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Peirce's Dragon-Head Logic (R 501, 1901)

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Abstract
Peirce wrote in late 1901 a text on formal logic using a special Dragon-Head and Dragon-Tail notation in order to express the relation of logical consequence and its properties. These texts have not been referred to in the literature before. We provide a complete reconstruction and transcription of these previously unpublished sets of manuscript sheets and analyse their main content. In the reconstructed text, Peirce is seen to outline both a general theory of deduction and a general theory of consequence relation. The two are the cornerstones of modern logic and have played a crucial role in its development. From the wider perspective, Peirce is led to these theories by three important generalizations: propositions to all signs, truth to scriptibility, and derivation to transformability. We provide an exposition of such proposed semiotic foundation for logical constants and point out a couple of further innovations in this rare text, including the sheet of assertion, correction as a dual of deduction and the nature of conditionals as variably strict conditionals.

1 Introduction

In this introductory section, we outline the content of a manuscript on logic—undated and unpublished—written by Charles Sanders Peirce (1839–1914) most likely in late 1901. We provide an editorial survey of the text and assess the place of the text in Peirce's logical and semiotic studies and highlight its main contributions. Albeit

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these pages have remained completely unknown in the literature, they present some of the main elements in the development of modern logical notions and theories, especially those of deduction and logical consequence. Peirce's text is reconstructed and transcribed in full in two appendices.

1.1 The Dragon Logic texts of 1901

Sometime in late 1901, and possibly continuing still in early 1902, American scientist, philosopher and logician Charles S. Peirce composed an interesting unpublished and undated text on formal logic that bears on the relation of logical consequence. The text, which Peirce did not date at all, consists of several autograph sequences and manuscript leaves scattered over three folders in the Charles S. Peirce Papers deposited in Harvard University's Houghton Library. In Robin numbers, those texts are R 9, R 11 and R 501.[1] Folder R 501 contains the bulk of the relevant material. Those sequences are arranged and the text reconstructed and transcribed here for the first time. The transcription is complete, and it includes both the later variant arranged and transcribed in "Appendix A" and its earlier draft provided in "Appendix B". There is some overlap between the two versions, but the many differences and divergent explorations justify the presentation of both in full and as two discrete versions of the text.

In both variants, Peirce advances a novel notation to express conditionals, which he terms the Dragon Head and the Dragon Tail (☊ and ☋, respectively). He uses the sign ☊ of the Dragon-Head type turned left on its side as the base and adds a circumflex to it to express the conditional: $\widehat{☊}$. This notation has not been found to appear in any other writing of his preserved in the archives. Far from being an incidental or casual exploration in some *ad hoc* notation which Peirce might have soon abandoned, the text reveals that Peirce is exercising a sustained effort to develop what turns out to be a strikingly modern approach by which to prove a number of theorems and corollaries concerning propositional logic, including logical consequence, conditionals and the meaning of logical constants. In doing so, Peirce is seen to make a noteworthy contribution to the philosophy of logic and its notation.

Given that Peirce presented these theories in such unusual Dragon-Head and Dragon-Tail notation of his own devising may have prevented earlier scholars who in the past might have stumbled across these manuscripts from paying sufficient account to their centrality in the development of modern theories of logic. Indeed, these texts have passed virtually unacknowledged in the previous historiography concerning contemporary logical theory. Equally notably, these writings have not been acknowledged in the Peirce scholarship, either.

Next, the present paper provides an analytic introduction to Peirce's Dragon Logic and sketches an account of its developmental and textual contexts. We also put Peirce's paper in the wider perspective, beginning with a summary of its contributions and some editorial head-notes on the transcriptions of the text as they occur in two appendices. We also justify the dating of the text to late 1901.

The introduction is followed by Sect. 2, a précis of Peirce's investigations in his logic in the years surrounding the turn of the century. We outline some main aspects of

[1] The references are to the Papers of Charles S. Peirce, as catalogued by Richard Robin (Peirce 1967).

his philosophy of logic, the logic of existential graphs, the game-theoretical semantics, and algebra of logic, as appears in writings roughly coeval to Peirce's 1901 composition of the Dragon Logic study.

Section 3 is an in-depth survey of Peirce's logical and semiotic notions as occurs in the text itself. These include signs and the status of their classification around 1901, the concept of the sheet of assertions as appropriated from the context of the logic of existential graphs invented five years earlier, and the notions of scriptibility and transformability as Peirce's proposed generalizations to ordinary semantic (truth-value based) and proof-theoretic (derivability) notions.

In our analytic representation of this text which is presented in Sect. 4, we translate Peirce's results in the notation of sequent calculus. Thus calling Peirce's innovation as "strikingly modern" receives its justification from Peirce's own formulation of the proofs as they appear in the paper in this proto-sequent calculus of his. According to our interpretation of the text, the central ideas of the sequent calculus—namely expressing logical proofs and properties in terms of illations in sequences of consequences—align well with Peirce's own intentions and goals as he was exploring the formulation of this logic, its notation and the proofs of its properties. It is this result that we call Peirce's Dragon Logic, DL for short.[2]

Appendices A and B provide a complete transcription of the texts.

1.2 The background of Peirce's contributions in Dragon Logic (DL)

Our analysis of the system of logic and the proofs Peirce presents in R 501 reveals that DL encapsulated both a *general theory of deduction* and a *general theory of consequence relation*. These theories are general in the sense that deductions are not limited to logical deductions but are extended to cover all statements of consequences. Indeed, what Peirce is in the process of working out is in the larger context of his *deductive logic* not only an account of *necessary inferences* but an account of the meaning of deduction in its full sense of characterizing the relations of consequence.

Peirce is led to his theories of deduction and consequence with the aid of a couple of important conceptual generalizations, which also occur in his other writings: first by generalizing *propositions* to all kinds of *signs*, and second, by generalizing *truth-values* to *scriptibility* and *derivation* to *transformability*.

The first generalization is the semiotic one. In these months and years, Peirce's attempts to break free from the confines of one particular mode of expression (such as the constraints imposed by linguistic and other linear notations upon meanings) are still in flux. In 1901, he investigates the possibility of having linear notations in place of the graphical, two-dimensional ones, to express logical notions (Bellucci et al. 2020). Moreover, while Peirce talks about propositions as the second category of symbolic signs (see below for a brief introduction to his 1901 classification of signs), it is only two years later, in the *Syllabus* of his 1903 Lowell Lectures (Peirce 2019–2021) that Peirce carries out a generalization of propositions to *dicisigns* (a sort of hybrid sign

[2] The allegiance between the two, Peirce's and Gentzen's (Gentzen 1934) theory of sequences, is argued in more length in Ma and Pietarinen (2017a, b, c, 2018a). Also Dipert (2004) is an exploration of Peirce's logic in relation to natural deduction and sequent calculus.

not confined to symbols, see Short 2007; Stjernfelt 2007; Bellucci 2017). Another two to three years later still this generalization proceeds from propositions to *phemes*, which Peirce intended to comprise the second category of signs of any kind that can express propositional content as well as illocutionary force. Ultimately, phemes were to encompass many similar features with what much later became the theory of speech acts (Peirce 2019–2021).

The second generalization, namely that of the values of truth and falsity of propositions to all values that the process of 'scribing' them could possible reveal, is another of Peirce's later attempts to broaden the scope of logic and to view it as coextensive with semiotics. This was to be achieved by showing logic's connections to neighbouring sciences that are likewise normative as logic is but precede logic in the scheme of classification of the sciences (see e.g. Kent 1987; Pietarinen 2006b, 2012, on Peirce's classification of the sciences). Those are the normative sciences of *esthetics* (Peirce's preferred spelling) and *ethics*. The values that the generalized propositions (as dicisigns or phemes) could take ought to include not only those of being true or false (and not even those of having gaps, that is, the value of *neither* as Peirce argues to happen in case of indefinite signs, or the gluts, that is those the value of which can be *both* true *and* false, as Peirce argues to happen in case of general signs), but also the values of *goodness* and *ugliness* or *right* and *wrong*. Peirce's prose describes such values in general as ideals that the utterer and the interpreter of generalized propositions have in their minds when making assertions in "conformity to the purpose" of those assertions. His idea of scriptibility, which in some sense adds to truth the value of 'goodness', and its dual of unscriptibility, which in some sense adds to the value of falsity the value of 'being bad', aim at preserving some vital nuances that otherwise may be lost. For example, his project of extending the scope of logic to encompass all of semiotics may not be possible if semantic values were phrased only in terms of truths and falsities.

The third generalization, namely that of derivations to transformations, is likewise anticipatory to these later aims of being better equipped to explore the full semiotic content of logic and reasoning. Here it is the subtle qualities of *continuity* of transformations of logical expressions and notations (in whatever form) of one's assertions (in whatever media) become of utmost importance. One wishes to embed continuity in the *illative* structure of proofs in order to capture the idea of proofs better than derivations do.

In addition to continuity, transformations of assertions from one logical form to another would better capture the *creative* nature of deductive reasoning, which Peirce around the same time had started to explore in terms of the distinction between "theorematic" and "corollarial" reasoning (see ms page 11, fn9, "Appendix A"). For it is the translations of diagrammatic forms in particular that might better show what, where and when insertions and erasures upon those forms take place in the course of the proof (Pietarinen and Stjernfelt 2021). In thus enriching the meaning of deducibility and its discrete derivations as a stepwise protocol to be followed in strict conformity to the rules that define the system of deduction, transformations and their logical analysis and representation in diagrammatic terms is emerging as a central issue of logic in Peirce's post-1900 writings.

In sum, it is especially the notions of scriptibility and transformability that play a fundamental logical role in the present context of DL, as it is in that context that Peirce now attempts to understand deduction and consequence afresh, from the point of view of what the Dragon-Head notation is intended to signify.

1.3 A summary of contributions in Dragon Logic

In the texts from late 1901 gathered in Appendices A and B, Peirce outlines both a general theory of deduction and a general theory of the consequence relation, the two cornerstones in the development of modern logic. As he now chooses to present them in a rather unusual notation ( ) that starts with the designs of the Dragon Head and Dragon Tail, it may have made earlier efforts to study these pages rather unappealing. Indeed, the texts have eluded any previous study by the historians of logic.

As noted, Peirce is led to the definitions of deduction and logical consequence relation with the help of a couple of important generalizations. These generalizations have, in turn, a strong semiotic motivation. First, Peirce is in these years embarking on a project that aims at expanding the meaning of propositions to take into account not only the meaning of symbols but also those of indices and icons. Second, he attempts to generalize values of truth and falsity to scriptibility and unscriptibility. Third, he is proposing to supplant deducibility of derivations with transformability.

The notions of 'scribing' and 'transforming' have been in full swing in Peirce's other writings, especially those that since late 1896 had their main concern in the development of the theory of logical graphs, though the range of the applicability of these notions extends to his general algebra of logic too. In the present text, the notions of scriptibility and transformability justify their fundamental role in understanding deduction and consequence relation in terms of the Dragon-Head signs.

Since the present texts have not been referred to in the literature before, we provide a complete reconstruction and transcription of these previously unpublished writings in the two appendices and analyse their main content in the introductory sections. "Appendix A" is the later and "Appendix B" a slightly earlier version of the text. Along with the analysis of the notions of deduction and consequence, we in this introduction point out a couple of further innovations that Peirce is making in these texts. In particular, he (i) uses and explores further the ideas of the *sheet of assertion*; (ii) presents *correction* as a dual of deduction; and (iii) seeks to establish *conditionals* ( ) as non-material, variably strict conditionals. Especially the second proposal, correction as the dual of deduction, is unique to the text and is only found in the earlier version ("Appendix B").

1.4 Appendix A: The mature version

Our analysis focusses on describing and explaining the later and the more mature version of R 501 and its variants. The diplomatic transcription of it is provided in "Appendix A". It reconstructs the complete sequence of the main body of that writing from R 501, ranging from the manuscript page $6\frac{1}{2}$ to the manuscript page 33. (All page numbers are according to Peirce's own pagination.) The preceding ms pages 1–9 that are included in the transcription are the ones that are most closely related to the main sequence, taken from R 9 together with some additional and closely related leaves from R 501.

These first nine manuscript pages bear the title of R 9, *On the Foundations of Mathematics*. That text is related to a number of adjacent attempts by Peirce (R 7–11) to write a paper on the topic of the foundations of mathematics. Entitling the entire Dragon Logic paper as *On the Foundations of Mathematics* might seem rather unfitting, however, given the concern of its main segments in logical theories—the nature of deduction and the relation of consequence in particular—and much less in what Peirce had called the "Simplest Mathematics", a logic of mathematics that he was sketching out around the same time in his other writings. Peirce (2010) contains selections of texts related to Peirce's *On the Foundations of Mathematics* project, such as "Sketch of Dichotomic Mathematics" (R 4, 1901) and "The Simplest Mathematics" (R 429, 1902); an extensive selection of Peirce's ambitious such project of *The New Elements of Mathematics* of 1901–1902 is included in NEM 4 (Peirce 1976); a topic of "certain extremely simple branches of mathematics which, owing to their utility in logic, have to be treated in considerable detail, although to the mathematician they are hardly worth consideration" (CP 4.227, Peirce 1931–1966). "Kaina Stoicheia" (R 517, Peirce 1901), which we will relate to the present context both substantially and chronologically below, is part of that larger projects of the "New Elements" which Peirce wanted to appear as a preface to the larger treatise on the foundations of mathematics.

Peirce's logical explorations in the context of DL are thus not without relevance to the foundational issues of mathematics. He does have the similar concern as he in many other writings had, namely to develop logic as a theory of the logical analysis of mathematical reasoning (Moore 2010; Peirce 2010), and he does refer to this job description of logic on pages 20–21 of the manuscript ("Appendix A"). Moreover, on manuscript page 19 of the same segment, he appears to be claiming that the notion of scriptibility is a prerequisite for the possibility of mathematical reasoning, and that consequently, whatever is unscriptible would not be part of the deductive framework of mathematical reasoning. Such issues do attest Peirce's text as bearing on the foundations of mathematics.[3]

The main sequence of the definite version contains a good number of definitions, axioms, postulates, clauses and scholia, and it goes on to derive eight theorems and thirteen corollaries from them. The results that Peirce claims in the text are explained in Sect. 3 of the present paper in terms of rewriting them in the modern notation of sequent calculus.

Several alternative and draft segments of the pages that appear in the body of the text are given as footnotes. The abundance of such variants and the somewhat incomplete status of this later version of the text suggest that Peirce may have wished to produce (or that he indeed did compose but those pages were subsequently lost) still some further versions of this paper. The pagination of the surviving fragments and the many lacunae that remain indicate that at least some pages, segments and alternative variants have been lost.

3 We thank the reviewer for raising this point on the relevance of Dragon Logic to the foundations of mathematics.

Apparently Peirce did not prepare any of the surviving manuscripts with a publication in mind, and he does not refer to these texts in his other surviving writings. There is no evidence of this version of the text being a final draft or a fair copy intended for a submission somewhere, as those manuscripts would usually contain instructions to the typesetter or to the printer. As far as the earlier version of the text, transcribed in "Appendix B", is concerned, there is only one indication that Peirce might have wanted to have the text typeset or even published, on manuscript page fn8: "Get a type that looks as near a Bull's head as may be".

1.5 Appendix B: The early version

"Appendix B" provides the transcription of a slightly earlier, discrete version of the text. Especially the main sequence as is found in R 11 and in R 501 antedates the pages $6\frac{1}{2}$ to 33 of R 9 of the more mature version.

This second reconstruction has an independent value of its own, however, as it provides a couple of notions, ideas and discussions that did not make their way to the later version. These include the type of reasoning which in a sense is a dual or mirror form of inference to that of deduction, and which Peirce calls "correction" (ms p. 6). We will explain it further in Sect. 3.

The earlier version from R 11 has a nearly complete sequence whose pagination runs from the manuscript page 6 to the manuscript page 24, with a possible discontinuity between the manuscript pages 17 and 18, and a definite discontinuity between manuscript pages 6 and 7. These later sequences too come along with a number of assorted and incomplete draft pages, which are provided here as footnotes.

The earlier variant contains five theorems and eleven corollaries, all of which are contained in the later version. Some theorems of the later variant do not appear in this earlier version. The results that Peirce achieved are explained in Sect. 3 in terms of the modern sequent calculus notation.

1.6 The dating of the manuscripts R 9, R 11 and R 501

The dating of these manuscript sheets to late 1901 is based on both textual and collateral evidence, including assessment of the type of paper, pen and Peirce's typical handwriting and style characterizing his texts around 1901. The closest resemblance in terms of content as well as the paper type which is Crane's 1900 Japanese Linen, may be his massive project *Minute Logic* (R 1578; R 425–435), for which he by early summer 1902 had amassed over 2500 manuscript pages and had many of them typeset, to be published as a substantial volume testifying the novelty of his logical developments. The Dragon Logic may well have arisen out of the explorations of the same project and may have been further motivated by the fact that Peirce was promised \$ 150 for each chapter completed for the *Minute Logic*, as well as assistance in their typesetting. However, nothing indicates either in R 501 or in the *Minute Logic* drafts or its plans and tables of contents that R 501 was intended to be included in that projected book.

This said, in the *Minute Logic*, R 430A, p. 66– (Peirce 1902), there are several pages in which the Dragon-Head and the Dragon-Tail signs are also used, but here they stand for propositions, not for the consequence relation. The key passage from that context shows the difference in the meaning of that notation:

> ~~Suppose we have two expressions, which may be indefinitely complex, when written out, but which we will denote by ☊ (Dragon's head) and ☋ (Dragon's tail). Suppose that, assuming ☊ to be scriptible, it follows from~~ [p. 67] the definition of the copula, in A, B, C, and from the fact that **v** [The letter **v** is Peirce's sign for the truth-value 'True'.] is definite, individual, and scriptible, that ☋ is scriptible. Then, the consequence ☊ −< ☋ (where we are to understand that ☊ and ☋ are written out), will necessarily be scriptible, whatever expressions be substituted for its different letters, except **v**.

Proof For there is nothing in the definitions of the algebra which is ~~special~~ limited to any particular letters, except that some additional assertions are made concerning **v**. When, therefore, it is found that, by those definitions the scriptibility of ☋ follows from the assumption of that of ☊, the corresponding conclusion will hold whatever expressions are substituted for those letters of ☊ that do not appear as **v**. Hence if any expression in the form of ☊ is ~~true~~ scriptible, the corresponding expression in the form of ☋ will be ~~true~~ scriptible, the letters taken to fill out the form in the case of ☊ being retained for ☋. Hence, [p. 68] by (65), is any expression in the form ☊ is scriptible the corresponding expression in the form ☊ −< ☋ will be scriptible. But, by (64), either the expression in the form of ☊ will be scriptible or the corresponding expression in the form ☊ −< ☋ will be scriptible. □

At the end, the support that Peirce had applied for the publication of the *Minute Logic* as the *Proposed Memoirs* was denied by the Carnegie Foundation, and Peirce was forced to leave aside these and many other valuable ideas that the applied funds could have helped him to further explore and bring to light. Maybe DL was abandoned for the same reason.

Other substantial connections of the text of DL are to R 516 (*On the Basic Rules of Logic Transformation*, Peirce 1901c), to a slightly earlier R 515 (*On the First Principles of Logical Algebra*, Peirce 1901b) and to R 530 (*A Proposed Logical Notation*, Peirce 1901a). Like R 501, these are also undated by Peirce but can be estimated to have been composed sometime in 1901 or early 1902. The text also bears some internal resemblance to R 484 *On Existential Graphs, F4* (Peirce 1898), which is definitely an earlier piece from August 1898 (Peirce 2019–2021). Especially R 516 is a study of scriptibility and transformability as analysed in the graphical language of existential graphs. R 484 applies transformability as permissible proof steps, demonstrating a number of logical consequences and theorems in the language of existential graphs. The theme is followed up in the *Logical Tracts* (R 491, R 492; Peirce 1903a, 2019–2021, LoF 2/1) and in the *Lowell Lectures of 1903* (*Some Topics of Logic Bearing on*

Questions Now Vexed, R 455, R 456, R S-29, R S-33, R 464, R 478; Peirce 1903b, c, d, e, 2019–2021; LoF 2/2), which were written in summer-autumn 1903.

The texts of the Dragon Logic are also connected, both substantially and compositionally, to the important piece of R 517 (*New Elements*, "Kaina stoicheia"). In fact, the writing of that piece was misdated to early 1904 in the previous literature. In EP 2 (Peirce 1998, pp. 300–324), it is said to have been "most probably written in early 1904, as a preface to an intended book on the foundations of mathematics" (EP 2: 300). Now R 517 is certainly connected to the set of manuscripts R 4-11 which can all be reasonably described as drafts of a preface to an intended book on the foundations of mathematics, and R 501 is certainly textually closely connected to R 11, which is transcribed here as the initial segment of R 501. However, another folder of many loose sheets and manuscript (R 1573) has evidence confirming the dating of R 501 (and hence R 11 and R 517, among others) to 1901. On one of the loose leaves Peirce writes: "Last year the CGS issued a quarto of nine hundred pages …" (R 1573). This and the verso page have many logic sketches in the dragon-tail notation, and such sketches do not appear anywhere else outside of R 11 and R 501. The CGS (Costal and Geodetic Survey) publication referred to is "The Transcontinental Triangulation …", *Special Publications* No. 4, 1900 (December), 871 pages. "The Transcontinental Triangulation…" "Last year" therefore refers to 1900, and thus the dating of the dragon-tail fragments in R 1573 (and consequently R 501 and the whole set of R 4-11) to 1901, and probably to late 1901, is correct.

1.7 Editorial notes

There are well-known problems in the editorial scholarship concerning Peirce's *Nachlass*, much of which have to do with Peirce's incessant rewriting and exploration of the material, resulting in the spawning of the text to multiple directions, with an abundance of incomplete, rejected and forgotten passages and sheets. This makes it hard and sometimes impossible to see what the definite version of the text—namely the thread closest to Peirce's creative, compositional and scientific intentions—may have been. The method adopted here is to altogether refrain from guessing at Peirce's authorial intentions and simply present the textual array in its full capacity.

The transcription of the manuscripts is divided into two appendices, which provide a complete diplomatic presentation of the material. Alternative segments are preceded by [Alt.]. Peirce's deletions and insertions are included in the text inline. Peirce's own manuscript pagination is enclosed in brackets. Other editorial emendations and minor textual notes are likewise bracketed. A couple of obvious typos have been silently corrected; other than those, the transcription preserves the original text, spelling and capitalization without editorial interventions. The text has been typeset by LaTeX EG-peirce.sty package developed by Jukka Nikulainen, which now also includes all the special characters needed in the present transcription, specifically C, Ĉ, ☊, ☋, ☾ and Ɔ.

2 A Précis of Peirce's logic

2.1 A general profile

Peirce was an accomplished scientist, philosopher, and mathematician, and above all, a logician. His contributions to the development of modern logic at the turn of the twentieth century have been far and wide, and although his overall influence upon the development of modern logic remained ill-understood for a long time (Fisch 1982; Dipert 1995; Hintikka 1996; Peirce 2019–2021; Putnam 1982), its impact is now being appreciated in a renewed fashion.

Peirce developed logic upon George Boole's algebra of logic and Augustus De Morgan's logic of relations. Peirce worked on the algebra of relatives (1870–1885), the theory of quantification (1880–1885), graphical and diagrammatic logic (1896–1913), triadic logic (1909), as well as higher-order and modal logics (1898–1911).[4]

Peirce also contributed significantly to the theory and methodology of science, including theory of probabilities and inductive reasoning. He discovered a third kind of reasoning, different from both deduction and induction, which he called abduction or retroduction, and which he identified with the logic of scientific discovery. Thus formal, or deductive, logic was just one of the branches in which Peirce worked on. Indeed, his work in deductive logic is an investigation into mathematical logic and, properly speaking, is part of mathematics, not of logic in general.

Philosophically, then, logic became for Peirce a broad and open-ended discipline with internal divisions and external architectonic relations to other parts of scientific inquiry. Logic depends upon, or draws its principles from, mathematics, phaneroscopy (phenomenology), aesthetics and ethics (phenomenology), while metaphysics and psychology depend upon logic.

One of the most important characters of Peirce's late logical thought was that logic becomes coextensive with semeiotic (his preferred spelling), namely the theory of signs. Peirce divided logic, conceived as semeiotic, into (i) *speculative grammar*, the preliminary analysis, definition, and classification of signs; (ii) *critical logic*, the study of the validity and justification of each kind of reasoning; and (iii) *methodeutic*, or *speculative rhetoric*, the theory of methods and an application of the methods of logical analysis to other fields of science, especially mathematics. Peirce's logical investigations intended to cover these three areas.

Known as the founder of American pragmatism, Peirce was one of the most creative and innovative philosophers of the late nineteenth and early twentieth century. In his later years, he coined "pragmaticism" in order to distinguish his logical and philosophical theory of meaning from the doctrines that he feared may be propounded by scholars who, according to Peirce, are tempted to kidnap the easier word "pragmatism" and dress it with quite a different meaning from what his logical and philosophical attitude towards its formulation had strived to encompass.

[4] These years are indicative only, given the constantly evolving nature of Peirce's explorations. For example, higher-order logic was algebraically investigated in his 1885 "On the Algebra of Logic: A Contribution to the Philosophy of Notation" paper (Peirce 1885) though presented in its graphical outfit beginning in 1898.

2.2 The emergence of quantification theory

In the 1880s, Peirce developed independently of Gottlob Frege a system of quantification theory in which quantifiers were treated as variable binding operators; thus, he can be regarded, alongside Frege, as a founder of contemporary formal logic. The standard notation used in contemporary logic is a variant of Peirce's notation rather than that adopted by Frege. As a part of his pragmaticist theory of meaning, Peirce developed a game-theoretic interpretation of logical constants, according to which their meaning is explained by means of a semantical zero-sum game between two parties, an utterer and an interpreter. Peirce also studied modal and many-valued logics and developed the basic ideas of the possible-worlds semantics for modal logic. In his general theory of reasoning, Peirce distinguished three main forms of reasoning, namely abduction, deduction, and induction, and revised the traditional account of non-deductive reasoning. In his work in general semiotics (the theory of signs) and the philosophy of language, he analysed the sign relation as a triadic relation involving a sign, an interpretant (meaning), and an object, and introduced the distinction between types and tokens into linguistics and the philosophy of language. He made a distinction between iconic, indexical, and symbolic signs and outlined an interestingly modern account of proper names (Pietarinen 2010). He developed a complex classification of signs involving several interpretants and objects, and his rich semiotic system provides a useful framework for the comparison of semiotic theories from the Stoics to the present. Summarizing, Peirce anticipated many significant developments in the later twentieth-century analytic philosophy and logic.

Unlike Frege, however, Peirce did not stick to any one formalism. He spent the rest of his logical life experimenting with alternative notations to serve the theory of logic and to advance scientific inquiry. The outcome of his notational researches was a system of logical graphs discovered in 1896, (Peirce 1896–7) which he termed the system of Existential Graphs (EGs).

2.3 Existential graphs

Peirce presented his theory of EGs in many papers, which also discussed various philosophical topics in semiotics and the philosophy of language. Much of this material remained unpublished during his lifetime, and some scholars became acquainted with it by studying his manuscripts. On the other hand, Peirce was able to get some of these works published, for example, his work *A Syllabus of Certain Topics of Logic* was published by Alfred Mudge & Son, Boston, 1903, and the long paper "Prolegomena to an Apology for Pragmaticism" appeared in the philosophical journal *The Monist* in 1906. However, Peirce's contemporaries ignored these works, perhaps because they were not able to see them as significant contributions to logic and philosophy. It might be said that Peirce was ahead of his times; his work on existential graphs began to receive serious attention only in the 1960s.

In the 1890s, Peirce reformulated quantification theory by expressing it in a language of diagrams which he called *existential graphs* (EGs). The switch from the algebraic notation to the language of graphs seems to have been motivated by his belief that the latter was more suitable for the purposes of logical analysis. According to Peirce, a system of logic can be used as a calculus, which helps to draw inferences as economically as possible, or it can be developed for the purpose of representing and analysing deductive processes. Peirce also thought that a graphical notation was more suitable for logical analysis than an algebraic notation because of its higher degree of *iconicity*. An iconic sign can be said to show what it means in the sense that it resembles its objects in some respect, that is, some features of the sign itself determine its interpretation. Peirce regarded the theory of EGs as one of his most important contributions to logical philosophy.

Sketchy presentations of EGs appeared in print in 1902 in the *Dictionary of Philosophy and Psychology* (DPP) edited by James Mark Baldwin (entry "Symbolic Logic" in Vol. 2, pp. 640–651; LoF 3; Peirce and Ladd-Franklin 1902), in *A Syllabus of Certain Topics of Logic*, a 23-page printed pamphlet that Peirce wrote to accompany his Lowell Lectures of 1903 and circulated in 100 copies, and in the 1906 *Monist* article (Peirce 1906). Most of his prolific writings on EGs remained unpublished in his lifetime.[5]

The diagrammatic nature of EGs consists in the relationship between forms of relations exhibited in the diagrams and the real relations in the universe of discourse. Peirce was convinced that, since these graphical systems exploit a *diagrammatic syntax*, they—together with extensions and modifications that would cover modalities, non-declarative expressions, speech acts, and so forth—can express any assertion, however intricate. Guided by the precepts laid out by the diagrammatic forms of expression, and together with the simple illative permissions by which deductive inference proceeds, the conclusions from premises can be "read before one's eyes"; these graphs present what Peirce believed is a "moving picture of the action of the mind in thought" (R 298, 1906; LoF 3):

> If upon one lantern-slide there be shown the premisses of a theorem as expressed in these graphs and then upon other slides the successive results of the different transformations of those graphs; and if these slides in their proper order be successively exhibited, we should have in them a veritable moving picture of the mind in reasoning. (R 905, 1907-8; Peirce 1908c, 2019–2021, LoF 3/1)

Peirce's work on such topics and questions highlights the importance of underlying ideas that were rediscovered significantly later, and often in different guises. In Peirce's largely unpublished works, one finds topics that later became known as, for example, multi-modal logics and possible-worlds semantics, quantification into modal contexts, cross-world identities (in R 490 he termed these special relations connecting objects in different possible worlds "references", see Pietarinen 2006b), and what is termed 'Peirce's Puzzle' (Dekker 2001; Pietarinen 2015a), namely the question of the meaning of indefinites in conditional sentences. Peirce himself proposed to analyse the latter in quantified modal extensions of EGs of his own devising.

[5] There are only a few references and hints to them in his other published papers from the early twentieth century, such as the "Some Amazing Mazes" series (Peirce 1908c, d, 1909a). The second *Monist* paper "Issues of Pragmaticism" (Peirce 1905c) makes one reference; the first, "What Pragmatism Is", does not (Peirce 1905a). Nor does the published version of the "Neglected Argument for the Reality of God" (Peirce 1908b) refer to EGs.

Peirce continued working on the theory of logical graphs for the rest of his life. On Christmas Day of 1909 he wrote to William James (1942–1910) that this graphical method "ought to be the Logic of the Future".

2.4 The game semantics of 1901

Peirce carried out his semantic insights in terms of what today is recognized as two-player zero-sum semantic games between the Graphist/Utterer and the Grapheus/Interpreter.[6] This is explained in a variant of "New Elements (Kaina stoicheia)" (R 517, c. late 1901), a text coeval to the Dragon Logic, as follows. The copulative is *general* and *definite*, as to assert A and B "is to assert a proposition which the interpreter is at liberty to take as meaning A or as meaning B". The disjunctive, on the other hand, is *vague* and thus *individual* in nature, as to assert A or B "is to assert a proposition which gives the utterer the option between defending it by proving A and defending it by proving B" (R 517, ms p. 50). Peirce continues on the strategic advantages gained when the order of the choices of selection is taken into account:

> The asserter of a proposition may be said to [be] *ex officio* a defender of it, or, in the old logical phrase, a respondent for it. The interpreter is, on the other hand, naturally a critic of it and quasi-opponent. Now if a proposition is in one respect vague, so that in that respect the respondent has the choice of an instance, while in another respect it is general, so that in that respect the opponent has the choice of an instance, whichever party makes his choice last has the advantage of being able to adapt his instance to the choice already made by the other. For that reason,
>
> Some woman is loved by all catholics,
>
> where the respondent is obliged to name the woman before the opponent has chosen his catholic, is harder to defend, and less apt to be true, than
>
> Every catholic loves some woman,
>
> where the opponent must instance his catholic, whereupon the respondent can choose his woman accordingly.
>
> It is a curious fact that when there are a number of ~~obvious~~ signified choosings of instances, it is not the later one which has the logical character of an operator upon the one already made, but the reverse. Thus, in the last example [end] (R 517, ms pp. 50–51)[7]

[6] Sometimes, and especially in relation to Peirce's *model-building games*, these roles are split so that the Grapheus and the Interpreter are playing separate roles (Pietarinen 2013). Hilpinen (1982) first identified Peirce's semantics as a species of the game-theoretic one. On games in logic, see e.g. Pietarinen (2003); Majer et al. (2009).

[7] Similar textual evidence for the game-theoretic interpretation occurs in numerous places, see e.g. R 238, R S-64 and the references in Pietarinen (2003).

Thus, Peirce not only prefigured those much later rediscoveries of game-theoretic semantics in logic, but actually created one.[8]

2.5 Peirce's later algebra of logic

Peircean semantic games were not limited to interpreting natural-language sentences or logical graphs. He often applied the same idea also to the interpretation of complex quantificational patterns and connectives in his general algebra of logic first presented in 1885.

2.5.1 Algebra and graphs as games

We find both algebra of logic and logical graphs considered in unison with regard to the game-theoretic interpretation, as revealed in the following fragment probably written sometime in 1893–1894 (R S-64):

> It will be found that the algebraic method is the more convenient; but some persons have such a difficulty with algebra that I add the graphical method.
>
> Given a proposition about two things *a* and *b*, if you are to select the thing to be represented by *a* with a view to making the proposition false, and I am to select *b* with a view to making the proposition true, it may be an advantage to me, and can be no disadvantage to know what your selection is to be, before I ~~determine~~ fix upon mine. Hence, if the proposition be such that it is true even if I make my selection first, much more will it be true if you make the first selection. Accordingly, if a proposition be written either in the algebraic or the graphical system, and that proposition be true, much more will it be true when any letter in a square or affixed to a Π is moved to the left. For a similar reason, of two letters both in circles or in squares, or both attached to Σ s or to Π s, it is indifferent which comes first. Thus, to say that every man loves every woman is the same as to say that every woman is loved by every man; and to say that some man loves some woman is the same as to say that some woman is loved by some man; but to say that some man loves every woman is to say much more than that every woman is loved by some man.
>
> **[Alt.]** There are other interesting systems of representing propositions; but it is not necessary to consider them here. The above algebraic system is the most convenient; but I add the graphical for the sake of the many readers who do not take kindly to algebra.
>
> Given a proposition about two subjects, A and B, if *you* are to select the subject A with a view to making the proposition false, if you can,—in which case, plainly, A is universal, for the proposition asserts itself to be true, and hence that you cannot succeed in this,—while *I* am to select B with a view to making

[8] How close Peircean semantic games come to contemporary ideas of game semantics has been explored in Pietarinen (2001, 2003, 2007, 2013).

> the proposition true,—so that B is particular,—then it may be of advantage to me, and can at any rate be no disadvantage, to know what your selection for A is to be, before I fix upon mine for B. That is, if the proposition be true though the particular subject be selected first, much more will it be true if the universal subject be selected first.

(The "circles" and "squares" Peirce talks about pertain to the notation of proto-graphs that preceded the discovery of the logical method of EGs; see Introduction to Part II, LoF 1; Peirce 2019–2021). Importantly, Peirce emphasizes the 'strategic' advantage to those who know what the earlier selections have been, which indeed is a standard property of semantic games (of perfect information) for classical logics.

2.5.2 Algebra and graphs

This raises the question of the relationship between algebraic and graphical implementations of logic.

Indeed the two are connected, both substantially and compositionally. Peirce's 1885 explorations and after in the algebra of logic led to the graphical method. These systems include his *five icons system* of 1885, the *qualitative logic* of 1886 and several of its later versions, the revisions he made on the algebra of the copula in 1891 (Peirce 1891a, b), and the new sign of consequence he introduced when reworking those system for the planned magnum opus of the *Grand Logic* in 1893-1894 (Peirce 1893a, b, c, 1894).

Ma and Pietarinen (2018a) (see also Sect. 4) takes up the issue of the graphical logic of the alpha graphs (a two-element Boolean algebra), which Peirce first proposed in 1896, and presents an algebraic remodelling of its rules. This leads to the following

Theorem. *For every α-graph G, $\vdash_\alpha G$ if and only if $\mathbf{A} \models G$ for all alpha algebras $\mathbf{A}$.* Conversely, one can ask how do the logical graphs affect algebras. Ma and Pietarinen (2018a) show that graphical systems admit of a reinterpretation in terms of algebraic rules.

This answers what might otherwise appear a rather enigmatic remark from Peirce in his reply to Christine Ladd-Franklin's letter of November 1901. Ladd-Franklin, one of Peirce's eminent students of logic at Johns Hopkins University in early 1880s had asked what is at stake with logical graphs when he already had a well-developed algebraic system of logic at hand. Peirce in his response admitted that logical graphs have not much bearing on the non-relative (Boolean) part of logic, "*except* in one highly important particular", namely "that they supply an entirely new system of fundamental assumptions to logical algebra" (Peirce to Ladd-Franklin, November 9, 1900, R L 237, Peirce 1900).

Peirce's algebraic studies reached by 1900 can be reinterpreted as culminating in a complete algebraic system **A** with an important proof rule of transformation, namely that of iteration/deiteration, which arises out of algebraic rather than graphical considerations. That algebra is nevertheless best studied in semantic (such as algebraic and game-theoretic) terms and with a consequence relation that agrees with the Boolean lattice order.

This links the status of algebra reached at this point with the Dragon Logic of late 1901. In fact Peirce appears to have started to draft his treatise right after he had sent his reply to his former student and collaborator.

In sum, a proof-theoretic analysis establishes the connection between Peirce's sequent calculus for Boolean algebras and the alpha system of graphs. For details on such proof analysis, see Ma and Pietarinen (2017a).

2.5.3 Rules of transformation

In the proof-theoretic sense, then, it speaks to the superiority of EGs over algebraic systems that in it deduction, as follows from Peirce's another gifted graduate student's work from Johns Hopkins University, Oscar Howard Mitchell (1851–1889). Mitchell's algebra of logic (Mitchell 1883) is reduced to a minimum number of permissive operations. Peirce termed such operations *illative rules of transformation*, and in effect they consist only of two: *insertions*, that is, permissions to draw a graph-instance on the sheet of assertion, and *erasures*, that is, permissions to erase a graph-instance from the sheet. Peirce later reports that

> The RULE OF OMISSION, in the form which it takes in the Universal Algebra of Logic, is that *Any component of any term of the Boolian can be struck out, provided some component remain*,—a rule first given by O. [H]. Mitchell. The correlative RULE OF INSERTION is that *Any term may have any term inserted as an integrant part of it*. Perhaps this had likewise never been stated before O. [H]. Mitchell. (R 253, 1905; Peirce 1905b)

In terms of the graphical representation of logic such as EGs, the *oddly enclosed* areas of graphs (areas within a non-even number of enclosures) permit inserting any graph in that area, while *evenly enclosed* areas permit erasing any graph from that area. A copy of a graph-instance is permitted to be pasted on that same area or any area deeper within the same nest of enclosures. This is the rule of *iteration*. A copy thus iterated is permitted to be erased by the converse rule termed *deiteration*. An interpretational corollary is that a *double enclosure* with no intervening graphs (other than the blank graph) in the middle area can be inserted and erased at will.

A more detailed exposition of these illative rules of transformation would demonstrate their application to quantificational expressions, namely applying insertions and erasures to ligatures. Roberts (1973) has the essential details.

In hindsight, Peirce explained that his "system of 'Existential Graphs' puts in a clearer light a truth first virtually enunciated by my student (afterward professor) O. H. Mitchell", who showed that deductive reasoning "can always be reached by *adding to the stated antecedents and subtracting from stated consequents*, being understood that if an antecedent be itself a conditional proposition, its antecedent is of the nature of a consequent" (R 905, 1908, Peirce 1908a). These two operations that adequately characterize logically necessary reasoning are much exploited in modern systems of proof sequents. The central notion that characterizes Peirce's logical investigations is the relation of a consequence. A sequent calculus is a theory about the consequence relation. It was such sequent calculus that Peirce was developing since at least 1880 for systems that agree with Boolean algebras (Peirce 1880; Ma 2018; Ma and Pietarinen 2018a).

2.6 A summary of Peirce's contributions to logic and its applications

During his long career, Peirce created a number of logical systems that largely coincide with Boole's algebra. One direction for future work is to ask which minimal modifications his principles and rules would permit non-classical calculi and graphs. These include intuitionistic and bi-intuitionistic logics, closure algebras, implicational and other fragments of propositional logic and substructural systems (Ma and Pietarinen 2018b, c). A closure algebra, for example, has an operator which arises from having another, strong cut in the system to interact with the classical, contradictory negation as a modal closure operator. Implicational fragments arise from omitting the fourth icon that Peirce used in 1885 in order to introduce a negation to the implicational system. Seeds of substructural logics, such as resource-bounded linear logics, were planted by Peirce's suggestion of a 'linear' type of negation (Pratt 1992).[9]

Far from just anticipating later findings, Peirce's logical innovations that arose out of the improvements on the algebra of logic have over the years been applied in a number of areas, including philosophical logic, formal semantics and pragmatics, mathematics, mind and language, AI, cognitive and computing sciences, biology, medical diagnosis and prognosis, astrobiology, physics, cosmology and geology, as much as in economics, game and decision theory, history and philosophy of science, archaeology, anthropology, musicology and art studies.[10]

3 Sign, scriptibility and deduction

3.1 The place of Dragon Logic among Peirce's logical projects

Peirce's introduction of the "Dragon-Head" and "Dragon-Tail" notations is situated within the same period of developments as his game-theoretic semantics and the *Kaina Stocheia* -style epistemology and philosophy of assertions. As far as the development of his mature logical theories is concerned, Peirce had recently completed (by August 1898) the first phase of the development of the theory of EGs, including a sound and semantically complete system of rules of transformations (Peirce 2019–2021). For the next couple of years, though, he had devoted most of his time to other researches and scholarly projects. The main logical outputs arising from the years around 1900 are his entries to DPP which he had completed by the end of 1901 (Peirce 2019–2021).

Between the summers of 1901 and 1902, Peirce had decided to invest his energies in the production of the *Minute Logic*, a massive work that like so many other works

[9] Such modifications and extensions of 'classical' logic may necessitate modifications to the structural rules of the graphical (sequent) calculus, such as taking the sheet of assertion to be a vector space.

[10] For some further work and applications along the lines, Peirce had set out to do see, for example, Bellucci and Pietarinen (2020); Bellucci et al. (2014); Pietarinen (2010, 2012, 2019); Sowa (2006); Zalamea (2012). For details on Peirce's deductive logic, see the collection of Houser et al. (1997). Hilpinen (2004) gives a good overview of Peirce's logic.

of his remained incomplete and entirely unpublished. By early 1903, when it was clear that he could not receive funds from the Carnegie Institute to aid publication of the work, Peirce decided to return to the topic of logical graphs and to work out a comprehensive presentation of EGs and their extensions to modal and higher-order logics, for the Lowell Lectures of the next fall (Peirce 2019–2021). Apparently he never returned to his 1901 exposition of the theory of deduction and consequence proposed in the Dragon-Head notation during these interim years.

The next couple of subsections contain a brief exposition of the key prefatory concepts assumed in the 1901 text, namely signs, sheets, scriptibility and deduction.

3.2 Signs

By 1901, Peirce's investigations in the semiotic side of logic had reached a rather definite disposition. The next major step was the 1903 revision in the context of the Lowell Lectures, which was to develop logic "into a general theory of signs" (EP 2:272; Pietarinen 2015b), with the three departments of Speculative Grammar (the classificatory part of reasoning into good and bad), Logical Critic (the theory of logic proper), and Methodeutic (the theory of the principles of how valuable courses of research and exposition are to be attained).

That revision, which the Dragon Logic text had not yet quite reached, was to add the third triad of signs, namely that of qualisign, sinsign (token) and legisign (type). But already here in 1901, Peirce does have an understanding of the type/toke distinction, albeit not explicitly delineated as a separate trichotomy of all signs, and not only those of symbols. What he did have was the famous distinction of signs into icons, indices and symbols, and the distinction of symbols into rhemas, propositions and arguments. In particular, the elements of these two trichotomies are understood as providing the categorial method of semiotics, in other words a way to the classification of signs in terms of the relational structure arising from many combinations with these elements (see Bellucci 2017 for details).

3.2.1 Sign, object, interpretant

In some of the coeval texts (e.g., R 599, c.1902; SE, p. 120), Peirce defines signs as "anything which in any way represents an object". The meaning of "representation", as indeed that of an "object", will be much elaborated in Peirce's texts on 1902 and 1903 in particular (Peirce 2019–2021, 2020, SW). In 1902, he held sign to be "something which brings another sign into objective relation to that sign which it represents itself, and brings it into that relation in some measure in the same respect or aspect in which it is itself a sign of the same sign …something which in some measure and in some respect makes its interpretant the sign of that of which it is itself the sign. It is like a mean function in mathematics" (SW, pp. 123–124).

3.2.2 Icon, index, symbol

The two trichotomies of signs operative in the preambles of the DL exposition are thus effectively those that Peirce had reached in 1901–1902, especially in the *Minute*

Logic text of R 425 as well as in R 599. The first, icon, index, and symbol, shows how radically different kinds of signs there may be.

Index (Peirce typically begins his explanations with the second category) is "a thing which having been forcibly affected by its object, forcibly affects its interpretant and causes that interpretant to be forcibly affected by the object, and to affect its interpretant in turn" (R 599, SW, p. 124). Icons, in turn, are "pure images" which, in contrast to indices, are not caused by their objects to represent them by any known or explicit cause. They represent their objects by virtue of their own qualities. Symbol differs from both index and icon types of signs in representing its objects "solely by virtue of being represented to represent it by the interpretant which it determines" (R 599, SW, p. 125). A symbol is a sign the meaning of which "can only be realized by the aid of its Interpretant" (R 425).

3.2.3 Term, proposition, argument

The second trichotomy is that of the term, proposition, and argument. In the 1901 text (R 501), Peirce conceives signs largely as symbols that can be repeatedly uttered and duplicated as different tokens of the same type of a sign. The interpretant of a sign is the effect of the sign that the sign gives rise to in relation to its correspondence with its object. If a sign separately represents its object, it becomes a *proposition* capable of being true or false. If a sign is separately to signify its intended interpretant, it becomes an *argument*. Having a proposition is one step in the perfection of a sign; having arguments is another.

Propositions are the second category of symbolic signs, and arguments are the third. The first category is the *term*, also called a rhema, which is an incomplete, unsaturated predicate term devoid of meaning, such as an expression "___ is red" that has a blank placeholder '___' attached to the predicate term. A term is what is left of a proposition stripped of its subject.

Beyond the elements themselves, what is important is the relational structure between the previous two classifications. In particular, certain relations of exclusion apply which imply that icons cannot be propositions and that only symbols can be arguments. This schema, albeit subject to significant later revisions, is from the point of view of Peirce's theory of signs the bedrock of Dragon Logic.

3.2.4 Vague, general, singular signs

A sign is applicable to different objects by virtue of its very nature. Even a proper name can stand for different objects. In particular cases, a sign represents a single object. Peirce explains that a sign can be used vaguely (non-definitely or non-determinately), generally (non-individually) or singularly. For example, the linguistic sign 'man' is used vaguely in the statement 'Some man sins', generally in 'Every man sins', and singularly in 'This man sins'.

In the DL text, Peirce introduces other special notations, namely acute and grave accents and a circumflex, to signify these three uses of a sign. The notations $\acute{a}$, $\grave{a}$ and $\hat{a}$ stand for the cases in which a is used generally, vaguely and singularly, respectively.

In §3 of the more mature version of the DL text in "Appendix A", Peirce introduces ☾ as a binary relational symbol and ☽ as the converse of that relation. We do not find Peirce using the moon character elsewhere in his preserved writings. Here he applies moon-relations to two signs with accents and circumflexes. The proposition 'Socrates loves Plato' might be formulated as *â* ☾ *ê*. Peirce then presents five laws of these two binary relations, which can be proven by the definition of the converse relation. In order to logically reason about signs, he in §4 formulates two rules of substituting vague and general signs for singulars. The first law is an informal description of an *existential generalization*, that is, a vague sign can be obtained from an individual sign. That a vague sign cannot be defined by establishing a general sign holds, as we now know, in first-order logic only under the assumption that the domain of individuals is non-empty. Peirce's second law is a formulation of a *universal instantiation*, that is, a general sign can be refuted by refuting an individual (definite) sign. A general sign cannot be established by establishing a vague or a singular sign.

3.2.5 Semantic games

In these two paragraphs referred to above, it is worth pointing out that the method that Peirce is applying is that of *game-theoretic semantics*. Textually, that method and its exposition come close to one that we encounter in R 517, *Kaina Stocheia*. The game of determining and choosing the objects of the propositions is that of between the utterer and the interpreter, in which the utterer defends the proposition and makes his choices according to that purpose. The interpreter is the protagonist (termed "critic" and "quasi-opponent" in the alternative draft version of the text provided in "Appendix B") who looks for the interpretation that could not be defended, that is, she aims at succeeding in the task of falsifying the proposition. Utterers and interpreters are agents in one's make-believe: utterers can be considered to be anything in which the sign could originate, while interpreters are anything in which the sign can be interpreted. Peirce then remarks that the game in question is that of perfect information, the property first noticed in Hilpinen (1982). This gives a strategic advantage to the player when planning his choices. Perfect-information games indeed assign every proposition one of the two truth-values and may be said to result in a classical, bivalent semantics of logic.

3.2.6 The sheet of assertion

The sheet of assertion is possibly the most consequential conceptual innovation in Peirce's EGs. However, it need not be exclusively confined to the graphical domain. In the theory of the alpha part of his logic of EGs (Ma and Pietarinen 2017a), the blank sheet of assertion indeed represents all truths. Algebraically it is the top element $\top$ of a lattice, such as representing tautology or *verum* in a Boolean algebra; assertorically it means being "irresponsible" (this term of Peirce's comes from coeval R 516). The denial of that blank sheet is the bottom element, or the *falsum* $\bot$.

We do find Peirce using the conceptual idea of the sheet prior to the 1896 invention of logical graphs in the context of the algebra of logic. As early as in the 1886 qualitative logic (W5: 331–2), Peirce had talked about "the sheet of conceivable states of things".

Indeed, DL does not explicitly avow a graphical notation in place of its algebraic notation; still, Peirce's use of the term sheet is quite manifest:

> We shall be supposed to be provided with a certain blank sheet, to be called the "sheet of assertion"; and in connection with this we are to be supposed to have a certain purpose, which remains vague, so that the writing of any given sign on the sheet of assertion may or may not accord with that purpose; but the purpose does not *require* any entire sign to be written ("Appendix A", § 5).

Indeed, the sheet of assertion can have many meanings, among which is the fundamental idea of it as catering for a place (a manifold, plane, ambient space, spread, or just a blank line), upon which assertions are being *scribed* (if graphical) or *written* (if linguistic/algebraic), and upon which logical *transformations* from one assertion to another are performed in a convenient and purposeful fashion. The important difference is that in the graphical notation, the sheet is a two- or multidimensional spread upon which assertions are scribed, and that sheet itself is a graph with a semantic value (such as tautology in alpha graphs). In DL, a blank sheet upon which the formulas of that logic are written has no such meaning.

3.3 Scriptibility

In early 1890s, Peirce had began to use the terminology of "scriptibility" and "non-scriptibility" in a couple of writings that also have remained mostly unpublished (R 579, R 1573, R 839; see W6: 208–209 on the publication of a two-page paper on the algebra of the copula where these notions do occur). This pair of terms is now a few years later fully exploited in the DL text drawn from R 501, R 516 and R 430A.

3.3.1 Scriptibility and the Dragon Logic notation

Dragon Logic, which we will expose in full in the next section, uses a special sign, the Dragon Head, notation ☊, from which the implicational sign is received. Implication or conditional is then used in a dual form, which Peirce terms the Dragon-Tail, notation ☋̂, which is the inverse of the head turned on its size and with a circumflex above it. It is not exactly a 'classical' duality of the implication in the manner in which the dual of the implication, notation $\overline{\prec}$, is the negation of implication in his 1880 algebra of logic, for example (Peirce 1880). The circumflex is used in ☋̂ given that ☋ is a singular sign (see above 3.2.4 for Peirce's classification of singular, vague and general sign).

From these, Peirce defines the key property of *scriptibility* as that which is "capable of being written conformably to the purpose". Since the context is deduction and the relation of consequence in their general senses, Peirce wrote "written" rather than "scribed": scribing is an act of asserting logical graphs on the sheet of assertion which were introduced some five years earlier (Bellucci and Pietarinen 2016, 2017; Pietarinen 2006a, 2019; Roberts 1973; Shin 2002; Zeman 1964). Scribing refers to anything partly written and partly drawn, such as graphs and diagrams. The "purpose" is a quality represented by the sheet of assertion.

3.3.2 Generalization of truth-values

As the semantic characterization of scriptibility suggests, Peirce is after a generalization of truth-values to some other values besides truth and falsity, or even their degrees or lack of values altogether (as in his 1909 triadic logic). He may be drawing motivation for this from his parallel interests in other kinds of normative sciences, namely those of *esthetics*[11] and *ethics*, from the kinds of values that could characterize the normativity of those disciplines (Pietarinen 2012). In aesthetics, the values are the ideals that are themselves their own justification, while in ethics they are found in the values of *good* and *bad*. While truth and falsity are common designations of values in ordinary logical domains, Peirce is looking for a possibility to generalize truth into its 'design features' that the idea of scriptibility could exhibit. Scriptible signs may be beautiful, good or true, while unscriptible ones may be ugly, bad or false, as he states in the present text. We might also want to recall that as early as 1886 (Peirce 1982-, W5, pp. 331–2), Peirce had made explicit appeals to modal concepts when talking about "the sheet of conceivable states of things": one could now regard these conceivable states as different ways of defining values beyond the extensional ones of truth and falsity. And in order to actually carry out this generalization, the first kind of generalization, that is from propositions to other kinds of signs, becomes a rather inevitable issue for Peirce to accomplish first.

3.4 Deduction: transformability and correction

3.4.1 Transformability

The second major concept is that of *transformability*. In contrast to scriptibility, defining transformability has its beginnings in syntactic and algebraic considerations. The property of soundness of reasoning may be obtained as soon as there is a well-defined deducibility relation in operation. Peirce had defined it in many occasions, such as when characterizing formulas and graphs of a given language that are "capable of being transformed without changing anything scriptible into anything non-scriptible" (R 430).[12] This perfectly defines the "permissibility of a deduction", namely deduction which is considered to be a transformation in which the entire original sign (e.g., the colligated premises) remain on the sheet while some others are added to it.

3.4.2 Correction

In the earlier variant of the text ("Appendix B"), Peirce also talks about an altogether different type of transformation, one which may result in a *retraction* of the signs that are scribed on the sheet. He terms is *correction*. Peirce cannot mean a permissive operation of omission, that is, a proof transformation that entitles erasing parts of the assertion on positive areas or accomplishing an erasure by virtue of deitera-

[11] Peirce's preferred spelling.

[12] See also R 516, *On the Basic Rules of Logical Transformation*, for similar definitions of 'scriptible' and 'transformable' in the context of the graphical method of the logic of existential graphs.

tion. Thus, correction appears as a rather noteworthy finding, as one could interpret Peirce to have identified in it an early idea of *non-monotonic* reasoning, which is strictly speaking not deductive but a process in which the necessary component of that inference is given as proceeding "from a sign which need not be written to a sign that must be written" (p. $6\frac{1}{2}$, fn7, "Appendix B"). The modalities of "need not" and "must" are central. The process of correction is in a sense a *converse* of deductive reasoning, reasoning in which something comes to be transformed from unscriptible to scriptible signs. But curiously, as unscriptible signs cannot be laid upon the sheet of assertion, the processes of correction cannot be performed upon the sheet, either. Peirce notices ("Appendix B", p. 7, fn7), however, that the *permissibility* of a correction can be expressed as 'Not ___ but ___ '. (This is similar to Peirce's "rhematic" way of expressing incomplete or unsaturated predicates.)[13] Since Peirce's preference is to study rules governing conditional forms of 'If ___ then ___ ', it is the latter—the conditional form following the procedure of illative transformations—and not the correction that follows the procedure of rebuttal which characterizes consequence relation, that is non-monotonic, namely one that allows for the generality of rules of transformation. Another way of putting this remark might be to attribute to Peirce a recognition of the context-dependent nature of non-monotonic forms of reasoning involved in correction.[14]

3.4.3 Permissibility of deduction

Peirce used the implicational statements 'If ___ then ___ ' to refer to the *permissibility of deductions*. These deductions are not limited to logical deductions, however, but are extended to cover all statements of consequences. Indeed, what Peirce is working out here is the general theory of deduction and not only an account of necessary inferences.[15] For example, the Dragon-Tail notation $\widehat{\mathbf{C}}$ is not purported to accommodate

[13] In the language of existential graphs, a graph that is not scriptible would be one that is surrounded by an oval, or a cut, such as P. But the cut itself is only a boundary and not something that strictly speaking can be scribed on the sheet. Hence, cuts alone are to be considered to be unscriptible, while enclosures (cuts and their contents) are scriptible.

[14] Or, one might add, possibly those of *adaptive* forms of reasoning as well (see e.g. Batens 2004). Moreover, one might be led to wonder whether Peirce had here hit upon an example of a form of reasoning which is none of the three of deduction, induction or abduction. Unless we interpret correction in strictly classical terms of being about negation in the classical sense and not about strong negation or negation as a 'stopping rule', 'failure' or similar interpretations familiar from non-classical logic and logic programming, or unless we embed correction and non-monotonic consequence into abductive reasoning, we might tentatively conclude that he might well have. In that case, the question remains why he seemed to have forgotten this 1901 invention of his when he in his later writings insisted that there are reasonable—though not conclusive—arguments that establish the presence of three and only three kinds of reasoning (see e.g. R 855–R 856, 1911; Peirce 2019–2021, LoF 1).

[15] For Peirce, and especially as we move towards the last decade of his life, necessary consequences do not exhaust what deduction consists of. Deduction is split into *logical analysis* (definition) and *demonstration*. Later he grew increasingly convinced that deduction ought to be characterized in terms of *compulsive* rather than necessary inferences:

> I have hitherto defined [deduction] as necessary reasoning; and no doubt much, perhaps most, possibly all deduction is necessary. But on reviewing the subject for this talk, it seems to me more correct to define Deduction as compulsive reasoning. Retroduction seduces you. Induction appeals

the generality of the deduction, given that Peirce states that "the antecedent refers to a single event or state of things without any reference to analogous cases" (ms p. 8). So what is going on?

4 The Dragon Logic

This section reformulates Peirce's Dragon Logic using Gentzen's sequent calculus. We show how to prove the theorems and corollaries of his second, more mature version of the text included in "Appendix A", in the formalism of the sequent calculus. The results that appear in the earlier version of the text transcribed in "Appendix B" can be translated to the Gentzen's calculus formulation very similarly. These translations help to clarify what Peirce's motivations were, what he indeed had accomplished in his papers, and how to put his results and insights into historical, logical and philosophical perspectives.

We use a, b, c etc. to denote propositions. Let X, Y, Z etc. denote finite sets of propositions. A *sequent* is an expression of the form $X \vdash Y$ which means that at least one proposition $a \in Y$ is *deducible* from X. This is, indeed, the notation for multi-succedent consequence relation. In particular, $a \vdash b$ means that b is deducible from a. Let $\vee$ be the connective of disjunction.

Definition 4.1 The sequent system DH for the logic of Dragon Head consists of the following axioms and rules:

$$\text{(Id) } a \vdash a \quad \text{(C1) } a \vdash b \,\widehat{\mathbb{C}}\, a \quad \text{(C2) } \vdash a \vee (a \,\widehat{\mathbb{C}}\, b) \quad \text{(C3) } a, a \,\widehat{\mathbb{C}}\, b \vdash b$$

$$(\vee\vdash)\ \frac{a, X \vdash Y \quad b, X \vdash Y}{a \vee b, X \vdash Y} \quad (\vdash\vee)\ \frac{X \vdash Y, a, b}{X \vdash Y, a \vee b} \quad (\vdash \widehat{\mathbb{C}})\ \frac{a, X \vdash Y, b}{X \vdash Y, a \,\widehat{\mathbb{C}}\, b}$$

$$(\mathrm{Wk}\vdash)\ \frac{X \vdash Y}{a, X \vdash Y} \quad (\vdash\mathrm{Wk})\ \frac{X \vdash Y}{X \vdash Y, a} \quad (\mathrm{Cut})\ \frac{X \vdash Z, a \quad a, Y \vdash U}{X, Y \vdash Z, U}$$

A *derivation* of a sequent $X \vdash Y$ in DH is a finite tree of formulas with the root node $X \vdash Y$. Each node of the tree is either an instance of an axiom, or derived from child node(s) by a rule. A sequent $X \vdash Y$ is *derivable* in DH, notation $X \vdash_{\mathsf{DH}} Y$, if there exists a derivation of $X \vdash Y$ in DH. In particular, a proposition a is *provable* in DH, notation $\vdash_{\mathsf{DH}} a$, if $\varnothing \vdash_{\mathsf{DH}} a$. The subscript DH is dropped if no confusion can arise.

Remark 4.2 The axiom (C1) is called the *definiteness* of $\widehat{\mathbb{C}}$, and (C2) is called the *individualness* of $\widehat{\mathbb{C}}$, in Peirce's definition of $\widehat{\mathbb{C}}$. Peirce provides the axioms (C1)–(C3) as the 'definition' of $\widehat{\mathbb{C}}$. The axiom (Id), weakening rules (Wk$\vdash$) and ($\vdash$Wk),

> to you as a reasonable being. But Deduction first points to the premises and their relation, and then shakes its fist in your face and tells you "Now by God, you've got to admit the conclusion". I beg your pardon, with all my heart, I meant to say, "Now by the eternal world forces spiritual and personal [*illeg.*]". Necessary reasoning is reasoning from the truth of whose premises it not only follows that the conclusion is true, but that it would be so under all circumstances. (R 754)

Deduction may be characterized as compulsive rather than necessary in that there is no room for *living doubt* that the conclusion follows from the premises.

and the rule (Cut) are basic properties of the logical consequence relation in classical propositional logic. Peirce draws (Id) as a corollary of (C1) and (C2). The rule $(\vdash \widehat{\mathbf{C}})$ naturally follows from the meaning of $\widehat{\mathbf{C}}$. The rules for $\vee$ correspond to the use of disjunction in proofs. These axioms and rules are not explicitly given in Peirce's present definition, but they are indeed used in his proofs of theorems.

Lemma 4.3 *The propositions* $a \,\widehat{\mathbf{C}}\, a$ *and* $(a \,\widehat{\mathbf{C}}\, b) \vee (c \,\widehat{\mathbf{C}}\, a)$ *are provable in DH.*

Proof One has the following derivations:

$$\dfrac{\vdash a \vee (a \,\mathbf{C}\, a) \qquad \dfrac{a \vdash a \,\mathbf{C}\, a \quad a \,\mathbf{C}\, a \vdash a \,\mathbf{C}\, a}{a \vee (a \,\mathbf{C}\, a) \vdash a \,\mathbf{C}\, a}\,(\vee\vdash)}{\vdash a \,\mathbf{C}\, a}\,(\text{Cut})$$

$$\dfrac{\vdash a \vee (a \,\mathbf{C}\, b) \qquad \dfrac{\dfrac{a \vdash c \,\mathbf{C}\, a}{a \vdash (a \,\mathbf{C}\, b) \vee (c \,\mathbf{C}\, a)}\,(\vdash\vee) \quad \dfrac{a \,\mathbf{C}\, b \vdash a \,\mathbf{C}\, b}{a \,\mathbf{C}\, b \vdash (a \,\mathbf{C}\, b) \vee (c \,\mathbf{C}\, a)}\,(\vdash\vee)}{a \vee (a \,\mathbf{C}\, b) \vdash (a \,\mathbf{C}\, b) \vee (c \,\mathbf{C}\, a)}\,(\vee\vdash)}{\vdash (a \,\mathbf{C}\, b) \vee (c \,\mathbf{C}\, a)}\,(\text{Cut})$$

This completes the proof. □

Lemma 4.4 *The following hold in DH:*

(1) *if* $\vdash a \,\widehat{\mathbf{C}}\, b$, *then* $a \vdash b$.

(2) *if* $a \vdash b$, *then* $c \,\widehat{\mathbf{C}}\, a \vdash c \,\widehat{\mathbf{C}}\, b$ *and* $b \,\widehat{\mathbf{C}}\, c \vdash a \,\widehat{\mathbf{C}}\, c$.

Proof (1) Assume $\vdash a \,\widehat{\mathbf{C}}\, b$. Clearly $a, a \,\widehat{\mathbf{C}}\, b \vdash b$. By (Cut), $a \vdash b$.

(2) Assume $a \vdash b$. One has the following derivations:

$$\dfrac{\dfrac{c, c \,\widehat{\mathbf{C}}\, a \vdash a \quad a \vdash b}{c, c \,\widehat{\mathbf{C}}\, a \vdash b}\,(\text{Cut})}{c \,\widehat{\mathbf{C}}\, a \vdash c \,\widehat{\mathbf{C}}\, b}\,(\vdash \widehat{\mathbf{C}}) \qquad \dfrac{\dfrac{a \vdash b \quad b, b \,\widehat{\mathbf{C}}\, c \vdash c}{a, b \,\widehat{\mathbf{C}}\, c \vdash c}\,(\text{Cut})}{b \,\widehat{\mathbf{C}}\, c \vdash a \,\widehat{\mathbf{C}}\, c}\,(\vdash \widehat{\mathbf{C}})$$

This completes the proof. □

One can derive several other properties which Peirce defines in his system on the manuscript pages 7–11 of "Appendix A". We can rewrite them as follows.

Theorem 4.5 *The following hold in DH:*

(1) *If* $\vdash x \,\widehat{\mathbf{C}}\, b$ *for every proposition* x, *then* $\vdash b$.

(2) *If* $b \,\widehat{\mathbf{C}}\, x \vdash a \,\widehat{\mathbf{C}}\, x$ *for every proposition* x, *then* $\vdash a \,\widehat{\mathbf{C}}\, b$.

(3) $a \,\widehat{\mathbf{C}}\, (b \,\widehat{\mathbf{C}}\, c) \vdash b \,\widehat{\mathbf{C}}\, (a \,\widehat{\mathbf{C}}\, c)$.

(4) $(a \,\widehat{\mathbf{C}}\, b) \,\widehat{\mathbf{C}}\, c \vdash a \vee c$.

(5) $(a \,\widehat{\mathbf{C}}\, b) \,\widehat{\mathbf{C}}\, c \vdash b \,\widehat{\mathbf{C}}\, c$.

(6) $a, b \,\widehat{\mathbf{C}}\, c \vdash (a \,\widehat{\mathbf{C}}\, b) \,\widehat{\mathbf{C}}\, c$.

(7) $a \vee c, b \,\widehat{\mathbf{C}}\, c \vdash (a \,\widehat{\mathbf{C}}\, b) \,\widehat{\mathbf{C}}\, c$.

(8) $a \vdash (a \,\widehat{\mathbf{C}}\, b) \,\widehat{\mathbf{C}}\, b$.

Proof (1) Assume $\vdash\ x \mathbin{\widehat{\mathsf{C}}} b$ for every proposition x. Then $\vdash\ (a \mathbin{\widehat{\mathsf{C}}} a) \mathbin{\widehat{\mathsf{C}}} b$. By Lemma 4.3, one has $\vdash a \mathbin{\widehat{\mathsf{C}}} a$. By (C3), $a \mathbin{\widehat{\mathsf{C}}} a, (a \mathbin{\widehat{\mathsf{C}}} a) \mathbin{\widehat{\mathsf{C}}} b \vdash b$. By (Cut), $\vdash b$.

(2) Assume $b \mathbin{\widehat{\mathsf{C}}} x\ \vdash\ a \mathbin{\widehat{\mathsf{C}}} x$ for every proposition x. Then $b \mathbin{\widehat{\mathsf{C}}} b\ \vdash\ a \mathbin{\widehat{\mathsf{C}}} b$. By Lemma 4.3, $\vdash b \mathbin{\widehat{\mathsf{C}}} b$. By (Cut), $\vdash a \mathbin{\widehat{\mathsf{C}}} b$.

(3) Obviously $b, a, a \mathbin{\widehat{\mathsf{C}}} (b \mathbin{\widehat{\mathsf{C}}} c)\ \vdash\ c$. By twice applying of ($\vdash \widehat{\mathsf{C}}$), one obtains $a \mathbin{\widehat{\mathsf{C}}} (b \mathbin{\widehat{\mathsf{C}}} c) \vdash b \mathbin{\widehat{\mathsf{C}}} (a \mathbin{\widehat{\mathsf{C}}} c)$.

(4) By (C2), $\vdash a \vee (a \mathbin{\widehat{\mathsf{C}}} b)$. Clearly $a \vee (a \mathbin{\widehat{\mathsf{C}}} b) \vdash a, a \mathbin{\widehat{\mathsf{C}}} b$. By (Cut), $\vdash a, a \mathbin{\widehat{\mathsf{C}}} b$. By (C3), $a \mathbin{\widehat{\mathsf{C}}} b$, $(a \mathbin{\widehat{\mathsf{C}}} b) \mathbin{\widehat{\mathsf{C}}} c \vdash c$. We have the following derivation:

$$\dfrac{\dfrac{\vdash a \vee (a \mathbin{\mathsf{C}} b) \qquad \dfrac{\dfrac{\vdash a, a \mathbin{\mathsf{C}} b \quad a \mathbin{\mathsf{C}} b, (a \mathbin{\mathsf{C}} b) \mathbin{\mathsf{C}} c \vdash c}{a, (a \mathbin{\mathsf{C}} b) \mathbin{\mathsf{C}} c \vdash a, c}\,(\text{Cut}) \qquad \dfrac{a \mathbin{\mathsf{C}} b, (a \mathbin{\mathsf{C}} b) \mathbin{\mathsf{C}} c \vdash c}{a \mathbin{\mathsf{C}} b, (a \mathbin{\mathsf{C}} b) \mathbin{\mathsf{C}} c \vdash a, c}\,(\vdash\text{Wk})}{a \vee (a \mathbin{\mathsf{C}} b), (a \mathbin{\mathsf{C}} b) \mathbin{\mathsf{C}} c \vdash a, c}\,(\vee\vdash)}{(a \mathbin{\mathsf{C}} b) \mathbin{\mathsf{C}} c \vdash a, c}\,(\text{Cut})}{(a \mathbin{\mathsf{C}} b) \mathbin{\mathsf{C}} c \vdash a \vee c}\,(\vdash\vee)$$

(5) By (C1), $b \vdash a \mathbin{\widehat{\mathsf{C}}} b$. By Lemma 4.4 (2), $(a \mathbin{\widehat{\mathsf{C}}} b) \mathbin{\widehat{\mathsf{C}}} c \vdash b \mathbin{\widehat{\mathsf{C}}} c$.

(6) One has the following derivation:

$$\dfrac{\dfrac{a, a \mathbin{\widehat{\mathsf{C}}} b \vdash b \quad b, b \mathbin{\widehat{\mathsf{C}}} c \vdash c}{a, a \mathbin{\widehat{\mathsf{C}}} b, b \mathbin{\widehat{\mathsf{C}}} c \vdash c}\,(\text{Cut})}{a, b \mathbin{\widehat{\mathsf{C}}} c \vdash (a \mathbin{\widehat{\mathsf{C}}} b) \mathbin{\widehat{\mathsf{C}}} c}\,(\vdash \widehat{\mathsf{C}})$$

(7) By (6), $a, b \mathbin{\widehat{\mathsf{C}}} c \vdash (a \mathbin{\widehat{\mathsf{C}}} b) \mathbin{\widehat{\mathsf{C}}} c$. One has the following derivation:

$$\dfrac{a, b \mathbin{\widehat{\mathsf{C}}} c \vdash (a \mathbin{\widehat{\mathsf{C}}} b) \mathbin{\widehat{\mathsf{C}}} c \qquad \dfrac{c \vdash (a \mathbin{\widehat{\mathsf{C}}} b) \mathbin{\widehat{\mathsf{C}}} c}{c, b \mathbin{\widehat{\mathsf{C}}} c \vdash (a \mathbin{\widehat{\mathsf{C}}} b) \mathbin{\widehat{\mathsf{C}}} c}\,(\text{Wk}\vdash)}{a \vee c, b \mathbin{\widehat{\mathsf{C}}} c \vdash (a \mathbin{\widehat{\mathsf{C}}} b) \mathbin{\widehat{\mathsf{C}}} c}\,(\vee\vdash)$$

(8) Clearly $a, a \mathbin{\widehat{\mathsf{C}}} b \vdash b$. By ($\vdash \widehat{\mathsf{C}}$), $a \vdash (a \mathbin{\widehat{\mathsf{C}}} b) \mathbin{\widehat{\mathsf{C}}} b$. □

Peirce then draws several corollaries from these theorems, followed by a long *Scholium* (a comment on the logical structure of the preceding assertion to counter possible objections) on the interpretation of the consequence relation 'If ___ then ___', which he compares to that of the conditional *de inesse*, namely the material conditional.

From these discussions, Peirce continues this manuscript segment by introducing the symbol '=' between propositions. For all propositions a and b, $a = b$ is a new proposition. The following axioms and a rule are given to define '=':

$$(=1)\ a, a = b \vdash b \quad (=2)\ b, a = b \vdash a \quad (\vdash=)\dfrac{a, X \vdash b \quad b, X \vdash a}{X \vdash a = b}$$

Let $\mathsf{DH}^{=}$ be the system obtained from DH by adding these axioms (=1) and (=2) and the rule ($\vdash=$). Then, one obtains the following proposition (Peirce's *Corollary 8*, "Appendix A"):

Proposition 4.6 *For all propositions a and b,* $a \vdash_{\mathsf{DH}^{=}} b = a \mathbin{\widehat{\mathsf{C}}} b$.

Proof One has the following derivation in $\mathsf{DH}^{=}$:

$$\dfrac{\dfrac{b \vdash a \,\widehat{\mathbf{C}}\, b}{a, b \vdash a \,\widehat{\mathbf{C}}\, b}\,(\mathrm{Wk}\vdash) \qquad a, a \,\widehat{\mathbf{C}}\, b \vdash b}{a \vdash b = a \,\widehat{\mathbf{C}}\, b}\,(\vdash=)$$

This completes the proof. □

Peirce proceeds to introduce two *constant symbols*, ♉ and ʘ. His definition of ♉ is 'If any thing is scriptible, ♉'. The corresponding axiom is 'Something is scriptible'. This can be formalized as follows:

$$(\text{♉})\ X \vdash \text{♉}.$$

Peirce's definition of ʘ is 'If ʘ, everything is scriptible'. The definition involves the assertion that 'Something is unscriptible'. Peirce prefers to call this a 'postulate' rather than an 'axiom': he holds the former to be not self-evident. A long discussion follows as to the historical meanings of an *axiom* and a *postulate*, as well as the presumed non-self-evidential nature of various attempts of trying to establish 'Something is unscriptible'. Interestingly, Peirce argues 'Something is unscriptible' to involve a paradoxical, *petitio principii* type of reasoning: since an interpretation is available to it according to which also unscriptible signs would have to be considered to be scriptible, the sentence 'Something is unscriptible' has a paradoxical sound to it in the context of the definitions of scriptibility and unscriptibility. In sequent calculus, the property of ʘ that something is unscriptible can be formalized as follows:

$$(\text{ʘ})\ \text{ʘ} \vdash X.$$

Remark 4.7 The notion of 'unscriptibility' obviously caused a pause in Peirce's flow of writing here. A spawning *scholium* follows in which Peirce discusses the possibility of a 'proto-logic'—which he here calls a "horse sense", or a reasoning of an infant—in which the conception of the negation and falsity has not yet been dawned upon an agent, but which are destined to so dawn as soon as 'Everything is true' is asserted, as that clause would at once suggest, even to an unreflective mind, that 'Not everything is true'—or that 'There has got be something that is unscriptible'. Peirce would make similar remarks on the possibility of such a 'paradisiacal logic' in a couple of other places, and mostly a couple of years later (see in particular R 493, c.1899; R S-30, 1906; R 669, R L 376, 1911). The idea was first mentioned somewhat earlier, though, in *The Logic of Relatives* (Peirce 1897b, p. 184). Paradisiacal reasoning is in an unstable state, since "it will soon be recognized that not every assertion is true; and that once recognized, as soon as one notices that if a certain thing were true, every assertion would be true, one at once rejects the antecedent that lead to that absurd consequence" (R 669; Pietarinen 2015a, p. 920; cf. Bobrova and Pietarinen 2019; Pietarinen et al. 2020).

We will move on and omit further exposition of the issues of this remark that would involve Peirce's ramblings about "the blot" (see Pietarinen et al. 2020). Such further issues implicate the possibility of a negation-free logic, such as what are known as positive implicational fragments of propositional logic or logics with one truth value only (Hamblin 1967), but these are tangential to the present matters.

Let DH° be the sequent system obtained from $\mathsf{DH}^{=}$ by adding the axioms (♉) and ($\boldsymbol{\odot}$). Then, we rewrite Peirce's *Corollaries 11–13* ("Appendix A", ms pp. 32–33) in the following way.

Proposition 4.8 *The following hold in* DH°*:*
(1) $\vdash a = \text{♉}\,\widehat{\mathfrak{C}}\,a$.
(2) $\vdash (a\,\widehat{\mathfrak{C}}\,\boldsymbol{\odot})\,\widehat{\mathfrak{C}}\,b = a \vee b$.
(3) $\vdash (a\,\widehat{\mathfrak{C}}\,b)\,\widehat{\mathfrak{C}}\,b = a \vee b$.

Proof (1) One has the following derivation:

$$\dfrac{a \vdash \text{♉}\,\widehat{\mathfrak{C}}\,a \qquad \dfrac{\vdash \text{♉} \quad \text{♉}, \text{♉}\,\widehat{\mathfrak{C}}\,a \vdash a}{\text{♉}\,\widehat{\mathfrak{C}}\,a \vdash a}\,(\text{Cut})}{\vdash a = \text{♉}\,\widehat{\mathfrak{C}}\,a}\,(\vdash=)$$

(2) By Theorem 4.5 (4), $(a\,\widehat{\mathfrak{C}}\,\boldsymbol{\odot})\,\widehat{\mathfrak{C}}\,b) \vdash a \vee b$. Then, one has the following derivation:

$$\dfrac{(a\,\widehat{\mathfrak{C}}\,\boldsymbol{\odot})\,\widehat{\mathfrak{C}}\,b) \vdash a \vee b \qquad \dfrac{\dfrac{\dfrac{a, a\,\widehat{\mathfrak{C}}\,\text{♉} \vdash \text{♉} \quad \text{♉} \vdash b}{a, a\,\widehat{\mathfrak{C}}\,\text{♉} \vdash b}\,(\text{Cut})}{a \vdash (a\,\widehat{\mathfrak{C}}\,\text{♉})\,\widehat{\mathfrak{C}}\,b}\,(\vdash \widehat{\mathfrak{C}}) \qquad b \vdash (a\,\widehat{\mathfrak{C}}\,\text{♉})\,\widehat{\mathfrak{C}}\,b}{a \vee b \vdash (a\,\widehat{\mathfrak{C}}\,\text{♉})\,\widehat{\mathfrak{C}}\,b}\,(\vee\vdash)}{\vdash (a\,\widehat{\mathfrak{C}}\,\text{♉})\,\widehat{\mathfrak{C}}\,b = a \vee b}\,(\vdash=)$$

(3) is shown similarly as the derivation of (2). □

All in all, Peirce has come around introducing the notion of *semantic consequence*. He notes in the *Scholium* on the manuscript page 12 ("Appendix A") that "[A]ll that logical criticism is concerned about is to secure the reasoner against passing from a true premiss, or antecedent, to a false conclusion, or consequent". By 'If x, then y' one means that "in passing from x, as antecedent, to y, as consequent, the reasoner will be so secure. That is, that y cannot be false and x true at the same time". Formally, y is a semantical consequence of x, notation $x \models y$, if y is true whenever x is true. More generally, $X \models Y$ if at least one proposition in Y is true whenever all propositions in X are true. Clearly, a proposition $x\ \widehat{\mathfrak{C}}\ y$ is true if either x is false or y is true.

Peirce then comments that the last conditional has a different meaning from ordinary conditional propositions: "An ordinary conditional proposition refers to a *general* range of possibilities of the fulfillment of the antecedent condition, from which the interpreter is at liberty to select any case he likes [...], and the truth of the consequent is guaranteed by the asserter in every case."

Remark 4.9 Peirce indeed provides an explication of conditionals which is similar to Lewis's semantics for counterfactuals (Lewis 1973). The meaning of $x\ \widehat{\mathfrak{C}}\ y$ makes no positive suggestion that x may not be true. This appears to be the only difference between $\widehat{\mathfrak{C}}$ and ordinary material conditionals, conditionals which Peirce typically terms conditionals *de inesse*.

We may view $\widehat{\mathfrak{C}}$ as what is known as *variably strict conditionals* in the subsequent literature (Stalnaker 1968; Lewis 1973; Hilpinen 2009). These are also connected to *defective conditionals* in the literature that has investigated human interpretations of conditional sentences. A defective conditional is one that is judged to be indefinite (vague) when its antecedent is false. Now this seems to be the common motivation between Peirce's generalization of truth-values to values of scriptibility and "conformity to the purpose" of what logical constants and especially the conditionals with their antecedent–consequent relationship represent, and contemporary treatises of defective conditionals over three-valued (or gappy) semantics.[16] In the theory of variably strict conditionals, the antecedent is a specification of a set of possible worlds; in Peirce's terms "a *general* range of possibilities of the fulfillment of the antecedent condition, from which the interpreter is at liberty to select any case he likes". The relevant set of worlds is, moreover, assumed to minimally depart from the worlds under consideration in evaluating the meaning of the conditional, that is, to preserve all other characteristics of the subject except those that the antecedent suggests to modify. The interpreter being "at liberty of selecting any case he likes" prefigures Stalnaker's *selection function* which selects a world most similar to the base world in which the proposition expressed by the antecedent is deemed true.[17] Non-monotonic effects can then be achieved by this variability in the set of possible worlds (a "range of possibilities") that the interpreter of the assertion in question has the liberty of choosing.

The system DH° is clearly sound and complete with respect to the semantics for classical propositional logic. That is, for all finite sets of propositions X and Y,

$$X \vdash_{\mathsf{DH}^{\circ}} Y \text{ if and only if } X \models Y.$$

Finally, we remark that using the constant $\boldsymbol{\odot}$, one can define negation and additional connectives. Let $\neg a$ be the abbreviation of $a \,\widehat{\mathfrak{C}}\, \boldsymbol{\odot}$. Then conjunction $\wedge$, disjunction $\vee$ and $=$ are defined as follows:

$$a \wedge b :=_{\mathbf{df}} \neg(a \,\widehat{\mathfrak{C}}\, b),\ a \vee b :=_{\mathbf{df}} \neg a \,\widehat{\mathfrak{C}}\, b \text{ and } a = b :=_{\mathbf{df}} (a \,\widehat{\mathfrak{C}}\, b) \wedge (b \,\widehat{\mathfrak{C}}\, a).$$

The system DH° can be simplified as the system $\mathsf{DH}^{\bullet}$ which is obtained from DH by removing ($\vee\vdash$) and ($\vdash\vee$), replacing (C2) with (C2$^{\bullet}$) $\vdash a, (a \,\widehat{\mathfrak{C}}\, b)$, and adding the axiom ($\boldsymbol{\odot}$). One can easily prove that $X \vdash_{\mathsf{DH}^{\circ}} Y$ if and only if $X \vdash_{\mathsf{DH}^{\bullet}} Y$.

16 A few years later still, in the Logic Notebook notes from February 1909, Peirce would famously present a sketch of his systems of triadic logic, whose underlying ideas have thus been simmering in the meaning of conditionals, among others, for quite some time.

17 The idea of a strict conditional, standardly attributed to C.I. Lewis (Lewis 1918) and which pairs the truth-conditions with modal necessity (a conditional is true when the antecedent is necessarily true) has as is well known its roots in Peirce's Philonian conditional (Peirce 1896; Sanford 1989).

5 Ethics of notation

Last, given the peculiarity of the new sign introduced in R 501, ⸦̈, a word should be said on Peirce's "the ethics of notation", the method of investigation which he adapted to formal sciences from biological taxonomies developed by Carl von Linné. The maxim of the ethics of notation, which Peirce had formulated around the same time, states that one should not introduce new notations when the old ones will do and that the justification of an introduction of new notation should be given in terms of the new meanings that the new pieces of notation introduce (LoF 1; R 515, R 516, R 530, 1901). His definition of the maxim of the ethics of notation was articulated in papers coeval to Dragon Logic in the following precise words:

> The person who introduces a conception into science has both the right and the duty of prescribing a terminology and a notation for it; and his terminology and notation should be followed except so far as it may prove positively and seriously disadvantageous to the progress of science. If a slight modification is sufficient to remove the objection, a much greater one should be avoided. (R 530, 1901; Peirce 2019–2021, LoF 1, p. 419)

We should assume that Peirce observed the validity of this maxim to hold in his own writings, too. So what then is the new meaning that the Dragon-Head and the Dragon-Tail notations add to the existing ones, namely to his usual signs of ' ⤙ ' or '—<', or even to the Box-notated sign of consequence ⋉ or its cursive form ∝ obtained from ⋉, which depending on their contexts stand for the signs of copula, implications or logical consequences, respectively?[18]

The Dragon Tail and Dragon-Tail symbols themselves are traditionally used to signify lunar nodes. In the medieval and early modern times, the ascending node was called the Dragon's Head (Lat. *caput draconis*) with the astronomical symbol ☊ and the descending node the Dragon's Tail (Lat. *cauda draconis*) with the astronomical symbol ☋. Peirce often used astronomical symbols in his logic, but the Dragon-Head notation turned to its side is unique to the present set of manuscripts. Other symbols such as the ascending and descending crescents ☾ and ☽ then naturally suggested themselves to Peirce to be used to signify an arbitrary binary relation and its converse.

We believe that Peirce was led to this new class of signs primarily not because the Dragon symbol resembles various implicational signs in its look, but because he desired at this time to complete his project of carrying out the transition of his logical systems and their notations from algebraic to graphical designs, and that because in this transition one has to exercise extreme caution in how the meanings get to be translated from one notation to another. In 1901, Peirce wants to maintain the algebraic notation and the sign of the copula which does not involve, unlike in the sign of the consequence from the qualitative logic (NEM IV, pp. 101–115; R 736; W5) with its vinculum or the 'streamer' ⌝⊹, the notion of *scope*. For in the parenthesis-free algebraic notation using signs of consequence with the streamer and the obelus, such signs also signify

[18] See, e.g. Fisch (1986) who coined the term "Box-X" notation. The system was studied in Clark (1997) and Zellweger (1997). Peirce's own presentation of the system of sixteen binary connectives has remained unpublished until Peirce (2019–2021), in LoF 1.

the scope of the propositions scribed underneath the vinculum, such as the antecedent x of the conditional as $\overline{x}$⅄ , thus dispensing with punctuation marks. But the constants of the Dragon-Head notation, like the usual signs for conditionals, do not have scope-denoting features incorporated in them.

Peirce's explications of the philosophy and ethics of notation emphasize the logical contribution of punctuation: "The Klammern, which are commonly spoken of in the elementary algebra-books as mere subsidiary signs, are in reality the very heart of algebraic notation" (R 530, c.1901; LoF 1). In this sense, Dragon Logic is a parallel study of logical systems which—while his studies on logical graphs were largely put on hold in these intervening years around 1901—continues the algebraic rather than the graphical line of investigation. The Dragon-Head sign for logical consequence is an algebraic one that allowed Peirce to carry out a fresh, and in some sense close to a complete, investigation of the algebra of the copula and its properties, in a notation which unlike signs of consequence from his earlier qualitative logic which express material conditionals in a parenthesis-free notation, are notationally classical (have parentheses) but still differ in meaning from them. As his Dragon-Head conditional Ĉ⁚ involves elements of variably strict conditionals, the introduction of this new piece of notation to denote the sign of consequence with its new meaning in R 501 is justified.

6 Conclusions

We have reconstructed and transcribed a text from previously unpublished manuscripts excavated from the Peirce Papers deposited in the Harvard's Houghton Library, folders R 9, R 11, R 501 (1901). We have provided an interpretative introduction to the text, with an analytic description of the central concepts and ideas that emerge out of the reconstructed unity of the text. In this piece, Peirce is seen to outline both a general theory of deduction and a general theory of the consequence relation. He presents them in an unusual notation of the Dragon Head and Dragon Tail (C⁚). Peirce does not use these signs elsewhere in his writings and for that reason their importance may have gone unnoticed in the earlier literature.

Peirce is led to these definitions of deduction and consequence aided by a couple of important generalizations that characterize the wider semiotic context of the present text: first, Peirce desires to generalize propositions to all kinds of signs, second, he proposes to generalize truth values to scriptibility and unscriptibility, and third, he generalizes derivation to transformability. Despite the fact that the notions of 'scribing' and 'transforming' were in full swing in his coeval writings that concern the logic of existential graphs as well as his general algebra of logic, the notions of scriptibility and transformability justify their fundamental role in understanding the general logical and philosophical meaning of deduction and consequence relation.

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our efforts not only in improving the presentation of the work but also in order to make better sense of the status of Peirce's logic as it obtained at the turn of the century.

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Declarations

Conflict of interest The authors declare they have no financial or non-financial interests.

Appendix A

[On the Foundations of Mathematics], R 9, R 11, R 501, c.1901

[R 9, p. 1] §1. A sign is not a real thing. The same sign may occur or as we may say, ~~may~~ can be *uttered*, over and over again. We may call these things embodying the same sign *replicas* of it. They need not be alike as things. Man, *homo*, ἄντροπος are the same sign. A sign is ~~supposed~~ intended to correspond to a real thing or fact, or to something relatively real; this *object* of the sign may be the very sign itself, as when a map is precisely superposed upon that which it maps. It is a perfection in a sign if it separately represent its object; in which case it becomes a *proposition* and is true or false. A sign is also ~~supposed~~ intended to determine, in a mind or elsewhere, a sign of the same object; this *interpretant* of the sign may be the very sign itself, but as a general rule it will be different. It is a perfection in a sign separately to signify its intended interpretant. If it does this, it becomes an *argumentation* or [p. 2] *argument*. (Some pedants insist on the former word; but the very best usage supports the latter.)

§2. A sign must, from the nature of it, be applicable to different objects, supposing there happen to exist any such objects. This [is] true even of a proper name. Phillip of Macedon may stand for Phillip drunk, or for Phillip sober, or for the collective Phillip. But, from a special point of view, a sign may be regarded as proper to a single object. If a sign is apt to represent many things, the option as to what single thing it shall be taken to represent may be reserved by the utterer of it, to whom it naturally belongs, in which case it may be said to be used *vaguely*, or *not definitely*. The utterer may, however, transfer this option to the interpreter, in which case the sign may be said to be used *generally*, or *not individually*. Obviously the option cannot, in the same respect, at once lie with both parties. Hence, a sign cannot be at once vague and general in the same respect. It may, [p. 3] however, be both definite and individual, and in that case may be said to be used *singularly*. 'Man' is used vaguely in 'Some man sins'; generally, in 'Every man sins'; singularly, in 'This man sins'. If we put Roman lower-case letters for ordinary signs, we may put an acute accent over a letter, as á, to show that it is taken generally; a grave accent, à, to show that it is used vaguely; and a circumflex, â, to show that it is used singularly.

§3. Let us use the sign of the moon's first quarter, ☾, to signify a dyadic relation, and ☽ for its converse. Thus, â☾ê might mean 'Socrates loves Plato'. Then we shall have,

â☾ê = ê☽â
â☾é = é☽â   à☾ê = ê☽à
á☾é = é☽á   à☾è = è☽à.

But á☾è may be understood in two senses. For here the utterer is to determine the object of e; the interpreter that of a. The utterer is essentially a defender of his own [p. 4] proposition and wishes to interpret it so that it will be defensible. The interpreter, not being so interested, and being unable to interpret it fully without considering to what extreme it ~~will~~ may reach, is *relatively* in a hostile attitude, or looks for the interpretation least defensible. Whichever of the two makes his choice of the object he is to choose, after the other has made his choice, is supposed to know what that choice was. This is an advantage to the defence or attack, as the case may be. If we enclose the earlier choice in parenthesis (á☾)è will signify more than á(☾è). We may omit the parentheses from the last.

§4. For purposes of logical criticism, *vague and general signs are to be treated as substitutes for singulars*.

A vague sign ~~gives~~ leaves the *utterer* the liberty of choosing the singular. Hence, it will be successfully *defended by establishing* a suitably *individual* sign. It cannot be *defended* by *establishing* a *general sign*, [R 501, p. 5] since giving the *interpreter* liberty to find a singular confers no power to do so.

A *vague* sign cannot be refuted by *refuting* any other kind of sign than a *vague* sign.

A general sign transfers to the *interpreter* the choice of the singular. Hence, it will be successfully *refuted* by *refuting* a suitable *definite* sign. It cannot be *refuted* by *refuting* a *vague* sign; [R 501, p. 5] since proving that the *utterer* cannot find a singular does not prove that the interpreter can.

A *general* sign cannot be *established* by *establishing* any other kind of sign than a *general* one.

§5. Let a *formal definition* be defined as a definition, not of the peculiar qualitative signification, or flavour, of its definitum, but of the essential relations between different applications of it.[19] I proceed to give formal definitions establishing a system of notation, beginning with a certain sign of relation, ☊ (a "dragon's head" turned on its side), and since, to begin with, this is to be taken in a singular sense, a circumflex accent shall surmount it. We shall be supposed to be provided with a certain blank sheet, to be called the "sheet of assertion"; in connection with this we are to be supposed to

[19] [R 9, p. 5, Alt.] Let a *formal definition* be a definition, not of the signification of a sign, but of all the necessary relations between different applications of it. I will proceed to give formal definitions of two ~~signs~~ relations which may be represented by the signs for dragon's head and tail turned upon their side, and the latter altered slightly, thus: ☊, ☋. Parentheses round the last two of three signs are omitted. Thus, we write ú☊à for ú(☊à).
We begin with the study of the former relation, which we will first take in a singular sense.
Whatever is written without any indication to the contrary is supposed to be written on a "sheet of assertion" and to be held for true, applicable, justified, etc. [p. 6]

have a certain purpose, which remains vague, so that the writing of any given sign on the sheet of assertion may or may not accord with that purpose, but the purpose does not *require* any entire sign to be written. Whether or not any or every sign rendered possible by the system of notation may be written, *as an entire sign*, on the [p. 6] sheet of assertion conformably to the purpose, remains to be seen.

A sign that may be so written as an entire sign will be termed *scriptible*, *good*, *true*, etc. Otherwise, it will be *unscriptible*, *bad*, *false*, etc. The mere writing of the sign, as an entire sign, conformably to the purpose of the notation is to be considered as, in effect, conclusive evidence that it is *scriptible*. This establishes a relation between propositions which we may really judge to be true concerning the notation, and signs which we may imagine to be written conformably to our imagined purpose upon the imagined sheet of assertion. In order that a notation should have an algebraic character, it must be that under certain conditions *transformations* should be permitted. Such transformation may either leave the original sign standing and add another, in which case it may be called a *deduction*, or it may involve the ~~erasure~~ crossing-out of the first sign, in which case, it may be called a *correction*. The necessary and sufficient condition of the *permissibility* of a *deduction* is that it should not proceed from a scriptible [p. 7] to an unscriptible sign; where I refer to the entire sign. I shall strictly confine my use of the conjunctions 'If ___ then ___ ' to the permissibility of deductions. But it is to be observed that I do not here ~~use~~ confine the word deduction to *logical* deductions, but extend it to all statements of consequences. That is to say the consequent may be true if the antecedent is true, merely as a matter of fact, not necessarily by virtue of the logical form, or because an analogous deduction would be true in every universe to which the language would be applicable at all. Ordinarily deductions are *general* with respect to their antecedent. Thus to say, 'If a man is ill, he will be excused from business', is to give the interpreter a right to apply the remark to any case of a man's being ill that he can find. Being general, it is, in the same respect, *definite*. The utterer does not reserve to himself any right to limit the breadth of application of the antecedent. But in speaking of $\widehat{\mathbf{C}}$, the generality of the deduction disappears. [p. 8] The antecedent refers to a single event or state of things without any reference to analogous cases. Therefore, we cannot say, as a universal rule, that a deduction is general with respect to its antecedent, but only that it is *definite*. In an

Footnote19 contiuned

Supposing that, whatever *u* may be,

if ú **C** à
then ú **C** è,

we shall have

á **C** è.

If ú **C** à
and á **C** è
then ú **C** è.

Supposing that, whatever *u* may be,

if é **C** ù
then á **C** ù,

we shall have

á **C** è.

If á **C** è
and é **C** ù
then á **C** ù.

The other definition may be postponed.
Theorem I. Whatever a may be, á **C** à.
Proof. For whatever u may be, if ú **C** à, then ú **C** à.

ordinary conditional proposition, there is a tacit implication that there is some *reason* for the consequence, which is not stated. In ~~this~~ respect to the reason of the antecedent, the proposition is *vague*. But this is not necessarily so: the reason may be fully stated in the antecedent. All that can be said universally, is that the interpreter is not allowed to ~~put into the utterer's~~ attribute to the utterer such reason as he likes. In respect to the reason of the antecedent, the proposition is individual. In regard to the consequent, the interpreter is at liberty to single out by abstraction any of the elements of the meaning of the consequent. 'If a man is ill he will be excused from business'. To be excused, he must be a moral being. Therefore, the interpreter may take the sentence as implying that if a man is ill, he remains a moral being. But since the [p. 9] consequent may conceivably be unanalyzable, we can only say that it is *definite* in its logical depth. For so far as it [is] indefinite in this respect, it is meaningless; as 'If a man is ill he has some resemblance to an excused man'. On the other hand in its application to an object, the consequent retains its natural *vagueness*. 'If a man is ill, he is excused etc.' does not say what individual man he is. But since, it may be definite in that respect, we can only say that it confers no liberty of choice of the object upon the inter[preter] [end]

[p. $6\frac{1}{2}$]

Definition of $\widehat{C}$

Clause 1. The relation expressed by $\widehat{C}$ is *definite* in such sense that, whatever a and i may be,

$$\text{If } a, \text{ then } i \,\widehat{C}\, a.$$

Clause 2. The relation expressed by $\widehat{C}$ is *individual* in such sense that, whatever a and e may be,

$$\text{Either } a \text{ or } a \,\widehat{C}\, e.$$

Clause 3. Whatever i and e may be,
If i and $i \,\widehat{C}\, e$, then e.

Corollary 1. Whatever a may be,

$$a \,\widehat{C}\, a.$$

For putting a for e in Clause 2,

$$\text{Either } a \,\widehat{C}\, a \text{ or } a.$$

But putting a for i in Clause 1,

$$\text{If } a \text{ then } a \,\widehat{C}\, a.$$

Corollary 2. From Clause 1 and Clause 2, whatever i, a, e, may be,

$$\text{Either } a \,\widehat{C}\, e \text{ or } i \,\widehat{C}\, a.$$

[*Corollary 3.* Crossed out.]

[p. 7]

Theorem I. If e be such that, whatever x may be, $x \,\widehat{C}\, e$, then e.

Proof For assume that, whatever x may be, $x \,\widehat{C}\, e$. Then I have only to prove e. Let x be $a \,\widehat{C}\, a$. Then, by hypothesis, $(a \,\widehat{C}\, a) \,\widehat{C}\, e$; whence, by Clause 3, if $a \,\widehat{C}\, a$ then e; whence, by Corollary 1, e. □

Theorem II. Whatever a and e may be, in case, whatever x may be,

$$\text{If } x \mathbin{\widehat{\mathrm{C}}} a \text{ then } x \mathbin{\widehat{\mathrm{C}}} e,$$

$a \mathbin{\widehat{\mathrm{C}}} e$ will be scriptible.

Proof For suppose a and e to be any such things that, whatever x may be, if $x \mathbin{\widehat{\mathrm{C}}} a$ then $x \mathbin{\widehat{\mathrm{C}}} e$. Then, I have only to show that $a \mathbin{\widehat{\mathrm{C}}} e$. Let x be a. Then, by hypothesis, if $a \mathbin{\widehat{\mathrm{C}}} a$ then $a \mathbin{\widehat{\mathrm{C}}} e$; whence, by Corollary 1, $a \mathbin{\widehat{\mathrm{C}}} e$. □

Theorem III. Whatever i and a may be, in case, whatever x may be,

$$\text{If } a \mathbin{\widehat{\mathrm{C}}} x, \text{ then } i \mathbin{\widehat{\mathrm{C}}} x,$$

$i \mathbin{\widehat{\mathrm{C}}} a$ will be scriptible.

Proof For suppose i and a to be any such things that, whatever x may be, if $a \mathbin{\widehat{\mathrm{C}}} x$ then $i \mathbin{\widehat{\mathrm{C}}} x$. Then I have only to show that $i \mathbin{\widehat{\mathrm{C}}} a$. Let x be a. Then, by hypothesis, if $a \mathbin{\widehat{\mathrm{C}}} a$, then $i \mathbin{\widehat{\mathrm{C}}} a$; whence, by Corollary 1, $i \mathbin{\widehat{\mathrm{C}}} a$. □

[p. 8]
Theorem IV. Whatever i, a, e may be, if

$$i \mathbin{\widehat{\mathrm{C}}} a \mathbin{\widehat{\mathrm{C}}} e,$$

then

$$a \mathbin{\widehat{\mathrm{C}}} i \mathbin{\widehat{\mathrm{C}}} e.$$

Proof For suppose i, a, e, to be any such things that $i \mathbin{\widehat{\mathrm{C}}} a \mathbin{\widehat{\mathrm{C}}} e$. Then I have only to show that $a \mathbin{\widehat{\mathrm{C}}} i \mathbin{\widehat{\mathrm{C}}} e$. By Clause 1,

(No. 1) If e, then $i \mathbin{\widehat{\mathrm{C}}} e$.
(No. 2) If $i \mathbin{\widehat{\mathrm{C}}} e$, then $a \mathbin{\widehat{\mathrm{C}}} i \mathbin{\widehat{\mathrm{C}}} e$. By Clause 2,
(No. 3) Either i or $i \mathbin{\widehat{\mathrm{C}}} e$.
(No. 4) Either a or $a \mathbin{\widehat{\mathrm{C}}} i \mathbin{\widehat{\mathrm{C}}} e$. By Clause 3,
(No. 5) If i and $i \mathbin{\widehat{\mathrm{C}}} a \mathbin{\widehat{\mathrm{C}}} e$, then $a \mathbin{\widehat{\mathrm{C}}} e$.
(No. 6) If a and $a \mathbin{\widehat{\mathrm{C}}} e$, then e. By (No. 5) and the hypothesis,

if i, then $a \mathbin{\widehat{\mathrm{C}}} e$; whence, by (No. 3), either $i \mathbin{\widehat{\mathrm{C}}} e$ or $a \mathbin{\widehat{\mathrm{C}}} e$; whence by (No. 6), either $i \mathbin{\widehat{\mathrm{C}}} e$ or if a then e; whence by (No. 1), if a, then $i \mathbin{\widehat{\mathrm{C}}} e$; whence, by (No. 2), if a, then $a \mathbin{\widehat{\mathrm{C}}} i \mathbin{\widehat{\mathrm{C}}} e$; whence, by (No. 4), $a \mathbin{\widehat{\mathrm{C}}} i \mathbin{\widehat{\mathrm{C}}} e$. □

[p. 9]
Theorem V. Whatever i, a, e may be, if $(i \mathbin{\widehat{\mathrm{C}}} a) \mathbin{\widehat{\mathrm{C}}} e$, then either i or e.

Proof For suppose i, a, e to be any such things that $(i \mathbin{\widehat{\mathrm{C}}} a) \mathbin{\widehat{\mathrm{C}}} e$. Then I have only to show that either i or e. By Clause 3, if $i \mathbin{\widehat{\mathrm{C}}} a$ and $(i \mathbin{\widehat{\mathrm{C}}} a) \mathbin{\widehat{\mathrm{C}}} e$, then e; whence, by the hypothesis, if $i \mathbin{\widehat{\mathrm{C}}} a$ then e. But by Clause 2, either i or $i \mathbin{\widehat{\mathrm{C}}} a$; whence either i or e. □

Theorem VI. Whatever i, a, e may be, if $(i \mathbin{\widehat{\mathrm{C}}} a) \mathbin{\widehat{\mathrm{C}}} e$, then $a \mathbin{\widehat{\mathrm{C}}} e$.

Proof For suppose i, a, e to be any such things that $(i \mathbin{\widehat{C}} a) \mathbin{\widehat{C}} e$. Then I have only to show that $a \mathbin{\widehat{C}} e$. By Clause 1,

(No. 1) If a, then $i \mathbin{\widehat{C}} a$.
(No. 2) If e, then $a \mathbin{\widehat{C}} e$. By Clause 2,
(No. 3) Either a or $a \mathbin{\widehat{C}} e$. By Clause 3,
(No. 4) If $(i \mathbin{\widehat{C}} a)$ and $(i \mathbin{\widehat{C}} a) \mathbin{\widehat{C}} e$, then e.

By (No. 4) and the hypothesis, if $(i \mathbin{\widehat{C}} a)$ then e; whence by (No. 1), if a then e; whence, by (No. 3), either $a \mathbin{\widehat{C}} e$ or e; whence by (No. 2), $a \mathbin{\widehat{C}} e$. □

[p. 10]

Theorem VII. Whatever i, a, e may be, if either e or both i and $a \mathbin{\widehat{C}} e$, then $(i \mathbin{\widehat{C}} a) \mathbin{\widehat{C}} e$.

Proof For, by Clause 1, if e then $(i \mathbin{\widehat{C}} a) \mathbin{\widehat{C}} e$. Consequently, it will be sufficient to prove that if i and $a \mathbin{\widehat{C}} e$ then either e or $(i \mathbin{\widehat{C}} a) \mathbin{\widehat{C}} e$. For this purpose, assume that i, a, e are any such things that; (1), i; and (2), $a \mathbin{\widehat{C}} e$. Then I have only to show that either e or $(i \mathbin{\widehat{C}} a) \mathbin{\widehat{C}} e$. By Clause 2,

(No. 1) Either $i \mathbin{\widehat{C}} a$ or $(i \mathbin{\widehat{C}} a) \mathbin{\widehat{C}} e$. By Clause 3,
(No. 2) If a and $(a \mathbin{\widehat{C}} e)$, then e.
(No. 3) If i and $(i \mathbin{\widehat{C}} a)$, then a.

By (No. 2) and the second assumption, if a then e; whence, by (No. 3), if i and $i \mathbin{\widehat{C}} a$, then e; whence, by the first assumption, if $i \mathbin{\widehat{C}} a$ then e; whence by (No. 1) either e or $(i \mathbin{\widehat{C}} a) \mathbin{\widehat{C}} e$. □

Corollary 3. Hence, *à fortiori*, if both $a \mathbin{\widehat{C}} e$ and either i or e, then $(i \mathbin{\widehat{C}} a) \mathbin{\widehat{C}} e$.

Scholium. This is the most difficult of all the theorems of this section and may be considered as the key to the subject.

[p. 11]

Theorem VIII. Whatever i and a may be, if i then $(i \mathbin{\widehat{C}} a) \mathbin{\widehat{C}} a$.

Proof. For, by Theorem VII, whatever i, a, e may be, if both i and $a \mathbin{\widehat{C}} e$ then $(i \mathbin{\widehat{C}} a) \mathbin{\widehat{C}} e$. Let e be a. Then, if both i and $a \mathbin{\widehat{C}} a$, then $(i \mathbin{\widehat{C}} a) \mathbin{\widehat{C}} a$; whence, by Corollary 1, if i then $(i \mathbin{\widehat{C}} a) \mathbin{\widehat{C}} a$.[20]

Corollary 4. By Clause 2, either i or $i \mathbin{\widehat{C}} e$. But by Clause 1, if e then $i \mathbin{\widehat{C}} e$. Hence, if from i follows e then $i \mathbin{\widehat{C}} e$.

Corollary 5. If $a \mathbin{\widehat{C}} e$, then $(i \mathbin{\widehat{C}} a) \mathbin{\widehat{C}} (i \mathbin{\widehat{C}} e)$. For, by Theorem VIII, if $a \mathbin{\widehat{C}} e$, then from i follows $(i \mathbin{\widehat{C}} a) \mathbin{\widehat{C}} e$; whence, by Corollary 4, if $a \mathbin{\widehat{C}} e$, then $i \mathbin{\widehat{C}} (i \mathbin{\widehat{C}} a) \mathbin{\widehat{C}} e$; whence, by Theorem IV, if $a \mathbin{\widehat{C}} e$, then $(i \mathbin{\widehat{C}} a) \mathbin{\widehat{C}} i \mathbin{\widehat{C}} e$.

[p. 12]

Corollary 6. From Corollary 5, it follows, by Clause 3, that if $i \mathbin{\widehat{C}} a$ and $a \mathbin{\widehat{C}} e$, then $i \mathbin{\widehat{C}} e$.

[20] The mathematician will say, with some disdain, that most of my "theorems" are corollaries; while some of my corollaries, by a liberal construction of the term, might be dignified with the title of theorems. I wish to say, therefore, that I have to propose, in future, to make a scientific distinction between two classes of mathematical inferences, to one of which the name 'corollarial' may be properly applicable, and to the other that of 'theorematic'. But this distinction is not yet thoroughly worked out, so that, while I attempt to apply it here, I may very likely have given the wrong titles to some of the propositions.

Corollary 7. From Corollaries 6 and 4, it follows that if $i \mathbin{\widehat{\mathrm{C}}} a$, then $(a \mathbin{\widehat{\mathrm{C}}} e) \mathbin{\widehat{\mathrm{C}}} (i \mathbin{\widehat{\mathrm{C}}} e)$.

Scholium. All that logical criticism is concerned about is to secure the reasoner against passing from a true premiss, or antecedent, to a false conclusion, or consequent. All that is meant, therefore, in this paper by 'If x, then y' is that in passing from x, as antecedent, to y, as consequent, the reasoner will be so secure. That is, that y cannot be false and x true at the same time. Thus, the conditional proposition has a different meaning from ordinary conditional propositions. For an ordinary conditional proposition refers to a *general* range of possibilities of the fulfillment of the antecedent condition, from which the interpreter is at liberty to select any case he likes (under assumptions supposed to be agreed upon, although they are commonly more or less vague), and the truth of the consequent is guaranteed by the asserter in every case. But Scotus [Wadding 1639] and other logicians consider a kind of conditional that occasionally [p. 13] occurs in ordinary speech, and which they term a "conditional proposition *de inesse*". This does not refer to any general range of possibilities, but only to a definite and individual state of things. Everything written on our "sheet of assertions" in the present paper is supposed to represent a definite and individual universe, "the Truth"; so that when a conditional proposition, say 'If i, then e' is written on the sheet, no illative connection is implied between i and e, which should mean that throughout a general range of possibilities the truth of anything analogous to i would be accompanied by the truth of something corresponding to e, but all that is meant is that taking the universe of Truth in the state in which it happens to be, although i may not be true (or applicable, or justifiable, or whatever is the appropriate variety of scriptibility), and should it not be so, nothing whatever is said about e, yet provided i is true, the interpreter may be sure that e is true. That is in passing from i to e, one would not pass from truth to falsity. For example, [p. 14] considered as a proposition *de inesse*, it would be true to say that 'If 6 is not divisible by 3, it is a prime number', or 'If a quadratic equation has two roots, a quadrangle has two dimensions', or 'If 7 is greater than 8, then 8 is greater than 7'. Perhaps it may aid to reconcile the reader to such language to call attention to a feature of conditionals of the ordinary kind. The proposition that 'If a sensible man were, under ordinary circumstances to deliberately dash a full inkstand upon his library floor, he would voluntarily ruin his carpet (supposing the room to be carpeted)' is certainly true, since an inkstand thrown on such a floor would inevitably ruin the carpet; and a sensible man would know this; so that if he did it, he would do it voluntarily. At the same time no sensible man would, under any ordinary circumstances, voluntarily ruin his library carpet. From this it follows that 'If a sensible man were, under ordinary circumstances, deliberately dash a full inkstand upon his library floor, he would so do it, or at least design so to do it as not to ruin his carpet, for he would not voluntarily ruin his carpet', for if a man [p. 15] were to do such a thing, he would be out of his senses. Here, then, are two contradictory propositions both true at once. This is possible because they refer to *general* conditions, and generals are not subject to the principle of contradiction. The truth is that no sensible man under ordinary circumstances would throw a full inkstand on his library floor. What reconciles us to a conditional proposition with a false antecedent is that, though it does not apply in the particular contingency, there are a great many cases where the general principle does apply; and in those cases it is true; and the utterer, disclaiming all attempt to find out whether the present case is one of those, but throwing the determination upon the interpreter, states what the interpreter will find when he meets such a case, if he ever does. Now the conditional *de inesse* is merely the limiting case when the general class of cases referred to is but a single state of things. The ~~utterer~~ asserter still ~~disclaims~~ refuses all responsibility for the antecedent. That is the concern of the interpreter. He simply guarantees him against all danger of passing from truth to falsity.

[p. 16] But such a proposition perplexes the ordinary interpreter, because in ordinary conditionals it is left to the interpreter's good sense, to guess what is the class of analogous cases to which the antecedent is meant to apply, and consequently, when he meets with a conditional *de inesse*, he straightway casts about for such a class of cases, although no such class is referred to. In these propositions, 'If A is true, then B is true' practically amounts merely to saying that 'Either A is not true or B is true (or both)'; the only difference being that the conditional makes no positive suggestion that A may not be true.

DEFINITION OF =.

CLAUSE 1. Whatever i and e may be, if $i = e$ and i, then e.
CLAUSE 2. Whatever i and e may be, if $i = e$ and e, then i.
CLAUSE 3. Supposing that if i then e, and if e then i, then, whatever i and e may be, $i = e$.

[p. 17]

Corollary 8. If i then $e = i \,\widehat{\mathrm{C}}\, e$. For assume, first, that i and e are any such things that if i then e. Then, since by Clause 2 of the Definition of $\widehat{\mathrm{C}}$, either i or $i \,\widehat{\mathrm{C}}\, e$, it follows that either e or $i \,\widehat{\mathrm{C}}\, e$. But, by Clause 1 of the same Definition, if e then $i \,\widehat{\mathrm{C}}\, e$. Hence e. Thus, it is proved that

(No. 1) Supposing that if i then e, it follows that $i \,\widehat{\mathrm{C}}\, e$.

Secondly, assume that i and e are any such things that $i \,\widehat{\mathrm{C}}\, e$. Then, by Clause 3 of the Definition of $\widehat{\mathrm{C}}$, it follows that if i, then e. Thus, it is proved that, whatever i and e may be

(No. 2) Supposing that $i \,\widehat{\mathrm{C}}\, e$, it follows that if i then e.

From (No. 1) and (No. 2), taken together with Clause 3 of the Definition of =, the corollary follows.

Scholium. The explanation of the meaning of the word *postulate* (ἀίτεμα) given by Aristotle (1831, 1844) (who must have had some acquaintance with mathematics to be admitted into the school of Plato, and must there have become familiar with the language of mathematics, [p. 18] as his writings show that he was with their discussions of that time) entirely agrees with the usage of Euclid in the following century, and with the established meaning of the word in English and in French. Aristotle, as is explained by Johannes Philoponus, draws an insignificant distinction between a postulate and a hypothesis, which latter word he uses in several allied senses, but disregarding idle distinctions, we may say that Aristotle gives to the word *postulate* the precise signification which Riemann attached to hypothesis when he wrote "Über die Hypotheses welche der Geometrie zu Grunde liegen" (Riemann 1854/1867). The point about which Aristotle is most particular is that a postulate is not evidently true.

The distinction which he makes between *hypothesis*, or supposition, in one of its senses, and *postulate*, in its only sense, is that the interpreter will already believe a hypothesis and will have no positive belief in a [p. 19] postulate. Two millennia of familiarity with the postulates of geometry seem to have bred in us a traditional and almost natural belief in most of them. Any close student of the first book of the *Elements* will, I believe, agree with me that Euclid purposely throws his fifth postulate (the one about parallels) into a form intended to bring about its non-evident character. This, then, is the meaning which we ought to attach to the word *postulate*. It is the only meaning upon which good usage, in English and French, can remain or ever be fixed. The Germans, under the influence of Baron Christian Wolff (1713), commonly attach to the word *Postulat* an abusive meaning of considerable antiquity; that of an indemonstrable practical proposition. Germans, generally, do not seem to feel that there is any such thing as the ethics of terminology. The mathematico-logical sense of the [p. 20] word axiom (ἀξίομα) is a secondary one. In general, it means approval, especially general approval, or what is approved or generally approved. Aristotle, in the same chapter X of the first *Posterior Analytics* just cited, speaks of τά κοινα λεγόμενα ἀξιόματα; and Euclid (1883) calls them κοιναί ἔννοιαι. A contemporary of Euclid was Zeno, the founder of the stoic school; among the stoics κοινέ ἔννοια meant a judgment of *common sense*, which seems to be an English translation of the phrase. Doubtless κοινέ was in this phrase understood to refer to the unanimity of men about such a judgment. (To a stoic ἀξίομα would mean no more than a proposition.) When Aristotle speaks of 'common axioms', the word 'common' must have the same meaning, since he refers to the expression as a familiar one. Otherwise, one might gather from what Aristotle says that the axioms are called common as being used in [p. 21] different branches of science. Very significant of the meaning attached by Aristotle to the word axiom is the remark of this true prince of logicians (76b16) which, though half-expressed, in the style of the Analytics, I understand to mean that a science does not fail in conclusiveness by omitting distinctly to note an axiom. For the axiom being self-evident, even when it is not noted it remains evident that the premisses cannot be true without the conclusion being true. It is to the sense of a principle to which this remark applies that the word *axiom* ought to be restricted.

Concerning *Definition*, also, Aristotle in this chapter makes a very essential remark, which acute logicians [Mill (1846) among them] have failed to give heed, to the effect that *a definition does not assert the existence of anything*.[21]

[p. 22]

Definition of ♉.[22]

If any thing is scriptible, ♉.

Axiom. Something is scriptible.

Scholium. Consider the following argument: By Corollary 1, $a \widehat{\mathrm{C}} a$ is scriptible. *Ergo*, Something is scriptible.

[21] ῾Ο μέν οὖν ὅδα ο΄νκ εἰσὶν ντοτέσεις ο΄νδὲνγὰς εἶναι ἢ μὴ λέγονται. (ον δέν is the reading of the two best MSS.; and perhaps λέγονται is one of Apellicon's conjectures for λέγονσι. But that need not be).

[22] ♉ is the sign of Taurus. The more the form of the character approaches mine, the better.

This is unexceptionable reasoning. Yet since Corollary 1 is based exclusively upon a definition, it must not be understood to assert, or imply, the existence of anything, while the Axiom does distinctly assert existence. Corollary 1 does assert, or imply, that $a \,\widehat{\mathbf{C}}\, a$ is scriptible. It simply writes $a \,\widehat{\mathbf{C}}\, a$, and then, the reasoner *observes* that $a \,\widehat{\mathbf{C}}\, a$ *is* written. When the argument says, 'By Corollary 1, $a \,\widehat{\mathbf{C}}\, a$ is scriptible', this is true in the sense of 'By observation of Corollary 1, etc.' This does not affect the validity of the argument. For if a reasoner were not permitted to observe facts about his premisses which those premisses do not themselves [p. 23] assert, no mathematical reasoning would be possible; not even

If A, then B;
But A;
∴ B.

As Aristotle well says, a science can reach its conclusions with perfect logic without referring to axioms or being distinctly aware of them. The only use that the expression of an axiom series is that of rendering the *why* of the conclusion, the source of it, more intelligible. When Aristotle calls axioms indemonstrable, he must not be understood to mean that an argument concluding an axiom is necessarily a bad argument. For every argument which proceeds from sound premisses and is of such a kind that no argument of its kind could have true premisses and a false conclusion, is a good argument. Therefore, an argument is not a *petitio principii*, nor has it any logical fault, merely because it assumes something [p. 24] absolutely self-evident. This follows from the very purpose of logical criticism, and not to recognize its truth is to be in imminent danger of falling into perhaps the most pestilential of all the swamps of metaphysics. Every axiom, therefore, is essentially capable of proof, provided anybody can be found who does not already see the truth of it. But it is quite true that there can be no real proof of what is not doubted, and therefore, so far as an axiom is self-evident, Aristotle is quite right in calling it indemonstrable. But it is possible that a man should be in a state of mind in which he cannot apprehend the meaning of an axiom; and for him it may be proved. [p. 25]

DEFINITION OF ʘ.
If ʘ, everything is scriptible.

POSTULATE. Something is unscriptible.

Scholium. The following argument might be proposed to prove this postulate:

For suppose it false. Then something is false, or unscriptible, contrary to the hypothesis. Therefore, the falsity of the postulate being impossible, it must be true.

This argument rests upon an entirely different sort of basis from the argument which proves the above axiom. For though the conclusion of this argument necessarily follows from the premiss; this premiss is quite undeniable, yet the argument is, as the reader assuredly perceives, a *petitio principii*.

For any argument which proceeds from a premiss such that a consistent opposition to its conclusion can be maintained by not accepting [p. 26] that premiss, and in no other way, is a *petitio principii*. Now a person who does not admit that anything is unscriptible is in one or other of two states of mind; namely, either the conceptions of falsity and of denial are unintelligible to him, or he holds that the system of signs to which "anything" refers does not contain any false or unscriptible sign nor afford any means of constructing such a sign. But in either of these cases he will not admit that there is any sign that signifies that 'Something is unscriptible'. But this is the essential premiss of the argument proposed to prove that something really is unscriptible. Moreover, if the person who should maintain that everything is scriptible were to admit that premiss, his position would at once become untenable. Consequently, the argument fulfils all the conditions of a *petitio principii*.

[p. 27] Were the proposition under consideration an *axiom*, it might perhaps be absurd to say that any argument concluding it was a *petitio principii*. But it is not an axiom: it is a *postulate*. For, antecedently to the admission of this proposition, the notation afforded no means of expressing any proposition that is necessarily false nor any means of denying any proposition expressible. Consequently, were the single letters restricted to signifying propositions necessarily true,—a restriction contrary to no convention of the notation, antecedently to the present proposition,—it would not have been true for the universe of signs of this system that 'Something is unscriptible'. Far from being self-evident, therefore, this proposition is a convention distinctly added to the conventions of the algebra.

This circumstance opens a line of thought which [p. 28] it is proper for me to notice, although it does not precisely concern me as a logician. We see, from the example of this notation, that a state of mind is conceivable which should be capable of making judgments and yet should not have any such ideas as falsity and denial. Does such a state of mind exist? It is very nearly, if not quite, the state of mind of a horse. The phrase "horse sense" testifies to the general conviction of those who are acquainted with horses that they make judgments. But we rarely, if ever, observe in the horse, what is common enough in the dog, a state of doubt and deliberation as to whether ~~a hypothesis~~ an idea is true or not. But admitting that there is such a state of mind, is it not presumable that every human mind passes through such a state in its development? It appears to be the [p. 29] state of mind of a baby at about the epoch of its first attempts to speak. If, however, we once admit that this is an inevitable stage in the development of every mind that develops sufficiently to make judgments at all, then we shall be forced to suppose that there was historically such a state of society. For society *is* a mind. We must, then, suppose that men once spoke without being distinctly aware of what they were doing, and without, ordinarily at least, thinking that a proposition could be denied or be false. Of course, now and then, it must have been forced upon them; so that this stage of linguistic development must have been brief. What we thus see reason to conjecture was true of the conception of logical evil, must have been true at some time of evil generally, and to a much later state of moral evil. Thus, the legend (or should we not say the philosophical hypothesis?)—of the Garden of Eden seems likely to have [p. 30] had a historical basis.

As far as logical evil, at any rate, is concerned, the mere question of whether anything is false, was no sooner put, than the ~~question~~ suggestion itself would have created something false. This was the Serpent in the Garden of Eden. He took the ~~form~~ shape of a mark of interrogation. Less fancifully, we may say that his dire *rôle* was enacted by the Mathematical Thought; since the precise difference between the Logician and the Mathematician is that while the former merely analyses the conception we already have, the latter so scrutinizes the relations of the signs already used, as to observe facts about them that had escaped notice, and thus introduces new conceptions. Whoever started the calumny that the 'Old Serpent' is the father of lies must have been one of those flat, unraised spirits who would call it a Deliberate lie to say that a sphere is a ruled surface, or that there is a single circle in which all spheres intersect.

[p. 31]

☞ N.B. Although it has been judged well to introduce this postulate immediately upon the Definition of ☉, and to discuss its nature, yet it will be ignored, until certain properties of ♉ and ☉ that do not depend upon it have been developed, lest the true character of these should be observed.

Corollary 9. ♉ = 'Something is scriptible'. For by Clause 1 of the Definition of ☊, if 'Something is scriptible', then ♉ ☊ 'Something is scriptible'. Hence, since evidently something is scriptible, it follows that ♉ ☊ 'Something is scriptible'; whence, by Clause 3 of the Definition of ☊, if ♉ then 'Something is scriptible'. On the other hand, by the Definition of ♉, if 'Something is scriptible', then ♉. Hence, by Clause 3 of the Definition of = follows the proposition.

Corollary 10. ☉ = 'Everything is scriptible'. For, by Clause 2 of the Definition of ☊, Either everything is scriptible or 'Everything is scriptible' ☊ ☉. But evidently, if everything is scriptible, 'Everything is scriptible' ☊ ☉ is scriptible. [p. 32] Hence 'Everything is scriptible' ☊ ☉; and by Clause 3 of the Definition of ☊, if everything is scriptible then ☉. But by the Definition of ☉, if ☉ then everything is scriptible. Hence, by Clause 3 of the Definition of =, the proposition follows.

Corollary 11. If anything is scriptible (as it evidently is), then whatever a may be, a = ♉ ☊ a.

For, by Clause 3 of the Definition of ☊, if ♉ ☊ a and if ♉ then a. But, by the Definition of ♉, if anything is scriptible, then ♉. Hence, if anything is scriptible and if ♉ ☊ a, then a. But by Clause 1 of the Definition of ☊, if a then ♉ ☊ a. Whence, by Clause 3 of the Definition of =, follows the proposition. [p. 33]

Corollary 12. (i ☊ ☉) ☊ e = Either i or e = (e ☊ ☉) ☊ i. For, by Theorem V, if (i ☊ ☉) ☊ e then either i or e; and by Theorem VII if either e or both i and ☉ ☊ e then (i ☊ ☉) ☊ e. But, by the Definition of ☉, ☉ ☊ e. Hence if either e or i, then (i ☊ ☉) ☊ e; and in the same way, if either e or i, then (i ☊ ☉) ☊ e.

Corollary 13. (i ☊ e) ☊ e = Either i or e = (e ☊ i) ☊ i. For, by Clause 3 of the Definition of ☊, if (i ☊ e) ☊ e and i ☊ e, then e. But, by Clause 2 of the same Definition, Either i or i ☊ e. Hence, if (i ☊ e) ☊ e then either i or e. But, by Theorem VII, if either e or i ☊ i then (e ☊ i) ☊ i. But, by Corollary 1, i ☊ i. Hence if either i or e then (e ☊ i) ☊ i; and in like manner (i ☊ e) ☊ e. Hence by Clause 3 of the Definition of = follows the proposition.

Appendix B

[Alternative draft version, R 11, R 501, c.1901]

[R 11, p. 1] A sign is supposed to have an *object* or meaning, and also to determine an *interpretant* sign of the same object. It is convenient to speak as if the sign originated with an *utterer* and determined its interpretant in the mind of an *interpreter*.

If a sign allows no latitude, either to utterer or to interpreter, as to what object or meaning it shall be regarded as ~~referring to~~ representing, it may be called a *singular sign*. Let us use the Roman capitals to represent singular signs. If a sign allows a latitude of choice to the utterer, in certain respects and within certain limits, as to what its object or meaning shall be, it may be called *vague*, or *non-determinate*. Let us use the Greek minuscules to represent vague signs. If a sign allows a latitude of choice [p. 2] to the interpreter, within certain limits and in certain respects, as to what its object or meaning shall be regarded as being, it may be called *general*, or *non-individual*. Let us use the Italic lowercase letters to represent general signs. It is evident that no sign can be at once vague and general in the same respect. For as long as the determination of the meaning depends upon the choice of one party, the other must follow it, or they will be at cross-purposes; the sign will fail to function as such.

Let ☾ stand for any dyadic relation, and ☽ for its converse.[23] Then if A☾ B, also B☽ A; if Socrates loves Plato, Plato is loved by Socrates. Likewise, if A☾ β, then β☽ A; if Socrates loves some man, some man is loved by Socrates. Further if A☾ b, then b☽ A; if Socrates loves every man, every man is loved by Socrates. So also α☾$\beta = \beta$☽ α and a☾$b = b$☽ a.

But a☾β may be understood in two senses. For here the utterer is to determine the meaning of one sign, the interpreter [R 501, p. 3] of the other. Whichever of the two has the last choice is supposed to know what the previous determinations were. Consequently, the utterer, who is essentially a defender of his own utterance, has an advantage in choosing last, while the interpreter, as not being necessarily a defender of that which he interprets, but rather a critic, and quasi-opponent, is as such, at a relative disadvantage. Consequently, a(☾β) is easier to defend, or signifies less than (a☾)β.

The key of the whole subject is in the above. Especially it will be observed that vague and general signs are, from the point of view of logical criticism, to be treated as mere substitutes for singular signs.

A vague sign gives the utterer liberty to choose the singular. Hence, it will be ~~supported~~ perfectly defended if a suitable individual sign is established. It cannot be defended by a general sign, since this, though it gives the interpreter liberty to [p. 4] choose an object affords no guarantee that there will be any object for him to choose. On the other hand, nothing but a general sign can refute a vague one.

A general sign on the other hand, in allowing the interpreter to adopt any meaning (within limits) that he likes, can be overthrown at once by overthrowing a single definite sign, whether singular or general.

If, whatever general sign x may be,[24]

If x(☊a)

[23] ☾ and ☽ are the signs for first quarter and last quarter of the Moon.

[24] [Here the columns are swapped. The Dragon Head occurs here in its original, upright position. This page is probably Peirce's earliest version and he is still unsettled on how this sign is to be used.]

then $x(☊\beta)$
we shall have
$a(☊\beta)$

If, whatever vague sign ξ may be,

If $\beta(☊\xi)$
then $a(☊\xi)$
we shall have
$a(☊\beta)$

[R 9, p. 5]

§5. Let a *formal definition* be defined as a definition, not of the qualitative signification of its definitum, but of the essential relations between different applications of it. I proceed to give formal definitions establishing a system of notation, beginning with a certain sign of relation, ☊ (a "dragon's head" turned on its side), and since, to begin with, this will be taken in a singular sense, a circumflex accent will be written over it. We shall be supposed to be provided with a certain blank sheet, called the "sheet of assertion"; a purpose is vaguely supposed. But whether or not every sign rendered possible by the system of notation may be written on this sheet as an entire sign remains to be seen. A sign that may be so written will be termed *scriptible*, or *good*, or *true*, etc. The mere writing of a sign, as an [R 501, p. 6] entire sign, conformably to the supposed purpose of the notation, is to be considered as, in effect, conclusive evidence that it is *scriptible*. If any two signs, i and e, are of such natures (formal ~~or~~ and material) that passage might always be made from the former to the latter, without passing from a scriptible[25] to an unscriptible sign, i is said to be *transformable* into e,

[25] [Alt., p. $6\frac{1}{2}$]…[scripti]ble to an unscriptible sign; where I speak of the entire sign. I shall use conjunctions 'If____ then____ ', strictly to express the permissibility of a deduction. Therefore, if the sign, i, is unscriptible, 'If i, then x' will express a necessarily [sic., contingently] permissible deduction; if the sign, e, is a scriptible sign, 'If y, then e' will express a *necessarily* permissible deduction. Further, this sort of transformation is possible upon the sheet of assertion; so that, supposing 'If i, then e' to express a permissible transformation (necessary or contingent), then supposing i to be written on the sheet of assertion, e may also be written. The necessary and sufficient condition of the *necessity* of a *correction* is that it should proceed from a sign which need not be written to a sign that must be written.

[Alt., p. 7] …[scripti]ble to an unscriptible sign, where I refer to the entire sign. I shall strictly confine my use of the conjunctions 'If____ then____ ' to the expression of the permissibility of deductions. A deduction, if permissible, may be performed on the sheet of assertion. That is, whatever ~~two~~ signs i and e may be, supposing 'If i, then x', and supposing i to be ~~written on~~ scriptible, e will also be scriptible. Supposing e to be scriptible, then, whatever sign x may be, 'If x, then e'. Supposing i to be unscriptible, whatever sign y may be, 'If i then ~~y~~'. In a *correction*, the crossing out of the original sign is supposed to signify the condemnation of it as *bad*. Consequently, the necessary and sufficient condition of the permissibility of a correction will be that ~~the original sign sh~~ it should proceed from an unscriptible to a scriptible sign. A correction cannot normally be performed on the sheet of assertion, because there will be no occasion for it, but its permissibility may be expressed upon the sheet of assertion or elsewhere. I will use the words 'Not____ but____ ' to express the permissibility of a correction. Consequently, [p. 8] 'Not(Not i, but e), but x', where x is a sign entirely vague, will be scriptible or not, in agreement with 'If e, then i'; although the latter does not directly express all that the former expresses; and 'If (if e, then i), then y', where y is a sign entirely general, will be scriptible or not, in agreement with 'Not i, but e'; although the former expresses an alternative that the latter does not leave open. Since the rules of the notation, which is what I am to consider, are all general, I shall have little occasion for the form of proposition 'Not____ , but____ '. I shall have much occasion to use the form 'If____ , then____ ', not only in its simple form, but also in the form 'If (if i, then e), then e'. This will be *bad*, *untrue*, or *unsriptible*, if, and only if, 'If i, then e' is scriptible while e is unscriptible. But that 'If i, then e' should be scriptible although e is unscriptible, requires that i should be unscriptible, lest the passage from i to e should be from the scriptible to the unscriptible. Therefore, 'If (if i, then e), then e' is scriptible if and only if i or e (one or [p. 9] other or both) is scriptible. I shall write, and then only ‘If i, then e’ becomes scriptible. It is evident that such a sign is *general*. If any two signs, i and e, are of such a nature, that whatever sign x be taken, supposing ‘If i, then x’ to be scriptible and ‘If e, then x’ to be scriptible, it will always be the case that x is scriptible, then, and then only, ‘Either i or e’ becomes scriptible. It is evident that such a sign is *vague*. [p. 6]

DEFINITION OF $\widehat{\mathrm{C}}$.

CLAUSE 1. The relation expressed by $\widehat{\mathrm{C}}$ is *definite* in such a sense that, whatever a and e may be,

If e, then $a \,\widehat{\mathrm{C}}\, e$.

CLAUSE 2. The relation expressed by $\widehat{\mathrm{C}}$ is *individual* in such a sense that, whatever a and e may be,

Either a or $a \,\widehat{\mathrm{C}}\, e$.

Theorem I. Something can be written on the sheet of assertion.

Proof For each clause of the above definition is so written. □

Scholium. It cannot be proved or disproved that there is anything that cannot be written on the sheet of assertion.

In place of Theorem I, we may substitute,

Theorem I′. Whatever a may be, $a \,\widehat{\mathrm{C}}\, a$.

Proof Putting a for e in Clause 2, unless $a \,\widehat{\mathrm{C}}\, a$ is scriptible, a is so. But in that case, by Clause 1, putting a in place of e, $a \,\widehat{\mathrm{C}}\, a$ is again scriptible. □

Theorem II. Whatever a, e, i may be, if $a \,\widehat{\mathrm{C}}\, i \,\widehat{\mathrm{C}}\, e$ is scriptible[26] then $i \,\widehat{\mathrm{C}}\, a \,\widehat{\mathrm{C}}\, e$ is scriptible.

[p. 7]

Theorem I. If, whatever x may be, $x \,\widehat{\mathrm{C}}\, e$, then e.

Proof For if, whatever x may be, $x \,\widehat{\mathrm{C}}\, e$, then, putting $a \,\widehat{\mathrm{C}}\, a$ for x, $(a \,\widehat{\mathrm{C}}\, a) \,\widehat{\mathrm{C}}\, e$ will be scriptible. Whence, by Clause 3, putting $a \,\widehat{\mathrm{C}}\, a$ for i,

If $a \,\widehat{\mathrm{C}}\, a$ and $(a \,\widehat{\mathrm{C}}\, a) \,\widehat{\mathrm{C}}\, e$, then e.

Whence, by Corollary 1, e will be scriptible. □

Corollary 4. By Corollary 1, something is scriptible. Indeed, either clause of the definition, since it holds good, is scriptible.

DEFINITION OF ♉.[27]

Footnote 25 continued
whatever i and e may be, ‘either i or e’ to express this; that is, ‘If i then either i or e’, and ‘If e, then either i or e’; and ‘either i or e’ is not scriptible unless so necessitated to be scriptible. The proposition ‘either i or e’ is thus vague, insofar as it does not declare i nor e, but only some one (or both) of the two.

[26] By the above convention, this means $a(\widehat{\mathrm{C}}[i \,\widehat{\mathrm{C}}\, e])$.

[27] ♉ is the sign of *Taurus*. Get a type that looks as near a Bull’s head as may be. [This annotation reads as an instruction to the typesetter or printer suggests that Peirce might in fact have contemplated to have this earlier version of the manuscript to be typeset or published, possibly along with his other *Minute Logic* drafts.]

♉ is scriptible; or, more briefly,

♉.

Scholium. It can neither be proved nor disproved that anything is not scriptible. The following sign therefore can never be written by itself on the sheet of assertion.

DEFINITION OF ☉.

CLAUSE 1. Whatever a may be

☉ Ĉ a.

CLAUSE 2. If, whatever x may be, x is scriptible, ☉ is scriptible.
[p. 8]

DEFINITION OF =.

CLAUSE 1. Whatever i and e may be,

If $i = e$ and i,
then e.

CLAUSE 2. Whatever i and e may be,

If $i = e$ and e,
then i.

CLAUSE 3. Supposing, that if i then e and if e then i, we have

$$i = e.$$

Scholium. A conditional proposition, expressing that if certain conditions are or were to be fulfilled, a certain consequent is or would be the case, usually refers to a *general* range of possibilities, and giving the interpreter liberty to imagine any case of the fulfillment of the condition, expresses that under assumptions supposed to be accepted, though they are usually more or less *vague*, the truth of the consequent would have to be admitted. But Scotus and other logicians consider a form of "conditional proposition *de inesse*", [p. 9] which does not refer to any general range of possibilities, but to a definite and individual state of things. Everything written on our sheet of assertions is supposed to ~~refer to~~ represent a definite and individual universe, "the Truth", so that when a conditional proposition is written on the sheet, say 'If i, then e', no illative connection between i and e is expressed that would imply that throughout a whole general range of possibilities the truth of i (where it is true) is accompanied with that of e, but all that is meant is that taking the universe of Truth as it happens to be, although i may not be true (applicable, justifiable, etc.), and should that be the case, nothing is said about e, yet i being true e will as a matter of fact be true also. All that is meant is that, in proceeding from i to e, we should not be proceeding from the scriptible to the unscriptible. As a proposition *de inesse*, it is true that 'if 7 is divisible by 3, it is a prime number'. For this proposition confines itself to the actual state of things, and since 7 is *not* divisible by 3, it really says [p. 10] nothing. Now a proposition is true so far as it is not false. It may contribute to a clear apprehension of the matter to compare the proposition about the number 7 with this: 'If a sensible man were (under ordinary circumstances) to throw a full inkstand upon his library floor, he would voluntarily spoil his carpet (supposing the room were carpeted)'. This is true, since an inkstand thrown on such a floor would certainly ruin the carpet; and a sensible man would know this beforehand; so that if he did it, he would do it voluntarily. At the same time, no sensible man would, under any ordinary circumstances voluntarily spoil his carpet. Consequently it is true that 'If a sensible man were, under ordinary circumstances, to throw a full inkstand upon his library floor, he would not voluntarily ~~spoil~~ ruin his carpet'; for if a man were to do such a thing, he would not be a sensible man. Here there area two contradictory propositions true at the same time, which is possible because they refer to *general* conditions, and generals are not subject to the principle of contradiction. The truth is that no sensible man under ordinary circumstances [p. 11] would throw a full inkstand upon his library floor, and thus, these two propositions are quite similar to that concerning the number 7, except that they are supported by general reasons and consequently refer to a general range of possibilities. In ordinary speech conditionals *de inesse* are rare, and they perplex the non-logician because he attempts to interpret them as referring to some general principle or reason. In logic they are quite indispensable to any thorough analysis of reasoning.

Corollary 5. Whatever i and e may be

$$\text{If } i, \text{ then } e \quad = \quad i \,\widehat{\mathbf{C}}\, e.$$

For by Clause 2 of the definition of $\widehat{\mathbf{C}}$, putting i in place of a, either $i \,\widehat{\mathbf{C}}\, e$ or i. But if i and 'If i, then e', then e, when by Clause 1 of the same definition, putting e in place of a, $i \,\widehat{\mathbf{C}}\, e$. Thus, supposing 'If i, then e' to be scriptible, so is $i \,\widehat{\mathbf{C}}\, e$. On the other hand by Clause 3 of the same definition, if $i \,\widehat{\mathbf{C}}\, e$ is scriptible, so is 'If i then e'. Hence, in Clause 3 of the definition of =, putting 'If i, then e' [p. 12] in place of i, and $i \,\widehat{\mathbf{C}}\, e$ in place of e, we have the proposition enunciated.

Scholium. Since our definition of $\widehat{\mathbf{C}}$ is purely formal, it follows that, in a conditional proposition *de inesse*, the 'If___ then___ ' has a purely formal signification.

Theorem II. Whatever a, e, i may be, if

$$i \,\widehat{\mathbf{C}}\, a \,\widehat{\mathbf{C}}\, e$$

then

$$a \,\widehat{\mathbf{C}}\, i \,\widehat{\mathbf{C}}\, e.$$

Proof For assume

$$i \,\widehat{\mathbf{C}}\, a \,\widehat{\mathbf{C}}\, e.$$

Then I have only to prove $a \,\widehat{\mathbf{C}}\, i \,\widehat{\mathbf{C}}\, e$. [p. 13] Hence, we have

$$\text{Either } i \,\widehat{\mathbf{C}}\, e \text{ or } a \,\widehat{\mathbf{C}}\, e.$$

But substituting a for i in the same clause, we have

$$\text{If } a \,\widehat{\mathbf{C}}\, e \text{ and if } a, \text{ then } e.$$

Whence

$$\text{Either } i \mathbin{\widehat{\mathrm{C}}} e \text{ or if } a, \text{ then } e.$$

But in Clause 2 of the same Definition, substituting $i \mathbin{\widehat{\mathrm{C}}} e$ for e we have

$$\text{Either } a \text{ or } a \mathbin{\widehat{\mathrm{C}}} i \mathbin{\widehat{\mathrm{C}}} e.$$

Hence

$$\text{Either } a \mathbin{\widehat{\mathrm{C}}} i \mathbin{\widehat{\mathrm{C}}} e, \text{ of } i \mathbin{\widehat{\mathrm{C}}} e, \text{ or } e.$$

But in Clause 1 of the same Definition, substituting e for a, we have

$$\text{If } e \text{ then } i \mathbin{\widehat{\mathrm{C}}} e.$$

Hence

$$\text{Either } a \mathbin{\widehat{\mathrm{C}}} i \mathbin{\widehat{\mathrm{C}}} e \text{ or } i \mathbin{\widehat{\mathrm{C}}} e.$$

But in the same clause substituting $i \mathbin{\widehat{\mathrm{C}}} e$ for a and a for i, we have

$$\text{If } i \mathbin{\widehat{\mathrm{C}}} e \text{ then } a \mathbin{\widehat{\mathrm{C}}} i \mathbin{\widehat{\mathrm{C}}} e.$$

Hence

$$a \mathbin{\widehat{\mathrm{C}}} i \mathbin{\widehat{\mathrm{C}}} e.$$ □

[p. 14]
Theorem III. Whatever i, a, e may be,

$$\text{If } (i \mathbin{\widehat{\mathrm{C}}} a) \mathbin{\widehat{\mathrm{C}}} e \text{ then either } i \text{ or } e.$$

Proof For assume

$$(i \mathbin{\widehat{\mathrm{C}}} a) \mathbin{\widehat{\mathrm{C}}} e$$

Then I have only to prove that either i or e.
By Clause 2 of the Definition of $\widehat{\mathrm{C}}$

$$\text{Either } i \text{ or } i \mathbin{\widehat{\mathrm{C}}} a$$

By Clause 3 of the same Definition

$$\text{If } (i \mathbin{\widehat{\mathrm{C}}} a) \text{ and } (i \mathbin{\widehat{\mathrm{C}}} a) \mathbin{\widehat{\mathrm{C}}} e, \text{ then } e$$

Whence by the assumption

$$\text{If } i \mathbin{\widehat{\mathrm{C}}} a \text{ then } e$$

Whence by the result of Clause 2,

$$\text{Either } i \text{ or } e.$$ □

[p. 15]
Theorem IV. Whatever i, a, e may be,

$$\text{If } (i \mathbin{\widehat{\mathrm{C}}} a) \mathbin{\widehat{\mathrm{C}}} e \text{ then } a \mathbin{\widehat{\mathrm{C}}} e.$$

Proof For assume

$$(i \mathbin{\widehat{\mathbf{C}}} a) \mathbin{\widehat{\mathbf{C}}} e.$$

Then I have only to prove that $a \mathbin{\widehat{\mathbf{C}}} e$.
By Clause 1 of the Definition of $\widehat{\mathbf{C}}$,

(A) If a then $a \mathbin{\widehat{\mathbf{C}}} a$,
(B) If e then $a \mathbin{\widehat{\mathbf{C}}} e$.

By Clause 2 of the same Definition,

(C) Either a or $a \mathbin{\widehat{\mathbf{C}}} e$.

By Clause 3 of the same Definition,

(D) If $(i \mathbin{\widehat{\mathbf{C}}} a)$ and $(i \mathbin{\widehat{\mathbf{C}}} a) \mathbin{\widehat{\mathbf{C}}} e$ then e.

By (D) and the assumption

If $(i \mathbin{\widehat{\mathbf{C}}} a)$, then e.

Whence, by (A),

If a, then e.

Whence, by (C),

Either $a \mathbin{\widehat{\mathbf{C}}} e$ or e.

Whence, by (B),

$a \mathbin{\widehat{\mathbf{C}}} e$. □

[p. 16]
Theorem V. Whatever i, a, e may be,

$$\text{If either } e \text{ or both } i \text{ and } a \mathbin{\widehat{\mathbf{C}}} e, \text{ then } (i \mathbin{\widehat{\mathbf{C}}} a) \mathbin{\widehat{\mathbf{C}}} e.$$

Proof I will first prove that if i and $a \mathbin{\widehat{\mathbf{C}}} e$, then either e or $(i \mathbin{\widehat{\mathbf{C}}} a) \mathbin{\widehat{\mathbf{C}}} e$. For assume

(A) i
(B) $a \mathbin{\widehat{\mathbf{C}}} e$.

Then, I am to prove that either e or $(i \mathbin{\widehat{\mathbf{C}}} a) \mathbin{\widehat{\mathbf{C}}} e$.
By Clause 1 of the Definition of $\widehat{\mathbf{C}}$,

(C) If e then $(i \mathbin{\widehat{\mathbf{C}}} a) \mathbin{\widehat{\mathbf{C}}} e$.

By Clause 2 of the same Definition,

(D) Either $i \mathbin{\widehat{\mathbf{C}}} a$ or $(i \mathbin{\widehat{\mathbf{C}}} a) \mathbin{\widehat{\mathbf{C}}} e$.

By Clause 3 of the same Definition,

(E) If a and $a \mathbin{\widehat{\mathbf{C}}} e$, then e
(F) If i and $i \mathbin{\widehat{\mathbf{C}}} a$, then a.

By (E) and (B)

If a, then e.

By this and (F)

If i and $i \,\widehat{\mathcal{C}}\, a$, then e.

[p. 17] By this and (A)

If $i \,\widehat{\mathcal{C}}\, a$, then e.

By this and (D)

Either e or $(i \,\widehat{\mathcal{C}}\, a) \,\widehat{\mathcal{C}}\, e$.

I have thus proved that if both i and $a \,\widehat{\mathcal{C}}\, e$, we get the result. But this is also true if e is true. Hence, it is true if either e or both i and $a \,\widehat{\mathcal{C}}\, e$. Now from this, with (C), we have

$(i \,\widehat{\mathcal{C}}\, a) \,\widehat{\mathcal{C}}\, e$ □

Corollary 6. Hence, *à fortiori*, if both $a \,\widehat{\mathcal{C}}\, e$ and either i or e, then $(i \,\widehat{\mathcal{C}}\, a) \,\widehat{\mathcal{C}}\, e$.

Corollary 7. Hence, if $a \,\widehat{\mathcal{C}}\, e$ then $(i \,\widehat{\mathcal{C}}\, a) \,\widehat{\mathcal{C}}\, (i \,\widehat{\mathcal{C}}\, e)$. For, by Theorem V, if $a \,\widehat{\mathcal{C}}\, e$ and if i then $(i \,\widehat{\mathcal{C}}\, a) \,\widehat{\mathcal{C}}\, e$; or by Corollary 5, if $a \,\widehat{\mathcal{C}}\, e$ then $i \,\widehat{\mathcal{C}}\, (i \,\widehat{\mathcal{C}}\, a) \,\widehat{\mathcal{C}}\, e$; or by Theorem II, if $a \,\widehat{\mathcal{C}}\, e$, then $(i \,\widehat{\mathcal{C}}\, a) \,\widehat{\mathcal{C}}\, i \,\widehat{\mathcal{C}}\, e$.

Corollary 8. From Corollary 7, by Theorem II: If $i \,\widehat{\mathcal{C}}\, a$ and $a \,\widehat{\mathcal{C}}\, e$, then $i \,\widehat{\mathcal{C}}\, e$.

Corollary 9. From Corollary 8, we have: If $i \,\widehat{\mathcal{C}}\, a$, then $(a \,\widehat{\mathcal{C}}\, e) \,\widehat{\mathcal{C}}\, (i \,\widehat{\mathcal{C}}\, e)$.

[p. 18][28]

Corollary 10. $a \,\widehat{\mathcal{C}}\, \odot$ = If a then everything is scriptible. For Clause 2 of the Definition of $\widehat{\mathcal{C}}$ gives

Either a or $a \,\widehat{\mathcal{C}}\, \odot$

Whence, if we assume

If a then everything is scriptible

we have

Either $a \,\widehat{\mathcal{C}}\, \odot$ or everything is scriptible, i.e. $\odot$.

But in Clause 1, putting $\odot$ and a for i, we get

If $\odot$ then $a \,\widehat{\mathcal{C}}\, \odot$.

From these two, it follows that

$a \,\widehat{\mathcal{C}}\, \odot$.

On the other hand, assuming this, Clause 3 gives

28 [The segment that follows here may be discontinuous with the above pages, as $\odot$ is used without its definition.]

If a then $\odot$ i.e. everything is scriptible.[29]

Scholium. A *postulate* (ἄιτεμα) as explained by Aristotle and as plainly understood by Euclid, is a hypothesis ~~necessary to the development which the utter~~ essential to a deductive theory, with the truth or falsity of which the utterer of the theory does not concern himself. When Riemann [p. 21] wrote about "die Hypotheses welcher zu Grunde der Geometrie liegen", he used the word *hypothesis* precisely in the sense in which Euclid understood ἀίτεμα. This is shown both by Aristotle's distinct statement that a postulate is not necessarily true but must be admitted before the propositions of geometry can be proved, and also by Euclid's pains in putting his fifth postulate into a shape in which its non-evidence should be striking. The German use of 'Postulat', as an indemonstrable practical proposition, may be very ancient; but it was brought into common use by Christian Baron Wolff. The English and French languages have, on the contrary, always followed the original and more useful meaning. An *axiom*, called ἀξίομα by Aristotle, and by Euclid by the term κοινέ ἔννοια, which afterward passed into stoical terminology, is a deliverance of common sense which nobody will [p. 22] call in question.[30] It is analogous to those public facts of which a law court will "take cognizance" without evidence.

Definition of $\Diamond$.[31]

[29] [Alt.] *Scholium.* To one who attaches an absolute value to logical analysis, it is a most important fact that a state of knowledge is possible from which the logician [p. 19] could not deduce the idea that anything is false. It is a Garden-of-Eden stage of development in which there is no knowledge of logical Evil; and since Truth, logical Good, consists merely in freedom from falsity, there can be no distinct knowledge of logical Good, although the idea is present in every idea. Though it does not concern me, as a logician to say so, I cannot doubt that every mind passes through this stage of development. If this be true, there must have been such a historical stage of society. For a society is a mind. What is true of logical evil must be true of moral and all other evil; for all are but varieties of logical evil. Therefore, the outline of the story of the Garden of Eden must be true. I have said that no pure logician could ever deduce from such knowledge the idea of falsity. But the Serpent was bound to appear in the guise of the mathematical mind, and suggest falsity. [p. 20] Now the false is no sooner named, than it manifestly exists. For if nothing else is false, it is so. I attribute this dire *rôle* to the Mathematician, because the precise difference between logical and mathematical thought is that the logician merely analyses the conceptions already had, while the mathematician looks at the situation and produces new conceptions. Whoever started the calumny that the Old Serpent is the "father of lies" must have been one of these flat unraised spirits who would call it a lie to say that a sphere is a ruled surface, or that all spheres have one circle in common.

[30] [Alt.] …call in question. It is analogous to those public facts of which a law-court will "take cognizance", without any proof.
Axiom. Something is non-scriptible.
Proof. For suppose this false. Then it is not scriptible on the sheet of assertion; and consequently, it is something non-scriptible, contrary to the hypothesis. Thus, we are driven to admit the axiom.
Although this demonstration is conclusive, it is not a deduction from any of our definitions, which allow the supposition that everything is scriptible. In point of fact, anything can be written on a scrap of paper; none of our definitions forbid it. Nor does the notation provide any means of writing anything contrary to the rules of the notation. It is not until we arbitrarily invent a sign which we define as being non-scriptible that we then create something that is non-scriptible. The "proof" rests upon the *observation* that [p. 23] the term "non-scriptible", which the notation had not afforded us any means of writing, is of such a character that as soon as it is admitted among the number of signs there comes to be something non-scriptible. Before that was done, the axiom was not demonstrable. It therefore involves a new premiss, which is, strictly speaking that it is possible to invent a non-scriptible sign. Given that [as] its objective expression, it becomes "something is non-scriptible", which ought therefore to be accepted as an axiom, the "proof" of which, though formally necessary, involves a *petitio principii*.

[31] [Peirce changed $\Diamond$ in the later versions of this paper to $\odot$.]

Whatever there may be that is unscriptible is $\Diamond$.

Axiom. Something is unscriptible.

Scholium. The following demonstration might be suggested:

Something is unscriptible. For suppose this to be false. Then, it is not scriptible. For suppose this to be false. Then, it is not scriptible, and consequently something is unscriptible, contrary to the hypothesis. The hypothesis must, therefore, be rejected, and we are forced to admit that something is unscriptible.

This argument is conclusive, but it is a *petitio principii*. For anybody who should maintain that everything is scriptible would mean by 'everything' either everything which he could conceive or everything that could be expressed in the notation used. In the one case, he would have no such conception as denial [p. 24] or falsity; in the other, he would point out that the notation provides no means of expressing that 'something is unscriptible' or of denying anything. Consequently, this person would in the one case pronounce the ~~proposition~~ 'unscriptible' unintelligible and in the other would say that it was not within the universe of signs intended. In either case he would deny that merely uttering this proposition afforded any proof of the truth of it. In this he would be right: the proposed demonstration begs the question.

It is true that the instant that the definition which I have connected with the axiom by a brace, to show that they are logically simultaneous, is admitted, so that there is a sign for the unscriptible, admitted to have that meaning, it becomes evident that there is something unscriptible. But a sound logic cannot admit the monstrous doctrine that a definition can declare the existence of anything. Although the propo- [end; version abandoned or remaining pages have not survived]

[p. 23] …demonstration proposed begs the question.

It is only when a sign for the non-scriptible is introduced into the notation and is admitted to have that meaning, that it becomes evident that something is non-scriptible. A mere definition may create the sign, but a definition does not, according to sound logic, declare the existence of anything. For that purpose, positive observation is required. In this case, the substance of that observation is evident. It ought, therefore, to be termed an axiom.

Corollary 11. ~~The sign~~ ⊙ ~~signifies~~ = the unscriptible, or false. For ⊙ signifies that everything is scriptible; since this could not be proved, it was before not positively scriptible. But the Axiom renders it non-scriptible. This proves that ⊙ is unscriptible. But to prove that ⊙ = the unscriptible it must further be proved that whatever is unscriptible is ⊙. That is, if Z is unscriptible Z $\widehat{\mathrm{C}}$ ⊙. But this follows from Clause 2 of the Definition [end, fragment abandoned]

[The following bibliography includes Peirce's references occurring in the text of Appendices A and B, with details, when known, of their provenance.]

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