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The eclectic content and sources of Clavius's *Geometria Practica*

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Abstract
We consider the *Geometria Practica* of Christopher Clavius, S.J., a surprisingly eclectic and comprehensive practical geometry text, whose first edition appeared in 1604. Our focus is on four particular sections from Books IV and VI where Clavius has either used his sources in an interesting way or where he has been uncharacteristically reticent about them. These include the treatments of Heron's Formula, Archimedes' *Measurement of the Circle*, four methods for constructing two mean proportionals between two lines, and finally an algorithm for computing *n*th roots of numbers.

1 Introduction

1.1 Clavius and the *Geometria Practica*

Christopher Clavius, S.J. (1538–1612) was the most accomplished and influential Jesuit mathematician of his era.[1] His mathematical outlook was essentially conservative and grounded firmly in the geometry of the *Elements* of Euclid, with some excursions into parts of algebra and what we would call discrete mathematics (Knobloch 1988, Sections III and IV). Yet his view of the subject was broad enough to acknowledge *both* the certainty of mathematical knowledge due the subject's reliance on strict standards of proof *and* the utility of mathematics for understanding the physi-

[1] See Knobloch (1988), Baldini (1983), Baldini (2003) for the original documentary sources for his life, career, and the activities of the Academy Clavius supervised at the Jesuit *Collegio Romano*. For a nuanced consideration of Clavius's place in the development of the teaching of mathematics in the Jesuit schools, see Romano (1999).

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cal world.[2] He was mostly a commentator, expositor, and evaluator of the mathematical work of others, not primarily an original mathematical researcher in the modern sense. Over most of his career, a large portion of his energy was devoted to the production of numerous influential source books or textbooks for the teaching of a wide range of mathematical subjects. These included his extensively augmented edition of the *Elements* of Euclid (first edition 1574), the *Epitome arithmeticae practicae* (Summary of Practical Arithmetic, 1583), this *Geometria Practica* (Practical Geometry, first edition 1604), and the *Algebra* (1608). Clavius also wrote a well-known commentary on the *Sphere* of Sacrobosco, books on the astrolabe and the construction of sundials, and more elementary treatments of plane and spherical triangles. Before the end of his life, his collected mathematical works Clavius (1611–1612) were published in five volumes.[3]

The *Geometria Practica* went through three editions within 10 years of its appearance. The first edition, Clavius (1604), was printed by the shop of Luigi Zanetti in Rome; just 2 years later, a second edition, Clavius (1606) was produced by the printshop of Johann Albin in Mainz. The first edition is slightly longer because of a different page format. However, there are no substantial differences between the texts. Moreover, very similar (but not identical) woodcut figures were used in both editions, so the overall appearance does not differ significantly. The version of the *Geometria Practica* included in Clavius (1611–1612) contains some corrections of typographical and mathematical errors in the previous editions, expanded discussions of the quadrant and geometric square constructed in Book I, and some other relatively minor additions. In this essay, page numbers refer to the page in the 1606 edition.

While a number of scholars have written recently on aspects of the book, the entire scope of Clavius's *Geometria Practica* seems not to have been studied in detail.[4] Yet this work has some rather surprising features that will be discussed in more detail shortly.[5]

We will not attempt any systematic analysis of how Clavius's book fits within the whole vast genre of practical geometry texts from the 16th through the 18th centuries as discussed, for instance, in Raynaud (2015). We have not examined enough of the at

[2] This is expressed most explicitly in Clavius's essay *In disciplinas mathematicas prolegomena* (*Prolegomena* on the mathematical disciplines) included in Volume I of the *Opera Mathematica*, (Clavius 1611–1612). Clavius sees mathematics as *intermediate* between metaphysics and natural philosophy, an idea that traces back at least to Proclus's Commentary on Book I of Euclid's *Elements*, (Morrow 1970), a text Clavius mentions several times. See, for instance, Rommevaux (2005, Chapter 1).

[3] These have been digitized, see: https://clavius.library.nd.edu/mathematics/clavius.

[4] In the past, this was perhaps a reflection of certain derogatory attitudes toward "applied" or "practical" mathematics in general. Raynaud proposes in fact that practical geometry has been "... doublement marginal, vis-à-vis des mathematiques savantes et vis-à-vis des traditions techniques," that is, " ... doubly marginal, with respect to theoretical mathematics, and with respect to technical traditions." (Raynaud 2015, p. 19)

[5] These features have also furnished the motivation for the author of this essay to undertake a translation of the entire *Geometria Practica* from the original Latin into English using the 1606 second edition. This translation is freely available at CrossWorks, the online faculty and student scholarship repository maintained by the Library of the College of the Holy Cross, at the URL: https://crossworks.holycross.edu/hc_books/57/. All quotations of passages from the *Geometria Practica* in English are taken from this translation. The original Latin text from the 1606 edition is provided in footnotes for purposes of comparison. The same will be done for quotations from other sources.

least 450 books involved (according to Raynaud) to come to any informed conclusions. Instead we will focus on Clavius's book itself—in particular on its features and sources.

First, the *eclecticism* of this work—the sheer range of different types of topics that fall under the category of practical geometry and its allied areas for Clavius and that make it into this book—is remarkable. This is a clear reflection of what Clavius set out to do in writing it. In his Preface, Clavius discusses previous works in the practical geometry genre, and how he wants his work to stand out from the works of Leonardo Pisano ("Fibonacci," ca. 1170–ca. 1250, Pisano 1862, 2008), Luca Pacioli (1447–1517, Pacioli 1494), Nicolo Fontana ("Tartaglia," ca. 1500–1557, Fontana 1560), Oronce Fine (1494–1555, Fine 1532), Girolamo Cardano (1501–1576, Cardano 1539), and Giovanni Antonio Magini (1555–1617, Magini 1592). After writing that Magini had done an especially excellent job of presenting methods for measuring lines (i.e., lengths, heights, depths, etc.),[6] Clavius continues:

> But truly, Magini concerned himself only with this one part of this subject, and the others, although they undertook to present all of those parts, have left out much in writing their books. I decided, if possible, to complete the subject, so that *whatever has been profitably handed down by others or found by myself in practical geometry is enclosed within the circle of one work*.[7] (Clavius 1606, Preface); emphasis added.

Thus, in this work, Clavius aims for *completeness* within the subject of practical geometry as understood by his contemporaries. The following rough outline of topics (organized by the division into books) will demonstrate how wide-ranging and encyclopedic this book truly is:[8]

I. Construction of a rudimentary proportional compass and a quadrant for measuring lengths and angles; summary of elementary plane trigonometry.
II. Measuring lengths, heights, and depths with the astronomical quadrant.[9]
III. This begins with a discussion of measuring lengths, heights, and depths parallel to the contents of Book II but using the geometric square, then continues to other methods for the same sorts of problems using additional instruments such as Jacob's staffs, planar mirrors, etc. The book concludes with an extended discussion of libration for aqueducts based on measurements of altitudes.
IV. Measuring areas of plane regions, including an augmented version of the *Measurement of the Circle* of Archimedes (ca. 287–ca. 212 BCE), and quoting from other works of Archimedes including the *Quadrature of the Parabola*.

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[6] The influence of Magini's work is especially evident in Books II and III of Clavius (1606).

[7] Verum quoniam & hic de vnica tantum parte fuit sollicitus: & alii, quamuis aggressi omnia, multa tamen inter scribendum praeterierunt: decreui, si qua possem, perficere: vt, quicquid vtiliter in Geometria practica ab aliis traditum, à me etiam inuentum est, vnius operis gyro clauderetur.

[8] At a higher degree of granularity, the complete list of chapter headings and propositions that serves as the table of contents, see pages iv–xx of Clavius (1606), is even more evidence here.

[9] Discussions of problems similar to those considered here can be seen in almost all practical geometry books, although inspired by Magini's treatment, Clavius's collection of problems is much more extensive than many. As Raynaud points out, these are part of a long and surprisingly stable tradition with connections to propositions 19–22 from Euclid's *Optics*, (Raynaud 2015, p. 15).

V. Measuring volumes of solid bodies, with extensive presentations of results from the stereometric books of Euclid, plus Archimedes' works *On the Sphere and Cylinder* and *On Conoids and Spheroids*.

VI. Geodesy, that is, the division of rectilinear surfaces of whatever sort, either by lines drawn through some point, or by parallel lines; see Knobloch (2015). How plane or solid figures are increased or decreased in a given ratio; several methods for finding two mean proportionals between two given lines selected from the commentary on Archimedes' *On the Sphere and Cylinder* by Eutocius of Ascalon (ca. 480–ca. 540 CE) and the *Mathematical Collection* of Pappus of Alexandria (ca. 290–ca. 350 CE); finally, an algorithm for extracting nth roots by hand calculations.

VII. Isoperimetric figures and questions (drawing on material in Pappus and the commentary on Ptolemy's *Almagest* by Theon of Alexandria (ca. 335–ca. 405 CE), inspired by earlier work of Zenodorus (ca. 200–ca. 140 BCE)), together with an appendix on the problem of squaring the circle via the *quadratrix* curve of Hippias, drawing from Pappus.

VIII. An extensive catalog of geometrical theorems and constructions, plus some number-theoretic results. Clavius claims these can be used to build mathematical power in problem-solving.[10] Several of them are drawn from Pappus, including one discussing the trisection of general angles using the *conchoid* curve of Nicomedes. A table of squares and cubes of all natural numbers less than 1000 is included at the end, together with a discussion of how the table can be extended using facts about the first and second differences of the sequences of squares and cubes, with applications to extraction of square and cube roots.

As is true in all of his other works, Clavius also has clarity of exposition as a second main goal. A striking example of this commitment to completeness and clarity of exposition is Clavius's treatment of Archimedes' *Measurement of the Circle* in Chapter 6 of Book IV, which will be examined in detail in Sect. 3.

The second feature that has seemed surprising, to this author at least, is the resolutely *dual theoretical and practical focus* of much of this text on practical geometry.[11] The practical side is signaled immediately in the Preface. After saying that his experience as a teacher has taught him that most students work and learn best when they understand that what they are learning will prove to be useful,[12] Clavius addresses how the contents of this book may find uses in the real world:

> For of course as long as the methods by which we must make measurements to understand the lengths of fields, the heights of mountains, the depths of valleys, and the distances between all locations are presented, it is clear to anyone (in my

[10] Neque vero hoc praeter institutum nostrum existimare quis debet: cum per eiusmodi demonstrationes Geometricas studioso Lectori via multiplex aperiatur ad inuestigandas similes speculationes in rebus Geometricis: quippe cum in iis ad exercendum ingenium amplissimum campum habeat. (Clavius 1606, p. 330)

[11] As Knobloch writes, Clavius's "... approche démontre les limites d'une division trop tranchée entre géométrie pratique et géométrie savante," (Knobloch 2015, p. 60). That is, Clavius's "... approach shows the limitations of a too-definite division between practical geometry and theoretical geometry." This applies to almost every section of the *Geometria Practica*, not just the discussion of geodesy.

[12] Et verò cum perpetua multorum annorum experientia comperentim, admodum paucos esse, qui non in Mathematicis exerceantur eo consilio, vt quae didicerint, ad aliquem vsum trahant. (Clavius 1606, Preface)

> opinion) how much that is of use in the construction of buildings, in agriculture, in the design of weapons, in the contemplation of the stars, and in all other arts and disciplines, can flow from the study of these things.[13] (Clavius 1606, Preface)

Clavius consistently uses numerical examples in many sections and he presents a number of purely calculational methods (e.g., the methods for extraction of roots in Book VI and the material on differences of squares and cubes at the end of Book VIII). He discusses the use of different mechanical tools for measurements and is even willing to countenance "*mechanical*," hence necessarily approximate, methods of measurement in geometric diagrams (Clavius 1606, p. 169).

In addition, Clavius discusses potential practical applications even for many of the more theoretical topics treated in Books IV through VIII. For example, in the course of discussing areas of planar figures, Clavius includes a section on methods used by surveyors in Book IV. At the end of Book V, after presenting results on volumes of solids from Euclid and Archimedes, he includes a section on measuring volumes of barrels or casks (Clavius 1606, p. 233).

However, once he gets past the very basic material in Books I, II, and III (where the theory draws almost exclusively on Books I–VI of Euclid and elementary trigonometry–subjects for which he had given comprehensive accounts in his other writings), Clavius's focus seems to shift to developing the required new mathematical theory along with a few practical applications. In the process, he usually provides full proofs for the most important results and references for the statements he does not prove. We will see striking examples of this in Sects. 3 and 4.

Another hallmark of Clavius's approach and theoretical orientation even within the practical discussions is his scrupulous attention to providing reasons for almost everything he writes. This applies even within the more overtly "practical" Books I, II, and III. Throughout the text, an elaborate system of marginal notes identifies justifications for assertions and for the individual steps in proofs or computations. Over the course of the whole book, the justifications for the steps in those proofs span almost all of the 13 books of the canonical version of Euclid's *Elements*, plus the 14th through 16th books added by later authors and included in Clavius's edition of the *Elements*, as well as some of Clavius's other works, several works of Archimedes, and Apollonius's *Conics* (once).

A third feature that is clearly visible in the table of contents, but that still might be surprising, is the extent to which Clavius draws on the geometrical works of Archimedes and Pappus, plus works of other ancient and medieval mathematicians and his contemporaries. Some of the work of the ancient Greeks was just coming back into the European mathematical mainstream at precisely this time due to the work of humanist scholars such as Federico Commandino (1509–1575) and others. Commandino's Latin translation of the surviving portions of Pappus's *Mathematical Collection*, for instance, only appeared in print in 1588.

[13] Etenim dum certa ratio traditur, qua camporum longitudines, altitudines montium, vallium depressiones, locorum omnium inaequalitates inter se, & interualla deprehendere metiendo debeamus: cuilibet liquet, vt arbitror, quantum commodi, vtilitatisque substructioni aedificiorum, cultui agrorum, armorum tractationi, contemplationi siderum, aliisque artibus, & disciplinis ex horum cogitatione manare possit.

1.2 This essay

Our plan in this essay is to flesh out this general description of the eclectic content and sources of Clavius's *Geometria Practica* by focusing on four particular sections dealing with topics of particular interest. We have restricted ourselves to parts of the text not covered in detail by other authors. So for instance, we have not included a discussion of the Appendix to Book VII giving Clavius's approach to the problem of squaring the circle via the *quadratrix* curve[14] because that is analyzed deeply in Bos (2001, Chapter 9). Similarly, we have not considered the discussion of *geodesy* at the start of Book VI, since Clavius's approach has been discussed in Knobloch (2015). The sections we do discuss are ones where (in our judgment) Clavius has either used his sources in an interesting way, or he has been uncharacteristically reticent about those sources.[15] We will look first at the beginning of his discussion of computing areas of triangles in Book IV, where Clavius presents what we now call *Heron's formula* before what we might think of as the usual method based on finding an altitude of the triangle.[16] He does not mention anything about his sources there, but by comparing what Clavius writes with the treatments in earlier works, some insight may be gained. Second, we will look at Clavius's treatment of Archimedes' *Measurement of the Circle* in Book IV. Here, we will see that Clavius has presented essentially a complete reworking of the received Archimedean text incorporating additional explanatory comments and details not found in other versions. Third, we will consider Clavius's discussion of some of the Greek constructions for finding two mean proportionals between two given lines. This involves a very deliberate selection of only a few of the methods discussed in the commentary on Archimedes' *On the Sphere and Cylinder* by Eutocius of Ascalon and by Pappus in the *Mathematical Collection*. Finally, we consider the discussion of an algorithm for extraction of nth roots discussed at the end of Book VI. Clavius does not explicitly identify his source here. But by considering what books would have been available to him, and comparing his treatment of extraction of roots with what appears in two of those books, we propose a very likely candidate. The similarity is so pronounced and Clavius is so definite that his account is "taken almost entirely" from an (unidentified) "remarkable book of German arithmetic"[17] that this

[14] A more extensive version of this also appears in Clavius's edition of Euclid.

[15] Clavius is often very careful to identify sources, and it stands out when he does not do so. Over the course of this book, the list of authors cited is quite extensive, including (but possibly not limited to) Apollonius, Archimedes, Archytas, Giovanni Battista Benedetti, Campanus de Novare, Girolamo Cardano, Federico Commandino, John Dee, Dinostratus, Diocles, Albrecht Dürer, Eratosthenes, Euclid, Eutocius of Ascalon, Oronce Fine, Fraçois de Foix, Comte de Candale, Niccolo Fontana ("Tartaglia"), Gemma Frisius, Marino Ghetaldi, Christoph Grienberger, Hippocrates, Hypsicles, Ioannes Pediasimos, Leonardo Pisano ("Fibonacci"), Ludolph van Ceulen, Mohammad of Baghdad, Odo van Maelcote, Giovanni Antonio Magini, Francesco Maurolico, Menaechmus, Nicholas of Cusa, Nicomedes, Latino Orsini, Luca Pacioli, Pappus, Georg Peuerbach, Proclus, Ptolemy, Joseph Justus Scaliger, Sporus, Simon Stevin, Theon of Alexandra, Juan Bautista Villalpando, Johannes Werner. A fuller listing of all the authors cited by Clavius across his whole written output is given in Knobloch (1990).

[16] But note that this would definitely be useful for measuring the area of a triangular plot of land or a triangular building where access to the interior and measurement of an altitude might not be possible.

[17] Hoc autem efficiam, si praescribam artem quandam generalem, qua cuiuscunque generis radicem extrahere possimus, ex libro eximij cuiusdam Arithmetici Germani depromptam fermè totam—(Clavius 1606, p. 276)

has no doubt been noticed before. However, we have not been able to locate an explicit reference proposing this source.

2 Clavius's treatment of "Heron's formula" for triangles in Book IV

In Chapter 2 of Book IV, Clavius discusses methods for finding the area of a plane triangle. He writes there are two ways of doing this and he will first present the most accurate or precise one. He states this as a *rule* or procedure for doing the computation rather than as a proposition:

> Let all the sides be added together in one sum; let each of the sides be subtracted from half of this sum, so that three differences between the semiperimeter and the sides are obtained; finally, let these three differences and the semiperimeter be multiplied together. The square root of the number produced will be the area of the triangle which is sought.[18] (Clavius 1606, p. 158)

In modern algebraic terms, the procedure can be collapsed into the single formula

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where a, b, c are the side lengths and $s = (a+b+c)/2$ is the semiperimeter of the triangle. This usually goes by the name *Heron's Formula* today, and indeed this is stated and proved in Proposition I.8 of the *Metrica* of Heron of Alexandria (ca. 10–ca. 70 CE(?)).[19] Clavius provides three numerical examples of triangles with integer side lengths similar to examples found in other works. One example leads to a product $s(s-a)(s-b)(s-c)$ that is not a square so the area is only found approximately according to techniques that will be discussed later. He then gives a complete, detailed proof that this rule or process does in fact produce the area of the triangle.

Somewhat unusually for him, Clavius does not provide any attribution for this result, so several natural questions arise. What source or sources would have been available to him? A number of the possibilities have been discussed extensively in Clagett (1964, 1967–1984), Volume 1, Chapter 4, Appendix IV). Recall that in Clavius's time Heron's *Metrica* was not known; it was considered lost until 1896, when Richard Schöne recognized it as part of a manuscript kept in a library in Istanbul.[20]

To help make some comparisons between various proofs, we begin with a version of the diagram in Heron's proof for a specific triangle.[21]

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18 Colligantur omnia latera in unam summam: Ex huius summa semisse subtrahantur singula latera, vt habeantur tres differentiae inter illam semissem, & latera singula: Postremo tres hae differentiae, & dicta semissis inter se mutuo multiplicentur. Producti enim numeri radix quadrata erit area trianguli quaesita.

19 The Islamic mathematician al-Bīrūnī (973–1048) thought that the result was originally proved by Archimedes, and C. M. Taisbak has recently provided a conjectural reconstruction of the way Archimedes might have stated the result. See Taisbak (2014).

20 This was first published in Schöne (1903). A modern study of this sole known surviving manuscript of the *Metrica* can be found in Acerbi and Vitrac (2014).

21 See Fig. 1. To generate these figures, we used the triangle with vertices at $B = (0, 0)$, $C = (5, 0)$ and $A = (1, 2)$ in the Cartesian plane. This happens to have a right angle at A so some of the line segments

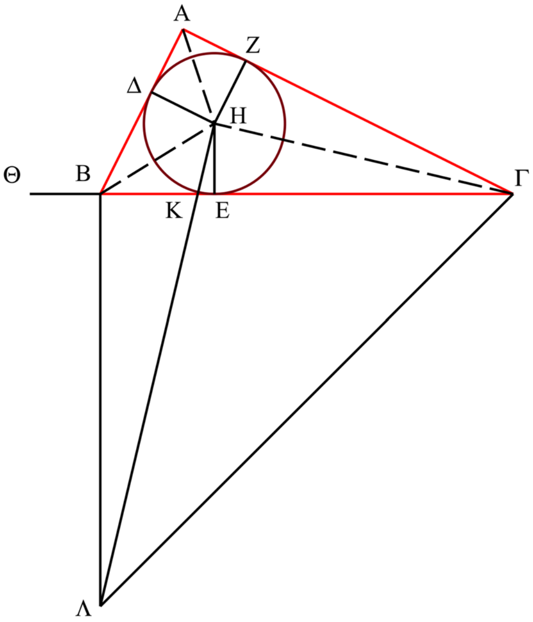

Fig. 1 Heron's diagram in the *Metrica*

Heron's proof in outline consists of the following steps. First, let the circle $Z\Delta E$ with center at H be inscribed in the triangle $AB\Gamma$. Proposition IV.4 of Euclid gives a construction for this where H is found as the intersection of two of the angle bisectors of the triangle, but Heron apparently takes this as known and does not mention it explicitly. Then, $H\Delta = HZ = HE$ since these are all radii of the same circle. Since HE, $H\Delta$ and HZ are perpendicular to the sides of the triangle, the area of triangle $AB\Gamma$ will equal one half times EH times the perimeter of the triangle. Heron then does further construction steps, first extending $B\Gamma$ to $\Theta\Gamma$, letting $\Theta B = A\Delta$. This makes $\Theta\Gamma$ equal to the semiperimeter of the triangle. Second, he takes $H\Lambda$ perpendicular to $H\Gamma$ and extends to Λ which is the intersection with the line through B perpendicular to $B\Gamma$. It follows that $HB\Lambda\Gamma$ is a cyclic quadrilateral and facts about the diagonals in such quadrilaterals imply triangle $B\Lambda\Gamma$ is similar to triangle ΔHA. The proportionality of corresponding sides implies the square of the area of the triangle $AB\Gamma$ is equal to the square on the perpendicular $HE = H\Delta = HZ$ above times the square on the semiperimeter (using the fact that $B\Gamma$ and $\Theta B = A\Delta$ together equal the semiperimeter).

One source for Clavius certainly could have been the *Verba filiorum Moysi filii Sekir, i.e., Maumeti, Hameti, Hasen*, (Clagett 1964, 1967–1984, Volume 1, p. 224). This work, also known as the *Liber trium fratrum de geometria*, is a Latin translation of an Arabic work on mensuration by the ninth century Banū Mūsā brothers made by Gerard of Cremona (1114–1187). The original authors were key figures in the early translation movement by which Greek mathematics was introduced into the Islamic

in the figures are in rather special positions that facilitated the plotting. However, this does not affect the arguments. None of the authors we consider would have done things this way, of course.

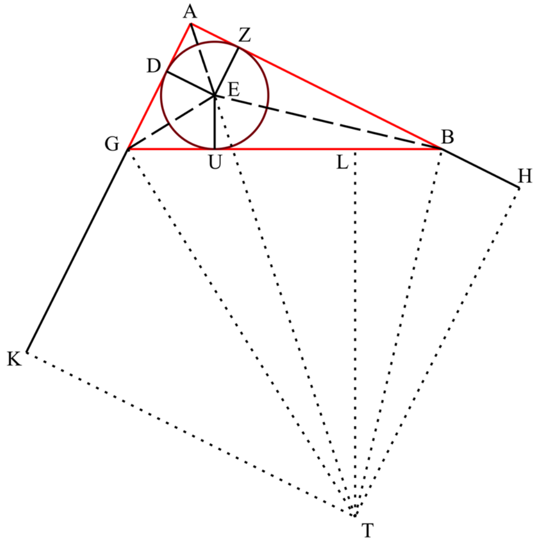

Fig. 2 *Verba filiorum* diagram

world and the Greek original of the *Metrica* may have been available in Baghdad in their time. But there are significant differences between the *Metrica* version and the *Verba filiorum* version. Figure 2 shows what the diagram in the *Verba filiorum* looks like for our triangle.

After identifying the points D, Z, U as the points of tangency of the inscribed circle, the side AB is extended to AH by making $BH = GU$, so AH is equal to the semiperimeter. Similarly, AG is extended to AK making $GK = BU = BZ$ and angle AKT is a right angle. The point T is chosen so that it lies on the extended angle bisector at A. It follows that triangle EBU is similar to triangle BTH and the proportionality of corresponding sides implies the square of the area of the triangle $AB\Gamma$ is equal to the square on the perpendicular $EU = EZ$ times to the square on the semiperimeter AH.

Several of the previous practical geometry texts that Clavius mentions in his preface (see above) also include proofs of Heron's process/formula for finding the area of a triangle. One of the earliest that does discuss this is the *De Practica Geometrie* of Leonardo Pisano ("Fibonacci"), (Pisano 1862, 2008). The later *Summa de arithmetica geometria proportioni et proportionalità*, (Pacioli 1494), does as well and Pacioli's treatment follows what Fibonacci writes very closely (but in the Tuscan dialect of Italian rather than in Latin).

Marshall Clagett has written that Fibonacci "borrows heavily and often in verbatim fashion" in the revised version of the *Practica Geometrie* from the *Verba filiorum* and he mentions that he believes Fibonacci's debt to the Banū Mūsā applies specifically to the treatment of Heron's formula in Fibonacci's work, Clagett (1964, 1967–1984, Volume 1, p. 224). However, a close analysis of the argument and the diagrams provided shows that while the proof of Heron's formula in Proposition VII of the *Verba filiorum* has many features in common with Fibonacci's proof, it also has other features in common with the proof from Heron's *Metrica* that do not occur in Fibonacci. Probably,

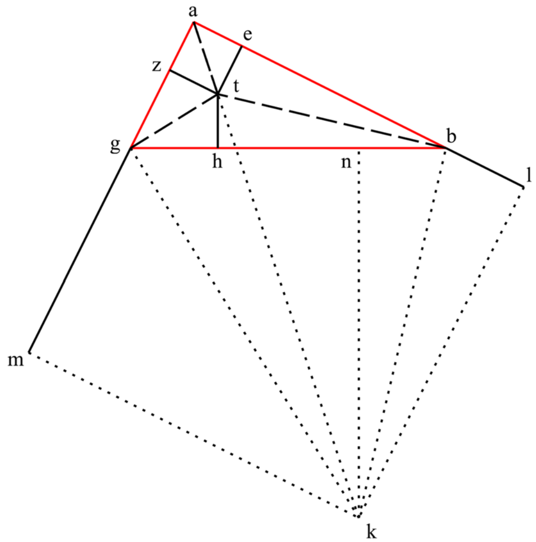

Fig. 3 Fibonacci's diagram

the major example here is that the whole first phases of the arguments in both the *Metrica* proof and in the *Verba filiorum* proof consider the inscribed circle in the triangle (as in Proposition IV.4 of Euclid). Fibonacci does not mention the inscribed circle; in fact, he ends up repeating a large portion of the Euclidean proof to show that if perpendiculars (or as Fibonacci writes, "cathetes") are dropped to the three sides from the intersection point of two angle bisectors in the triangle, then the three perpendicular segments are equal.[22] There are also some less drastic differences in the way that similar triangles within the figure are used to deduce that the square of the area is equal to the square on the perpendicular above times the square on the semiperimeter (and the square on the semiperimeter equals the semiperimeter times the product of the three excesses of the semiperimeter over the sides). So it is not entirely accurate to characterize Fibonacci's proof (at least as a whole) as "verbatim" borrowing even if the overall strategies of the proofs are similar and the final sections of the proofs do more or less converge.

On the other hand, Clavius's version of the proof of Heron's formula is different again, but significantly closer to the proof in Fibonacci than it is to the proof in the *Verba filiorum*. To discuss this in more detail, it will be necessary to consider the diagrams from these two proofs. (See Figs. 3, 4.) These two figures show Clavius's and Fibonacci's constructions applied to the same particular triangle as in the previous figures.

[22] The difference in the diagrams is also noted by the translator B. Hughes in Pisano (2008). See the footnote on p. 83. But there is an unfortunate mistranslation at the start of the proof of Heron's formula in Pisano (2008). At the start of the first full paragraph on p. 81, Hughes has, "To prove this: in triangle *abg* bisect the two equal angles *abg* and *agb* ... ." This would make the proof apply only to isosceles triangles. But that is not correct. The Latin text of Pisano (1862, p. 40) at this point is: "Ad cuius rei demonstrationem adiaceat trigonum *abg*: et dividantur in duo equa anguli, qui sub .abg. et .agb. a rectis .bt. et .tg. ... ." That is, "To prove this: in the triangle *abg*, let the angles *abg* and *agb* each be divided into two equal angles by the lines *bt* and *tg* ... ." Fibonacci is definitely *not* restricting his discussion to isosceles triangles.

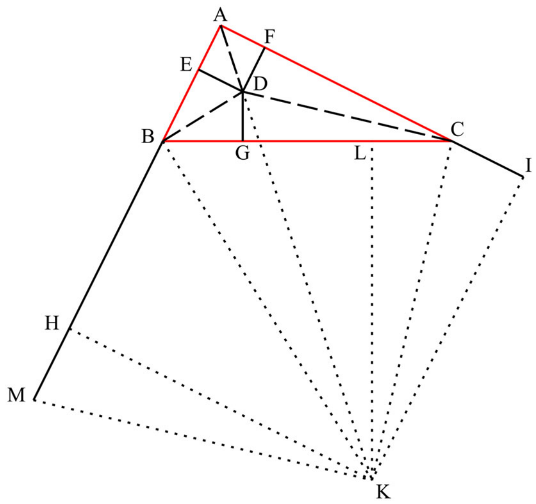

Fig. 4 Clavius's diagram

Both Clavius and Fibonacci start by considering angle bisectors (these are two of the dashed lines) for the two vertices on the horizontal side in the diagram and their intersection point (t, and D, respectively). They both drop perpendiculars (th, tz, te, and DG, DE, DF, resp.) and use facts about congruent triangles in the figure to show that the three perpendiculars are equal that at (resp. AD) also bisects that angle and, moreover, the two segments closest to each vertex are equal—that is $ae = az$, $be = bh$, $gz = gh$ (resp. $AE = EF$, $CF = CG$, and $BE = BG$). Neither mentions the inscribed circle, which would be tangent to the sides of the triangle in the points e, z, h (resp. E, F, G). This implies that any one of the sides, together with one of the equal segments not meeting that side are together equal to the semiperimeter of the triangle—for example, side ab (resp. AC) together with gh or gz (resp. BG or BE). In addition, the three excesses of the semiperimeter over the sides of the triangle that feature in Heron's formula coincide with the segments: ae or az, be or bh, gh or gz (resp. AE or AF, BG or BE, CG or CF).

Then, in further parallel constructions, the sides ag, ab (resp. AB and AC) are extended to am, bl (resp. BH, AI) by making $gm = hb$ and $bl = gh$ (resp. $BH = GC$ and $CI = BG$). As noted before, this makes both am and al (resp. AH and AI) equal to the semiperimeter of the triangle, hence equal. At this point, Fibonacci produces the third angle bisector at until it meets the segment lk making a right angle with ab at k. Clavius, on the other hand (literally), produces AD to K where it meets the line through H perpendicular to AH. But either way, the next deduction is that by the SAS criterion, the triangles amk and alk (resp. AHK and AIK) are congruent, so angle amk (resp. AIK) is also a right angle, and moreover $mk = lk$ (resp. $HK = IK$).

In the final constructions, Fibonacci cuts off the segment bn from gb so that $gn = gm = bh$, and hence $bn = bl = gh$. Clavius does the parallel operations making $BL = BH = CG$, and hence $CL = CI = BG$. But now Clavius does one further step that Fibonacci does not: He extends AH to AM, making $HM = CL = CI$. With k, resp. K joined to all of the newly constructed points, both proofs proceed to show that the lines kn (resp. KL) meet the horizontal side in a right angle. The additional triangle HMK introduced in Clavius's argument is congruent to triangles CIK and CLK (resp. bnk, blk in Fibonacci's figure), and hence it is somewhat redundant. But what we have here would seem to be a typical kind of procedure for Clavius. At the cost of a few more steps, he furnishes a reader of his proof with another triangle HKM that gives a perhaps easier way to understand why the angle at n or L must be a right angle. This is not really clear visually in Clavius's original diagram, where it seems that no attempt has been made to show all the right angles accurately.

As in all of the proofs of Heron's formula we have discussed, similar triangles can be identified in the figure such that the proportionality of corresponding sides implies that the square of the perpendicular times the semiperimeter is equal to the product of the three excesses of the semiperimeter over the sides. Fibonacci uses the triangles ebt and kbl; Clavius uses the triangles AED and AHK. The details in this step are somewhat different, but the idea is analogous. The final step in both of these proofs is to note that by the usual "one half base times height" way of computing areas of triangles, the sum of the areas of the three triangles atb, btg, gta (resp. ADC, CDB, BDA), which equals the area of the whole triangle, is also equal to the product of any one of the three equal perpendiculars and the semiperimeter.

It seems very probable that Clavius was looking at Fibonacci's proof (or perhaps other proofs derived from that one such as the proof in Pacioli's text), but his version is not a verbatim copy, any more than Fibonacci's was a verbatim copy of the proof in the *Verba filiorum*. As was often the case, Clavius reworked and amplified what he found in other sources so that his version has additional or alternate features intended to heighten clarity or to increase convenience for his readers. Why Clavius chose not to say this explicitly at this point in his book is somewhat mysterious, however. It is possible, of course, that since neither Fibonacci nor Pacioli attribute this statement to a particular mathematician, Clavius simply did not have a way to refer to a primary source. However, in analogous situations, Clavius did sometimes say explicitly how his account of a proof would differ from what was found in his source or sources even when they were probably also referring to unmentioned previous works.[23]

3 Clavius's treatment of Archimedes' *Measurement of the Circle* in Book IV

After Greek versions of this work of Archimedes (including summaries from Book V of the *Mathematical Collection* of Pappus and the commentary on Ptolemy's *Almagest* by Theon) were intensively studied in the Islamic world and the resulting Arabic translations were retranslated into Latin, the *Measurement of the Circle* was surely the best-known and most-copied Archimedean text throughout the medieval period in western Europe. A major part of the reason was certainly the utility of its results

[23] For example, he was very explicit about this in the introduction to Book VI on *geodesy* (divisions of figures) where the material in question might ultimately derive from a lost work of Euclid, as discussed, for instance, in Knobloch (2015).

for practical questions, namely the following three propositions making up the whole contents of the received text.[24]

Proposition 1 *Every circle is equal [in area] to a right triangle, one of whose sides containing the right angle is equal to the radius of the circle while the other side containing the right angle is equal to the circumference.*

Proposition 2 *The ratio of the area of any circle to the square of its diameter is the ratio* 11 *to* 14.

Proposition 3 *The circumference of a circle exceeds three times its diameter by a quantity less than* $\frac{1}{7}$ *of the diameter and greater than* $\frac{10}{71}$ *of the diameter.*[25]

Clavius explains what he aims to do in the following introductory paragraph in Chapter 6 of Book IV and makes the connection with Archimedes explicit from the start:

> It will not be a digression, therefore, if I include his [i.e., Archimedes'] truly most acute and precise book, partly because it is very brief (indeed, it consists of only three propositions), partly so that the student, in order to understand something so useful and so widely applied in the works of all authors, should not be forced to go to Archimedes himself, and finally mostly because the writings of Archimedes, as a result of their brevity, are somewhat obscure, and we hope to bring some light to them.[26] (Clavius 1606, p. 182)

Note in particular that Clavius writes that he is including his version of *the whole book of Archimedes* and not just discussing the results contained in it. Clavius proceeds immediately to a detailed account of the rather subtle exhaustion proof of Proposition 1 found in many of the versions of the *Measurement of the Circle* mentioned above.[27] For the convenience of the reader, we will outline this argument. The plan is to show that assuming the area of the circle is either greater than or less than the area of a right

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24 The brevity of the work and its somewhat sketchy form have led Dijksterhuis to conjecture that "it is quite possible that the fragment we possess formed part of a larger work" (Dijksterhuis 1956, p. 222), and Knorr to judge that the versions we have represent "at best an extract from the original composition" (Knorr 1989, p. 375). Clagett (1964, 1967–1984, Volume 1) reproduces two translations of this work from Arabic into Latin, the first made ("perhaps") by Plato of Tivoli (fl. twelfth century), and the second made by Gerard of Cremona. Clagett also reproduces six additional "emended" versions as well as the treatment of the results of this work in the *Verba Filiorum*, following the Banū Mūsā. Part III of Knorr (1989) contains a more complete study of the transmission including additional versions. By Clavius's time, many versions of this work of Archimedes were available, including the Greek *editio princeps* published by Thomas Gechauff in 1544 and the Latin translation in Commandino's edition of works of Archimedes (1558).

25 That is, in modern terms, $3\,\frac{10}{71} < \pi < 3\,\frac{1}{7}$. Archimedes may well have used methods similar to the ones to be discussed to produce tighter estimates for the ratio of the circumference to the diameter. But if so, no text doing this has survived.

26 Non abs re ergo erit, si eius libellum de circuli dimensione acutissimum sane, & subtilissimum hic interferam, tum quia breuissimus est, quippe qui tribus duntaxat propositionibus constet: tum ne studiosus, vt rem tam vtilem, atque apud omnes artifices peruulgatam intelligat, Archimedem ipsum adire cogatur: tum vero maximè, quod cum Archimedis scripta ob affectam breuitatem, sint paulo obscuriora, illis nos lucem aliquam allaturos speramus.

27 This proof has much in common with the proof of Proposition XII.2 of Euclid. It is also very closely related to the isoperimetric problems that Clavius will discuss later in Book VII.

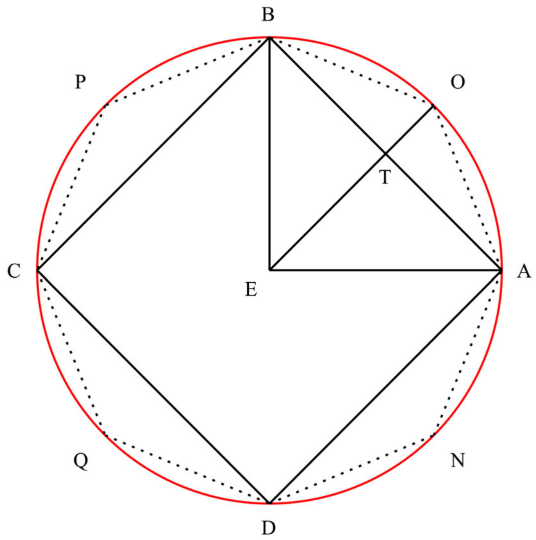

Fig. 5 Essential portions of Clavius's diagram for the proof of the first part of Proposition 1

triangle as in the statement leads to a contradiction. A key role is played by a statement introduced by Euclid in the proof of Proposition X.1 of the *Elements*. Clavius uses this in the form: *If an area at least half the area of a figure is taken away from the figure, and from the residual area again an area at least half of that remaining area is taken away, and so on, there will eventually remain an area less than any positive magnitude z.*

First suppose the circle is larger than the stated triangle by a certain positive magnitude (Clavius calls this z). Let a sequence of non-overlapping areas be removed from the interior of the circle.

Clavius's account incorporates quite a few explanatory comments and justifications for the individual steps in the reasoning not found in many other versions presented in Clagett (1964, 1967–1984). Here is the relevant paragraph for the first part of the proof, referring to Fig. 5:

> This sort of continual subtraction of areas will be done, if in the first place the inscribed square $ABCD$ is removed from the circle. For this is half of the square $IKLM$, circumscribed about the circle, as we have shown in the scholium to Proposition 9 in Book IV of Euclid. Since the circle is a portion of this square $IKLM$, the inscribed square will be more than one half of the circle. Then, let four isosceles triangles AOB, BPC, CQD, DNA, with sides drawn from the midpoints of the arcs, be removed from the four segments. These together are greater than the halves of the segments together, since any one of them is greater than half of the segment in which it lies. For having completed the rectangle AR, [marginal note: Book I, Proposition 41.] its half will be the triangle AND, and therefore the triangle will be greater than half of the segment AND. The same reasoning applies to the other segments. By the same reasoning, if from the eight

remaining segments eight other isosceles triangles are removed, constructed in them, and so forth.[28] (Clavius 1606, p. 182)

It is not hard to see that this pattern continues indefinitely.[29] So eventually the remaining region between a 2^m-gon and the circle will have area less than the positive magnitude z. However, this leads to a contradiction if the construction is applied sufficiently many times. The area of the circle is supposed to equal (area of triangle) $+ z$, but the area of the circle also equals: (area of inscribed polygon) + (remaining area) which is strictly less than (area of triangle) $+ z$, since the perimeter of the polygon is smaller than the circumference of the circle, and the apothem—the perpendicular from the center to the side of the inscribed polygon—is less than the radius. Hence, the area of the circle cannot be greater than the area of the triangle.

Next it is assumed that the circle is less than the stated triangle by a certain magnitude (again denoted z). Starting from the square circumscribed about the circle, areas are removed: first the circle, then four exterior triangles (such as KXV in Fig. 6) with base tangent to the circle, then again eight exterior triangles with base tangent to the circle at the midpoints of the arcs BO, OV, and so forth. The remaining regions now are collections of what Clavius calls "mixed triangles," with one side an arc of the circle.

Again, more than half the remaining area is removed at each step, and the remaining area is eventually less than z. But this also leads to a contradiction. On the one hand (area of circumscribed polygon) $>$ (area of triangle) because the perimeter of the polygon is greater than the circumference and the apothems are now all equal to the radius of the circle. But on the other hand, by the process described above, (area of circumscribed polygon) is strictly less than (area of circle) $+ z =$ (area of triangle). Since the triangle is neither greater nor less than the circle, it can only *equal the circle*.

An amusing sidelight in the form of a long Scholium follows, in which Clavius refutes the claim of Joseph Justus Scaliger (1540–1609)[30] that Archimedes must have

[28] Haec autem detractio continua fiet, si primo loco auferatur ex circulo quadratum inscriptum $ABCD$. Hoc enim cum dimidium quadrati $IKLM$, circulo circumscripti, vt in schol. propos. 9. lib. 4. Eucl. ostendimus: circulus autem ipsius quadrati $IKLM$ pars sit, erit quadratum inscriptum $ABCD$ maius quam dimidium circuli. Deinde si auferantur à residuis quatuor segmentis quatuor triangula Isoscelia AOB, BPC, CQD, DNA, ductis rectis ad media puncta arcuum. Haec enim simul maiora sunt, quam dimidium quatuor segmentorum simul, cum vnum quodque maius sit, quam dimidium segmenti in quo existit. Completo enim rectangulo AR, [marginal note: 41. primi.] erit eius dimidium triangulum AND: ac proinde idem triangulum maius erit quam dimidium segmenti AND. Eademque ratio est de aliis. Pari ratione, si à residuis octo segmentis auferantur octo alia triangula Isoscelia in illis constituta, &c. atque ita deinceps.

[29] Clavius stops with the octagon so he has a diagram that has components virtually identical with the ones presented by Fibonacci and Pacioli. But many of Clavius's point labels and additional lines constructed in a very "busy" diagram have been omitted at this stage for clarity.

[30] Scaliger, the eminent French Protestant classical philologist and historian, also fancied himself a mathematician. His mathematical *magnum opus*, grandly titled *Cyclometrica Elementa*, was published in 1594 in a lavish edition with statements of theorems in both Latin and ancient Greek. But more competent mathematicians quickly recognized that it was largely erroneous. In 1609, Clavius published an 84-page pamphlet *Refutatio Cyclometriae Iosephi Scaligeri* (Refutation of Joseph Scaliger's claims about measurement of circles), giving a blow-by-blow analysis of all of the (numerous) errors in Scaliger's work. This is contained as an appendix in Volume V of Clavius (1611–1612).

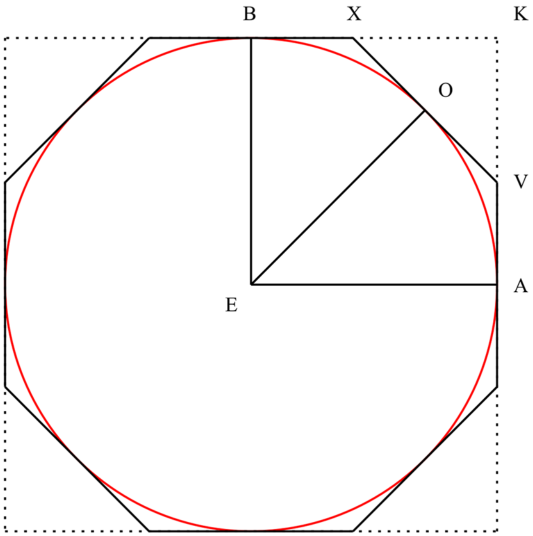

Fig. 6 Essential portions of Clavius's diagram for the proof of the second part of Proposition 1

been mistaken in this proof. Indeed, Clavius addresses Scaliger directly and takes him to task rather savagely over his misunderstandings. A small sample:

> And I am really astonished that you, *Mathematicus* that you are, deny that some quantity is equal to another when it is neither greater nor less. For if it is not equal to the other, then it will be unequal to the other, therefore either greater or less, against that hypothesis. Or don't you see that not only Archimedes but also Euclid used this way of arguing most frequently in Book XII of the *Elements*?[31] (Clavius 1606, p. 185)

Scaliger had had another "run-in" with Clavius over the Gregorian calendar reform in which Clavius had taken a leading role, as well as ongoing controversies with other Jesuits on various subjects, so there was ample bad blood between them. Some of that is manifest in the scathing polemical tone of Clavius's comments.

Following this, Clavius notes that the usual Proposition 3 is used in the proof of Proposition 2, and hence he has decided to reverse the order of Propositions 2 and 3 as found in other versions to maintain the chain of logical implications.[32] In his account of this famous proof, Clavius essentially follows the plan used in most other versions. Two aspects of his version of the proof of his Proposition 2 (the usual Proposition 3), as compared with other versions, are notable. First, although he presents the same calculations, Clavius provides *more details*, more *justification for individual steps*, and *generally a fuller treatment* leading to the estimates than Archimedes (or whoever

[31] Et sane miror, te, Mathematicus, cum sis, negare quantitatem aliquam illi esse aequalem, qua neque maior est, neque minor. Si enim aequalis non est, erit inaequalis. Igitur vel maior vel minor, contra hypothesim cum dicatur neque maior esse, neque minor. An non vides, non solum Archimedem, sed etiam Euclidem lib. 12. hunc argumentandi modum frequentissimè vsurpare?

[32] Clavius writes: Haec est Archimedis propositio 3. quam nos secundam facimus, vt doctrinae ordo servetur, quando quidem sequens propositio 3. quam ipse 2. facit, hanc nostram propositionem 2. in demonstrationem adhibet. (Clavius 1606, p. 185)

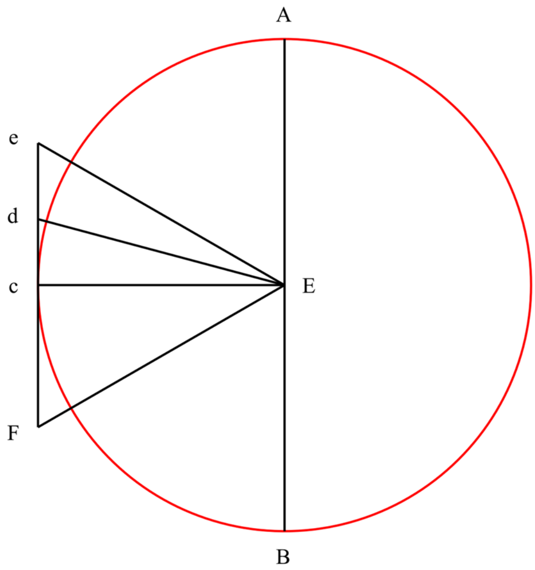

Fig. 7 Proof of Clavius's Proposition 2, first bisection step

wrote the versions of the Archimedean text that we have) did. Since the bisection steps in each half of the proof follow exactly the same plan, some versions work out the first step in detail, and then just present the numerical results for the subsequent steps.[33] Here is the set-up and the first bisection step in Clavius's version. Taking eF as one side of the regular hexagon circumscribed about the circle ABc with center E (see Fig. 7 below) and

> ... *ce* to have length 153 ... Ee will equal 306. If the square of this *ce*, 23409, is subtracted from 93636, the square of Ee, [marginal note: Book I, Proposition 47] the difference will make the square of Ec, 70227, whose square root is slightly larger than 265. [marginal note: Book V, Proposition 8] From this, the ratio of Ec to *ce* will be greater than 265 to 153.
> Having bisected the angle eEc by the line Ed, [marginal note: Book VI, Proposition 3] eE is to Ec as *ed* is to dc. *Componendo*, eE and Ec together are to Ec as *ec* is to dc. *Permutando*, eE and Ec together are to *ec* as Ec is to cd. Since eE and Ec together are more than 571 (since indeed eE is 306 and Ec slightly larger than 265), and *ec* is taken to be 153, [marginal note: Book V, Proposition 8] the ratio of eE and Ec together to *ec* will be greater than 571 to 153, hence the ratio of Ec to *cd* will be greater than 571 to 153, and it follows that if *cd* is set equal to 153,[34] [marginal note: Book V, Proposition 10] then Ec will be slightly larger than 571. Therefore, the square of Ec will be slightly larger than 326,041, and since the square of *cd* is 23,049, the square of Ed, [marginal note: Book I, Proposition 47] which is equal to the sum of the squares of Ec, cd, will

[33] The author used this expedient, in fact, in his translation of Clavius's proof. In mathematical discussions, extreme verbosity is sometimes the unfortunate corollary of complete explicitness.

[34] Note that this essentially rescales the whole figure from the dimensions given before. Since it is the ratio of the lengths that is important, this is harmless.

> be slightly greater than 349,450. The square root of this is greater than 591 $\frac{1}{8}$, for indeed the square of this number is only 349,438 $\frac{49}{64}$. [marginal note: Book V, Proposition 8] Therefore, the ratio Ed to dc will be greater than 591 $\frac{1}{8}$ to 153.[35] (Clavius 1606, pp. 185–186)

The second aspect is perhaps less mathematically significant, but still interesting. Namely, by showing the bisections from the first phase of the proof to the left of a vertical diameter in the circle, and the bisections from the second phase to the right, Clavius manages to condense all the steps of the constructions for both phases of the proof into a single diagram in a way that is somewhat clearer than the figure from Pisano (1862, p. 90) or (2008, p. 155). Many accounts of this proof provide separate diagrams for each phase.[36]

Another quite significant deviation from sources such as Fibonacci is that in his Proposition 3 (the usual Proposition 2), Clavius explicitly adds the qualifier *approximately* (proximè in the Latin) to the usual statement that the ratio of a circle to the square on the diameter is the ratio 11 to 14.[37]

Finally, it is interesting to note that after his account of Archimedes' results, Clavius also includes closer approximations to π later in Book IV, quoting results of his contemporary Ludolph van Ceulen (1540–1610), and his student, Jesuit colleague, and successor as professor of Mathematics at the *Collegio Romano*, Christoph Grienberger, S.J. (1561–1636). Clavius states the equivalent of the bounds

$$3\,\frac{14159265358979323846}{100000000000000000000} < \pi < 3\,\frac{14159265358979323847}{100000000000000000000}$$

and tries to give a practical "spin" on how these might be useful. If one of these estimates is used (for instance, the upper bound to parallel the 22/7 value), then

> ... the area of the circle will differ less from the true value than the area found from the Archimedean ratio. But since it is more difficult to compute with large numbers [i.e., numbers with more digits] than with small ones [i.e., numbers with fewer digits], the practice of craftsmen has persisted so that the Archimedean

[35] Posita igitur ce, 153. erit Ee, 306. Et si quadratum ipsius ce, 23409. dematur ex 93,636. quadrato ipsius Ee, reliquum fiet quadratum ipisus Ec, 70,227. cuius radix est paulo maior quam 265. ac proinde Ec, ad ce, maiorem habebit proportionem quam 265. ad 153. Secto iam angulo eEc, bifariam per rectam Ed, erit eE, ad Ec vt ed, ad dc. Et componendo, eE, Ec simul ad Ec, vt ec, ad dc. Et permutando, eE, Ec, simul ad ec, vt Ec, ad cd. Quia verò eE, Ec simul maiores sunt, quam 571. (quippe cum Ee sit 306. & Ec, paulo maior, quam 265.) & ec, posita est 153. habebunt eE, Ec, simul ad ec, maiorem proportionem quam 571. ad 153. ideoque & proportio Ec, ad cd, maior erit, quam 571. ad 153. ac proinde si cd, ponatur 153. erit Ec, paulo maior quam 571. Igitur quadratum ipsius Ec, paulo maius erit, quam 326,041. atque idcirco, cum quadratum ipsius cd, sit 23,409. erit quadratum ipsius Ed, quod quadratis rectarum Ec, cd, est aequale, paulo maius, quam 349,450. eiusque radix maior quam 591 $\frac{1}{8}$. quippe cum huius radicis quadratum sit tantum 349,428 $\frac{49}{64}$. Habebit igitur Ed, ad, dc, maiorem proportionem quam 591 $\frac{1}{8}$. ad 153.

[36] See the figure inserted for the first time at Clavius (1606, p. 186) and repeated several times thereafter for the convenience of the reader. For comparisons, look at figures on folios 87 and 88 of Fine (1532) and the facsimiles in Knorr (1989, pp. 460, 463). Setting up the figure this way could have been a purely practical decision to reduce the number of different figures that had to be produced in printing the book.

[37] Circulus quilibet ad quadratum diametri proportionem habet, quam ad [sic] 11. ad 14. proximè. (Clavius 1606, p. 191)

ratio is applied to the calculation. However, when more accurate values are desired, the Ludolphine ratio above should be used, especially for large circles.[38] (Clavius 1606, p. 199)

Clavius is clearly thinking in terms of an absolute error rather than a relative error (a more common way for us to judge how serious the error would be). It must be said, though, that the times when values this precise (20 decimal digits) would really be needed seem very rare.

It is interesting to compare Clavius's treatment of these results with those in other practical geometry texts. One of the earliest, the *Practica Geometriae* attributed to Hugh of St. Victor (ca. 1096–1141), (Homann 1991), omits this topic entirely. In his *De Practica Geometrie*, Fibonacci addresses the content of all three propositions in turn. See Pisano (1862, pp. 86–91) and (2008, pp. 152–158), paragraphs [191]–[200] of Chapter 3. However, he only mentions Archimedes when he begins the discussion of Proposition 3. As was true for the discussion of Heron's formula considered in the previous section, Luca Pacioli's discussion of the results of the *Measurement of the Circle* follows Fibonacci very closely, although he mentions Archimedes in connection with Proposition 1, rather than waiting for Proposition 3.[39]

Fibonacci does not attempt to present the full proof for Proposition 1 that is found in Clavius and in other versions. Instead, first considering a regular polygon circumscribed about the circle, Fibonacci argues that the product of the radius and the perimeter of the polygon is greater than the area of the circle by considering the triangles formed by joining the center and the vertices of the polygon. Hence, the product of the radius and a number greater than half the circumference of the circle gives an area greater than the circle. Fibonacci apparently takes it as obvious that the perimeter of the circumscribed polygon is greater than the circumference of the circle.[40] Next, Fibonacci considers an inscribed n-gon and adds vertices bisecting the arcs between successive vertices of the n-gon to form an inscribed regular $2n$-gon. Fibonacci notes that the product of the radius and half the perimeter of the n-gon is equal to the area of the $2n$-gon, hence less than the area of the circle. Therefore, multiplying the radius by a number less than half the circumference of the circle gives an area less than the area of the circle. "Whence it is concluded that the product of the radius of the circle and half its circumference equals its area."[41] Although it is certainly intuitively clear that the areas of the inscribed

[38] ... quae quidem area minus à vera distabit, quam illa, quae ex proportione Archimedis inuenitur. Sed quia difficilius est per magnos numeros calculum instituere, quam per minores, vsus artificum obtinuit, vt proportio Archimedis ad calculum adhibeatur. Quando tamen desideratur accuratior calculus, vtendum erit posteriori hac proportione Ludolphi, praesertim in maioribus circulis.

[39] About the proof of Proposition 3, Pacioli writes: "Ancora eglie da mostrare comme e fo trouata da Archimenide la linea circonferentiale essere .3. volte .$\frac{1}{7}$. del diametro: la quale inventione fo bella e sotile in questo modo, bene che con breuita se dica," (Pacioli 1494), Pars Secunda, Distinctio quarta, Capitulum secundum, folio 31. Compare with Fibonacci's introductory remarks given below. Clagett (1964, 1967–1984, Vol. 3, Part III) gives a closer comparison of Fibonacci's version and Pacioli's version.

[40] Clavius returns to this point in Book VIII of the *Practical Geometry*, discussing arguments by Archimedes from *On the Sphere and Cylinder*, and an alternate treatment by Girolamo Cardano.

[41] Quare concluditur, quod ex multiplicatione semidiametrij circulj in dimidium lineae circumferentis prouenit embadum ipsius. See Pisano (1862, p. 87).

and circumscribed n-gons converge to the area of the circle as n increases without bound, it seems fair to characterize what Fibonacci writes as more of a plausibility argument than a complete proof because he has not shown that the difference between a circumscribed polygon and an inscribed polygon can be made arbitrarily small.[42]

Second, Fibonacci argues by way of Euclid XII.2 that the ratio of the square of the diameter to the area is the same for all circles. He essentially then does a "proof by numerical example"[43] for Proposition 2, using the result of Proposition 3 and effectively taking $\pi = 22/7$. There is no indication that this ratio is only an approximation and that no actual circle has the square of the diameter exactly equal to 196 and area exactly 154.

Finally, Fibonacci turns to the content of Proposition 3. He writes that he is not going to follow Archimedes' proof exactly because smaller numbers will suffice to make the point.[44] Fibonacci expresses the lengths in this proof in particular units. Referring back to Fig. 7, he effectively starts with $Ee = 30$ rods and $Ec = 15$ rods. Hence by the Pythagorean theorem, $ec^2 = 30^2 - 15^2 = 675$. He then finds an approximate value of $\sqrt{675}$ as "very close to 26 rods less $2\,\frac{1}{3}$ inches, a rod being 108 inches" (Pisano 2008, p. 155). The approximation $25\,\frac{51}{52} \doteq \sqrt{675}$ is indeed accurate to four decimal places after rounding. However, as Clagett (1964, 1967–1984, Volume 3, part III, p. 432) points out, Fibonacci's approximation yields an overestimate for $Ec/ec = \cot(\pi/6) = \sqrt{3}$, rather than the underestimate needed to carry through Archimedes' proof. By using Archimedes' actual numerical values, Clavius is also demonstrating a surer understanding of the details of the Archimedean argument.

To conclude our discussion of this section, it is surely significant that Clavius characterized what he was doing here as "including the book" of Archimedes on the measurement of the circle and not just discussing ways to find areas of circles via the ratio of the circumference to the diameter. He was relying, of course, on the fact that Archimedes had earned tremendous prestige and name recognition. But the way Clavius framed his intention also meant that he had obligated himself to present full accounts of Archimedes' versions of the proofs involved. At the same time Clavius deployed all of his pedagogical skill to make Archimedes more accessible. So this section is one of the clearest examples of the mixture of the scholarly/theoretical and the practical in this work, as pointed out in the Introduction.

42 According to Clagett (1964, 1967–1984, Volume 3, Part III, p. 427), and referring to Pacioli's version of this argument: "As given, this is very loose indeed."

43 fuit enim quadratum dyametrij suprascripti 196. et embadum ipsius 154. quorum proportio est sicut 14. ad 11. ... See (Pisano 1862, p. 88).

44 Ostendendum est etiam quomodo inuentum fuit lineam circumferentem omnis circulij esse triplum et septima sui dyametrij ab Archimede philosopho, et fuit illa inuentio pulcra et subtilis ualde: quam etiam reiterabo non cum suis numeris, quibus ipse usus fuit demonstrare; cum possibile sit cum paruis numeris ea que ipse cum magnis ostendit plenissime demonstrare. See Pisano (1862, p. 88). Note that again there is no indication that 22/7 is only an approximation to the ratio in question.

4 Clavius's discussion of methods for finding two mean proportionals between two given lines in Book VI

In this section and the next, we will discuss two connected topics from Book VI of the *Geometria Practica*. The first is the extended solution of the following problem: *To find two mean proportionals between two given lines approximately*,[45] (Clavius 1606, pp. 266–272). This, the closely connected problem of duplicating the cube, plus the problems of squaring the circle and trisecting an arbitrary angle (which are discussed by Clavius in Books VII and VIII) were tremendous stimuli to Greek geometry over hundreds of years.[46] As Clavius writes at this point in Book VI,

> We will first report what the ancient geometers have left to us in their writings concerning this problem. For this drove and tormented the talents of many, although up to this day, no one will truly and geometrically have constructed two mean proportionals between two given lines.[47] (Clavius 1606, p. 266)

Clavius has included what might seem to be a surprising amount of the Greek work on this construction problem in the *Geometria Practica*. Although the works of the authors involved did not survive from antiquity in their original forms, they were summarized and hence preserved in the commentary on Archimedes' *On the Sphere and Cylinder* by Eutocius of Ascalon (Netz 2004, pp. 273–306). This is Clavius's stated primary source for this material although he may also have consulted Book III of the *Mathematical Collection* of Pappus where some of the same methods are surveyed.

45 Inter datas duas rectas, duas medias proportionales prope verum inuenire

46 For an extensive modern study of the surviving sources and the historical development, see Knorr (2012). Hippocrates of Chios (ca. 470–ca. 410 BCE) was traditionally credited with the reduction in the problem of duplicating the cube to the problem of constructing two mean proportionals between given lines. If AB and CD are the lines, two other lines XY and ZW are said to be two mean proportionals (in continued proportion) if

$$AB{:}XY{::}XY{:}ZW \quad \text{and} \quad XY{:}ZW{::}ZW{:}CD.$$

Representing the lengths by numbers and using algebra, this becomes the string of equations

$$\frac{AB}{XY} = \frac{XY}{ZW} = \frac{ZW}{CD},$$

from which it follows that

$$\left(\frac{ZW}{CD}\right)^3 = \frac{AB}{CD}.$$

So for instance if $CD = 1$ and $AB = 2$ in some units, a construction of the two mean proportionals gives the line ZW which has length $\sqrt[3]{2}$, and that is the edge length of the cube with twice the volume of the cube with edge length $CD = 1$.

47 Quocirca prius in hac propos. in medium afferemus, quae antiqui Geometrae nobis hac de re scripta relinquerunt. Multorum enim ingenia res haec exercuit, atque torsit, quamuis nemo ad hanc vsque diem, verè, ac Geometricè duas medias proportionales inter duas rectas datas inuenerit.

Note that Clavius explicitly writes "approximately" in the statement of the problem.[48] The solutions he will present all involve either limiting operations relying on the senses of the geometer (so-called *neusis* (νεῦσις) constructions) or the use of auxiliary curves such as *cissoid* of Diocles (ca. 240–ca. 180 BCE) or the *conchoid* of Nicomedes (ca. 280–ca. 210 BCE) that cannot be drawn as a whole using only the straightedge and compass. Because these solutions use more than the traditional Euclidean tools, they do not qualify as what Clavius means by "geometric" or exact solutions. It is understood today that no such purely "geometric" solutions are possible for the three problems mentioned above and it is primarily this *methodological question*—are the three problems solvable under the most severe restriction to the use of only the Euclidean tools?—that has come to dominate many modern discussions.

But for Clavius, as for at least some of the Greeks before him, although the methodological question might be interesting, it was also important to find *some reasonably accurate method* for constructing the two mean proportionals even if it meant using an approximate method rather than an exact, "geometrical" solution. Perhaps surprisingly, this is actually a quite *practical* problem that had applications in architecture, military science and some of the other areas Clavius mentions in his Preface. It gives a method for determining the linear dimensions of a solid figure similar to a given figure whose volume has a given ratio to the volume of the given figure. Just as a procedure for finding one mean proportional lets one rescale a plane figure in a given ratio, a solution for this problem lets one rescale solid figures in any given ratio, and Clavius points this out explicitly a number of pages later, after Proposition 17 in the same Book VI:

> This establishes the method by which a cube is not only to be duplicated (which the ancients were seeking), but also increased or decreased in any given ratio. It also gives the method by which bores of cannons are to be made larger or smaller according to a given ratio.[49] (Clavius 1606, p. 274)

In this connection, we also point out the first part of the heading of the first method, where Clavius writes: "Method of Heron in the introduction to Mechanics and the Making of Missile-throwing Machines," (Clavius 1606, p. 266), following the heading in Eutocius, (Netz 2004, p. 275).

We note that Fibonacci also discusses methods for finding two mean proportionals in his *De Practica Geometrie*, (Pisano 2008, Chapter 5, paragraphs 12–15). We will return to this point shortly and compare his approach with Clavius's approach.

In introducing his discussion, Clavius writes that he is making a very deliberate choice from among the many solutions presented in Eutocius's commentary:

> Although they are most elegant and acute, the solutions of Eratosthenes, Plato, Pappus of Alexandria, Sporus, Menaechmus by means of the hyperbola and parabola, then with the help of two parabolas, and Archytas of Tarentum will be omitted and we will explain only the four solutions from Heron and Apollonius

[48] In the Latin: *prope verum*, literally "near the truth".

[49] Constat ex his, qua ratione Cubus non solum duplicandus sit (quod veteres inquirebant) sed etiam augendus minuendusue in quacunque proportione: Item quo pacto pylae bombardarum maiores, aut minores fieri debeant secundum proportionem datam.

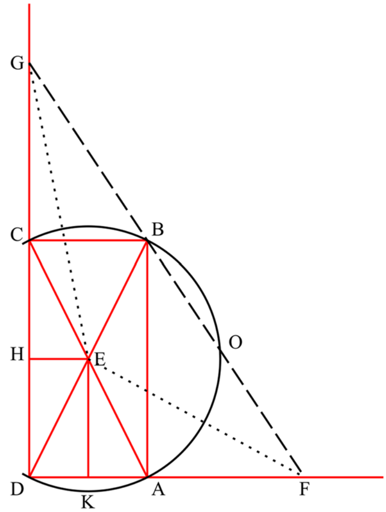

Fig. 8 Essential portions of Clavius's diagram for the first two methods. For Heron's method and Philo's method, the dashed line $GBOF$ would need to be rotated about B to reach the final desired position with $EG = EF$ or $OF = BG$

> of Perga, Philo of Byzantium and Philoponus, Diocles, and Nicomedes. *We have judged these to be more useful, easier, and less prone to error.* Anyone who should want the other methods will be able to read them in the commentary of Eutocius of Ascalon in the second book of *On the Sphere and Cylinder* of Archimedes, and in the book of Johannes Werner of Nuremburg[50] on the conic sections.[51] (Clavius 1606, p. 266); emphasis added.

In other words, the methods discussed here are sufficient for the applications Clavius has in mind and they are the ones he thinks are easiest and best suited for practical implementation.

By way of contrast, Fibonacci makes a different selection and presents only the methods ascribed to Archytas, Philo, and Plato by Eutocius. Hence, there is very little overlap between his account and Clavius's account of this topic. Moreover, he presents the method of Archytas (which relies on some quite involved solid geometry) first, after saying that finding the two mean proportionals "... is not a thing that can be done easily, but this is how it must be done."[52]

Turning now to the details of Clavius's account, the first method presented actually combines two very closely related approaches, ascribed to Heron and Apollonius and discussed separately by Eutocius.

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50 German mathematician, 1468–1522.

51 Praetermissis autem modis Eratosthenis; Platonis; Pappi Alexandrini; Spori; menechmi [sic] tum beneficio Hyperbolae, ac parabolae, tum ope duarum parabolarum; & Architae Tarentini, quamuis acutissimis, subtilissimisque; solum quatuor ab Herone, Apollonio Pergaeo, Philone Bysantio, Philoppono, Diocle, & Nicomede traditos explicabimus, quos commodiores, facilioresque, & errori minus obnoxios iudicauimus. Qui aliorum rationes desiderat, legere eas poterit in Commentarijs Eutocij Ascalonitae in librum 2. Archimedis de Sphaera, & Cylindro: Item in libello Ioannis Verneri Norimbergensis de sectionibus Conicis.

52 ... hoc facili operari non possit, tamen, qualiter hoc fieri debeat. (Pisano 1862, p. 153).

Clavius's version is a very close copy of Eutocius's text for Heron's method, with the variation represented by Apollonius's method inserted at one point. In Fig. 8, suppose we wish to find two mean proportionals between the lines AB and BC, which have been arranged as two sides of the rectangle $ABCD$. For Heron's method, Clavius writes:

> With sides DA, DC extended, it is understood that a straightedge [represented by the dashed line in the figure] placed at B should be moved until it meets DA, DC, produced, in points F and G such that the lines EF and EG are equal.[53] (Clavius 1606, p. 267)

When this is true, consideration of the various similar triangles in the figure shows that AF and CG are the two desired mean proportionals between AB and BC. Apollonius's variation of this method consists of finding a circle with center at E, cutting the extended line DC in a point G and the extended line DA in a point F such that the chord GF passes through B, and hence $EF = EG$ again.[54] Clavius includes a brief description of a trial-and-error method for finding the required circle not found in Eutocius.

The second method, ascribed to Philo and Philoponus, has been reworked and greatly simplified by Clavius based on the realization that it is again very closely related to the first one (in fact Clavius sets up the discussion so that the same diagram applies). Namely, with the circle $DABC$ described with center E and radius $EA = EB = EC = ED$, the ruler at B (that is, the dashed line in the figure) is moved until $BG = OF$, where O is the second intersection with the circle above. Then, it is easy to see we are back in exactly the same configuration as in the other methods, so the same reasoning applies to give the two mean proportionals.

This family of methods would certainly be among the easiest to apply and probably accurate as well. Note that the geometer is only required to rotate the line through B or adjust the radius of a circle centered at E until a certain condition is satisfied. As presented by Clavius, this involves approximation processes making use of the senses of the geometer, as we said earlier. The next two methods will be somewhat different in that they are set up to make use of *auxiliary curves* whose description (that is, the description of the whole curve and not just a finite set of points on the curve) would require tools besides the straightedge and compass.

The next method Clavius discusses is ascribed by Eutocius to Diocles, and specifically to a book called *On Burning Mirrors*. The Greek original has not survived so this was known from fragments preserved in other texts such as Eutocius's commentary. But an Arabic translation of the whole has survived and this has been translated into English by G. J. Toomer, (Diocles 1976). Clavius covers essentially the same ground as in the corresponding section from Eutocius's commentary. However, as usual, he has reworked and augmented his source material significantly. Clavius begins by isolating what he calls the "Lemma of Diocles," which identifies a geometric configuration containing two mean proportionals between two lines.

[53] Protractis autem lateribus, DA, DC, intelligatur circa punctum B, moueri regula hinc inde, donec ita secet DA, DC productas in F, & G, vt rectae emissae EF, EG, aequales sint.

[54] The required circle is not shown in Fig. 8.

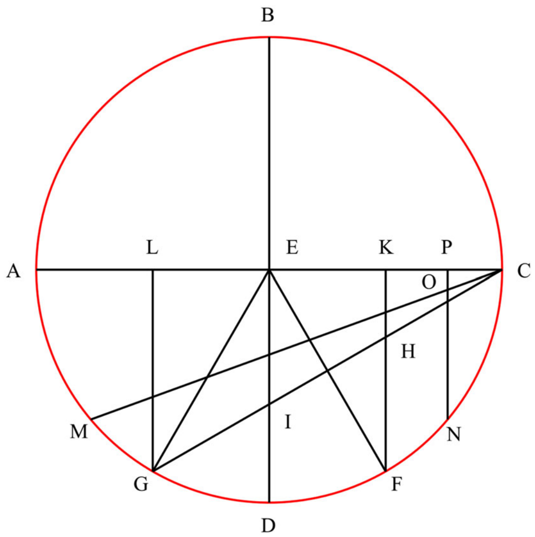


Fig. 9 Clavius's figure for the "Lemma of Diocles."

See Fig. 9. An equivalent figure with Greek letter labels appears in Eutocius. Let AC and BD be diameters of the circle meeting at right angles at E. Let arcs DG and DF be equal and join CG. Let GL and KF be drawn parallel to BD. Let CG meet FK at H. Then by considering relationships of the lines in the figure, Clavius essentially follows the proof given by Eutocius to show that FK and KC are two mean proportionals between AK and KH. Similarly, if the arcs DM and DN are equal, then drawing CM cutting the vertical line PN in O, it follows that NP and PC are two mean proportionals between AP and PO.

Now given two lines $AB > BC$, the "Lemma of Diocles" applies in the following way: First construct a circle with center B and radius AB. Lay off BC along the perpendicular diameter as in Fig. 10.

Provided that a point H and the vertical segment KH (parallel to EF) are found so that the intersection L of the extended line AC and KH makes the arcs EH and EM (formed by the line through D and L) equal, then the "Lemma of Diocles" will imply that KH and DK are two mean proportionals between AK and KL. But the triangles ABC and AKL are similar, and hence all four lines can be rescaled by the ratio $AB : AK$ to get two mean proportionals between AB and BC as desired.

Finding the required point H could be done by the same sort of approximate trial-and-error processes we saw in the previous methods. But Diocles and Clavius now actually take this idea one step farther. Namely, start by considering the circle with radius AB as before. If the locus of all points L as in this figure for all possible arcs EM is considered, the so-called *cissoid* of Diocles (a cuspidal cubic algebraic curve) is obtained.[55] Namely, for each possible M in the quadrant AE, consider the line DM and then take K so that the vertical line KH makes the arcs EH and EM equal. Take

[55] Clavius does not use this name, though. In the coordinate system suggested by placing the diameters along the coordinate axes and taking the circle to have radius 1, the equation of the cissoid is $(x^2+(y+1))^2(y+1) = 2x^2$.

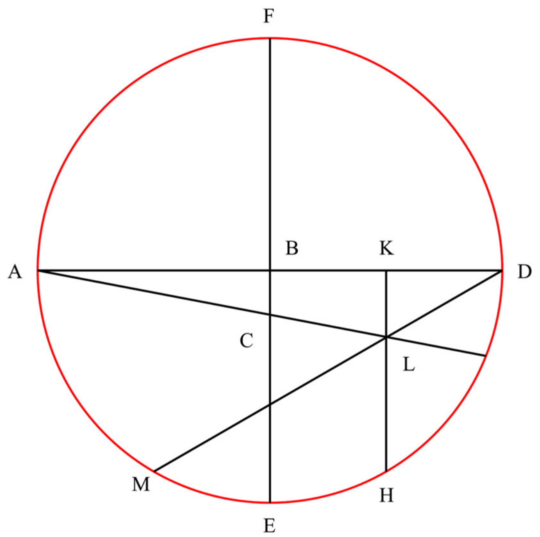


Fig. 10 Configuration for finding two mean proportionals between AB and BC

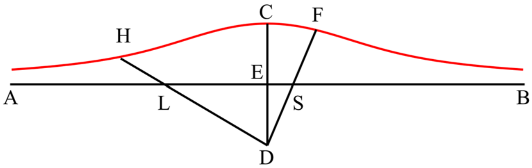
the point L corresponding to that choice of M as the intersection of the lines KH and MD.

Then for each point C on the radius BE, the line AC, when extended, will intersect the cissoid at some uniquely determined L and hence produce a line KH making the arcs EM and EH equal. Then, two mean proportionals between AB and AC will be found as above by rescaling KH and DK, which are mean proportionals between AK and KL. Thus, the cissoid in effect solves the problem for all possible pairs of the fixed AB and smaller segments BC *simultaneously*. As usual, compared to Eutocius or Diocles, Clavius provides a much more specific method for constructing as many points on the cissoid curve as are desired. But having only a finite collection of points on the curve, some approximation or judgment of the geometer would still be needed to connect the points into a continuous curve and find an appropriate point L for an arbitrary given line BC as above. Hence, the qualifier "approximately" (the *prope verum* in the Latin) still applies.

The final method for constructing two mean proportionals between given line segments addressed by Clavius is the one attributed to Nicomedes, using the *conchoid* curve. This discussion is probably the closest Clavius comes to simply reproducing what he finds in Eutocius, or more precisely, parts thereof. Clavius starts by writing that the conchoid can be drawn with a certain instrument which is described in the first section of Eutocius's version of this method. But since Clavius does not have the instrument, he writes that it will be enough to give a construction by which as many points on the conchoid as desired can be produced.[56] So let AB be a line and let CD be another perpendicular line meeting AB at a right angle at E. Taking D as

[56] Nicomedes construit prius instrumentum quoddam, quo lineam inflexam describit, quam Conchilem, vel Conchoideos appellat. Sed nos omisso eo instrumento, eandem, (quod ad nostrum institutum satis est) per puncta delineabimus, hac ratione (Clavius 1606, p. 270). Note the parallel with the discussion of the cissoid.

Fig. 11 Conchoid of Nicomedes



a *pole*, consider all straight lines passing through D. All lines except the parallel to AB through D will intersect AB (extended if necessary). Say the line DS meets AB at S. Then extending the line again in the direction of S, there will be another point F on the line with $SF = EC$. The locus of all such points F is the curve known as the conchoid.[57]

Next, following Eutocius, Clavius proves two "remarkable properties"[58] considered by Nicomedes. First, the farther the point S is from E, the smaller the vertical distance is from F to the line AB and second, the conchoid meets every line lying above AB, no matter how close.[59] Clavius then shows how the conchoid gives a solution of the following problem that Eutocius credits to Nicomedes:

> Given any rectilinear angle, and a point outside the lines making up the angle, to construct from this point, a line intersecting the lines containing the given angle, so that the portion of the line intercepted between the lines is equal to a given line.[60] (Clavius 1606, p. 272)

Finally, Eutocius and Clavius show how the solution of this problem lets one construct the triangle GDF in Fig. 9 for which AF and CG are the two mean proportionals between the sides AB and BC of the rectangle as in that figure. There are several additional constructions of lines made starting from the rectangle and the problem above is used to produce a line intersecting two other lines such that the line intercepted is equal to one half of AB. Here, Clavius adds a sort of mnemonic diagram, (Clavius 1606, p. 272), intended to help the reader visualize some of the proportionalities between sides of similar triangles in the rather complicated figure, which reproduces the fifth figure in Eutocius's account (Netz 2004, p. 305).

In this section, the choices Clavius makes in deciding which methods to include certainly do address his criteria of *usefulness*, *ease of application*, and *lower susceptibility to error*. Moreover, the methods of Diocles and Nicomedes are definitely more involved than the previous ones, so there is a very clear (and pedagogically desirable) progression from simpler methods to more complicated ones.

57 More precisely, if we introduce coordinates placing the x-axis along the line AB and E at the origin and take $CE = ED$, Nicomedes' conchoid is one of the connected components of the real algebraic quartic curve defined by $(x^2 + (y+1)^2)y^2 = (y+1)^2$. There is also a second connected component below the line AB with a cusp at the point $(0, -1)$, namely the point D. If some other ratio between the lengths CE and ED is specified, other conchoids with nodes at the point corresponding to D will be produced (Fig. 11).

58 duas proprietates huius lineae insignes, (Clavius 1606, p. 270).

59 In modern terms, the line AB is a *horizontal asymptote* of the conchoid.

60 Dato quouis angulo rectilineo, & puncto extra lineas angulum datum comprehendentes: Ab illo puncto educere rectam secantem rectas datum continentes angulum, ita vt eius portio inter illas rectas intercepta aequalis sit datae rectae.

5 Clavius's presentation of an algorithm for extracting roots in Book VI

While Clavius was a strong proponent of the geometrical methods found in Euclid's *Elements*, he understood very well that many geometrical constructions correspond to *algebraic or numerical* calculations. In numerical terms (and also not too anachronistic terms by Clavius's time), finding a mean proportional XY between two lines AB and CD is closely related to finding a *square root* and finding two mean proportionals is closely related to finding a *cube root*, (Clavius 1606, Book VI, Proposition 18). As a result many texts on practical geometry, including the texts of Fibonacci and Pacioli mentioned earlier, included extensive discussions of numerical algorithms for computing square and cube roots (at least). Clavius's book is no exception. He points out this connection in Proposition 18 immediately following the material discussed in Sect. 4 above and he devotes the final section of his Book VI to this topic, starting with Proposition 19.

As we mentioned in the Introduction, this is another case where Clavius does not acknowledge a source explicitly. Indeed, he is notably coy about this.[61] However, it seems very likely that Clavius's treatment of this algorithm is drawn from an algorithm presented by Michael Stifel (1487–1567) in the *Arithmetica Integra*, (Stifel 1544, folios 39–46), and later summarized in the revised and much expanded edition of the first textbook of algebra in the German language, *Die Coss*, (Rudolff and Stifel 1553). This also went through several later editions. There, the discussion of root extractions appears in the *Anhang* (Appendix) to Chapter 4 of Part I written by Stifel, found starting on folio 46 and going to folio 59.[62]

As usual, and in its entirety, Clavius's account is not directly copied from either of Stifel's versions: Clavius's explanations are rewritten and expanded. He also presents different numerical examples. Our conjecture that Clavius was consulting this source is based on the fact that the overall outline of the method Clavius presents is essentially exactly the same as what Stifel presents:

1. Very similar terminology for the different species of roots is used, e.g., "zensizenic" roots are fourth roots, "surdesolidic" roots are fifth roots, and so forth. Variations of this terminology are found in many sixteenth century works dealing with algebra, though, so this is only a start.
2. The digits from the number whose root is being found are arranged into groups called "points" in both Clavius and Stifel. This is done in essentially exactly the same way in both accounts by marking certain digits with dots; each "point" will yield one digit of the root. Clavius writes the dots below the corresponding digits, while Stifel writes them above.
3. Tables of nth powers of the digits $1, 2, 3, \ldots, 9$ are provided so that the largest nth power that can be subtracted from the first (leftmost) "point" can be identified.

[61] The relevant passage from the original has been quoted in the Introduction.

[62] From its title, it might be expected that (Stifel 1545) is another candidate for Clavius's source. But this is not the case. Even though this last book is written in German, unlike (Stifel 1544), its discussion of extracting roots (see folios 45 and 46) deals with abacus or counting board calculations, not the hand calculation algorithm that Clavius discusses.

4. The essential role of collection of "special numbers" for each species of root to be used in preparing the "divisor" at each step of the algorithm is the same in both. In our terms, these "special numbers" are binomial coefficients times powers of 10, since the algorithm works with numbers of the form $(10d_k + d_{k+1})^n$, where d_k and d_{k+1} are successive digits of the root. For example,

$$(10d_k + d_{k+1})^3 = 1000d_k^3 + 300d_k^2 d_{k+1} + 30d_k d_{k+1}^2 + d_{k+1}^3,$$

so the coefficients 300 and 30 are the "special numbers" used to compute cube roots. Clavius provides a table containing the binomial coefficients up to $n = 17$ that coincides verbatim (except for a typographical error in one entry) with the table from Stifel (1544, folio 45).[63]
5. The calculations are laid out in a very similar tabular format.

The method used here has close connections with root extraction algorithms presented by many medieval and Renaissance authors, as discussed for instance in Johansson (2011). The novelty is the very convenient tabular format for the computation of each successive digit of the root that will be illustrated in the following examples.

Probably, the best way to convince the reader of this identification of Clavius's source is to quote from two extractions of cube roots, one from Clavius and one from Stifel's *Anhang*. The process described finds the root decimal digit by decimal digit. The steps all follow the same pattern after the determination of the left-most digit. So the point will be made if we look at the determination of the first two digits of the root in the examples. We begin with the first two steps of this example from Clavius:[64]

63 Clavius's table is also pointed out in Knobloch (1988, p. 351), but Knobloch attributes this to the influence of Cardano and Tartaglia on Clavius. The identical and somewhat unusual formatting of the tables in both Clavius and Stifel is very suggestive.

64 See Clavius (1606, p. 280). Sit ex numero

$$\begin{array}{ccccccccc} 2 & 3 & 9 & 4 & 8 & 3 & 1 & 9 & 0 \\ & & \bullet & & & \bullet & & & \bullet \end{array}$$

extrahenda radix cubica.
Primvm ex puncto 239. subtraho cubum 216. qui est maximus in eo

$$\begin{array}{rcr} 36 & -- & 300 \\ 6 & -- & 30 \end{array}$$

contentus, cuius radicem 6. scribo in Quotiente ad marginem. Et quia relinquitur numerus . 23. erit sequens punctum 23483. Deinde paro diuisorem hoc modo. Supra radicem inuentam 6. pono eius quadratum 36. Et ad dextram colloco duos numeros peculiares radicis cubicae, nimirum 300. & 30. vt hic vides. Multiplico superiores duos numeros 36. & 300. inter se, & producto 10800. addo productum 180. ex multiplicatione numerorum inferiorum 6. & 30. inter se. Nam summa 10980. erit Diuisor. Satis etiam esset productus ex duobus superioribus inter se multiplicatis, nimirum 10800. pro Diuisore. quod in alijs extractionibus intelligendum quoque est. Diuido ergo punctum meum 23483. per diuisorem inuentum 10980. & Quotientum 2. scribo post figuram 6. prius inuentam.
Pingo post haec figuram huiusmodi. Ad dextram numerorum 36. & 300.

$$\begin{array}{rcrcr} 36 & -- & 300 & -- & 2. \\ 6 & -- & 30 & -- & 4. \\ & & & & 8. \end{array}$$

Let it be required to extract the cube root of

$$\begin{array}{c} 2\,3\,9\,4\,8\,3\,1\,9\,0 \\ \quad\bullet\quad\bullet\quad\bullet \end{array}$$

First, from the point 239, I subtract the 216 which is the largest cube contained in it. I write its cube root 6 in the margin in the quotient. And since 23 is left over, the next point will be 23483.

$$\begin{array}{rcr} 36 & -- & 300 \\ 6 & -- & 30 \end{array}$$

Next I provide a divisor in this way. Over [the digit] 6 of the root found above, I put its square, 36. And on the right, I place the two particular numbers for cube roots, namely 300 and 30. I multiply the numbers on the first row, yielding a product of 108,00 and I add the product from multiplying the two numbers on the second row, 180. The sum 10,980 will be the divisor. (It would be enough to take the product of the two numbers on the first row as the divisor, namely 108,00, as must be understood in other root extractions.) I divide the point 23,483 by 10,980 and write the quotient 2 next to the digit 6 found first. I treat what comes after this digit as follows. At the right

$$\begin{array}{rcrcl} 36 & -- & 300 & -- & 2. \\ 6 & -- & 30 & -- & 4. \\ & & & & 8. \end{array}$$

of the numbers 36 and 300, I add this digit 2 [found in the quotient] and below it, its square, 4, and its cube, 8. Now, the three numbers on each of the first two rows are multiplied, and the products are 21,600 and 720. Adding the cube 8 makes 22,328. I subtract this from the point, leaving 1155, and the next point will be 1,155,190. (Clavius 1606, pp. 280–281)

Clavius continues to find the (approximate) cube root 621 for 239,483,190. Note that $621^3 = 239483061$, so this value is 129 "short." Later in this section, Clavius also shows how to compute additional decimal digits in the fractional part, obtaining closer approximate cube roots.

We now translate a step of the computation from folios 47-49 in Stifel's *Anhang* to Chapter 4 in Part I of Rudolff and Stifel (1553)[65]

colloco inuentam figuram 2. & infra eam eius quadratum 4. & sub hoc cubum eiusdem 8. Nam si tam superiores tres numeri 36. 300. & 2. quam inferiores tres 6. 30. & 4. inter se multiplicentur, & productis 21600. & 720. addatur cubus 8. fiet numerus 22328. quem si ex meo puncto 23483. subtraham, remanent 1155. atque adeo puncto sequens erit 1155190.

[65] We have chosen to translate this passage from the German of Rudolff and Stifel (1553) rather than one of the parallel computations in Stifel (1544), which are written in Latin. Clavius insists on the *German* provenance of the method, so he may well have learned it first from here, although from various features of his later *Algebra* text (1608), it is clear that he was also familiar with Stifel (1544).

Example.

$$\begin{matrix} \bullet & & \bullet & & \bullet & & \bullet & & & \\ 8\,0 & 6 & 2\,1 & 5 & 6\,8 & 0 & 0\,0 \end{matrix}$$

First I subtract the largest cube that I can from the leftmost point (that is, from 80). That is 64, leaving 16, which then belongs to the next point, which is composed of the digits 16,621. So now I set the cube root of 64, or 4, in the quotient, and the first point is decided. So then I take the next point, namely 16,621. I divide that by 4800 (that comes from 300 times 16) and the division gives only 3 in the quotient. And so the new digit is found. I put this next to the two numbers 300 and 30 with the accompanying numbers in this way:

16 –– 300 –– 3
4 –– 30 –– 9

Since the digit 4 was found [first] in the quotient, that is placed next to the 30 on the left, and above, next to 300 goes its square, namely 16. On the right next to the 300 goes the next digit 3, and its square 9 goes below next to the 30, as you clearly see.
So now I multiply and say 16 times 300 times 3 makes 14,400, and 4 times 30 times 9 makes 1080. I add those and obtain 15,480. I subtract that from the 16,621 as from the other points. The number 1141 remains. Last, I take the cube of the newly found digit 3. Namely, 3 times 3 times 3 makes 27. I also subtract this and 1114 remains. This belongs to the following point.

Exemplum.

$$\begin{matrix} \bullet & & \bullet & & \bullet & & \bullet & & & \\ 8\,0 & 6 & 2\,1 & 5 & 6\,8 & 0 & 0\,0 \end{matrix}$$

Erstlich subtrahir ich von dem hindersten puncten (das ist von 80) die aller gröste cubic zal/ die ich subtrahiren kan. Die selbig ist 64. so bleybett nach vbrig davon 16 die gehören denn sum nehisten puncten hernach/ der selbig uverkompt denn dise figuren 16621. So setz nu die cubic würzel von 64 in den quotient. facit 4. und is also der erst punct aufsgericht.
So nehme ich nu fur mich den andern punct/ nemlich 16221. Den dividir ich mit 4800. (das kompt von 300 mal 16) Nu gibt das gedacht dividiren nur 3 in den quotient. Und ist also die newe figur gefunden.
Dem selbigen nach stehn die zwo zalen 300 und 30. mit jren zugethonen zalen also.

16 –– 300 –– 3
4 –– 30 –– 9

Denn erstlich ist gefunden in den quotient de figur 4. die steht neben 30 zur lincken hand/ vnd drob neben 300 steht jr quadrat/ nemlich 16.
So is nu darnach gfunden in den quotient die figur 3. Die steht oben neben 300 zur rechten hand/ vnd darunder steht jr quadrat 9. neben 30. wie du alles vol sihest.
So multiplicir ich nu/ vnd sprich. 16 mal 300 mal 3. facit. 14,400. vnd 4 mal 30 mal 9. facit 1080. Das addir ich/ so kompt 15,480. Das subtrahir ich von 16,621. Als vom andern puncten diser operation/ so bleyben denn 1141.
Auffs letzt multiplicir ich die newe gefundne figur Cubice. Nemlich 3 mal 3 mal 3. facit 27. die subtrahir ich auch/so bleyben 1114. die gehören zu volgenden punct.

Since there were four “points” in the original number, Stifel’s cube root will contain four decimal digits. After two more steps of the process, he finds the value 4320, an exact cube root of 80,621,568,000.

Stifel’s significant influence on Clavius’s thinking about arithmetic and algebraic topics has been noted, for instance in Rommevaux (2012). If Clavius was following Stifel’s presentation of a root extraction algorithm here (and I hope I have proved the point with the quotations above), then there remains the question why Clavius did not make an explicit attribution.

It is certainly possible that Clavius thought he did not need to say any more to identify his source because Rudolff’s *Die Coss*, at least, was extremely well-known in German-speaking areas. Standards for citations in scholarly works were also considerably looser at this time. Of course, it is also true that Clavius’s citation methods were controversial for at least one of his contemporaries, namely Viète (Knobloch 1988, pp. 334–335). One superficially plausible explanation, namely the fact that Stifel had started out as an Augustinian monk, but later became a Protestant minister and an outspoken supporter of Martin Luther, was evidently *not* a factor here.[66] As pointed out in Knobloch (1988, p. 351), in other contexts Clavius seems to have had no qualms about mentioning Stifel explicitly, even praising his works highly and proposing one of Stifel’s books as a candidate textbook for teaching algebra in the Jesuit schools.[67] From Knobloch (1990), we find that Clavius also mentioned Stifel by name once in his Euclid and twice in the *Algebra*.

6 Conclusions

Clavius presented a tremendous amount of interesting and useful mathematics in his *Geometria Practica* and is certainly arguable that he achieved his stated goal of presenting the whole range of practical geometry as understood in his time in a form that would be useful for his readers. In assembling the material for this book, he drew on an extremely broad range of ancient, medieval, and contemporary sources. At the same time, his typical procedure was to rework, augment, and clarify the mathematical texts he dealt with. In many cases, he was quite careful about informing the reader of how he was using his sources, as we have seen in Sects. 3 and 4. But he was also not entirely consistent about that as we given in Sects. 2 and 5. For any more definite conclusions about Clavius’s choices regarding citations of his sources for Heron’s

[66] It is true, though, that Stifel’s works appeared several times in the *Indices* of works to be forbidden or expurgated prepared by Catholic authorities in various locations in the mid-to-late 16th and early 17th centuries [including Venice in 1554, Antwerp in 1571, (Gijón 2015, p. 80), and Spain in 1632 (and possibly earlier)]. What this entailed in the case of the 1632 *Expurgatorio* of the Spanish Inquisition is visible in the digitized copy of Stifel (1544) from the library of the Universidad Complutense de Madrid at https://babel.hathitrust.org/cgi/pt?id=ucm.5323774127. This book shows extensive hand-written emendations and striking-out of honorifics, etc. In particular, the hand-written notation “autore damnato, opus permissum” appears next to Stifel’s name on the title page. While Stifel himself had been anathematized, it was still permissible to read the mathematical contents of the work.

[67] In the *Ordo Servandus in Addiscendis Disciplinis Mathematicis*, essentially a position paper on the role of mathematics instruction prepared during the development of the *Ratio Studiorum*, see Gatto (2006, p. 252).

Theorem and the *n*th root algorithm, a more specific understanding of when and why he did choose to make explicit citations in his writings would be necessary. But that lies beyond the scope of what we have tried to do here.

The quality of this work was recognized very soon after it appeared, as evidenced (for instance) by the fact that mathematicians such as Mersenne Knobloch (1988, p. 356) and Kepler[68] mentioned sections of this book in their writings. Recognition of Clavius's work was also evident in other ways. In the Jesuit mission in China, one of Clavius's former students in the *Collegio Romano*, Mateo Ricci, S.J. (1552–1610), together with his Chinese collaborator Xu Guangqi (1562–1633), made translations of not only the first six books of Euclid's *Elements* from Clavius's version, but also material from the *Geometria Practica*. Later, Giacomo Rho, S.J. (1593–1638) made Chinese translations of additional sections of this work (Martzloff 1995, pp. 318–319).

But Clavius's devotion to the synthetic Euclidean tradition in geometry would shortly come to seem very old-fashioned. The recovery of Pappus's treatment of the Greek tradition of geometric *analysis* in Book VII of the *Mathematical Collection*, combined with the ever-growing influence of algebraic thinking was the impetus for an explosion of work starting in the late sixteenth century and continuing into the first half of the seventeenth century, (Bos 2001). But this was largely orthogonal to the ways that Clavius approached geometry and he seemingly had little interest in or taste for the treatment of geometric analysis in Pappus's writings (Sasaki 2003, Chapter 2, Section 3). Within 30 years of his death, the introduction and systematic use of analytic, or coordinate geometry by Descartes and others was well under way. That new way of harnessing the power of algebra to discover new geometrical results and prove them was fundamentally changing the practice of mathematics.

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[68] In Kepler's *Harmonice Mundi*. The reference was to Clavius's discussion of various approximate constructions of regular heptagons in Book VIII.

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