsot discusses the applications of his results on the central axis to the theory of the invariable plane. In fact, as Poinsot remarked, the areas swept over by the radius vector in the movement of a point are simply proportional to the moments of forces. Using the geometrical representation of physical entities by means of directed segments, Poinsot is also able to demonstrate the principles of conservation of momentum and of moment of momentum. These results follows immediately from the third law of mechanics and from the fact that the moment of a force acting on a particle is the same as the moment of momentum it produces in unit time.[12]

A comparison with the *Mécanique céleste* shows that the third part of Poinsot's paper is almost a translation of some parts of the first section of Laplace's treatise, the one devoted to the general principles of mechanics, into the language of vectors. The exposition is obscured by the looseness of the language (Poinsot uses the word *force* to denote both the *quantité de mouvement* and the *force motrice*) and by the fact that he does not use mathematical symbols, but writes down in words the operations that he performs on vectorial entities. This last defect shows the difficulties that Poinsot encountered in trying to express geometrical operations between different kinds of vectors without the aid of an adequate system of symbolism.

While in the first edition of this paper Poinsot had not said anything about the results obtained by Euler, in the subsequent editions,[13] published as an appendix to the *Éléments*, Poinsot adds an observation about the formula $G \cos\theta = L \cos\lambda + M \cos\mu + N \cos\nu$, which furnishes the value of the projection of the couple G on the axis whose cosines are $\cos\lambda$, $\cos\mu$, $\cos\nu$ with respect to the coordinate axes:

> [elle est une] formule très-simple qu'Euler a donnée dans le tome VII des *Nouveaux Actes de Petersbourg*, mais à laquelle il n'était parvenu que par de longs circuits d'analyse. [Poinsot, 1842, p. 355]

The *Éléments de Statique* was the first work in which statics was put entirely in terms of the geometrical composition of directed segments. Even if Poinsot's work had no immediate effect on the development of the mathematical theory of vectors, its importance for the history of vector calculus had been greatly undervalued. Most of the mathematicians of the nineteenth century who used vectorial methods in their works were deeply influenced by Poinsot's writings.

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12 See my [Caparrini, 1999].

13 See *e.g.* the 8th edition [1842].

8. Poisson and the geometric representation of moments by means of plane surfaces (1808)

In 1808 a new approach to the geometrical theory of moments was developed by Siméon Denis Poisson (1781–1840). His ideas were first expounded in a short paper entitled "Note sur differentes proprietés des projections" [1808]. Here he employed only the most elementary parts of algebra and analytic geometry; by using modern vector algebra his reasonings could be reduced to a few lines.

Poisson begins by considering a purely geometrical question, to which at first he does not attach any dynamical significance. He considers a system of plane surfaces and studies the sum of their projections on the coordinate planes of an orthogonal Cartesian system. He proves that the sum of the squares of the projections is an invariant with respect to the system of coordinates. From this result he demonstrates that it is always possible to find a plane with respect to which the sum of the projections is a *maximum*.

The relations obtained by Poisson are identical with some well-known formulae related to the components of a force with respect to a system of rectangular axes. Even taking into account the danger of attributing to Poisson things that he did not intend, we see that there is a new geometrical concept adumbrated in this paper: by considering the sum of the projections of a system of plane surfaces on three orthogonal planes, Poisson comes close to defining a *sum of plane surfaces*, by means of their components, analogous to the sum of directed segments.

Poisson's theory of the projection of plane surfaces became quite well-known at the beginning of the 19$^{\text{th}}$ century, as can be deduced from the fact that it was included in several important treatises of analytic geometry, like J. P. N. Hachette's *Traité des Surfaces du second degré* [1813] and L. Puissant's *Recueil de diverses propositions de Géométrie*, *résolues et démontrées par l'Analyse algébrique* ... [1824].

Having obtained these geometrical theorems, Poisson turns to their application to mechanics. He remarks that the moment of a force about a point is numerically equal to the double of the area of a triangle having the vertex in the point and the force itself as its basis. Thus, implicitly, Poisson assumes that the moment of the force can be represented geometrically by this triangle. The moments of a system of forces are therefore plane surfaces, and their components with respect to a fixed axis are simply their projection on a plane perpendicular to the axis. By choosing that plane so as to have the greatest projection one obtains Laplace's theory of the invariable plane. Moreover, Poisson easily deduces Euler's formula from his theory.

Though very different from Poinsot's theory of couples, Poisson's approach to moments was not basically new. It was clearly inspired by Laplace's theory of the invariable plane, as Poisson himself notes in the last lines of his paper:

> Ces théorêmes sur le plan invariable et sur la composition des momens, sont dus a M. Laplace. En le faisant dépendre de quelques propriétés des projections, nous avons cherché à les démontrer de la manière la plus simple et la plus appropriée à l'enseignement de l'Ecole. [Poisson, 1808, p. 594]

Moreover, Poisson was probably aware of the contemporary research of L. Carnot[14] and S. Lhuillier[15] on the projections of plane surfaces. Finally, it is difficult not to think that he had been somewhat influenced by the works of Poinsot, for the two theories can be considered complementary. However, it must be observed that he cites neither Euler nor Poinsot.[16]

Poisson incorporated his theory of moments in in his *Traité de Mécanique* [1811], the book from which every scientist of the 19th century learned the elements of rational mechanics. Since it was read by everyone, we may safely assume that Poisson's ideas became widely known.[17]

In the *Traité* Poisson reproduced, except for the wording, every part of his earlier paper, but he also made one addition to the principles which is worth noting. To make the theorems meaningful, the sum of the projections of several plane surfaces had to be taken algebraically. However, in the original paper he had not defined clearly the conventions for the signs. Now Poisson considers the perpendicular to the planes that he is studying, and defines the angles between the planes by means of the angles between the normals:

> Pour déterminer les inclinaisons respectives des plans que nous aurons à considérer, rapportons-les à trois plans rectangulaires, que nous nommerons les plans primitifs de projection, ou simplement les *plans primitifs*; appelons de même *axes primitifs*, les intersections de ces plans, et soient *mA*, *mB*, *mC*, ces trois axes rectangulaires.
>
> Si l'on considère un plan mené par le point *m*, intersection de ces trois axes, il est évident que sa position sera determinée en même tems que celle de la perpediculaire à ce plan, élévée par ce point; or, la position de cette ligne dépend des trois angles aigus ou obtus, qu'elle fait avec les axes; lors donc que nous regarderons un plan comme donné de position, ce sera toujours au moyen de ces trois angles, comptés comme il a été dit dans le n° 5. [Poisson, 1811, t. I, liv. I, chap. III, n° 77, p. 101]

Thus, in effect, Poisson distinguishes between the two sides of a plane surface by considering an oriented straight line normal to it. As far as I know, this is the first appearance of *oriented surfaces* in mathematics.

Most important for our present purpose are the application of Poisson's geometric theory of moments to mechanics. He considers again the moments of forces [1811, t. I, liv. I, chap. III, nos 86–87, pp. 111–114] and the theory of the invariable plane [1811, t. II, liv. III, chap. VIII, § I, nos 462–466, pp. 278–284]. This time he emphasises the fact that

[14] See the *Géométrie de position* [1803, artt. 253–268, pp. 303–313].

[15] [Lhuillier, 1789; 1806; 1811–12; 1824; 1828].

[16] Poisson's dislike for the theory of couples was well-known to his contemporaries: "Poisson enfin vit avec un certain chagrin la belle théorie des couples de M. Poinsot, et sembla craindre que des pareilles méthodes ne rendissent trop facile l'étude des mathématiques" [Parisot, 1845, p. 350].

[17] Poisson's geometric theory of moments is expounded, for example, in Sir W. Thomson and P. G. Tait's *Treatise on Natural Philosophy* [1890, part I, chap. I, §§ 231–233].

by means of his theory the laws of transformation of moments in different coordinate systems are easily found:

> Les quantités que nous avons désignées dans le chapitre précédent, par L, M, N, dépendent de la position des plans des coordonnées par rapport aux forces P, P', P'', etc.; souvent on a besoin de changer la direction de ces plans, dans la vue de simplifier, et même de rendre possible, la solution d'un problème; or, il existe entre les valeurs de L, M, N, relatives à un même système de de forces et à des plans différens, des relations d'après lesquelles on déduit de ces valeurs les unes des autres, et qui renferment des théorèmes remarquables. C'est la démonstration de ces théorèmes que je me propose de donner dans ce chapitre. [Poisson, 1811, t. I, liv. I, chap. III, n° 74, p. 99]

New is Poisson's attempt to apply his methods to the basic laws of statics [1811, t. I, liv. I, chap. III, n^{os} 88–91, pp. 114–118] and the introduction of the invariable plane in the dynamics of rigid bodies [1811, t. II, liv. III, chap. IV, § I, n° 378, pp. 130–133]. The former would be remarkable for its unification of the principles of statics under a single mathematical point of view, had it not been anticipated by Poinsot. The latter is now classic, and is reported in every textbook on mechanics. Given the evidence I am aware of, its merit should be ascribed to Poisson.[18]

In the second edition of Poisson's *Traité* [1833], the chapter on moments is copied nearly word for word from the first edition, with some slight changes here and there. Thus, for example, Poisson remarks:

> Ces différentes équations nous montrent que les projections des surfaces planes sur différens plans, suivent les même lois que celles des lignes droites sur des droites différentes. [Poisson, 1833, t. I, liv. III, chap. II, n° 277, p. 540]

The substance of this passage occurs also a few pages later:

> Ces théorèmes remarquables sont dus à Euler. Ils établissent une parfaite analogie entre la composition des moments et celle des forces; analogie qui tient à ce que les forces étant représentées par des lignes droites, les momens sont exprimées par des surfaces planes, qui se projettent sur des plans différens, de la même manière que les lignes sur des droites différentes. [Poisson, 1833, t. I, liv. III, chap. II, n° 281, p. 544]

Be it noted that all these developments in the theory of moments are here attributed to Euler alone. It seems that by this time Poisson had lost his earlier admiration for Laplace's theory of the invariable plan.

9. Lagrange's rediscovery of the vector representation of small rotations (1811)

In Sect. III of his *Méchanique analytique* [1788], Joseph-Louis Lagrange (1736–1813) takes up the problem of determining the conditions of equilibrium of a general mechanical system. To this end, he applies the principle of virtual work: if a system is in equilibrium, the total work of the external forces is zero for any virtual displacement of

[18] Probably Poisson introduced the invariable plane in the theory of rigid bodies in his "Mémoire sur la variation des constantes arbitraires dans les questions de Mécanique" [1809, p. 270].

the system. (We recall that a virtual displacement is any displacement which is consistent with the constraints.) Lagrange remarks that the necessary conditions of equilibrium should be independent of the internal motions of the system. Thus he confines himself to the case of a rigid body, and studies the two most general movements which it can take: a translation and a rotation about a fixed point. His proof is not much different from what is currently found in our textbooks. Here it is sufficient to say that it immediately yields the balance of forces and the balance of moments.

This procedure leads Lagrange to consider the kinematics of a rigid body. In considering the motion of a rigid body with a fixed point, he resolves a general infinitesimal rotation in three rotations about the axes of a rectangular system of coordinates. Thus he is able to demonstrate the existence of the instantaneous axis of rotation and to find the law of composition of small rotations:

> On doit conclure de-là en général, que des rotations quelconques $d\psi$, $d\omega$, $d\varphi$ autour de trois axes qui se coupent perpendicuilerment dans un point, se composent en une seule, $d\theta = \sqrt{d\psi^2 + d\omega^2 + d\varphi^2}$, autour d'un axe passant par le même point d'intersection & faisant avec ceux-là des angles λ, μ, ν, tels que $\cos.\lambda = d\psi/d\theta$, $\cos.\mu = d\omega/d\theta$, $\cos.\nu = d\varphi/d\theta$, & réciproquement, qu'une rotation quelconque $d\theta$ autour d'un axe donné, peut se décomposer en trois rotations partielles, exprimées par $d\theta \cos.\lambda$, $d\theta \cos.\mu$, $d\theta \cos.\nu$, autour de trois axes qui se coupent perpendiculairement dans un point de l'axe donné, & qui fassent avec cet axe les angles λ, μ, ν; ce qui fournit, comme l'on voit, un moyen bien simple de composer & de décomposer les mouvements de rotation. [Lagrange, 1788, p. I, sect. IV, art. 9, p. 33; = (with slight changes) *Oeuvres de Lagrange*, t. XI, p. 61]

It is easy to see how close Lagrange comes to establishing the vectorial character of small rotations, yet he fails to do so.[19]

In 1811 Lagrange published the first volume of the new edition of his treatise of mechanics, whose title was now changed to *Mécanique analytique*. Thus, after a lapse of twenty-three years, he came to consider again some old problems. The new edition is very different from the old one: it is almost double in size, and many parts had been written anew to take into account the advances in mechanics that had been made in the intervening years. The changes also affect the chapter that we are now studying. We will not detail all the new additions to the statics and the kinematics of rigid bodies, for here we are concerned only with the problem of rotations.

After the end of the passage reported above, Lagrange adds a completely new analysis of the decomposition of a given rotation along the coordinate axes of two different systems of rectangular Cartesian coordinates. He is able to demonstrate that the partial rotations around the coordinate axes transform as the components of a displacement. It is the same kind of proof that we could find in a modern textbook, where a vector

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[19] Lagrange's theory of the composition of small rotations is expounded, for example, in Prony's *Nouvelle architecture hydraulique* [1790–1796, t. I, art. 154, pp. 67–71] and G. Venturoli's *Elementi di Meccanica e d'Idraulica* [1817–1818, vol. I, cap. XIII: *Della composizione de' moti rotatorj*].

is defined by means of the laws of transformation of its components. Thus Lagrange succeeds at last in formulating the vectorial representation of small rotations:

> On voit par là que ces compositions et décompositions des mouvements de rotation sont entièrement analogues à celles des mouvements rectilignes.
>
> En effet si, sur les trois axes des rotations $d\psi$, $d\omega$, $d\varphi$, on prend, depuis leur point d'intersection, des lignes proportionelles respectivement à $d\psi$, $d\omega$, $d\varphi$, et que l'on construise sur ces trois lignes un parallélépipède rectangle, il est facile de voir que la diagonale de ce parallélépipède sera l'axe de la rotation composée $d\theta$ et sera en même temps proportionelle à cette rotation $d\theta$. De là, et de ce que les rotations autour d'un même axe s'ajoutent ou se retranchent suivant qu'elles sont dans le même sens ou dans des sens opposés, comme les mouvements qui ont la même direction ou des directions opposées, on doit conclure en général que la composition et la décomposition des mouvements de rotation se fait de la même manière et suit les même lois que la composition ou décomposition des mouvements rectilignes, en substituant aux mouvements de rotation des mouvements rectilignes, suivant la direction des axes de rotation. [Lagrange, 1811, p. I, sect. III. § III, art. 15; = *Oeuvres*, t. XI, p. 61]

From these results Lagrange is immediately able to obtain the equivalent for small rotations of Euler's formula [1811, p. I, sect. III, § III, art. 16; = *Oeuvres*, t. XI, p. 63]. He flatters himself with having obtained it more elegantly, from purely analytic reasonings.

Now a question naturally arises: Did Lagrange plagiarize Frisi? It is possible that he did, for he knew Frisi's works well.[20] Moreover, we now know that sometimes Lagrange did not cite some of his sources.[21] On the other hand, as we have seen, the vectoriality of small rotations follows immediately from his treatment of the problem. It is likely that Lagrange, having read the then recently published writings of Poinsot on the theory of couples, came to realize the analogy between his analytic theory of rotations and the representation of physical quantities by means of directed segments.[22] However it may be, his contemporaries attributed to him all the merits of the discovery, as can be seen, for example, from the works of Poisson [1827] and Chasles [1837, *Notes*, p. 412]. Undoubtedly, it is through the *Mécanique analytique* that the vectorial theory of angular velocity made its way into the modern literature.

10. Laplace, Poisson and the angular velocity vector

A modern reader may find it difficult to believe that more than twenty years had to pass before Lagrange could see that his formulae for small rotations were in effect the same as those that were commonly used to describe forces and velocities. To persuade oneself of this fact, it is sufficient to turn the pages of the most important treatise on

[20] Frisi is cited several times in the correspondence between Lagrange and d'Alembert, published in vol. XIII of Lagrange's *Oeuvres*.

[21] Truesdell [1960] examines at length several examples of Lagrange's tendency to underestimate some contributions of his predecessors.

[22] We know, from an article by J. Bertrand [1872], that Lagrange had discussed with Poinsot the new developments in vector mechanics around 1806.

mechanics of that period. Consider, for example, the first volume of Laplace's *Traité de Mécanique celeste* [1799], where the basic laws of mechanics are laid down. The angular velocity is described by these words:

> Les quantités p, q, r, que nous avons introduites ... ont cela de remarquable, qu'elles déterminent la position de l'axe réel et instantanée de rotation du corps, par rapport aux axes principaux. ... [Cette droite forme] avec les axes des x'', des y'' et des z'' des angles dont les cosinus sont
>
> $$q/\sqrt{p^2+q^2+r^2};\ r/\sqrt{p^2+q^2+r^2};\ p/\sqrt{p^2+q^2+r^2}.$$
>
> Cette droite est donc en repos, et forme l'axe réel de rotation du corps.
> ... on aura donc $\sqrt{p^2+q^2+r^2}$ pour la vitesse angulaire de rotation.
>
> On voit par là que, quel que soit le mouvement de rotation d'un corps autour d'un point fixe ou consideré comme tel, ce mouvement ne peut être qu'un mouvement de rotation autour d'un axe fixe pendant un instant, mais qui peut varier d'un instant à l'autre. La position de cet axe par rapport aux trois axes principaux et la vitesse angulaire de rotation dépendent des variables p, q, r, dont la détermination est très-importante dans ces recherches, et qui, exprimant des quantités indépendantes de la situation du plan des x' et des y', sont elles-mêmes indépendantes de cette situation. [Laplace, 1799, liv. I, n° 28; = *Oeuvres*, t. I, p. 90]

Here the analogy with forces is, if possible, even clearer than in Lagrange's text. The same could be said of the first edition of Poisson's *Traité de Mécanique*:

> Supposons que la droite *OI* représente [l'axe instantanée de rotation] à un instant quelconque; les équations (2) sont celles de ses projections sur les plans des coordonnées x', y', z'; d'où l'on conclut, par les formules connues,
>
> $$\cos.IOx' = p/\sqrt{p^2+q^2+r^2},$$
>
> $$\cos.IOy' = q/\sqrt{p^2+q^2+r^2},$$
>
> $$\cos.IOz' = r/\sqrt{p^2+q^2+r^2}.$$
>
> Lors donc que les trois quantités p, q, r, seront connues, on pourra assigner la position de l'axe instantanée, par rapport aux trois axes mobiles Ox', Oy', Oz'. ... par conséquent la vîtesse angulaire cherchée est simplement égale à $\sqrt{p^2+q^2+r^2}$. [Poisson, 1811, t. II, n^os^ 374–375, pp. 125–126]

When the second edition of the *Traité* appeared in 1833, Poisson added a new paragraph to the preceding section, to show that his formulae could be interpreted as a demonstration of the vectorial character of angular velocity:

> On appelle p, q, r, les composantes rectangulaire de la vitesse de rotation autour des axes Ox', Oy', Oz'; et l'on dit aussi que chacune de ces trois quantités est la vitesse angulaire du mobile autour de l'axe correspondant.
>
> Or, les équations (3) peuvent être remplacées par
>
> $$p = \varpi\cos(IO, x');\ q = \varpi\cos(IO, y');\ r = \varpi\cos(IO, z')$$
>
> et l'on peut écrire les équations (4) sous cette forme:

$$\varpi \cos(IO, x) = ap + bq + cr$$

$$\varpi \cos(IO, y) = a'p + b'q + c'r$$

$$\varpi \cos(IO, z) = a''p + b''q + c''r$$

d'où l'on conclut que la décomposition des vitesses de rotation suit les même lois que celles des vitesses de translation, en remplaçant les directions de celles-ci par les directions des axes de rotation. [Poisson, 1833, t. II, n° 407, p. 127]

We note that here the three quantities *p*, *q*, *r*, are called *composantes*, thus making it clear that Poisson now visualizes the angular velocity as a directed segment.

Interestingly enough, in the second edition of his *Traité* Poisson inserted also his own version of Lagrange's proof of the vectorial properties of small rotations:

On conclut de là que si un point *m* tourne successivement autour de trois axes rectangulaires, avec des vitesses angulaires *p*, *q*, *r*, et pendant des instans égaux, son déplacement final sera le même que s'il eût tourné pendant un de ces instans, avec une vitesse angulaire ω, autour d'un seul axe, faisant avec les trois premiers des angles dont les cosinus sont p/ω, p/ω, p/ω. Cette remarque, relative aux trois vitesses de rotation *p*, *q*, *r*, qu'on appelle les *composantes* de la vitesse ω (n° 407), s'applique également aux composantes d'une vitesse de translation.

La composition des vitesses de rotation suit les même lois, et est comprise dans les mêmes formules que celle des vitesses de translation; en partant de cette analogie de ces deux sortes de mouvement, on en peut déduire l'identité de la composition des momens et de la composition des forces, que nous avons conclue (n° 281) d'une semblable analogie entre les projections des lignes droites et les projections des surfaces. [Poisson, 1833, t. II, n. 543]

It is curious, of course, that Poisson had to give two different demonstrations for what is in effect a single theorem. Perhaps this fact indicates lack of confidence.[23] We may conjecture that before the discovery of vector calculus it was difficult for a mathematician to visualize these new results as part of a general theory, as the slow reception of Poinsot's vector theory of statics seems to show.

In passing, we note that it has been sometimes asserted that the celebrated Poisson's equations for the motion of a rigid body were given for the first time in the *Traité de Mécanique*.[24] Let us recall that, in modern notation, they are usually written in the form

$$d\boldsymbol{i}/dt = \omega \times \boldsymbol{i}, \quad d\boldsymbol{j}/dt = \omega \times \boldsymbol{j}, \quad d\boldsymbol{k}/dt = \omega \times \boldsymbol{k},$$

[23] This fact had been already noted by the Italian mathematician and astronomer Giovanni Plana (1781–1864) in some unpublished notes probably composed around 1835. They are currently preserved in the archives of the Academy of Science of Turin [Mss 0273, *cc. 224v–226v*]. For a more complete analysis of this material see my book on Plana's manuscripts [Caparrini, 2000, p. 92].

[24] See, for example, T. Levi-Civita and U. Amaldi's *Lezioni di Meccanica razionale* [1949, vol. I, cap. III, § 15].

where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, are unit vectors along the coordinate axes of a rotating frame of reference and ω is the angular velocity of the rotating frame. They are used to express the rate of change of a rotating vector. While it is true that they can be found in Poisson's *Traité* [1811, t. II, liv. III, chap. IV, n^{o} 377, p. 128], their origins can be traced far back in time, for they appear in Euler's memoir "Du mouvement de rotation des corps solides autour d'un axe variable" [1765b, § 12; = *Opera*, s. II, VIII, p. 206]. This fact confirms what R. A. Raimi, in a semi-serious tone, had already noted: "There is ample precedent for naming laws and theorems for persons other than their discoverers, else half of analysis would be named for Euler" [1976, p. 522]. He should have added: "and more than half of mechanics."

11. J. F. Français and the angular velocity vector (1812–13)

Only a year after publication of the first part of the new edition of the *Mécanique analytique*, Jacques Frédéric Français (1775–1833) arrived at analogous conclusions concerning the vectorial properties of angular velocity. His short paper, entitled "Théorèmes nouveaux sur la rotation des corps solides" [1812–13], is in effect an *abregé* of a longer work, for it contains only a collection of results on the motion of a rigid body, which are simply stated but not proved. Here is the relevant passage:

> IV. Si pour chaque position de l'axe instantanée, on prend, sur sa direction, une longuer proportionelle à la vitesse de rotation, pour representer cette vitesse, à chaque instant; l'extrémité de l'axe instantané, ainsi determinée, décrira une courbe plane, située dans un plan parallèle à celui du couple d'impulsion primitive, quelle que soit l'oscillation de cet axe. [Français, 1812–1813, p. 211]

The work of Français has not been studied in detail, except for his contribution to the geometric theory of complex numbers. Thus it is hard to say if he dealt with the subject in an original way, and on which foundations his analysis is based. However, even from the few lines reported above it is clear that Français' study is a forerunner of Poinsot's theory of the rotation of rigid bodies.

Shortly afterwards Français published a small book on the dynamics of rigid bodies, the *Mémoire sur le mouvement de rotation d'un corps solide libre autour de son centre de masse* [1813]. It is possible that it contains a more detailed treatment of the angular velocity vector, but I have not yet been able to see this work.

12. Lagrange on the geometrical theory of moments

We are not surprised to discover that in the second edition of the *Mécanique analytique* Lagrange takes notice of the new results on the geometric theory of moments. One should not think of a detailed treatment of the entire question, but there are several changes and additions that are worth noticing.

First of all, there are some slight changes in his definition of the moment of a force about an axis; for, in the new edition, Lagrange remarks that it is the double of the area of a triangle [1811, p. I, sect. III, art. 5; = *Oeuvres de Lagrange*, t. XI, p. 51]. We note that Poisson is not cited here.

Among the new parts there is the following comment:

> . . . on entend aujourd'hui, en Mécanique, par *moment* d'une force ou puissance par rapport à une ligne, le produit de cette force estimée parallèlement à un plan perpendiculaire à cette ligne, et multipliée par son bras de levier, qui est la perpendiculaire menée de cette ligne sur la direction de la puissance rapportée au même plan. [1811, p. I, sect. III, art. 6; = *Oeuvres de Lagrange*, t. XI, p. 52]

To understand this passage, we recall that for Lagrange the moment of a force is in effect the virtual work of that force.[25] Obviously, the "new" terminology was due to the works of Euler, Poinsot and Poisson.

As we have seen in Sect. 9, Lagrange had demonstrated in the second edition that infinitesimal rotations admit a vector representation. In the remainder of this section of his book, Lagrange presents a new derivation of the vector representation of moments which makes use of the above result [1811, p. I, sect. III, art. 17; = *Oeuvres de Lagrange*, t. XI, p. 64]. His proof follows immediately from the assumption that the moment of a force about an axis is proportional to the small rotation of the system about that axis. This approach was justly criticized by Poinsot [1827, p. 564; see also *Oeuvres de Lagrange*, t. XI, p. 64, footnote], who remarked that it does not take into account the role of the moments of inertia. It is, in fact, astonishing that a mathematician of Lagrange's stature could have blundered so badly, for the rôle of the moments and the products of inertia had been clarified more than thirty years before.[26]

13. Binet on the theory of moments (1814, 1818)

After the appearance of the second edition of the *Mécanique analytique*, a change in character of the research on vector quantities took place. The works of Euler, Poinsot, Poisson and Lagrange had shown that there were many remarkable similarities between forces and moments of forces and between momenta and moments of momenta. The next logical step was to develop this line of reasoning further.

This new tendency found its clearest expression in two papers of Jacques Philippe Marie Binet (1786–1856), written in 1814 and 1818 but published, respectively, in 1815 and 1823.[27] The first paper, entitled "Mémoire sur la composition des forces et sur la composition des moments" [1815], collects several different results concerning the theory of moments. In the first section Binet demonstrates several formulae valid for

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[25] "Nous nommerons chaque terme de cette formule, tel que *Pdp*, le *moment* de la force *P*, en prenant le mot de moment dans le sens que Galilée lui a donné, c'est-à-dire, pour le produit de la force par sa vitesse virtuelle." [Lagrange, 1788, partie I, section II, art. 1, p. 15; = *Oeuvres de Lagrange*, XI, p. 29].

[26] Se, for example, Euler's "Recherches sur la connoissance mécanique des corps" [1765a]. The early history of the general theory of rigid bodies has been discussed by C. Wilson [1987].

[27] These two papers were studied also by I. Grattan-Guinness [1990; vol. 1, § 6.2.4, pp. 368–370].

an arbitrary system of forces. Denote the forces by $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3, \ldots, \boldsymbol{F}_n$; then Binet's relations may be written, in modern notation, as follows:

$$R^2 = \sum_i F_i^2 + 2\sum_{i \succ k} F_i F_k \cos(\boldsymbol{F}_i \boldsymbol{F}_k);$$
$$R\cos(\boldsymbol{RF}_i) = \sum_k F_k \cos(\boldsymbol{F}_i \boldsymbol{F}_k);$$
$$R = \sum_i F_i \cos(\boldsymbol{RF}_i);$$

where $\boldsymbol{R} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \boldsymbol{F}_3 + \cdots + \boldsymbol{F}_n$, is the vector sum (*i.e.*, the resultant) of the system, $(F_i F_k)$ denotes the angle between the vectors F_i and F_k, and the sums are taken over all the forces of the system.

Obviously, these formulae would hold even if $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{F}_3, \ldots, \boldsymbol{F}_n$ were arbitrary vectors. While a modern reader may be justified in thinking that this derivation is no more than a routine exercise, no earlier occurrence of these identities can be found.[28] It is interesting to note that, according to Binet, their most important feature consists in their being independent from the coordinate system.

Binet was able to demonstrate the usefulness of his formulae by applying them to the statics of a rigid body to obtain a new form of the general conditions of equilibrium of a rigid body [1815, p. 342]. It differs from the usual form in being expressed by only two equations, whereas six were previously necessary. The new equations are

$$\sum_i F_i^2 + 2\sum_{i \succ k} F_i F_k \cos(\boldsymbol{F}_i \boldsymbol{F}_k) = 0$$
$$\sum_i M_i^2 + 2\sum_{i \succ k} M_i M_k \cos(M_i M_k) = 0$$

where F_i and M_i are espectively the forces and the moments acting on the body. At that time they were considered an interesting result in rational mechanics.

Next comes a new geometric theory of moments, the third after those of Poinsot and Poisson. Binet substitutes to every applied force F a second force having the same moment about some fixed point O. Thus he defines the moment of $\boldsymbol{F}$ about O as a force whose line of action is situated at a unitary distance from O, and whose moment about O is the same as that of $\boldsymbol{F}$. It is clear that this procedure is valid only for a rigid body with a fixed point. Binet's definition is nonetheless another step toward the modern theory: it is sufficient to turn Binet's *momens* by ninety degrees to obtain our usual moments.

The remainder of the memoir contains another interesting result. From "Newton's law" of mechanics $\boldsymbol{F} = m\boldsymbol{a}$ Binet easily obtains, for a system of mass points, the equations

[28] They were included in the second edition of Poisson's *Traité de Mécanique* [1833; t. I, livre I, chap. I, nn. 33–34, pp. 58–60]; there is no trace of them in the first edition. In a different context, they were demonstrated anew by G. Giorgini [1820] and M. Chasles [1829, p. 102].

$$\sum_i m_i \left(y_i \frac{d^2 z_i}{dt^2} - z_i \frac{d^2 y_i}{dt^2} \right) = \sum_i F_i \, (y_i \cos\gamma_i - z_i \cos\beta_i)$$
$$\sum_i m_i \left(z_i \frac{d^2 x_i}{dt^2} - x_i \frac{d^2 z_i}{dt^2} \right) = \sum_i F_i \, (z_i \cos\alpha_i - x_i \cos\gamma_i)$$
$$\sum_i m_i \left(x_i \frac{d^2 y_i}{dt^2} - y_i \frac{d^2 x_i}{dt^2} \right) = \sum_i F_i \, (x_i \cos\beta_i - y_i \cos\alpha_i)$$

where m_i are the masses, α_i, β_i, γ_i are the angles of the forces with the coordinate axes, F_i represents the forces and the sums are taken over the points. Then he substitutes the sums of the moments with three other sums:

$$\sum_i M_i \cos\lambda_i, \quad \sum_i M_i \cos\mu_i, \quad \sum_i M_i \cos\nu_i,$$

where M_i represents the moments, and λ_i, μ_i, ν_i are the angles formed by the planes of the moments with the three coordinate planes. Thus, for the first time the law of rotational momentum is written in a form which takes into account the geometric representation of moments. Let us note that in this formulation the similarities between the two fundamental laws of mechanics are quite evident.

In the second paper, entitled "Sur les principes généraux de Dynamique, et en particulier sur un nouveau principe de mécanique générale" [1823], Binet introduces the *vîtesse aréolaire*, remarking that it is a vector quantity:

> Les vîtesses aréolaires se combinent entre elles, d'après des règles analogues à celles de la composition et de la décomposition des mouvements linéaires: je n'ai pas dû insister sur cet objet, que les théorèmes d'Euler et les recherches de M. Poinsot sur les moments ont mis hors de doute, puisque nos vîtesses aréolaires sont précisément les momens des vîtesses ordinaires. [Binet, 1823, p. 164]

The aim of Binet in this memoir is to obtain a new theorem, analogous to the *théorème des forces vives*, where forces and velocities are substituted by moments of forces and areal velocities.

In this work occurs, perhaps for the first time, the derivation of the principle of moment of momentum for a system of mass points from the law $\boldsymbol{F} = m\boldsymbol{a}$ and from the equality of action and reaction. This is a fundamental theorem of elementary mechanics and can now be found in every textbook. In a well-known article[29] concerning the history of the principle of moment of momentum, C. Truesdell had conjectured that this theorem was mainly due to Poisson [1833], who had demonstrated that the sum of the moments of internal forces for a system of mass points is zero. Since the complete demonstration, in the form used in modern expositions, can be found in Binet's memoir, written fifteen years before, Truesdell's conjecture is wrong.[30]

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[29] Truesdell [1964].

[30] The question is discussed in [Caparrini, 1999]. Another reply to Truesdell's query is that of B. L. van der Waerden [1983].

14. Bordoni and the generalization of Euler's formula (1822)

We have seen that by 1820 Euler's formula for moments and its significance was fairly well known. The matter was taken up again in an interesting paper written by the Italian mathematician Antonio Bordoni (1788–1860), entitled "Sopra de' Momenti ordinarj" [1822]. The greater part of this work is dedicated to the resolution of different forms of the following problem: Given four concurrent straight lines in space and the moments of a system of forces about three of them, to find the moment about the fourth line. Thus, in effect, Bordoni is studying the generalization of Euler's formula to non-orthogonal Cartesian axes.

While Bordoni uses the terminology of Poisson's representation of moments by means of plane surfaces, he formulates some of his theorems in terms of the composition of directed segments; here is an example:

> Osservazione 2. Se dell'asse del momento Q si fissasse una porzione, che rappresentasse il momento Q stesso, ed avesse un termine nel punto comune agli assi dei tre lati A, A', A''; e dall'altro termine di essa porzione si calassero le perpendicolari agli assi dei medesimi momenti A, A', A'', le porzioni di questi assi intercette fra i piedi di siffatte perpendicolari ed il punto comune ai medesimi assi rappresenterebbero le grandezze degli stessi momenti A, A', A''. [1822, p. 85]

Let us note that Bordoni's moments are not Poinsot's couples, nor Binet's *momens*: they are our modern vectorial moments, whose theory was created by Cauchy in 1826. This is perhaps their first explicit appearance.

15. Cauchy and the theory of the *moments linéaires* (1826)

As is well known, Augustin Louis Cauchy (1789–1857) wrote a sequence of five papers in 1826 in which he brought the theory of moments to its final form.[31] Cauchy's treatment does not differ in essence from the theory that we find today – in vector formulation – in every textbook. From the historical point of view, it is a fourth geometric theory of moments.

To comprehend fully the real place of these papers in the history of mechanics it is sufficient to read the first page of the first paper. Here Cauchy writes:

> La théorie des *moments linéaires* se lie intimement, d'un côté, à la théorie des moments des forces, pris par rapport à un point fixe, et représentées par des surfaces planes; de l'autre, à la théorie des couples établie par M. Poinsot, et fournit, comme cette dernière, les moyens de simplifier la solution d'un grand nombre de problèmes de Mécanique. Elle a d'ailleurs l'avantage de faire disparaître les difficultés que présente, dans certains cas, le choix des signes qui doivent affecter les surfaces désignées sous le nom de *moments*. Enfin elle s'applique non seulement aux forces, mais encore à toutes les quantités qui ont pour mesure des longueurs portées sur des droites, dans des directions déterminées,

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[31] [Cauchy, 1826b; 1826c; 1826d; 1826e; 1826f]. They were all published in consecutive pages of vol. I of the *Exercices de Mathématiques*.

> par exemple aux vitesses et aux quantités de mouvement. [Cauchy, 1826b; = *Oeuvres de Cauchy*, (2)9, p. 89]

From these lines we can see what Cauchy really did for the theory of moments: he took the best parts out of the three theories of moments then in existence and used them to form his own theory. Cauchy's moments are vectors, like Poinsot's couples and Binet's *momens*, that represent Poisson's plane surfaces. There are no ambiguities of sign and the theory can be described either from a geometrical or from an analytic point of view.

It is not necessary here to detail Cauchy's theory of moments. A student of today would be able to translate it at once into the modern formulation, by simply replacing the long verbal descriptions with some elementary formulae of vector algebra.

Many years later, when the new methods of geometric calculus began to appear, Cauchy recognized that his old results on the moments of forces could be expressed using the new formalisms that had been created in the meantime. In 1853 he wrote:

> J'ai developpé, depuis plus d'un quart de siècle, non seulement dans mes *Exercices de Mathématiques*, mais aussi dans mes Leçons données à l'École Polytechnique et à la Faculté des Sciences, la théorie des *moments linéaires*. Comme j'en ai fait la remarque, cette théorie se lie intimement, d'un côté, à la théorie des *moments* des forces, pris par rapport à un point fixe, et representés par des surfaces planes, de l'autre, à la théorie des *couples* établie par M. Poinsot. Elle a d'ailleurs, l'avantage de s'appliquer non seulement aux forces, mais encore à toutes les quantités qui ont pour mesure des longueurs portées sur des droites dans des directions déterminées, par exemple aux vitesse et aux quantités de mouvement. Il en résulte qu'elle peut être très utilement employée dans la détermination du mouvement d'un système de points matériels, et en particulier dans la détermination des deux mouvements de translation et de rotation d'un corps solide. D'ailleurs, les théorèmes auxquels on est alors conduit s'énoncent plus facilement, lorsqu'avec MM. Mœbius et Saint-Venant on appelle *somme géométrique* de deux longueurs données une troisième longueur représentée en grandeur et en direction par la diagonale du parallélogramme construit sur les deux premières. [Cauchy, 1853; = *Oeuvres*, (1)12, p. 5]

In this paper Cauchy tries to develop the first principles of elementary vector mechanics using the new terminology related to vector concepts. He cites the works of several mathematicians that had used vectorial methods in their works: Poinsot, Poisson, Binet, etc.

16. Polemics and controversies

By 1826 three different mathematicians, all of them working in Paris, had laid out the basis of somewhat similar theories of moments. Obviously, this fact led to some priority controversies. These polemics contributed little to mechanics, but they allow us to judge how these mathematicians viewed their own work.

The first controversy arose in 1827 and the disputants were Cauchy and Poinsot.[32] It began when Cauchy published an analysis of his own work on the theory of moments.[33]

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[32] This polemic is briefly cited by B. Belhoste [1991, p. 254].

[33] "Exercices de mathématiques; par M. Cauchy" [1827a].

This article was strongly attacked by Poinsot,[34] who considered Cauchy's results to be merely repetitions of his theorems on couples of forces disguised under a different notation:

> Actuellement, donnez à la résultante *R* le nom de force principale, et au couple résultante *G*, le nom de moment linéaire principal, et vous avez le théorème dont il s'agit. Faites partout le même changement, ou plutôt rétablissez les dénominations connues, et vous ne verrez rien, dans la théorie des momens linéaires, qui ne soit beaucoup plus clair dans la théorie des couples. [Poinsot, 1827a, p. 255]

Cauchy's reply to Poinsot's article was published immediately:[35]

> Je ne m'arretêrai point à discuter cette proposition de M. Poinsot que s'il y avait un moyen *de faire rétrograder la science*, *ce serait de compliquer ce qui avait été rendu simple*, *et de remettre un voile sur ce qui était découvert*. Je ne rechercherai point si la théorie des momens linéaires doit être préférée à celle des couples, dont j'aime à reconnaître la grande simplicité. Je crois que ces deux théories ont chacune leurs avantage particuliers, et je remarquerai que la première peut être appliquée, non seulement aux forces, mais encore à toutes les quantités représentées par des longueurs portée sur des droites dans des directions données, par exemple aux vitesses et aux quantités de mouvement. [Cauchy, 1827b; p. 336]

A second controversy began in 1827,[36] when Poisson published a brief article on the history of the theory of moments.[37] The central issue of the debate was the originality of Poinsot's concept of couples of forces; Poisson claimed that the discovery of the vectorial properties of the moments was entirely due to Euler and Laplace:

> La relation qui existe entre ces differentes sommes de momens d'un même système de forces, est la même que celle qui a lieu entre les composantes d'une même force; et cette identité de composition des forces et des momens est vraiment un beau théorème de mécanique. [Poisson 1827]

Poinsot replied with a lengthy paper[38] in which he discussed many points related to the theory of moments:

> Mais il faut bien remarquer ici que ces théorèmes ne constituent point la composition proprement dite des moments. Cette composition n'a été, et je dirai même, n'a pu être connue que par la théorie des *couples*. Et en effet, ce qu'on appelait le moment d'une force par rapport à un point, ou un axe fixe, n'était jusque-là, pour les géomètres, qu'une simple expression de calcul, un produit abstrait de deux nombre, dont l'une marque une certaine force, et l'autre une certaine ligne; et il me semble qu'il ne pouvait venir à personne l'idée

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[34] "Note de M. Poinsot sur l'article n°. 6 du Bulletin dernier, relatif aux *Exercices mathématiques* de M. Cauchy", 5^e^ et 6^6e^ livraisons" [1827a].

[35] "Note de M. Cauchy, sur l'article du Bulletin des Sciences du mois d'avril, n°. 178" [1827b].

[36] By the kindness of Prof. Patricia Radelet de Grave I have seen a first version of her paper "La composition des moments en mécanique, ou la querelle des couples" (to be published in *Sciences et techniques en perspective*), where this controversy is studied.

[37] "Note sur la composition des momens" [1827].

[38] "Mémore sur la composition des moments en mécanique" [1827].

> de chercher des lois d'*équilibre* entre de tels produits. Que si, par la propriété connue du levier, on pouvait voir, dans ces produits, comme une certaine expression des efforts que font les puissances pour faire tourner autour du point fixe, il est claire que cette idée disparaissait entièrement quand il n'y avait ni point ni axe fixes dans le corps ou système sur lequel les forces étaient appliquées: de sorte que ces produits ne restaient que comme des expressions de calcul, et n'avaient pu conserver, dans leur définition précise, aucune trace de cette espèce de signification que leur donne la la présence d'un axe fixe, et qui les avait fait nommer *moments* par les anciens géomètres. Pour découvrir la composition des moments, il fallait donc découvrir ce que le moment exprime dans la science des forces considérée en elle-même: il fallait une *notion statique*, qui manquait alors aux géomètres, et cette notion est celle du *couple*, dont le moment n'est que la mesure. Par cette idée nouvelle, les momens devinrent des *couples*, qu'on avait sous les yeux, et qu'on pouvait songer à composer ou à mettre en équilibre entre eux, exactement comme des simples forces autour d'un point. Voilà ce qui nous a donné la composition des moments, ou pour mieux dire, la composition des *couples*, qui sont, en statique, la *chose même* que l'on compose, dont le moment n'est que la mesure dans le calcul, et qu'on n'avait point encore découverte. [Poinsot, 1827b; p. 559]

There were two more articles, one by each of the disputants, which did not add anything new to the discussion.[39] The debate took a new direction when Poinsot published [1828] a brief résumé of his latest work,[40] where he corrected the calculations of Laplace on the determination of the invariable plan by considering the rotation of the planets about their axes. Poisson objected[41] that the new formulae were not useful, since the corrections were small and the distribution of the mass in the interior of the planets is not known. In the last article of the series[42] (written by a *rédacteur*, probably by Férussac), the author simply remarked that the equations obtained by Poinsot were exact, regardless of the fact that they were useful or not.

17. Poinsot and the rotation of rigid bodies (1834, 1851)

The most notable imperfection in Poinsot's theory of couples was its inability to explain the dynamics of rotation of rigid bodies. Clearly the movement of rotation of a rigid body was entirely due to the applied couple, but the connection between the two phenomena was not known.

This part of the theory was developed in the *Théorie nouvelle de la rotation des corps* [1834–1851], which is now considered a classic of rational mechanics.[43] The first part of this work is the most significant for our study. Herein Poinsot gives a detailed study of the vectorial properties of the angular velocity, which closely follows the methods that he

[39] Poisson: "Addition à la Note sur la composition des Momens" [1827]; Poinsot: "Note de M. Poinsot sur l'article 296 du tome VIII du *Bulletin*, relatif à la composition des momens" [1828].

[40] "Sur le système du monde" [1828]. The memoir here summarized was the "Mémoire sur la théorie et la détérmination de l'équateur du système solaire" [1830].

[41] "Note sur le plan invariable" [1828].

[42] "Note du rédacteur sur les trois articles précédens" [1828].

[43] In 1834 Poinsot published a short version of his work; the complete memoir was published in 1851.

had developed in the *Éléments de statique*. For example, he studies the couples of small rotations (*couples de rotation*), which are pure translations, and introduces the *couple accélérateur*, which is similar to the *force accélératrice* (as the acceleration was then called) of elementary mechanics. He remarks that any proposition regarding the composition of forces has its counterpart in the composition of small rotations; for example, the theory of central axis is precisely the same as that of instantaneous axis of rotation. Thus Poinsot clearly recognizes that there is a formal dualism between infinitesimal rotations and forces.

Poinsot further developed these ideas in several memoirs.[44] They had an indirect but important influence on the development of vector calculus.

18. Chasles and the *principe de dualité* between translations and rotations (1838)

Without pretending to study the connections between our subject and the history of pure geometry,[45] we remark that in the years 1820–40 some experts in geometry directed their attention to the writings on mechanics discussed above. To Michel Chasles (1793–1880), for example, we owe several works in which he tries to take advantage of the new developments in mechanics for geometry or to combine the two subjects. He was a great admirer of Poinsot, and it is interesting to note that his opinion on Poinsot's discoveries in mechanics are included in his *Rapport sur les progrès de la géométrie* [1870, *Introduction*, § IV, pp. 13–17], a classic in the historiography of geometry.

The appendix to Chasles'celebrated *Aperçu historique sur l'origine et le développement des méthodes en Géométrie* [1837], entitled *Sur la dualité dans les sciences mathématiques*, deserves particular notice. Here Chasles tries to establish, in rather vague terms, a principle of duality for mechanics, which asserts that there are strong similarities between the properties of translations and rotations. In this way he hoped to unify the study of mechanics under a single general methodology, much in the same way that Binet had tried to do in his two papers of twenty years before. This program partly came to be realized with the advent of vector mechanics.

19. Rotating frames of reference

In a memoir written by Robert Baldwin Hayward[46] (March 7, 1829–February 2, 1903) in 1856, we can see an example of the influence of the ideas of Poinsot, Lagrange and Cauchy on the development of vector methods.

The paper is devoted to the general theory of motion of rigid bodies, and is divided into two parts. In the first section Hayward develops the purely mathematical theorems

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44 "Théorie des cônes circulaires roulants" [1853], "Précession des équinoxes" [1858].

45 See R. Ziegler's *Die Geschichte der geometrischen Mechanik im 19. Jahrhundert* [1985].

46 He was fellow of St. John's College and Reader in Natural Philosophy in the University of Durham. In 1892 he published a book on vectors: *The Algebra of Coplanar Vectors and Trigonometry* (London, New York, Macmillan and co., 1892).

that he will use in the second part of his work. What he achieves here is simply the formula which relates the derivatives of the vector $\boldsymbol{u}$ obtained in two different frames

$$d\boldsymbol{u}/dt = \dot{\mathbf{u}} + \Omega \times \boldsymbol{u}$$

where $d\boldsymbol{u}/dt$ is the derivative in a non-rotating frame, $\dot{\mathbf{u}}$ is the derivative in a rotating frame and Ω is the angular velocity of the rotating frame. Hayward obtains this formula in rectangular Cartesian coordinates, but he uses purely geometric methods: he considers the angular velocity as a vector, and makes full use of the fact that it can be decomposed along different directions as if it were a force.

The originality of Hayward's methods can be evaluated by comparing them with those used in two papers by E. Bour [1856; 1863], written in the same period, where similar results are obtained in a purely analytic way. Even if Hayward's work shows a deep understanding of vector methods there is no trace in this paper of any formal algorithm, such as those employed by Hamilton[47] or Grassmann.

Hayward's formula is used in modern textbooks to obtain Euler's equations for the rigid body easily: this derivation appears for the first time in § 26, p. 13 of his paper.

A similar result was obtained shortly afterwards, by G. M. Slesser [1858].

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[47] Hayward uses the word "quaternion" just once, when he makes the interesting remark that the inertia of a rigid body is represented by a quaternion whose scalar part is the mass of the body and the vector part his angular velocity.

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