Vitali's generalized absolute differential calculus

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Abstract

The paper provides an analysis of Giuseppe Vitali's contributions to differential geometry over the period 1923–1932. In particular, Vitali's ambitious project of elaborating a generalized differential calculus regarded as an extension of Ricci-Curbastro tensor calculus is discussed in some detail. Special attention is paid to describing the origin of Vitali's calculus within the context of Ernesto Pascal's theory of forms and to providing an analysis of the process leading to a fully general notion of covariant derivative. Finally, the reception of Vitali's theory is discussed in light of Enea Bortolotti and Enrico Bompiani's subsequent works.

1 Introduction

The name of Giuseppe Vitali (1875–1932) is generally associated with noteworthy contributions to analysis and integration theory, such as the discovery of quasi-continuity of measurable functions (1905), the proof that a function is absolute continuous if and only if it is an integral function (1904–1905), the first exhibition of a set that is not Lebesgue measurable (1905) and the so-called Vitali's covering theorem (1908), to cite only some of them.[1]

Less well known and less studied are Vitali's contributions to differential geometry to which he turned in the last decade of his life, over the period 1923–1932. Probably, the scientific impact of Vitali's geometrical production cannot parallel the resonance of his previous works in the realm of analysis. Nonetheless, the lack of attention towards Vitali's geometrical investigations on the part of historians appears to a certain extent to be unjustified. Indeed, his attempts at providing a generalization of Riemannian geometry are worth considering for at least two reasons. An understanding of Vitali's contributions to differential geometry can offer a more adequate and complete com-

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[1] On Vitali's contributions to real analysis and integration theory, see, e.g., (Pepe 1984) and (Borgato 2012).

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prehension of Vitali's scientific figure as a whole, and it can also offer some new insight into the historical development of infinitesimal geometry over the first half of twentieth century.

As will be seen, Vitali mainly moved into two directions: by exploring alternative notions of parallelism and by providing a higher-order extension of Ricci's absolute differential calculus, which he called the generalized absolute differential calculus. The contributions of Vitali into both research directions will be analyzed in this paper. Section 2 provides a general description of the context of Vitali's geometrical work; Sect. 3 discusses the introduction of an absolute parallelism, the so-called Vitali-Weitzenböck parallelism, characterized by vanishing curvature and non-vanishing torsion, along with its reception. The subsequent sections are devoted to a presentation of Vitali's generalized differential calculus. Special attention is paid to describing its origin within the context of Ernesto Pascal's theory of forms and to providing a diachronic analysis of the emergence of a fully general notion of covariant derivative. In the concluding section, the reception of Vitali's work is discussed in light of Enea Bortolotti and Enrico Bompiani's subsequent investigations.

2 Beyond Riemann geometry

Over the first quarter of the twentieth century, the field of differential geometry experienced a process of deep and rapid evolution that resulted in an extraordinary variety of interconnected research themes and different technical approaches. Undoubtedly, one of the most important (and most widely studied) episodes in this transformation was the discovery of the notion of parallel transport in a Riemannian manifold by Tullio Levi-Civita and Jan Schouten. The geometrical interpretation of Christoffel symbols that the notion of parallel transport brought about produced a proliferation of studies devoted to extend the Riemannian framework by exploring the possibility of defining connections independent of the notion of a metric. (This research direction will be referred to as research line A.)

Before the publication of (Levi-Civita 1917) where the concept of parallelism for Riemannian manifolds was first introduced, investigations into other directions such as those concerning projective differential properties and the study of higher-order elements (osculating spaces) of manifolds played a prominent role in shaping the development of the discipline, beyond the realm of the theory of connections. (This second set of research lines will be collectively referred to as research line B.)

On a methodological level, the new geometrical horizons disclosed both by research lines *A* and *B* often required the use of techniques more refined than those of Ricci's absolute differential calculus, which had been designed mainly to deal with Riemannian manifolds.

A most interesting response to this challenge was offered by Élie Cartan who, starting from 1910s, elaborated a peculiar approach to both Riemannian and non-Riemannian geometry based upon the notion of Pfaffian forms and a generalization of the method of moving frames. Cartan's position toward the calculus of Ricci was somehow skeptical, and he went so far as to affirm that the formalism of absolute

differential calculus often obscured the intuitive content of the geometrical theories to which the calculus was applied.[2]

A less radical view was expressed by René Lagrange, who got his doctorate in 1923 under Cartan's supervision. He devoted his dissertation (Lagrange 1926) to providing a generalization of the calculus of Christoffel, as he called it, that could be regarded as an attempt at harmonizing it with the emerging theory of moving frames. The central idea consisted in the observation that the formal rules of Ricci's calculus maintain their validity when the differentials of a given coordinate system, $dx^i, i = 1, \ldots, n$, are replaced by general Pfaffian forms, $\omega_i, i = 1 \ldots, n$, which are not exact differentials.

At that same time, the evolution of differential geometry experienced a period of intense transformation in Italy too. The variety of research topics was widely expanded in such a way as to cover, for example, the theory of hypersurfaces in Euclidean n-dimensional spaces, projective differential geometry and the study of higher-order Riemannian geometries.[3] In particular, the emergence of a projective approach in the realm of infinitesimal geometry played a crucial role in fostering the elaboration of new methods and techniques, aptly designed to deal with higher-order properties of manifolds. In this respect, one should at least cite the works by Luigi Bianchi, Umberto Sbrana, Eugenio Elia Levi, Guido Fubini and Enrico Bompiani. Bianchi and Umberto Sbrana, a student of Bianchi at Pisa University, completely solved to problem of determining those hypersurfaces in $\mathbb{R}^n$, with $n \geq 4$ that admit non-trivial deformations; see (Bianchi 1905), (Sbrana 1909). In his dissertation degree (Levi 1908), written under the supervision of Bianchi, Levi tackled the study of higher-order properties of surfaces immersed in n-dimensional Euclidean spaces. Fubini focused mainly on projective properties of manifolds, i.e., those properties that are left invariant by the action of the projective group (see Fubini and Čech 1926-1927). Bompiani, who also contributed to projective differential geometry, developed a most peculiar approach to metric geometry, dubbed by him "geometrie riemanniane di specie superiore", which is particularly relevant for the reception process of Vitali's generalized calculus, see (Bompiani 1935) and Sect. 7. For a general overview on Bompiani's scientific biography, see (Ciliberto and Sallent Del Colombo 2012).

In this very general context, in which various attempts were made to extend the theoretical framework of Riemannian geometry in different directions, Vitali started to cultivate the proposal to provide an extension of both Riemannian geometry and Ricci's calculus. His hope was to elaborate a general algorithmic method suitable for dealing with manifolds of arbitrary (finite) dimension that were regarded as immersed in the Hilbert space of square-integrable functions of a real variable.

At the same time, as will be seen, the viability of Vitali's project was encouraged by research of a purely analytical tenor undertaken by Ernesto Pascal, who had succeeded in providing a generalization of covariant tensors to differential forms of arbitrary order and degree.

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[2] In this respect, see (Cogliati 2018).

[3] According to Chern, the main problem of projective differential geometry "is to find a complete system of local invariants of a submanifold under the projective group and interpret them geometrically through osculation by simpler geometrical figures. The main difficulty lies in that the projective group is relatively large and invariants can only be reached through high order of osculation." (Chern et al. 1992, p. 62).

3 Vitali(-Weitzenböck) parallelism

In early 1920s, the scientific interests of Vitali, who had previously concentrated on analysis and integration theory, experienced a distinct turn which oriented his research activity mainly to differential geometry. Vitali contributed both to research line A and B, and he also tried to establish a connection between the two. We first turn to discussing two closely related memoirs, (Vitali 1923a) and (Vitali 1925), that can be ascribed to research line A. The next sections will be devoted to analyzing Vitali's extensive contributions to research line B.

The subject of (Vitali 1923a) consisted in the introduction a new type of covariant derivative that analytically describes a parallelism with vanishing curvature and non-vanishing torsion.[4] The underlying idea was simple. Vitali endowed a given $n-$dimensional manifold V_n with a set of independent 1-forms $\omega_k = \sum_{i=1}^n X_i dx^i$, $k = 1, \ldots, n$, and introduced the coefficients Γ_{ij}^k by putting:

$$\Gamma_{ij}^k = \sum_{r=1}^n X_r^k \partial_{x^j} X_r{}_i, \qquad i, j, k = 1, \ldots, n. \tag{1}$$

where X_i^k denote the coefficients of the inverse matrix of $\mathbf{A} = [A_{is} := X_i{}_s]$ so that $\sum_{i=1}^n X_k{}_i X_i^j = \delta_k^j$. In order to prove that these n^3 functions actually define a connection,[5] one should check that Γ_{ij}^k are transformed in an appropriate way under arbitrary coordinate transformations.[6] Vitali chose a more indirect strategy by first showing that the derivation of a covariant 1-form $\omega_0 = X_s{}_0 dx^s$ associated to (1) is actually covariant.

Indeed, by analogy with the classical definition of covariant derivative of Christoffel and Ricci, Vitali set:

$$D_t X_s{}_0 := \partial_{x_t} X_s{}_0 - \sum_{k=1}^n \Gamma_{st}^k X_k{}_0, \tag{2}$$

and showed that the system $D_t X_s{}_0$ is a covariant system of the second order.

We can legitimately ask how Vitali came up with definition (2). Indeed, whereas the verification of the covariant character of (2) is a matter of trivial computation, to obtain an explicit expression containing the derivatives of $X_s{}_0$ with the requested covariance property is far less obvious. A plausible reconstruction (partially suggested by Vitali

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[4] The ordinary notion of parallelism in Euclidean space is a kind of Vitali's parallelism with vanishing curvature and torsion.

[5] Our use of the term "connection" is improper. Vitali did not employ the word "connection" to designate the mapping between tangent spaces, defined by the set of coefficients Γ_{ij}^k, $i, j, k = 1, \ldots, n$. When he wanted to attribute a geometrical meaning to the symbols Γ_{ij}^k, he spoke of the parallelism corresponding to them.

[6] This is the point of view adopted for example in (Eisenhart 1927, §2).

himself[7]) leading to (2) goes as follows. Consider first the transformation laws for the covariant tensors ω_i, $i = 1, \ldots, n$ and ω_0:

$$\underset{i}{\bar{X}_s}(y) = \sum_{m=1}^{n} \partial_{y^s} x^m \underset{i}{X_m}(x), \qquad \underset{0}{\bar{X}_s}(y) = \sum_{m=1}^{n} \partial_{y^s} x^m \underset{0}{X_m}(x). \tag{3}$$

Upon derivation of both sides of these equations with respect to $\dfrac{\partial}{\partial y^t}$, one obtains:

$$\partial_{y^t} \underset{i}{\bar{X}_s}(y) = \sum_{m=1}^{n} \partial^2_{y^t y^s} x^m \underset{i}{X_m}(x) + \sum_{k,m=1}^{n} \partial_{y^s} x^m \partial_{y^t} x^k \partial_{x^k} \underset{i}{X_m}$$

$$\partial_{y^t} \underset{0}{\bar{X}_s}(y) = \sum_{m=1}^{n} \partial^2_{y^t y^s} x^m \underset{0}{X_m}(x) + \sum_{k,m=1}^{n} \partial_{y^s} x^m \partial_{y^t} x^k \partial_{x^k} \underset{0}{X_m} \tag{4}$$

$i = 1, \ldots, n$. The first equation of (4) can be solved with respect to the second-order derivatives $\partial^2_{y^t y^s} x^m$ to get:

$$\partial^2_{y^t y^s} x^m = \sum_{i=1}^{n} \underset{i}{X^m} \partial_{y^t} \underset{i}{\bar{X}_s} - \sum_{k,r,i=1}^{n} \underset{i}{X^m} \partial_{y^s} x^r \partial_{y^t} x^k \partial_{x^k} \underset{i}{X_r}. \tag{5}$$

By replacing this expression for $\partial^2_{y^t y^s} x^m$ in the second equation of (4) and observing that $\sum_{m=1}^{n} \underset{i}{\bar{X}^m} \underset{0}{\bar{X}_m} = \sum_{m=1}^{n} \underset{i}{X^m} \underset{0}{X_m}$, $i = 1, \ldots, n$, one can finally obtain the required relation:

$$\sum_{k,m} \left(\partial_{x_k} \underset{0}{X_m} - \sum_{l=1}^{n} \Gamma^l_{mk} \underset{0}{X_l} \right) \partial_{y^q} x^k \partial_{y^r} x^m = D_q \underset{0}{\bar{X}_r}, \tag{6}$$

where the right-hand side denotes the quantity:

$$D_q \underset{0}{\bar{X}_r} = \partial_{y^q} \underset{0}{\bar{X}_r} - \sum_{l=1}^{n} \bar{\Gamma}^l_{rq} \underset{0}{\bar{X}_l} = \partial_{y^q} \underset{0}{\bar{X}_r} - \sum_{l=1}^{n} \sum_{u=1}^{n} \left(\underset{u}{\bar{X}^l} \partial_{y^q} \underset{u}{\bar{X}_r} \right) \underset{0}{\bar{X}_l}$$

More generally, the coefficients Γ^k_{ij} could be employed to construct covariant and contravariant systems of higher order by considering the following definition which represents a natural extension of the classical notion of covariant derivative for tensors of arbitrary order:

$$D_t X^{s_1,\ldots,s_k}_{r_1,\ldots,r_h} = \partial_{x^t} X^{s_1\ldots s_k}_{r_1\ldots r_h} - \sum_{i=1}^{h} \sum_{l=1}^{n} \Gamma^l_{r_i t} X^{s_1\ldots s_k}_{r_1\ldots r_{i-1},l,r_{i+1}\ldots r_h}$$

$$+ \sum_{i=1}^{h} \sum_{l=1}^{n} \Gamma^{s_i}_{lt} X^{s_1\ldots s_{i-1},l,s_{i+1}\ldots s_k}_{r_1\ldots r_h} \tag{7}$$

Vitali's treatment focused mainly on algorithmic procedures and invariant properties. Nonetheless, he also considered the geometric consequences of the choice for Γ^k_{ij},

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[7] See also (Weitzenböck 1923, pp. 317–320).

$i, j, k = 1, \ldots, n$. The most significant ones are the following: (i) since the curvature tensor associated to Γ_{ij}^k is identically equal to zero, the connection defines a type of parallelism that is independent of the selected path; (ii) the covariant differentiation preserves the metric associated to the 1-forms ω_i, $i = 1, \ldots, n$ in the following sense: if one defines $a_{ik} = \sum_{r=1}^n \underset{r}{X_i} \underset{r}{X_k}$, then $D_t a_{ik} = 0$, $i, k, t = 1, \ldots, n$.

Almost at the same time, the idea of constructing a covariant derivation based upon the assignment of n independent 1-forms was set forth by Weitzenböck in his monograph devoted to the theory of invariants, (Weitzenböck 1923). However, it should be observed that Weiztenböck's exposition was limited to an analytical treatment. Contrary to Vitali, Weitzenböck made no attempt at providing a geometrical interpretation of his covariant derivation in terms of parallelism of vectors (at least on that occasion).

A notable reaction to Vitali's memoir came from Ricci Curbastro, to whom Vitali sent a copy of his work in early 1924. Ricci's remarks are interesting in many respects. It seems that Ricci was not particularly impressed by the geometrical consequences of Vitali's definition. He failed to appreciate the novelty produced by the new parallelism introduced by Vitali probably because he was not particularly interested in exploring non-Riemannian geometries. As a consequence of this, he preferred to focus on the algorithmic aspects of Vitali's covariant differentiation by proposing an alternative, more natural, approach. As shown by a letter to Vitali dating back to February 1924, Ricci thought that Vitali's treatment could be highly simplified upon consideration of absolute invariants.[8] The main idea can be easily described as follows: given any tensor $A_{r_1 \ldots r_h}^{s_1 \ldots s_k}$ and a system of covariant tensors $\underset{r}{X_k}$ with the associated contravariant tensors $\underset{r}{X^k}$, we can construct n^{k+h} absolute invariants:

$$J_{i_1 \ldots i_k i_{k+1} \ldots i_{k+h}} = \sum_{j,s} A_{j_1 j_2 \ldots j_h}^{s_1 s_2 \ldots s_k} \underset{i_1}{X_{s_1}} \underset{i_2}{X_{s_2}} \ldots \underset{i_k}{X_{s_k}} \underset{i_{k+1}}{X^{j_1}} \underset{i_{k+2}}{X^{j_2}} \ldots \underset{i_{k+h}}{X^{j_h}}$$

and consider the system of coefficients $A_{r_1 \ldots r_h, t}^{s_1 \ldots s_k}$ produced by (ordinary) differentiation of these invariants:

$$A_{r_1 \ldots r_h, t}^{s_1 \ldots s_k} := \sum \frac{\partial J_{i_1 \ldots i_k i_{k+1} \ldots i_{k+h}}}{\partial x^t} \underset{i_1}{X^{s_1}} \underset{i_2}{X^{s_2}} \ldots \underset{i_k}{X^{s_k}} \underset{i_{k+1}}{X_{j_1}} \underset{i_{k+2}}{X_{j_2}} \ldots \underset{i_{k+h}}{X_{j_h}}. \qquad (8)$$

By construction, it is clear that $A_{r_1 \ldots r_h, t}^{s_1 \ldots s_k}$ are coefficients of a tensor with $h+1$ covariant indices and k contravariant indices. Ricci's crucial observation consisted in identifying them with the coefficients of Vitali's covariant derivative, thus providing a much simpler proof of the covariant character of (7). Incidentally, it is interesting to observe that similar observations were also communicated to Vitali by Tullio Levi-Civita, as is testified by a letter that he addressed to Vitali in February 1924.[9]

Some months later, Vitali decided to return to the subject in another brief memoir (Vitali 1925) in which he thoroughly developed the point of view that Ricci had communicated to him, along the lines described above.

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[8] (Vitali 1984, p. 486).

[9] (Vitali 1984, p.487).

Despite its innovative content, the publication of (Vitali 1923a) went almost unnoticed in the short term. It took some time before its importance could finally be appreciated. In 1927 Enea Bortolotti exploited Vitali's parallelism to provide a systematic analysis of special types of absolute parallelisms recently introduced by Cartan and Schouten.[10] Furthermore, he could prove that Vitali's parallelism can be characterized as the most general parallelism associated to a zero-curvature connection which preserves both angles and lengths of vectors.

Quite unexpectedly, the type of absolute parallelism studied by Vitali found a physical application in the attempt at developing a unified theory of gravitational and electromagnetic interactions which was pursued by Albert Einstein over the period 1928-1931. According to this approach, which Einstein dubbed Fernparallelism (distant parallelism), the physical fields of the theory are identified with the components of 4 linearly independent contravariant first-order tensors, corresponding to Vitali's 1-forms $\omega_i = \sum_k X_{i\ k} \, \mathrm{d}x^k \ (i = 1, \ldots, n = 4)$.[11]

The publication of Einstein's papers on Fernparallelism and the consequent resonance of the mathematical notions employed by him triggered a lively priority debate. On his part, Vitali tried to propagate his research on the subject and to obtain recognition for the discovery of the notion that he had introduced in (Vitali 1923a). Indeed, following Levi-Civita's advice, he decided to write to Einstein to inform him of his works on absolute parallelism.[12]

Undoubtedly, beyond individual claims, the discovery of the connection associated to (7) can be considered as a collective achievement to which various mathematicians—-Weitzenböck, Cartan and Vitali himself—contributed in different ways, by proposing

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[10] (Cogliati and Mastrolia 2018).

[11] On Einstein's Fernparallelism approach, (Sauer 2006).

[12] Vitali's letter (11 Febbraio 1929) is preserved at Einstein Archives, Jerusalem.

Illustrissimo Signor Professore.
Il mio carissimo amico Prof. Tullio Levi-Civita mi informa di aver ricevuto da Lei copie delle note (a cui hanno fatto cenno recentemente anche i giornali politici), e mi comunica come Ella faccia ricorso ad una specie di parallelismo assoluto, che fu già considerato in una mia nota "Una derivazione Covariante formata con l'ausilio di n sistemi covarianti del 1°."
Ella forse non ha avuto occasione di conoscere tale nota che è pubblicata in una rivista poco diffusa, e mi permetto di inviargliela, insieme con altra che con essa ha qualche attinenza.
Esiste pure sull'argomento un lavoro di R. Weitzenbock [sic!] del 1923, e più recentemente un lavoro, di Enea Bortolotti, il quale, pregato da me, Le invierà fra giorni un estratto.
Le sarei molto grato se Ella volesse inviarmi una copia delle Sue ultime note.
Gradisca, Illustre Professore, con l'espressione della mia considerazione, i miei più devoti ossequi.

Illustrious Mr. Professor. My dear friend Prof. Tullio Levi-Civita informs me that he has received copies of some papers from you (to which political newspapers have also recently referred), and tells me that you have recourse to a kind of absolute parallelism that was already considered in the note of mine "Una derivazione Covariante formata con l'ausilio di n sistemi covarianti del 1°."
Perhaps you did not have the opportunity to know this paper, which is published in a little-known journal. I take the liberty of sending it to you, together with others that have some connection with it. There is also a work on the subject by R. Weitzenbock [sic!] dating back to 1923, and more recently a work by Enea Bortolotti, who, at my request, will send you an extract in a few days.
I would be very grateful if you would like to send me a copy of your most recent articles. I send, most distinguished Professor, my most devoted respects with the expression of my consideration.

complementary perspectives. An accurate and reliable reconstruction of the historical process leading to the introduction of the notion of absolute parallelism was offered by Cartan in (Cartan 1930), a paper written for *Mathematische Annalen* at Einstein's encouragement. In addition to providing a detailed list of his own works, Cartan recognized the relevance of Vitali's insight by emphasizing the difference between Weiztenböck's analytical treatment and Vitali's geometrical interpretation.

4 Pascal's higher-order differential forms

Almost at the same time when exploring alternative definitions of parallelism and covariant differentiation, Vitali began to cultivate the ambitious project of constructing an extension of absolute differential calculus capable of embracing tensors of a new type, characterized by a peculiar multi-index structure.

On some occasions,[13] Vitali recognized the role played by previous investigations in influencing his research in this field. He praised with special emphasis the achievements of Ernesto Pascal for his contributions as a continuator of the path disclosed by Ricci's calculus and his research on general differential forms. Thus, in order to put the emergence of Vitali's calculus in an appropriate historical perspective, it is first necessary to analyze in some details some aspects of Pascal's theory of differential forms.

Since early 1890s, Pascal had pursued investigations that led him to develop a generalization of the theory of differential forms. He set out to study invariant quantities associated to higher-order differential expressions for the purpose of creating a theory that could be regarded as an extension of both the classical integration theory of Pfaffian forms ($\mathbf{X} = \sum_k X_k \mathrm{d}x_k$) and the theory of quadratic differential forms ($\tilde{\mathbf{X}} = \sum_{ij} X_{ij} \mathrm{d}x^i \mathrm{d}x^j$).

In (Pascal 1907), Pascal introduced the notion of "higher-order differential form." This is a complicated expression of the following type:

$$\mathbf{X}^{(r_1,\ldots,r_k)} = \sum_{m=1}^{r_1} \cdots \sum_{p=1}^{r_k} \sum_{j\ldots i} X_{j_1\ldots j_m;\ldots;i_1\ldots i_p} \delta_{j_1\ldots j_m}^{(r_1)} \cdots \delta_{i_1\ldots i_p}^{(r_k)}. \tag{9}$$

Here, the symbols $\delta_{j_1\ldots j_l}^{(r_j)}$ denote appropriate combinations of the differentials of the variables $x^1, \ldots, x^n$ defined by:

$$\delta_{j_1\ldots j_m}^{(r_j)} = \frac{1}{m!} \sum_{(j_1\ldots j_m)} \sum_{i_1\ldots i_m} [i_1 \ldots i_m] \mathrm{d}^{i_1} x_{j_1} \ldots \mathrm{d}^{i_m} x_{j_m}, \quad \sum_{s=1}^{m} i_s = r_j,$$

where $[i_1, \ldots, i_m]$ are numerical coefficients to be determined. Furthermore, the summation symbol $\sum_{(j_1\ldots j_m)}$ is extended to all permutations of $(j_1 \ldots j_m)$ and the summation $\sum_{i_1\ldots i_m}$ covers all integer partitions of r_j. The differential form $\mathbf{X}^{(r_1,\ldots,r_k)}$ was said by Pascal to be a general differential form of $r = \sum_{i=1}^{k} r_i$ order and k degree.

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[13] See (Vitali 1923b) and the next section.

The simplest example of a differential expression of this new type was investigated in (Pascal 1902). Here, Pascal considered the following form of second order (i.e., containing second-order differentials) and first degree:

$$\mathbf{X}^{(2)} = \sum_{k=1}^{n} X_k d^2 x^k + \sum_{i=1}^{n} \sum_{j=1}^{n} X_{ij} dx^i dx^j. \tag{10}$$

Here, X_k, X_{ij} are functions of the variables $(x^1, \ldots, x^n)$ that are symmetric in the indices i, j: $X_{ij} = X_{ji}$, $i, j = 1, \ldots, n$. It should be observed that if X_k are taken to be zero then $\mathbf{X}^{(2)}$ is a quadratic differential form of the classical type that can be geometrically interpreted as a metric of a Riemannian n-dimensional manifold. En route to the search for invariants and differential parameters associated to (10), Pascal introduced a rich set of new quantities, including a sort of generalization of the Christoffel symbols. One should mention the following functions:

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\begin{aligned}
(ij) &:= \frac{\partial X_i}{\partial x^j} - \frac{\partial X_j}{\partial x^i} \\
((ij)) &:= \frac{\partial X_i}{\partial x^j} - X_{ij} \\
\{ij\} &:= \frac{\partial X_i}{\partial x^j} - \frac{\partial X_j}{\partial x^i} - 2X_{ij},
\end{aligned}
\tag{11}
$$

by means of which two covariant differential forms could be defined:

$$A = \sum_{i,j=1}^{n} ((ij)) dx^i dx^j, \qquad B = \sum_{i,j=1}^{n} \{ij\} dx^i dx^j. \tag{12}$$

It is to be noticed that the coefficients X_k, X_{ij} are characterized by a specific transformation law, which is different from that of covariant tensors. Indeed, by observing that, under a change of coordinates $x \mapsto y(x)$ (a diffeomorphism), one has:

$$d(dy^k) = \sum_{s,r=1}^{n} \frac{\partial^2 y^k}{\partial x^s \partial x^r} dx^s dx^r + \frac{\partial y^k}{\partial x^r} d^2 x^r,$$

it can be easily verified that if Y_k, Y_{ij} denote the coefficients of $\mathbf{X}^{(2)}$ with respect to the coordinates $(y^1, \ldots, y^n)$, then the following relations hold true:

$$X_{ik} = \sum_{l=1}^{n} Y_l \frac{\partial^2 y^l}{\partial x^i \partial x^k} + \sum_{q,t=1}^{n} Y_{qt} \frac{\partial y^q}{\partial x^i} \frac{\partial y^t}{\partial x^k}, \quad i, k = 1, \ldots, n;$$

$$\tag{13}$$

$$X_k = \sum_{l=1}^{n} Y_l \frac{\partial y^l}{\partial x^k} \quad k = 1, \ldots, n.$$

Interestingly enough, Pascal also introduced the symbols

$$\{ijk\} := \frac{\partial^2 X_k}{\partial x^i \partial x^j} + \frac{\partial X_{ij}}{\partial x^k} - \frac{\partial X_{ik}}{\partial x^j} - \frac{\partial X_{jk}}{\partial x^i}, \quad i, j, k = 1 \ldots, n;$$

which are easily proven to be equal to

$$\{ijk\} = \frac{1}{2}\frac{\partial}{\partial x^i}\{kj\} + \frac{1}{2}\frac{\partial}{\partial x^j}\{ki\} - \frac{1}{2}\frac{\partial}{\partial x^k}\{ij\} \quad i,j,k = 1\ldots,n. \tag{14}$$

He regarded them as a generalization of the Christoffel symbols; indeed, their expression as given by (14) coincides with the classical formula for Christoffel symbols of the first kind, computed with respect to the form B. According to Pascal, this formal analogy could be exploited in order to extend the classical theory of differential parameters to forms such as $\mathbf{X}^{(2)}$.

The study of differential forms of arbitrary order r and arbitrary degree k was carried on in a series of subsequent papers. In (Pascal 1910), a long memoir amounting to almost one hundred pages, a systematic and comprehensive treatment of this new theory was offered for the first time.

Two years before, the subject was the topic of a conference that Pascal delivered on the occasion of the ICM held in Rome in 1908. The introduction to his speech provides a clear insight into both his aims and his underlying motivations.

> The aim of this contribution is to draw the attention of mathematicians to a new theory of differential forms of arbitrary order and degree. I developed this theory in recent years as an extension of the ancient theory of Pfaffian forms and differential quadratic forms.
> All the most important results in the realm of these two theories, whose contributors are among the greatest analysts of the nineteenth century Pfaff, Jacobi, Grassmann, Riemann, Clebsch, Lie, Lipschitz, Frobenius, Christoffel, Beltrami, etc., are but the simplest and most obvious among other results, so far unnoticed, much more general and of a wider nature; the general theory about which I am about to speak, although at first glance it might appear difficult due to the complicated formulas that it produces. Nonetheless, with appropriate devices, it can be rendered more manageable and thus acquire symmetry and elegance, virtues for which I dare to ask for hospitality for this brand new theory among the disciplines of modern analysis.[14]

Indeed, Pascal succeeded in developing a general theory that, in addition to providing a formidable extension of classical theories, could offer a theoretical framework in which previously unrelated notions, such as the bilinear covariant of a Pfaffian form and the Christoffel symbols of a quadratic differential form, could be subsumed under a common concept.

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[14] Lo scopo di questa mia Comunicazione è di richiamare l'attenzione dei matematici sulla nuova teoria delle forme differenziali di ordine e grado qualunque, che io sono andato formando in questi ultimi anni, come estensione dell'antica teoria delle forme Pfaffiane e di quella delle forme differenziali quadratiche. Tutti i più brillanti risultati nel campo di queste due particolari teorie al cui sviluppo sono legati i nomi dei maggiori analisti del secolo scorso, Pfaff, Jacobi, Grassmann, Riemann, Clebsch, Lie, Lipschitz, Frobenius, Christoffel, Beltrami, etc., non sono che i casi più semplici e più ovvii di risultati, rimasti finora inosservati, assai più generali, e di una natura molto più ampia; e la teoria generale di cui vi parlo, per quanto a prima vista possa apparire irta di difficoltà, per la complicazione delle formule cui sembra dar luogo, pure, con opportuni artifizii e congegni, è capace di perdere ogni eccessiva complicazione, e di acquistare quella simmetria e quella eleganza, che sono le doti in omaggio alle quali io mi permetto di domandare ospitalità anche per questa nuova teoria fra i capitoli dell'Analisi moderna. (Pascal 1909, p. 138)

To this end, the study of the transformation property of the coefficients and the discovery of a covariant algorithm for derivation turned out to be essential.

In order to tackle the first problem, Pascal introduced the symbols (symmetric both with respect to the indices j and h):

$$\begin{pmatrix} j_1 \dots j_m \\ h_1 \dots h_r \end{pmatrix}_{xy}, \tag{15}$$

which he defined as follows.

Consider a (sufficiently regular) function F of the variables $(x^1, \dots, x^n)$ and suppose that the x^i's may be regarded as functions of other variables $(y^1, \dots, y^n)$. The symbols (15) are implicitly defined (in addition to the request of symmetry with respect to the indices j and h) by:

$$\frac{\partial^r F}{\partial y^{h_1} \dots \partial y^{h_r}} = \sum_{m=1}^{r} \sum_{j_1 \dots j_m = 1}^{n} \frac{\partial^m F}{\partial x^{j_1} \dots \partial x^{j_m}} \begin{pmatrix} j_1 \dots j_m \\ h_1 \dots h_r \end{pmatrix}_{xy}.$$

It is clear then that (15) are sums of products of the partial derivatives of $x = (x^1(y), \dots, x^n(y))$ until the r-th order.[15] As a consequence of the structure of the differentials $\delta_{j_1 \dots j_m}^{(r_j)}$, the coefficients of $\mathbf{X}^{(r_1, \dots, r_k)}$ transform as follows under an arbitrary change of coordinates:

$$Y_{h_1 \dots h_\mu; \dots; l_1 \dots l_\pi} = \sum_{m=1}^{\mu} \dots \sum_{p=1}^{\pi} \sum_{j \dots i} X_{j_1 \dots j_m; \dots; i_1 \dots i_p} \begin{pmatrix} j_1 \dots j_m \\ h_1 \dots h_\mu \end{pmatrix}_{xy} \dots \begin{pmatrix} i_1 \dots i_p \\ l_1 \dots l_\pi \end{pmatrix}_{xy} \tag{16}$$

Clearly guided by an analogy with Ricci's calculus, Pascal called a system of functions with k sets of indices $X_{j_1 \dots j_m; \dots; i_1 \dots i_p}$ characterized by the transformation law (16) a covariant system with k sets of indices. Clearly, the usual notion of covariant system (covariant tensor of k-th order) can be obtained by choosing $r_1 = \dots = r_k = 1$, thus assuring that the theory of generalized differential forms comprises the classical notion of covariant tensors too.

The above-mentioned project consisting in developing a unitary approach to differential forms of different kinds was regarded by Pascal as a priority for his research and, at the same time, as one of the main achievement of his theoretical work. This ambitious plan could be made viable thanks to a technical algorithm, called the operation of "deducing", which Pascal introduced in 1906. It was a sort of covariant differentiation that allowed one to produce covariant systems with $k+1$ sets of indices starting from a given one with k sets of indices.

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[15] By way of an example, it is easily seen that $\begin{pmatrix} j_1 j_2 \\ h_1 h_2 \end{pmatrix}_{xy}$ is equal to:

$$\frac{1}{2!} \left(\frac{\partial x^{j_1}}{\partial y^{h_1}} \frac{\partial x^{j_2}}{\partial y^{h_2}} + \frac{\partial x^{j_2}}{\partial y^{h_1}} \frac{\partial x^{j_1}}{\partial y^{h_2}} \right).$$

Let $X_{j_1\ldots j_m;\ldots;i_1,\ldots i_p}$ be a covariant system with k set of indices; consider another set of q indices $g_1\ldots g_q$ and the corresponding partial derivative:

$$DX_{j_1\ldots j_m;\ldots;i_1,\ldots i_p} = \frac{\partial^q X_{j_1\ldots j_m;\ldots;i_1,\ldots i_p}}{\partial x^{g_1}\ldots\partial x^{g_q}}. \tag{17}$$

By following (Pascal 1910, p. 26), we introduce a simplified notation to denote the right-hand side of (17) by defining:

$$\frac{j_1\ldots j_m;\ldots;i_1,\ldots i_p}{g_1\cdots g_q} := \frac{\partial^q X_{j_1\ldots j_m;\ldots;i_1,\ldots i_p}}{\partial x^{g_1}\ldots\partial x^{g_q}} \tag{18}$$

Now, let us construct all the partial derivatives of the $(q-1)$-th order that are obtained from (17) by "moving" each index of $(g_1\ldots g_q)$ to each group of indices $(j_1\ldots j_m);\ldots;(i_1,\ldots i_p)$, i.e.:

$$\frac{j_1\ldots j_m g_1;\ldots;i_1,\ldots i_p}{g_2\cdots g_q},\ldots,\frac{j_1\ldots j_m;\ldots;i_1,\ldots i_p g_q}{g_1\cdots g_{q-1}} \tag{19}$$

We denote by $\boldsymbol{\Omega}D$ the operation consisting in summing all the derivatives obtained in this way from D. The operator $\boldsymbol{\Omega}$ can be extended by linearity to sums of derivatives $DX+D'X+\ldots$. With these preliminaries, Pascal introduced the so called *fundamental symbols*:

$$((j_1\ldots j_m;\ldots;i_1,\ldots i_p;g_1\ldots g_q)) = \sum_{k=1}^{q}\frac{(-1)^k}{k!}\vec{\Omega}^{(k)}DX_{j_1\ldots j_m;\ldots;i_1,\ldots i_p};$$

he also called the quantities $((j_1\ldots j_m;\ldots;i_1,\ldots i_p;g_1\ldots g_q))$ the q-th covariant deduced (function) of the coefficients $X_{j_1\ldots j_m;\ldots;i_1,\ldots i_p}$. The attribute "covariant" was aptly chosen since these symbols transform according to the transformation rule that is characteristic of a covariant system with $k+1$ sets of indices.

Despite their complexity, the introduction of the fundamental symbols for the general case of coefficients such as $X_{j_1\ldots j_m;\ldots;i_1,\ldots i_p}$ must have appeared quite natural. Pascal clearly followed a reasoning by analogy. Indeed, similar expressions had already presented themselves in previous works where Pascal's attention had focused on the study of differential expressions of first degree and second order.

The role of the fundamental symbols consisted mainly in producing additional functions by means of which a thorough study of the invariant properties of forms $\mathbf{X}^{(r)}$ of first degree could be attained. Their definitions as provided in (Pascal 1910, §10) read as follows:

$$\left(j_1\ldots j_m;i_1\ldots i_p\right)_X = ((j_1\ldots j_m;i_1\ldots i_p)) - (-1)^{m+p}((i_1\ldots i_p;j_1\ldots j_m))$$

$$\left\{j_1\ldots j_m;i_1\ldots i_p\right\}_X = ((j_1\ldots j_m;i_1\ldots i_p)) + (-1)^{m+p}((i_1\ldots i_p;j_1\ldots j_m))$$

These quantities, dubbed principal symbols of the first and second kind, respectively, can be seen as a direct generalization of the functions already introduced for cases $r = 1, 2$. Indeed, this can easily be checked by performing the relevant computations with respect to forms $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}$.

To this end, one has first to compute the symbols $((i; j_1))$, $((i; j_1 j_2))$ and then deduce the corresponding principal ones. One easily obtained for example:

$$((i; j_1)) = \frac{\partial X_i}{\partial x^{j_1}} - X_{ij_1}, \qquad ((i; j_1 j_2)) = \frac{\partial^2 X_i}{\partial x^{j_1} \partial x^{j_2}} - \frac{\partial X_{ij_1}}{\partial x^{j_2}} - \frac{\partial^2 X_{ij_2}}{\partial x^{j_1}}, \qquad (20)$$

and consequently:

$$(j; i) = \frac{\partial X_j}{\partial x^i} - \frac{\partial X_i}{\partial x^j}, \qquad (21)$$

$$(j_1 j_2; i) = \frac{\partial^2 X_i}{\partial x^{j_1} \partial x^{j_2}} - \frac{\partial X_{j_1 i}}{\partial x^{j_2}} - \frac{\partial X_{j_2 i}}{\partial x^{j_1}} + \frac{\partial X_{j_1 j_2}}{\partial x^i}; \qquad (22)$$

thus showing that the bilinear covariant of a differential form $\mathbf{X}^{(1)}$ and the Christoffel symbols of a quadratic differential form $\mathbf{X}^{(2)}$, with $X_k = 0, k = 1, \ldots, n$, can both be considered as principal symbols of the first kind.

A thorough examination of Pascal's contributions to the theory of general differential forms would go well beyond the scope of this paper. Nonetheless, in order to gain a general idea of the motivations at the basis of his research, it is useful to mention some of his most noteworthy results:

1. He provided an extension of the theory of Pfaff's reduction problem, by finding conditions for a given differential form of order r (and degree 1), in the variables $x^1, \ldots, x^n$, $\mathbf{X}^{(r)}$, to be written as $\Phi \cdot \tilde{X}$, where Φ is a function of n variables and $\tilde{X}$ depends on $n - 1$ variables, only;
2. He generalized the notion of completely integrable system to a system of differential forms of arbitrary order and first degree.

These results notwithstanding, the reception of Pascal's works in the short run was scarce. His theory of general differential forms did not attract much interest or even achieve a widespread appreciation. No doubt, the algorithmic complications imposed by the high degree of generality of his treatment were judged as excessive and inadequately balanced by the advancements that they produced. This point of view was publicly expressed, for example, on the occasion of the two competitions for the Royal Prize for Mathematics in 1901 and 1907. In both circumstances, Pascal was not awarded the prize. The motivation at the basis of the negative outcome may be summarized as followed: Pascal's theory had still to prove its fecundity by displaying some noteworthy application in the realm of integration theory of PDEs, specifically second-order ones.

The following passage taken from the report of the Commission responsible for the attribution of the 1907 prize provides a clear representation of the attitude of the Italian mathematical milieu (or at least of a part thereof) towards Pascal's theoretical constructions.

Pascal's theory of higher-order differential forms is very remarkable, especially from the formal point of view, for the great generality of the results and for their relative and unexpected simplicity. The constant industriousness and the singular algorithmic ability displayed by the author in the discovery of the simple laws for invariant formations are truly admirable. By means of them, the author solved the fundamental problems posed by this new theory. Of course, if the new theories of differential forms constructed by Pascal show their effectiveness in dealing with problems concerning partial differential equations of order higher than the first, […] then the value of his research will be greatly elevated.[16]

Quite surprisingly, in spite of his original motivations, the techniques introduced by Pascal first displayed and proved their fecundity in geometrical investigations rather than in purely analytical studies of the integration theory of second-order PDEs. As will be seen, it was Vitali's merit to recognize the usefulness of Pascal's theory in dealing with the new problems emerging in recent developments of differential geometry.

5 Vitali's calculus: origins and first definitions

The introduction to (Vitali 1923b), the first paper that Vitali devoted to the edification of his calculus, contains noteworthy comments concerning the original motivation of his work. On this occasion, he was very explicit in acknowledging the influence of Pascal's investigations on his own research, by praising the latter especially for his discovery of the operation of "deducing" and the proof of its covariant character.

In this memoir, I set out to expose in a very simple way the foundations of a generalized absolute calculus according to ideas that can be found in some important contributions by the most illustrious professor Pascal. In these works, whose results were collected in a valuable memoir,[17] notions such those of "deducing" and of principal symbols were discovered that are so fundamental for this theory that one can consider Pascal as of the best continuator of Gregorio Ricci-Curbastro.[18]

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[16] Questa teoria delle forme differenziali d'ordine superiore, costruita dal Pascal, è molto notevole, specialmente dal punto di vista formale, per la grande generalità dei risultati e per la relativa ed inattesa semplicità loro. È veramente ammirevole la costante operosità e la singolare abilità algoritmica spiegata dall'A. nella scoperta delle semplici leggi per le formazioni di carattere invariantivo, col sussidio delle quali vengono a risolversi i problemi fondamentali della nuova teoria. Certo, se le nuove teorie sulle forme differenziali costruite dal Pascal manifesteranno la loro efficacia nella trattazione dei problemi concernenti le equazioni a derivate parziali d'ordine superiore al 1° […] grandemente elevato ne verrà il valore di queste sue ricerche. (Segre 1907, p. 421).

[17] Here Vitali referred to (Pascal 1910).

[18] Nella presente Memoria io mi propongo di esporre in modo molto semplice i fondamenti di un calcolo assoluto generalizzato quale è suggerito da vari e importanti lavori del Ch.mo Prof. Ernesto Pascal. In questi lavori, i cui risultati furono raccolti dall'Autore in una bella Memoria, sono stati trovati degli elementi, come le *dedotte* e i *simboli principali* che sono fondamentali per questa teoria, e tanto importanti, a mio avviso, che si può considerare il Pascal come il migliore continuatore dell'opera di Gregorio Ricci-Curbastro. (Vitali 1923b, p. 157).

Despite Pascal's efforts, his calculus of differential forms had attracted little attention in the years that followed. Unsurprisingly, the reception process of his work was bound to the fortune of Ricci's calculus itself. It was only after the emergence of a favorable attitude towards the latter, mainly due to the discovery of General Relativity (1915-1916), that attempts at generalization, such as those brought about by Pascal, could be regarded as viable and even longed for. From this perspective, the birth of Vitali's absolute calculus can be seen as a by-product of the ongoing process of re-evaluation of Ricci's calculus techniques following the publications of Einstein's papers.

On a technical level, Vitali introduced two main innovations with respect to both Ricci's and Pascal's theories. They are the use of a functional representation for manifolds, which are regarded as immersed in a Hilbert space, and the recourse to a multi-index notation that greatly simplified the execution of intricate calculations.

Possibly out of his deep interest in functional analysis, to which he had widely contributed, Vitali chose a Hilbert space as the natural setting for the geometrical objects he set out to study, by supposing that an n-dimensional manifold could be described, in a sense to be explained in what follows, by giving a representative function $F(t; u^1, \ldots, u^n) \in L^2(\mathbb{R})$. The idea was to provide an extension of the notion of a parametrized vector $F(u^1, \ldots, u^n) \in \mathbb{R}^N$ in terms of $F(t; u^1, \ldots, u^n) \in L^2(\mathbb{R})$. Since $L^2(\mathbb{R})$ can be regarded as an infinite-dimensional generalization of $\mathbb{R}^N$, Vitali considered it to be the natural ambient space for the study of manifolds. In particular, he thought that this highly general setting could be particularly suitable for research on projective differential properties. As for the second innovation that Vitali introduced, a highly effective multi-index notation (Bortolotti spoke of "ingenious technique") allowed him to define a generalization both of tensors and covariant derivatives that was particularly useful in order to emphasize similarities with respect to Ricci's calculus. Starting from 1923, Vitali produced various presentations of his calculus in a series of publications that culminated in the monograph Geometria nello spazio hilbertiano where he provided a first systematic treatment of the subject.

A first version of the multi-index notation was introduced in (Vitali 1923b) and later refined in his Geometria. We will mainly follow the presentation provided in the latter. By analogy with Ricci's classical treatment, Vitali defined his generalized tensors (Vitali referred to them as "Pascal systems" or as "absolute systems") as a set of functions of the coordinates of a given manifold with covariant and contravariant indices that transform according to prescribed laws. From a synthetic point of view one can regard these generalized tensors as multilinear maps defined on osculating spaces (and their dual counterpart) of a given manifold; precisely as ordinary tensors can be regarded as multilinear maps on tangent spaces and their dual counterpart.

First examples of this general notion had been introduced a few years before Vitali's investigation in a number of works by Eugenio Elia Levi and by Pascal himself. Indeed, the coefficients of a differential form of degree k and order r are the coefficients of a generalized tensor (to be defined later) in Vitali's sense. In order to write down the transformation laws of these general objects, the introduction of a multi-index notation turned out to be essential.

Let us see some definitions proposed in (Vitali 1929), by limiting ourselves to the ones most relevant to our approach. First, Vitali considered scalar functions, which

he called invariant, on a manifold V_n. They are regular functions on V_n, $I : V_n \to \mathbb{R}$ whose representations in different coordinates systems $u^1, \ldots, u^n$, $v^1, \ldots, v^n$, are denoted by $I[u] = I(u) = I(u(v)) = I[v]$. He then introduced what he called the set (*campo*) Ω defined as follows. Let n be a fixed positive integer and let Ω be the set of combinations of elements $\{1, 2, \ldots, n\}$ with repetitions taken k at a time, regardless of their order, with $k = 1, 2, 3, \ldots$. An index is a variable taking value on Ω, which is denoted by a single letter of the Greek alphabet $\alpha, \beta, \ldots$. A given value α of an index is said to be a *state* of the index. Moreover, the number of digits of a given state, denoted with ρ_α, is called the *rank* of the state. Finally, Vitali defined an index α to be of class ν if it takes on all the states for which $1 \leq \rho_\alpha \leq \nu$. By way of example, an index α of class 2 is a variable index that can take on all values in the set $I_2 := \{1, 2, \ldots, n, 11, 12, \ldots, 1n, 22, 23, \ldots, nn\} \subset \Omega$.

In light of these definitions, Vitali introduced his own version of Pascal's symbols (15), $\dfrac{\partial u^\beta}{\partial v^\alpha}$, $\alpha, \beta \in \Omega$, is implicitly defined by the following equations that express the derivatives $I_\alpha[v] := \dfrac{\partial^r I}{\partial v^{i_1} \ldots \partial v^{i_r}}$, $\alpha = (i_1, \ldots, i_r)$ with respect to $I_\beta[u] = \dfrac{\partial^s I}{\partial u^{j_1} \ldots \partial u^{j_s}}$, $\beta = (j_1, \ldots, j_s) \in \Omega$:

$$I_\alpha[v] = \sum_{\beta \in \Omega} \frac{\partial u^\beta}{\partial v^\alpha} I_\beta[u],$$

where by definition $\dfrac{\partial v^\alpha}{\partial u^\beta} = \dfrac{\partial v^i}{\partial u^j}$, if $\alpha = i$, $\beta = j$ and $\dfrac{\partial u^\beta}{\partial v^i} \equiv 0$, when $\rho_\beta > 1$. In general, as was proven in (Vitali 1929, pp. 156–158), these expressions are polynomials in the derivatives of the functions $u^k = u^k(v^1, \ldots, v^n)$.

Finally, on the basis of these results, Vitali was able to propose his own generalization of Pascal's systems. He defined[19] an *absolute system* to be a set of functions of $(u^1, \ldots, u^n)$, $H_{\alpha_1, \ldots, \alpha_r}^{\beta_1, \ldots, \beta_s}$ that, under an arbitrary (invertible and sufficiently regular) change of coordinates $(u^1, \ldots, u^n) \mapsto (v^1(u), \ldots, v^n(u))$ transform as:

$$\tilde{H}_{\alpha_1, \ldots, \alpha_r}^{\beta_1, \ldots, \beta_s}(v) = \sum_{\alpha', \beta'} H_{\alpha'_1, \ldots, \alpha'_r}^{\beta'_1, \ldots, \beta'_s}(u) \prod_{h=1}^{r} \frac{\partial u^{\alpha'_h}}{\partial v^{\alpha_h}} \prod_{k=1}^{r} \frac{\partial v^{\beta_h}}{\partial u^{\beta'_k}}, \tag{23}$$

where the summation is extended to all states of indices in the classes of α'_h, β'_k which coincides with the classes of α_h and β_k, respectively. It is clear that an absolute system is characterized by transformations laws formally identical to those that occur in the classical Ricci calculus. Nonetheless, it should be borne in mind that the symbols α and β actually represent groups of indices and the expressions $\dfrac{\partial u^\alpha}{\partial v^\beta}$ and $\dfrac{\partial v^\alpha}{\partial u^\beta}$ are polynomials that contain derivatives until orders depending on the classes of the indices α, β.

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[19] See (Vitali 1929, pp. 166). See also (Vitali 1927–1928, p. 418), for an earlier occurrence of this notion.

Vitali could exhibit many examples of absolute systems by considering the notion of a manifold immersed in the Hilbert space $L^2(\mathbb{R})$ and the related notion of parametrized point-function, $F(u^1, \ldots, u^n; t) \in L^2(\mathbb{R})$. An absolute system that extends the concept of Riemannian metric could be introduced by means of the scalar product on $L^2(\mathbb{R})$, just as in the case of a manifold immersed in a (finite-dimensional) Euclidean space $\mathbb{R}^N$, whose fundamental tensor is induced by the ordinary scalar product. Vitali defined the absolute system $a_{\alpha;\beta}$ (his own generalization of a Riemannian metric) as follows:

$$a_{\alpha;\beta}(u^1, \ldots, u^n) = \int_{\mathbb{R}} F_\alpha(u; t) F_\beta(u; t) \mathrm{d}t,$$

where α, β are derivation indices of a given class, say ν.

Operations generalizing the classical notions of summation, subtraction, and product were introduced by Vitali without any serious difficulty, by analogy with corresponding definitions of the Ricci calculus. As will be seen, the task of providing an adequate definition of covariant derivative for generalized tensors turned out to be much more problematic.

6 The search for covariant differentiation

The discovery of the notion of covariant differentiation represented a landmark in the historical development of both Riemannian geometry and tensor calculus. Unsurprisingly, the challenge consisting in finding a generalization of this notion played a major role in the process of elaboration of Vitali's new calculus too. However, it took some time before this problem could find a satisfactory and general solution, which indeed Vitali achieved only in 1930, after many unsuccessful attempts.

A first step into these investigations was taken in 1927 in a memoir presented by Fubini to the Accademia dei Lincei. Here, Vitali succeeded in defining a covariant differentiation for Pascal systems with class index 2 and differentiation index of class 1.

The heuristic process that Vitali might have pursued in the course of his search after a completely general differentiation algorithm was succinctly described by Angelo Tonolo, one of his students at Padua University, on the occasion of the obituary notice that he wrote soon after Vitali's death in February 1932.

> By studying the normal directions that lie in the second osculating space at a point of a manifold immersed in a Hilbert space, he obtained Ricci's covariant derivative. For this reason, he thought that the study of the normal directions to the n-th osculating space at a point of the manifold, would lead him to find the expression of the generalized derivative. And so it happened. Indeed, the aforementioned research, guided by a suitable choice of notations, led him to write an operation which is precisely the derivative that he was looking for.[20]

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[20] Studiando le normali che giacciono nel secondo spazio osculatore in un punto di una varietà immersa nello spazio hilbertiano, Egli s'imbattè proprio con la derivata covariante del Calcolo di Ricci. Allora Egli

The remarks here proposed appear convincing. We can gain a better appreciation of Tonolo's reconstruction by a close examination of (Vitali 1927–1928). Let us see in some detail the main idea at the basis of Vitali's discovery. For the sake of simplicity, when it is possible, we avoid the use of the functional representation by limiting ourselves to considering manifolds immersed in $\mathbb{R}^N$.

The starting point consisted of a new geometrical characterization of the classical notion of covariant differentiation in terms of the normal directions to the tangent space at a given point of an immersed manifold.

To this end, Vitali made recourse to the notion of q-th-order osculating space, already introduced and employed by Del Pezzo in 1886.[21] Let the map $(u^1, \ldots, u^n) \mapsto F = (z^1(u), \ldots, z^N(u)) \in \mathbb{R}^N$ provide an analytical representation of a given immersed manifold $V_n \subset \mathbb{R}^N$. The partial derivatives of F, at a given point $P = (\bar{z}^1, \ldots, \bar{z}^N)$ with respect to the variables $u^1, \ldots, u^n$ until a given order q, define a linear space, called q-th-order osculating space (or, alternatively, q-th fundamental space), which Vitali denoted with σ_q. It is clear that σ_1 coincides with the tangent space of V_n at the point P.[22]

In order to determine all the directions X that belong to σ_2 and are normal to σ_1, we can write X as a linear combination of first- and second-order derivatives of the functions F, i.e.:

$$X = \sum_{i=1}^{n} \lambda^i F_i^{(1)} + \sum_{i,j=1}^{n} \lambda^{ij} F_{ij}^{(2)}, \tag{24}$$

where $F_i^{(1)}, F_{ij}^{(2)} \in \mathbb{R}^N$ are defined by: $F_i^{(1)} := \left. \dfrac{\partial F}{\partial u^i} \right|_P$, $F_{ij}^{(2)} := \left. \dfrac{\partial^2 F}{\partial u^i \partial u^j} \right|_P$, $i, j = 1, \ldots, n$. Upon scalar multiplication of both sides of (24) by $F_k^{(1)}$, in virtue of the orthogonality condition, one easily obtains:

$$F_k^{(1)} \cdot X = \sum_{i=1}^{n} \lambda^i F_k^{(1)} \cdot F_i^{(1)} + \sum_{i,j=1}^{n} \lambda^{ij} F_k^{(1)} \cdot F_{ij}^{(2)} = 0, \qquad k = 1 \ldots, n. \tag{25}$$

Since $F_k^{(1)} \cdot F_i^{(1)}$ can be regarded as the coefficients of the Riemannian metric $\sum_{k,i=1}^{n} a_{ki} du^k du^i$ induced on V_n by the Euclidean scalar product in $\mathbb{R}^N$, $F_k^{(1)} \cdot F_{ij}^{(2)}$ are equal to $\Gamma_{ij,k}$.[23] Consequently, equations (25) can be rewritten as follows:

Footnote 20 continued

pensò che lo studio delle normali all'n-esimo spazio osculatore in un punto della varietà in discorso, Lo avrebbe condotto a trovare l'espressione della derivata generalizzata. E così avvenne, perché la ricerca anzidetta, guidata con opportuna scelta di notazioni, Lo portò a scrivere un'operazione che è proprio la derivata che Egli cercava. (Tonolo 1932, p. 75).

[21] See (Del Pezzo 1886).

[22] In (Vitali 1929), the denomination "spazio fondamentale di ordine q" was employed to refer to σ_q. In this respect, one can see a letter of Enrico Bompiani to Vitali (May 1929). (Vitali 1984, p. 503).

[23] Indeed, $\Gamma_{ij,k} = \dfrac{1}{2} \left[\dfrac{\partial}{\partial u^i} \left(F_j^{(1)} \cdot F_k^{(1)} \right) + \dfrac{\partial}{\partial u^j} \left(F_i^{(1)} \cdot F_k^{(1)} \right) - \dfrac{\partial}{\partial u^k} \left(F_i^{(1)} \cdot F_j^{(1)} \right) \right] = F_k^{(1)} \cdot F_{ij}^{(2)}$.

$$\sum_{i=1}^{n} \lambda^i a_{ki} + \sum_{i,j=1}^{n} \lambda^{ij} \Gamma_{ij,k} = 0, \tag{26}$$

or equivalently as:

$$\lambda^j = -\sum_{ik} \Gamma_{ik}^j \lambda^{ik}. \tag{27}$$

This implies that the directions of σ_2 that are perpendicular to σ_1 are linear combinations of the covariant derivatives of the scalar components of $F_k^{(1)}$.

A similar procedure can be applied to determine the directions of σ_3 that are orthogonal to σ_2. By doing this, Vitali found that such directions are linear combinations of the following expressions:

$$\mathcal{F}_{ijk} := F_{jhk}^{(3)} - \sum_{\alpha \in I_2} \begin{bmatrix} jhk \\ \alpha \end{bmatrix} F_\alpha^{(1,2)} \tag{28}$$

where

$$F_\alpha^{(1,2)} = \begin{cases} F_k^{(1)} = \dfrac{\partial F}{\partial u^k}, & \alpha = k, \\ F_{ij}^{(2)} = \dfrac{\partial^2 F}{\partial u^i \partial u^j}, & \alpha = ij \end{cases}, \qquad \begin{bmatrix} jhk \\ \beta \end{bmatrix}_2 = \sum_{\alpha \in I_2} A^{\alpha;\beta} \begin{pmatrix} jhk \\ \alpha \end{pmatrix}$$

with

$$A_{\alpha;\beta} = A_{\beta;\alpha} = \begin{cases} F_{pq}^{(2)} \cdot F_{rs}^{(2)}, & \text{if} \quad \alpha = pq, \quad \beta = rs \\ F_{pq}^{(2)} \cdot F_r^{(1)}, & \text{if} \quad \alpha = pq, \quad \beta = r \\ F_p^{(1)} \cdot F_r^{(1)}, & \text{if} \quad \alpha = p, \quad \beta = r \end{cases},$$

$$\begin{pmatrix} jhk \\ \alpha \end{pmatrix} = \begin{cases} F_{jhk}^{(3)} \cdot F_{mn}^{(2)}, & \text{if} \quad \alpha = mn \\ F_{jhk}^{(3)} \cdot F_m^{(1)}, & \text{if} \quad \alpha = m \end{cases}$$

The coefficients $A^{\alpha;\beta}$ denote the coefficient of the inverse matrix of $\mathbf{A} = [A_{\alpha;\beta}]$.

The quantities $\mathcal{F}_{ijk}$ could be regarded as an extension of Ricci's covariant derivation since it could be shown through a direct computation that they transform as a third-order covariant tensor (in the classical sense). This result suggested to Vitali a promising strategy to tackle the problem of constructing a general differentiation algorithm suitable for generalized tensors with second class indices. Indeed, it was precisely by following this route that Vitali succeeded in proving the following:

Theorem 1 *(Vitali 1927) Let $z = F(u^1, \ldots, u^n)$ be an immersed manifold and let H_α be a generalized tensor with second class index, then the system:*

$$M_{\alpha;k} := \begin{cases} M_{j,k} := \dfrac{\partial H_j}{\partial u^k} - H_{jk}, & \alpha = j; \\[2ex] M_{jh;k} = \dfrac{\partial H_{jh}}{\partial u^k} - \sum_\beta \begin{bmatrix} jhk \\ \beta \end{bmatrix}_2 \cdot H_\beta, & \alpha = jh; \end{cases} \tag{29}$$

is a generalized tensor with two indices α and k of second and first class, respectively.

Interestingly, Vitali made use of this result also to prove the covariant character of the coefficients $\mathcal{F}_{ijk}$. By doing this, he was inverting, as it were, the order of the heuristics: the new calculus could also be employed to solve problems concerning Ricci's classical notions.

We will briefly examine the relevant computations. To this end, let us consider the generalized tensor H_α defined by setting:

$$H_\alpha := F_\alpha^* = \begin{cases} F_k^{*(1)} = \dfrac{\partial F^*}{\partial u^k}, & \alpha = k, \\[2ex] F_{ij}^{*(2)} = \dfrac{\partial^2 F^*}{\partial u^i \, \partial u^j}, & \alpha = ij, \end{cases}$$

where F^* denotes any given scalar component of the vector $F \in \mathbb{R}^N$. As a consequence of theorem (1), the quantities $M_{\alpha;k}$ transform according to:

$$M_{\alpha;k}(u) = \sum_{\beta \in I_2} \sum_{r=1}^n \tilde{M}_{\beta;r}(v(u)) \frac{\partial v^\beta}{\partial u^\alpha} \frac{\partial v^r}{\partial u^k}, \qquad \alpha \in I_2, \, k = 1, \ldots, n. \tag{30}$$

Furthermore, it is easily seen that, as a consequence of the first of (29), $M_{j;k}$ vanishes identically. Incidentally, it should be observed that the $M_{j;k}$ coincide with the coefficients of the first "dedotta" of Pascal's theory. Thus, equations (30) can be rewritten as follows (here $I_2' = I_2 \setminus \{1, 2, \ldots, n\}$):

$$M_{\alpha;k}(u) = \sum_{\beta \in I_2'} \sum_{r=1}^n \tilde{M}_{\beta;r}(v(u)) \frac{\partial v^\beta}{\partial u^\alpha} \frac{\partial v^r}{\partial u^k}, \qquad \alpha \in I_2, \, k = 1, \ldots, n. \tag{31}$$

By observing that, as a consequence of the definition of $\dfrac{\partial v^{st}}{\partial u^{ij}}$, the following relations hold true:

$$\frac{\partial v^{st}}{\partial u^{ij}} = \begin{cases} \dfrac{\partial v^s}{\partial u^i} \dfrac{\partial v^t}{\partial u^j} + \dfrac{\partial v^t}{\partial u^i} \dfrac{\partial v^s}{\partial u^j}, & s \neq t \\[2ex] \dfrac{\partial v^s}{\partial u^i} \dfrac{\partial v^t}{\partial u^j}, & s = t; \end{cases}$$

one easily deduces that the $\mathcal{F}_{ijk}$ transform as:

$$\mathcal{F}_{ijk} = \sum_{r,s,t} \tilde{\mathcal{F}}_{rst} \frac{\partial v^r}{\partial u^i} \frac{\partial v^s}{\partial u^j} \frac{\partial v^t}{\partial u^k};$$

this indeed proves that $\mathcal{F}_{ijk}, i, j, k = 1, \ldots, n$ are the coefficients of a covariant tensor of the third order.

The covariant differentiation defined according to (29) was further extended first to generalized tensors of an arbitrary number of indices of first and second class only and then to generalized tensors of any number of indices of arbitrary class.

A further step towards the discovery of a general notion of covariant differentiation was the extension to the case in which the derivation index is arbitrary too. In Vitali's words:

> It seemed strange that the covariant derivative could not be defined with a derivation index of rank > 1; but only recently have my attempts in this direction been successful. […]
>
> Trying to simplify this proof, I have come to a surprising result. The covariant derivative of an absolute system can be written in a concise form […] which highlights its absolute character.
>
> At first I saw the form to be assigned to the covariant derivative of a covariant system H_α also for states of rank two of the derivation index, and on 5th April, in my lecture of Higher Analysis I proved its absolute character, by taking as a model the demonstration published in GH [i.e. (Vitali 1929)], pp. 186–187. The synthetic form of the covariant derivative not only has the advantage of avoiding a long proof, but also allows one to define the covariant derivative of an absolute system with indexes and superscripts of integer classes for any state of the derivation index chosen in the set Ω.[24]

The main idea at the basis of this generalization was the introduction of an operation, which Vitali called "reciprocity", that extended the procedure, ubiquitously employed in Ricci's calculus, of raising and lowering the indices of a given tensor by means of the coefficients of the metric. In order to follow closely the original treatment, it is necessary to restore the functional representation employed by Vitali and thus to consider an $n-$dimensional manifold as being immersed in the Hilbert space $L^2(\mathbb{R})$.

As a consequence of this functional setting, according to which the manifold V_n is represented by means of an n-dimensional parametrization of points of $L^2(\mathbb{R})$,

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[24] Pareva strano che non si potesse definire il derivato covariante anche con indice di derivazione di rango > 1; ma solo recentemente i tentativi da me fatti in questo senso hanno avuto successo. […]

Cercando di semplificare questa dimostrazione sono giunto ad un risultato sorprendente. Il derivato covariante di un sistema assoluto può essere scritto in una forma sintetica […] che mette in evidenza il suo carattere assoluto.

In un primo momento ho intravisto la forma da assegnarsi al derivato covariante di un sistema covariante H_α anche per gli stati di rango 2 dell'indice di derivazione, ed il 5 aprile u. s. nella mia lezione di Analisi Superiore ne ho dimostrato il carattere assoluto, prendendo come modello la dimostrazione che figura in GH, a pp. 186–187. La forma sintetica del derivato covariante non ha solo il vantaggio di far risparmiare una lunga dimostrazione, ma consente di definire il derivato covariante di un sistema assoluto con indici ed apici di classi intere per qualunque stato dell'indice di derivazione scelto nel campo Ω. (Vitali 1930, p. 47).

$F(t, u^1, \ldots, u^n)$, the coefficients of the generalized metric $a_{\alpha;\beta} = F_\alpha \cdot F_\beta$ are to be seen as the result of the scalar product in $L^2(\mathbb{R})$ of the (real) functions $F_\alpha(t) = F_\alpha(t, u^1, \ldots, u^n)$

$$a_{\alpha;\beta}(u^1, \ldots, u^n) := F_\alpha \cdot F_\beta = \int_{\mathbb{R}} F_\alpha(t) F_\beta(t) \mathrm{d}t,$$

where the indices α, β belong to the same class, say ν. Under the hypothesis that the determinant $\det(a_{\alpha;\beta}) \neq 0$, one can consider—provided a stipulation on the order among different states of the indices is made—the inverse matrix of $\mathbf{A} = [a_{\alpha;\beta}]$ whose coefficients Vitali denoted with the symbols $a^{\alpha;\beta}_{\nu}$, which include an indication of the class of α, β. Precisely by means of $a^{\alpha;\beta}_{\nu}$, Vitali introduced the notion of reciprocity. To this end he considered an absolute system $H^{\beta_1,\ldots,\beta_s}_{\alpha_1,\ldots,\alpha_r}$ and defined the reciprocal system with respect to the index α_h, $h = 1, \ldots, r$ (respectively, β_k) as the absolute system $\sum_{\alpha'_h \in I_\nu} a^{\alpha_h,\alpha'_h}_{\nu} H^{\beta_1,\ldots,\beta_s}_{\alpha_1,\ldots,\alpha'_h,\ldots,\alpha_r}$ $(\sum_{\beta'_k \in I_\nu} a_{\beta_k;\beta'_k} H^{\beta_1,\ldots,\beta'_k,\ldots,\beta_s}_{\alpha_1,\ldots,\alpha_r})$. When this procedure is applied to each one of the $r + s$ indices, the result that is obtained was called the reciprocal system with respect to $H^{\beta_1,\ldots,\beta_s}_{\alpha_1,\ldots,\alpha_r}$.

We can now analyze the definition of covariant differentiation as illustrated by Vitali in (Vitali 1930, Sect. 2). Let $H^{\beta_1,\ldots,\beta_s}_{\alpha_1,\ldots,\alpha_r}$ be an absolute system (α_h and β_k are indices of class ν_h and μ_k, respectively, $h = 1, \ldots, r$; $k = 1, \ldots, s$). Consider the following system associated to H:

$$U^{\beta_1,\ldots,\beta_s}_{\alpha_1,\ldots,\alpha_r} = \prod_{h=1}^{r} F_{\alpha_h}(t_h) \prod_{k=1}^{s} F^{\beta_k}_{\nu_k}(\tau_k), \tag{32}$$

and the corresponding reciprocal system $V^{\alpha_1,\ldots,\alpha_r}_{\beta_1,\ldots,\beta_s}$. Although Vitali did not employed the notion of tensor product of Hilbert spaces, it is evident that U and V can be regarded as elements of

$$\mathcal{H} := \underbrace{L^2(\mathbb{R}) \otimes \ldots \otimes L^2(\mathbb{R})}_{r+s \text{ times}}.$$

Now, let γ be any index in Ω and let $[H, F]$ be defined as

$$\sum_{\alpha_h \in I_{\nu_h}, \beta_k \in I_{\mu_k}} H^{\beta_1,\ldots,\beta_s}_{\alpha_1,\ldots,\alpha_r} V^{\alpha_1,\ldots,\alpha_r}_{\beta_1,\ldots,\beta_s};$$

Vitali introduced the covariant derivative of $H^{\beta_1,\ldots,\beta_s}_{\alpha_1,\ldots,\alpha_r}$ with respect to the index γ, to be denoted with $H^{\beta_1,\ldots,\beta_s}_{\alpha_1,\ldots,\alpha_r,\gamma}$, as follows:[25]

$$H^{\beta_1,\ldots,\beta_s}_{\alpha_1,\ldots,\alpha_r,\gamma} := (U, \Delta_\gamma[H, F])_{\mathcal{H}}. \tag{33}$$

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[25] Here Δ_γ denotes the operator $\dfrac{\partial^{\rho_\gamma}}{\partial u^{i_1} \ldots \partial u^{i_{\rho_\gamma}}}$.

The absolute character of this object is essentially due to the fact that $[H, F]$ is an invariant and U is an absolute system. Furthermore, it should be observed that this notion is sufficiently general so that it comprehends the definitions of covariant differentiation elaborated over the period 1927–1930.

As Vitali himself explained in the above quotation, he first discovered the form to be attributed to the covariant differentiation in the case in which $\rho_\gamma = 2$. However, he provided no hint concerning the idea underlying the synthetic form expressed in (33). It could be argued that, in this circumstance too, Vitali's reasoning might have proceeded by analogy starting from Ricci's classical notion of covariant derivative, which indeed admits a representation of type (33). In order to see this, it is sufficient to replace $L^2(\mathbb{R})$ with $\mathbb{R}^N$ and the inner product $(\cdot, \cdot)_{\mathcal{H}}$ with the ordinary Euclidean dot product in $\mathbb{R}^N$. Indeed, let V_n be a Riemannian manifold immersed in R^N: $F : (u^1, \ldots, u^n) \to F(u) = (z^1(u), \ldots, z^N(u)) \in \mathbb{R}^N$; let H_j, $j = 1, \ldots, n$ denote a covariant (ordinary) tensor of first order and let $\gamma = k$, $k \in \{1, 2, \ldots, n\}$, be a first class index. Then equation (33) can be rewritten as:

$$H_{j,k} = \sum_{t=1}^{N} \frac{\partial}{\partial u^k} \left(\sum_{r,s=1}^{n} H_r F_s^t a^{rs} \right) F_j^t, \qquad j, k = 1, \ldots, n; \qquad (34)$$

where $F_s^t = \dfrac{\partial z^t}{\partial u^s}$ and $a_{ik} = \sum_{t=1}^{N} F_i^t F_k^t = F_i \cdot F_k$. It is easily proved that $H_{j,k}$ coincides with the classical covariant derivative of H_j, with respect to the metric a_{ik}. It is interesting to observe that the structure of the definition (33) essentially coincides with formula (8), which was suggested to Vitali by Ricci in 1924. Indeed, one first constructs the absolute invariant $[H, F]$, applies the ordinary differentiation operator Δ_γ and then restores the appropriate index structure by multiplication with respect to $U = U_{\alpha_1, \ldots, \alpha_r}^{\beta_1, \ldots, \beta_s}$.

A noteworthy consequence of this further extension was investigated in one of Vitali's last works, (Vitali 1932), a collection of lectures held in Bologna and gathered by a student of his at the local University. Here Vitali set out to reinterpret the covariant derivation that he had introduced in (Vitali 1923a) in light of the theoretical framework of his new calculus. In order to do that, he simply replaced the absolute systems $F_\alpha(t, u)$ with a covariant system ϕ_α, $\alpha \in I_\nu$, deduced from fixed orthogonal directions of a given linear space.

However, as Bortolotti aptly remarked in (Bortolotti 1933, p. 220), such an attempt at providing a unitary treatment of the absolute parallelism introduced in (Vitali 1923a) in the context of his generalized absolute differential calculus was not completely successful, since the representation of the covariant derivative corresponding to the Weitzenböck-Vitali parallelism by means of (33) remained problematic to a certain extent.

The search for other geometrical applications was more fortunate. A detailed outline of Vitali's contributions to both projective and metric differential geometry would be well beyond the scope of this paper. We refer the reader to (Bortolotti 1933, Sect. 4) which contains useful information on Vitali's mathematical production as a whole. We will limit ourselves to discussing an example of geometrical application leading

to an analytical characterization of quasi-asymptotic lines, a special kind of curves drawn upon surfaces that were recently introduced in (Bompiani 1914) and could be regarded as a generalization of the classical notion of asymptotic lines.

Let us first consider the case of asymptotic lines. We will follow the discussion that Vitali offered in (Vitali 1927–1928). To this end, let V_2 be a two-dimensional manifold (surface) immersed either in $\mathbb{R}^N$ or in $L^2(\mathbb{R})$. Let us suppose that the second-order osculating space σ_2 is a three-dimensional linear space. If V_2 is represented by a parametrized function $F(u^1, u^2; t) \in L^2(\mathbb{R})$, this is the vector space generated by the elements $F_i^{(1)}$, $F_{jk}^{(2)}$, $i, j, k = 1, \ldots, 2$. Asymptotic curves on V_2 can be defined as those curves $\gamma : s \mapsto \gamma(s) = F(u^1(s), u^2(s); t) \in V_2$ whose σ_2 (to be denoted by $\sigma_2^{(\gamma)}$), the linear space generated by $\dfrac{dF}{ds}, \dfrac{d^2F}{ds^2}$, is contained (at each point of γ) in the tangent space σ_1 (at corresponding points) to V_2. To this end, it is both necessary and sufficient that $\sum_{ik} F_{ik} \dfrac{du^i}{ds} \dfrac{du^k}{ds} = 0$, where $F_{ik} = D_k F_i^{(1)}$ is the covariant derivative of $F_i^{(1)}$. Indeed, by computing the derivatives $\dfrac{dF}{ds}, \dfrac{d^2F}{ds^2}$, one easily obtains:

$$
\frac{dF}{ds} = \sum_{i=1}^{2} F_i^{(1)} \frac{du^i}{ds},
$$
$$
\frac{d^2F}{ds^2} = \sum_{i,j=1}^{2} F_{ij}^{(2)} \frac{du^i}{ds} \frac{du^j}{ds} + \sum_{k=1}^{2} F_k^{(1)} \frac{d^2u^k}{ds^2}.
$$

Now, if we introduce the (ordinary) covariant derivative of F_i, $F_{ik} = F_{ik}^{(2)} - \sum_{j=1}^{n} \Gamma_{ik}^j F_j^{(1)}$, we get

$$
\frac{d^2F}{ds^2} = \sum_{i,j=1}^{2} F_{ij} \frac{du^i}{ds} \frac{du^j}{ds} + \sum_{k=1}^{2} F_k^{(1)} \left(\frac{d^2u^k}{ds^2} + \Gamma_{il}^k \frac{du^i}{ds} \frac{du^l}{ds} \right);
$$

one can conclude that asymptotic lines are indeed characterized by the condition

$$
\sum_{i,j=1}^{2} F_{ij} \frac{du^i}{ds} \frac{du^j}{ds} = 0,
$$

which is a consequence of the fact that $\sigma_2^{(\gamma)} \subset \sigma_1$.

Vitali pursued a similar path in order to derive an analytical description of quasi-asymptotic lines in the special case of a surface V_2 such that $\dim(\sigma_2) = 5$, $\dim(\sigma_3) = 6$. A curve on V_2 is said to be quasi-asymptotic if its third-order osculating plane σ_3 is contained in the corresponding tangent plane σ_1 to V_2. A reasoning similar to the one adopted in the case of asymptotic lines, led Vitali to a characterization of these lines as those curves $s \mapsto \gamma(s) = F(u^1(s), u^2(s), t)$ for which the following equation hold true:

$$\sum_{ijk=1}^{2} \mathcal{F}_{ijk} \frac{\mathrm{d}u^i}{\mathrm{d}s} \frac{\mathrm{d}u^j}{\mathrm{d}s} \frac{\mathrm{d}u^k}{\mathrm{d}s} = 0, \tag{35}$$

where $\mathcal{F}_{ijk}$, $i, j, k = 1, \ldots, n$, denote the system defined by (28).

7 Further developments and concluding remarks

Despite the variety of geometrical applications investigated by Vitali and his disciples, the power of the new calculus had still to be proved and tested in light of applications. In this respect, noteworthy advancements were achieved in (Bortolotti 1931).

In this extensive piece of work, Bortolotti succeeded in making important improvements, especially by investigating geometrical aspects of Vitali's theory. More precisely, he provided an extension of Vitali's techniques to the case, not addressed by Vitali, in which the osculating spaces σ_q of an n-dimensional manifold do not exhibit the maximal dimension, i.e., $\dim(\sigma_q) \leq \sum_{m=1}^{q} \binom{n+m-1}{m}$.

Interestingly, Bortolotti cherished the hope of providing a proof of the fecundity of Vitali's calculus by testing its effectiveness in tackling problems in the realm of the so-called geometrie riemanniane di specie superiore, a sort of generalization of classical Riemannian geometry that consisted in studying the group of isometries $\Phi_m : V_n \rightarrow \tilde{V}_n$ between two n-dimensional Riemannian manifolds (immersed in $\mathbb{R}^N$) that preserve the curvatures, up to a given order, say $(m-1)$-th, of *every* curve drawn upon V_n. Isometries of this kind were said to be isometries of m-th type. This new branch of research was initiated in (Bompiani 1914) in a successful attempt at providing a geometrical interpretation of the conditions assuring that a given hypersurface V_{n-1} (with $n \geq 4$) immersed in the n-dimensional Euclidean space $\mathbb{R}^n$ admits non-trivial deformations.[26]

Bortolotti found out that Vitali's calculus could be profitably applied to the study of the analytical conditions that characterize the kind of isometries mentioned before. Indeed, if $a_{\alpha;\beta}$ denote the coefficient of the generalized fundamental tensor associated to $(u^1, \ldots, u^n) \mapsto F(u) \in V_n \subset \mathbb{R}^N$, α and β are indices such that $\rho_\alpha, \rho_\beta \leq m$ and $\tilde{a}_{\gamma;\delta}$ denote the corresponding coefficients of the fundamental tensor of $(\tilde{u}^1, \ldots, \tilde{u}^n) \mapsto \tilde{F}(\tilde{u}) \in \tilde{V}_n \subset \mathbb{R}^N$, then the necessary and sufficient conditions for $\Phi_m(u^1, \ldots, u^n) = (\tilde{u}^1(u), \ldots, \tilde{u}^n(u))$ to be an isometry of the requested type can be written as follows:

$$a_{\alpha;\beta}(u) = \sum_{\gamma,\delta \in I_m} \frac{\partial \tilde{u}^\gamma}{\partial u^\alpha} \frac{\partial \tilde{u}^\delta}{\partial u^\beta} \tilde{a}_{\gamma;\delta}(\tilde{u}(u)).$$

It is clear that for $m = 1$ isometries of type m are ordinary isometries of the kind investigated by Riemann, Christoffel and Ricci.

In spite of Bortolotti's efforts, the range of applications of Vitali's calculus remained somehow limited. The new techniques introduced by Vitali and later refined by Bortolotti were regarded at best as an interesting tool that could nonetheless also be

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[26] For a first explicit definition of isometries of m-th type, $m > 1$, see (Bompiani 1916).

dispensed of. In this respect, the attitude of Bompiani is particularly significant for evaluating the impact of Vitali's research among his contemporaries. As the following quotation suggests, he probably considered Vitali's calculus as an unnecessary, though elegant, instrument for carrying out his plan to study isometries of type $m > 1$.

> Thanks to Bortolotti, it has been shown that the proper object of Vitali's absolute calculus is precisely the geometry of deformations of type ν of a V_m in the normal case (this case has been extended by Bortolotti to other cases). At the basis of the absolute calculus there is a fundamental tensor whose coefficients are exactly the symbols introduced by Levi and widely used by me […]. […] the transformation law of these symbols with respect to a transformation of parameters varies according to the derivatives that appear there: these laws can be formally written in a unique way by means of appropriate symbolic derivatives […] but still these symbolic derivatives depend on actual derivatives of the ancient parameters with respect to the new ones, which are of different orders for different symbols. […] Now it is precisely this drawback that I overcame with the introduction of the fundamental invariant forms (1919). To put it in a more geometrical form, the difference between the two standpoints can be expressed as follows: I showed (in 1919) that while a manifold V_m undergoes a deformation of type ν the manifold W [locus] of its osculating planes is deformed by ordinary applicability: Vitali's calculus deals with ordinary applicability theory, i.e., Riemannian (ordinary) geometry of W. The covariant derivation of Vitali coincides with that of Ricci for W.[27]

Bompiani did not fail to appreciate the importance of Vitali and Bortolotti's achievements. Nonetheless, he thought that the complications imposed by the cumbersome algorithmic apparatus of the theory could be bypassed by considerations of a more intuitive character, consisting in replacing the Riemannian manifold V_m with a higher-dimensional object to which Ricci's ordinary differential calculus could be applied.

In some sense, it may be said, Bompiani's remarks reflected a general methodological conviction asserting preference for geometrical intuition over abstract algorithmic procedures: a point of view, tenaciously mantained, for example, by Bianchi in the

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[27] E' merito del Bortolotti di aver posto bene in luce che l'oggetto proprio del calcolo assoluto del Vitali è precisamente la geometria delle deformazioni di specie ν di una V_m qualora questa presenti il *caso normale* (e quel caso è stato esteso dal Bortolotti agli altri casi). A base di quel calcolo assoluto sta un "tensore fondamentale" i cui elementi sono esattamente i simboli introdotti dal Levi e largamente usati da me […]. […] la legge di trasformazione di questi simboli per una trasformazione di parametri è differente a seconda delle derivate che vi compariscono: queste varie leggi si possono formalmente scrivere in modo unico con l'introduzione di opportune derivate simboliche […] ma rimane il fatto che queste derivate simboliche dipendono da derivate effettive degli antichi parametri rispetto ai nuovi di ordini differenti per simboli differenti. […]
Ora è proprio questo inconveniente che io avevo superato con l'introduzione delle forme fondamentali invarianti (1919). In forma più geometrica si può esporre la differenza dei punti di vista così: avevo dimostrato (dal 1919) che mentre una varietà V_m subisce una deformazione di specie ν la varietà W [luogo] dei suoi $S(\nu - 1)$ osculatori si deforma per ordinaria applicabilità: orbene il calcolo del Vitali per la V_m rispecchia esattamente le ordinarie applicabilità, cioè la geometria riemanniana (ordinaria) di W. La derivazione covariante del Vitali è quella del Ricci per W. (Bompiani 1935, pp. 278-279).

judgments he had expressed towards Ricci's calculus some decades before, that had been prevalent within the Italian mathematical community.[28]

Similar convictions had guided the work of the commission (consisting of Castel-nuovo, Pascal, Severi and Fubini) charged with the assignment of the Royal Prize for Mathematics (1931). While praising the whole of Vitali's mathematical production, the commission had to acknowledge that Vitali's most recent investigations aimed at providing a generalization of Ricci's calculus were not sufficient to win him the attribution of the prize. Indeed, in the report written by Fubini, we read:

> [The commission] had to recognize that the introduction by Vitali of absolute systems and their derivatives, while undoubtedly constituting a valuable work, does not justify its great formal complication in view of the results that have been obtained.[29]

This judgment can be seen as a further indication of the slight impact produced by Vitali's calculus within already existing geometrical theories. After all, in contrast to the reception process of Ricci's calculus, Vitali could not count on external supports such as the one carried about by the discovery of General Relativity. This circumstance, together with Vitali's untimely death in February 1932, which prevented any further advancements of the theory, inevitably relegated the calculus to a marginal condition.

Still, the historical significance and the mathematical value of Vitali's contributions to geometry should not be questioned. Indeed, not only do they represent an important episode in the development of metric and projective differential geometry over the period 1920-1935, but they also allow us to gain a more complete picture of the scientific figure of Vitali himself. Indeed, as our analysis has suggested, they are a testimony to Vitali's extraordinary algorithmic skills and outstanding mathematical creativity.

Declarations

Conflict of interest The author states that there is no conflict of interest.

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[28] In this respect, see (Cogliati 2019).

[29] […] [the commission] ha dovuto riconoscere che la introduzione fatta dal Vitali dei sistemi assoluti e delle loro derivate, pur costituendo opera senza dubbio pregevole, non giustifica, con l'importanza dei risultati ottenuti, la sua grande complicazione formale. Quoted in (Vitali 1984, p. 22).

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