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# Frigyes Riesz and the emergence of general topology

## The roots of 'topological space' in geometry

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**Abstract** In 1906, Frigyes Riesz introduced a preliminary version of the notion of a topological space. He called it a mathematical continuum. This development can be traced back to the end of 1904 when, genuinely interested in taking up Hilbert's foundations of geometry from 1902, Riesz aimed to extend Hilbert's notion of a two-dimensional manifold to the three-dimensional case. Starting with the plane as an abstract point-set, Hilbert had postulated the existence of a system of neighbourhoods, thereby introducing the notion of an accumulation point for the point-sets of the plane. Inspired by Hilbert's technical approach, as well as by recent developments in analysis and point-set topology in France, Riesz defined the concept of a mathematical continuum as an abstract set provided with a notion of an accumulation point. In addition, he developed further elementary concepts in abstract point-set topology. Taking an abstract topological approach, he formulated a concept of three-dimensional continuous space that resembles the modern concept of a three-dimensional topological manifold. In 1908, Riesz presented his concept of mathematical continuum at the International Congress of Mathematicians in Rome. His lecture immediately won the attention of people interested in carrying on his research. They promoted his ideas, thus assuring their gradual reception by several future founders of general topology. In this way, Riesz's work contributed significantly to the emergence of this discipline.

## 1 Introduction

The problem of characterising the continuity of space is rooted in the study of the foundations of geometry. Although in the history of geometry space has been commonly

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assumed to be continuous, continuity has been understood over the centuries in different ways, depending on the methods at hand.

In the second half of the nineteenth century, the study of the foundations of geometry was challenged by the emergence of new geometries. An extensive discussion arose that went hand in hand with a philosophical reflection on the relation between the axioms of geometry and physical space. The discussion was motivated, in particular, by Bernhard Riemann's notion of an $n$-dimensional manifold because it opened diverse possibilities for a mathematical description of physical space. In the last decade of the nineteenth century, the discussion reached a climax with the so-called Riemann-Helmholtz-Lie space problem, a problem that had implications for the postulates of geometry and for the treatment of the continuity of space. Continuous space was understood by Riemann and Hermann von Helmholtz as a differentiable manifold, by Sophus Lie simply as a number manifold (Zahlenmannifaltigkeit), i.e. $\mathbb{R}^n$ or $\mathbb{C}^n$. To reduce continuous space to Riemann's notion of an $n$-dimensional manifold brought into view another aspect of space: continuity as a local property. This aspect was highlighted by Lie's treatment of the concept of motion as an infinitesimal continuous transformation. The characterisations of the local continuity of space given by Riemann, Helmholtz and Lie all rested on analytical methods, which were the common ones at that time.

Around 1900, new mathematical techniques were available that seemed likely to be helpful in grasping the notion of the continuity of space. They arose from the development of analysis situs and from recent contributions to the foundations of analysis that were closely related to the establishment of Georg Cantor's point-set topology.[1] Hence, when Hilbert (1902b, 1903) reconsidered Lie's treatment of the foundations of geometry, he used a point-set topological approach and resorted to concepts of analysis situs for the characterisation of continuity as a local property. Hilbert succeeded in providing an axiomatic foundation for two-dimensional geometry that started with only continuity assumptions. Part of his success was due to his definition of the plane as a two-dimensional manifold because it allowed him to use the point-set topology of $\mathbb{R}^2$. But Hilbert's achievements still left unanswered the question of how to characterise three-dimensional continuous space in a study of geometry that starts with continuity assumptions only. It is precisely this question, the challenge of extending Hilbert's concept of the plane to a three-dimensional manifold, that I will refer to as Hilbert's continuity of space problem.

This challenge was taken up in 1904 by Frigyes Riesz, a young Hungarian mathematician, who later became best known for his work on functional analysis. First as a student and then in the early years of his career, Riesz investigated questions related to Hilbert's work on geometry. In fact, Riesz's early mathematical work (1904–1908) was marked as much by his involvement with Hilbert's work on geometry and analysis as by his interest in the developments of analysis in France. He was particularly

[1] I will use the name '(abstract) point-set topology' to designate the historical research area concerned with the study of topological properties of sets of points such as closed, open, dense. In this way, I attempt to present a compromise between the historical name '(abstract) point-set theory' and the modern name 'general topology'. Thus, the terms 'topology' and 'topological' will refer here to a research area rather than to a system of neighbourhoods.

interested in the work carried out by Emile Borel, Henri Lebesgue, René Baire and Maurice Fréchet. The fact that point-set topological methods had assumed a central role in both Hilbert's work on geometry from 1902 and contemporary French analysis was the core of Riesz's motivation in approaching Hilbert's continuity of space problem. Hilbert's technical approach motivated Riesz to search for a point-set topological characterisation of continuous space that might be useful for both geometry and (functional-) analysis.

The present essay traces back Riesz's early work on set theory and abstract point-set topology to Hilbert's continuity of space problem through the following sequence of events: in November 1904, Riesz wrote to Hilbert about his research on set theory and its application to the concept of continuous space. In January 1905, he submitted an article on the pure set-theoretical part of his investigations to the *Mathematischen Annalen*, a journal which was at that time co-edited by Hilbert. The paper appeared that same year (1905c). In July 1905, he attended a meeting of the Göttingen Mathematical Society and presented the research he had communicated to Hilbert in his November letter.[2] However, his results concerning the space problem were never published in Göttingen. They appeared instead in Hungary in 1906. Thus, it would seem that Riesz failed to gain Hilbert's support for this aspect of his work. The original Hungarian version of his article was entitled 'The origin of the notion of space' (Riesz 1907a). Fortunately, its German version 'Die Genesis des Raumbegriffs' (the *Genesis*, for short) appeared soon in an Hungarian international journal (Riesz 1906b). It is thanks to the German version that I was able to understand that Riesz intended to approach Hilbert's continuity of space problem. In order to do so, he had started to develop an abstract point-set theory, a theory that today is known as general topology.

It is not a new discovery that the *Genesis* contains a pioneer contribution to the development of an abstract point-set topology. Prior to this essay, other historians have referred to the *Genesis* in this way.[3] The first relevant reference to the *Genesis* was made by Taylor (1982) in a paper dedicated to Frechét's early work on point-set theory.[4] There Taylor not only summarises Riesz's rudimentary theory of mathematical continua but also rightly points out that Riesz did not use the word 'topology', a term which together with today's notion of topological space was first introduced by Felix Hausdorff (1914). Referring to the *Genesis* Taylor (1982, p. 268) remarks:

> This paper is not, in the main, about general point-set topology, either concrete or abstract. It is a quasi-philosophical paper in which Riesz attempts to construct a mathematical model for the geometry of space as needed or used in physics. In so doing he formulated a notion of what he called a mathematical continuum. This notion is, in fact, that of an abstract space with a rudimentary topology defined axiomatically.

2 See the report of the 12th meeting from 25 July 1905 in *Jahresbericht der DMV* **14**(1905).

3 E.g. Manheim (1964), Johnson (1981), Taylor (1982), Thron (1997) and more recently Purkert et al. (2002).

4 Manheim (1964) discusses the *Genesis* very shortly.

Taylor's emphasis on the 'quasi-philosophical' character of Riesz's paper indicates that he, and with him all other historians of mathematics who have discussed the *Genesis* after him, overlooked Riesz's genuine interest in geometry. Because of its complexity many sections of the *Genesis* were never read. Only its introduction and two of 14 sections found attention. Historians have approached the *Genesis* only from the viewpoint of history of general topology. Therefore, they have focused on Riesz's rudimentary theory of mathematical continua, recognising it rightly (as well as Fréchet's works from 1904 to 1906) as a pioneer's contributions to abstract point-set topology.

Thus, it has been ignored until now that Riesz's contribution emerged in a broad scientific context that included not only current philosophical and scientific discussions about space but also research on the foundations of geometry at that time.[5] In order to understand Riesz's aims in the *Genesis*, it is worth going through the other twelve sections, which are indeed difficult to read, and whose contents will appear weird to the modern reader. In fact, Riesz's project is as brilliant and fascinating as Kepler's 'Mysterium cosmographicum', but, just as the latter, it does not work. It is the project of a mathematician attempting to find a mathematical model for the space of our sensations.[6]

Unfortunately, most of the fascinating passages of the *Genesis* will be discussed only superficially, if at all, because they fall beyond the scope of this essay. Here, I investigate how Riesz's concept of a 'mathematical continuum' evolved from Hilbert's work on the foundations of geometry, and especially from Hilbert's continuity of space problem.[7]

We start in Sect. 2 with Hilbert's work on the foundations of geometry and its relation to his fifth Paris mathematical problem. Then, I present Hilbert's notion of a two-dimensional continuous space and the problem of generalising it to higher dimensions (Hilbert's continuity of space problem). I then analyse Hilbert's use of topological methods on an abstract set, which he presented at a time when the concept of topology did not yet exist.[8] This analysis plays a key role in my attempt to show the relation of Riesz's concept of mathematical continuum with Hilbert's continuity of space problem from a technical mathematical viewpoint.

In Sect. 3, I summarise Riesz's early contributions to analysis situs, point-set topology and theory of multiple order types pointing to two important aspects of Riesz's research: (1) his interest in applications of Cantor's point-set theory to the development

[5] The exceptions are Johnson (1981) who gives a brief but accurate summary of the *Genesis*, and a few remarks in Gray's (2008) *Plato's Ghost* where the doctoral thesis of Rodríguez (2006) is acknowledged.

[6] For an extensive description of the *Genesis*, see my thesis (Rodríguez 2006, chap. 5–8).

[7] The present article extends certain results that I presented in my thesis (Rodríguez 2006). Besides, it offers a focused discussion on both the relation of Riesz's early work to Hilbert's work on the foundations of geometry and Riesz's treatment of Hilbert's continuity of space problem in the *Genesis*.

[8] Hilbert's concept of two-dimensional continuous space has repeatedly been object of study in the history of mathematics, principally by Scholz (1999). In Purkert et al. (2002), Hilbert's concept is discussed in connection with the history of the concept of manifold and the history of Hausdorff's concept of topological space. However, as far as I know, such a technical analysis of Hilbert's way of doing point-set topology on the plane has not yet been presented in published form.

of analysis in France and (2) his interest in problems related to Hilbert's foundations of geometry.

Following the chronological flow of events, Sect. 3.4 provides an insight into two letters addressed to Hilbert in which Riesz communicated his recent research. Finally, in Sect. 4 I discuss Riesz's concept of a 'mathematical continuum' and the rudimentary abstract point-set topology he developed from it. There I explain how Riesz intended to apply that framework to Hilbert's continuity of space problem. In Sect. 5, the methods and concepts used by Hilbert and Riesz in their own attempts to characterise continuous space are compared, as well as their notions of continuous space.

Section 6 is dedicated to the reception of Riesz's ideas in the context of general topology. In preparation, I briefly discuss the lecture Riesz gave at the International Congress of Mathematicians in Rome in 1908. This lecture was published in the congress proceedings (Riesz 1909). That was the only occasion in which Riesz presented the work he started in the *Genesis*. From now on, I will refer to this paper as Riesz's Rome lecture.

## 2 Hilbert's continuity of space problem

In 1900, David Hilbert emerged as a leading mathematician in Germany. At the second International Congress of Mathematicians in Paris, where he was invited to give a plenary lecture, he presented his famous list of 23 mathematical problems. The fifth focused on the foundations of geometry and specifically, the continuity of space problem.

In response to this problem, Hilbert produced two papers (1902b; 1903). They contain an approach to the foundations of geometry that was entirely different from the one in his celebrated book 'Grundlagen der Geometrie' of 1899 (Hilbert 1956). Both papers are called 'On the Foundations of Geometry' ('Ueber die Grundlagen der Geometrie'). The first appeared in the *Göttinger Nachrichten* and consisted essentially of a sketch of his second paper which was published in the *Mathematischen Annalen*. I will refer to the former as *memoir*, to the latter as his *Annalennote*, and to his book as *Festschrift*.[9]

### 2.1 Hilbert's fifth mathematical Paris problem

Hilbert's fifth Paris problem was based on the so-called Riemann–Helmholtz–Lie space problem. From the perspective of mechanics, Hermann von Helmholtz had investigated the question of which of all the possible spaces opened to exploration by Riemann's notion of three-dimensional manifold could best describe physical space.[10] He was of the opinion that in those optimal spaces, measurement has to be possible, and that only these spaces deserved the name of geometry. Helmholtz postulated that the possibility of measurement rests upon the existence of rigid bodies that can be trans-

[9] Hilbert let his *Annalennote* appear in the appendix of his *Festschrift* from its third edition (1903) on (e.g. Hilbert 1956, pp. 178–230).

[10] It was Lie who called the task set forth by Helmholtz 'Riemann–Helmholtz space problem'.

ported everywhere without suffering any deformations. This is known as Helmholtz's postulate of the free mobility of rigid bodies. Starting from this postulate, Helmholtz succeeded in limiting the class of geometries to the now so-called simply connected three-dimensional manifolds with constant curvature, namely the Euclidean, the non-Euclidean Bolyai–Lobachevsky, the spherical and the elliptical geometries. Although his ideas were essentially valid, his mathematics was rather deficient.[11] Around 1890, Sophus Lie managed to correct Helmholtz's ideas by formalising the notion of motion using his concept of an infinitesimal transformation group that he had used in his general theory of continuous groups. However, Lie's approach depended on certain differentiability assumptions concerning the functions that defined the group. Hilbert found these assumptions unsatisfactory.

In Paris, Hilbert discussed his fifth problem as follows:

> It is well known that Lie, with the aid of the concept of continuous groups of transformations, has set up a system of geometrical axioms and, from the standpoint of his theory of groups, has proven that this system of axioms suffices for geometry. But since Lie assumes, in the very foundation of his theory, that the functions defining his group can be differentiated, it remains undecided in Lie's development, whether the assumption of the differentiability in connection with the question as to the axioms of geometry is actually unavoidable, or whether it may not appear rather as a consequence of the group concept and the other geometrical axioms.[12]

Thus, Hilbert was wondering whether the differentiability assumptions imposed by Lie had to be included. He suspected that it was possible to find both a more suitable system of axioms for geometry and a more appropriate characterisation of the continuity of the group of transformations so that the differentiability properties would follow as a necessary consequence and would not need to be included among the assumptions. Hence, from the perspective of Hilbert's famous axiomatic approach in his *Festschrift* of 1899, the task Hilbert was setting forth concerned the mutual independence of the elementary assumptions. This issue, typical in Hilbert's foundations of geometry, can be summarised in the following question: how far can geometry be established using a group-theoretical approach and starting merely from continuity assumptions?

It is precisely in these terms that Hilbert explained his new approach to the foundations of geometry in his *Annalennote*:

> There [in his *Festschrift*, LR] the axioms were arranged in such a way that continuity is assumed after all the rest of the axioms as the last one, so that naturally the question arises as to what extent are the known theorems and results of elementary geometry dependent on any continuity assumptions. In the present

[11] This summary of Riemann–Helmholtz–Lie problem up to Lie's contribution is based on the one given by Torretti (1978, pp. 155–178). In the discussion of Hilbert's contribution, I distance myself from Torretti's following my own interpretation.

[12] Translated for the Bulletin of the AMS, with the author's permission, by Dr. Mary Winston Newson', Hilbert (1902a, p. 451); original German version in Hilbert (1900, p. 269).

> paper, continuity is assumed on the contrary before any other axiom, from the very beginning, via the definitions of plane and motion, so that the main task consists of finding out the least possible number of assumptions from which (and making extensive use of the continuity property) the elementary objects of geometry (circle and line) can be obtained as well as those properties that are necessary to establish geometry.[13]

Thus, in this new approach 'continuity' is a fundamental concept and as such, Hilbert intended to make extensive use of it as a property of both the plane and motion in a plane, so as to find out the least possible number of further assumptions necessary to establish geometry.

From a technical point of view, two questions need to be addressed: how is the concept of a continuous space to be defined? and how is the concept of a continuous group to be defined? This last question is not of further interest to us here. It is sufficient to mention that its solution is known as the solution of Hilbert's fifth mathematical problem. Its importance is associated with the emergence of a whole new branch of mathematics: the theory of topological groups.[14] The question I will deal with is: how to define the notion of a continuous space?

## 2.2 Hilbert's concept of continuous space

As a first approach to the concept of a continuous space, Hilbert chose Riemann's notion of an $n$-dimensional manifold. Until the end of the nineteenth century, this notion had not been clearly defined but was commonly understood as a 'number manifold', i.e. a subset of $\mathbb{R}^n$ or $\mathbb{C}^n$. Hilbert did not resort to Lie's and Helmholtz's concept of an $n$-dimensional *differentiable* manifold. Instead he formulated his own definition of a two-dimensional manifold. His notion corresponds to today's concept of a topological two-dimensional manifold.

Hilbert methodically explored a new approach to continuity by incorporating recent point-set topological practices in analysis. As a result, Hilbert succeeded in formulating the most accurate definition of the concept of two-dimensional manifold at the time.[15] From Cantor's point-set theory, he took elementary concepts such as open, closed, dense and perfect sets. Most importantly, he resorted to the characterisation of continuous functions by means of the notions of convergence of a sequence, limit point and accumulation point. From analysis situs, Hilbert used the concept of a closed Jordan curve, the Jordan curve theorem and a converse of the Jordan curve theorem (a characterisation of a closed Jordan curve using methods of both point-set topology and analysis situs). Analysis situs was a research area that primarily concerned with the study of qualitative relations of space that are independent of any metric relation characteristic of geometry, e.g. connectedness and the reciprocal position of bodies in space.

13 Hilbert (1903), 422; reprint in Hilbert (1956), 230, my translation.

14 For an overview, see Browder (1974), Aleksandrov (1998), Rowe (1995). On the history of topological groups in conjunction with Hilbert's fifth problem, see Hawkins (1999).

15 On the history of the concept of manifold, see Scholz (1999).

## 2.3 Neighbourhoods and Jordan domains

The fundamental concept in Hilbert's characterisation of continuous space was the notion of a neighbourhood. He defined it in terms of Jordan domains in $\mathbb{R}^2$ and of what today are called coordinate functions.

By a Jordan domain, Hilbert understood the interior of a closed Jordan curve in $\mathbb{R}^2$, i.e. the interior of a plane, continuous curve without multiple points.[16] Then, he defined a system of neighbourhoods in the plane by means of a system of bijective images of Jordan domains (under the coordinate functions) that satisfy certain axioms.

Clearly, Hilbert was relying on the Jordan curve theorem that asserts that every closed Jordan curve separates the plane into two domains: the interior, a bounded simply connected domain in Riemann's sense, and the exterior.[17]

## 2.4 Axiomatising the system of neighbourhoods

In his *memoir*, Hilbert presented two definitions of the plane. The first one is rather more general than the second, which is the only one that he actually used in his *Annalennote*. According to Hilbert's *memoir*, the definition of the plane as two-dimensional manifold reads in modern notation as follows:

The plane $E$ is a point-set in which to each point $p$ there corresponds a family of so-called neighbourhoods formed by sets $U \subseteq E$ containing $p$. The system of neighbourhoods fulfils six axioms.

1. For every neighbourhood, there exists a one-to-one mapping onto a Jordan domain [In line with modern terminology I call these mappings coordinate functions.].
2. Let $V$ be a Jordan domain such that $j^{-1}(V)$ is a neighbourhood of $p$ and let $U \subseteq V$ be any other Jordan domain with $j(p) \in U$, then the converse image $j^{-1}(U)$ is also a neighbourhood of $p$.
3. If a neighbourhood has two different images, then the corresponding coordinate change is a continuous mapping, i.e. let $\phi : U \to V$ and $\varphi : U \to W$ be two coordinate functions of the same neighbourhood $U$ onto the Jordan domains $V$ and $W$, then the coordinate change $\varphi \cdot \phi^{-1} : V \to W$ is a bijective continuous mapping from $V \subset \mathbb{R}^2$ onto $W \subset \mathbb{R}^2$.

[16] Hilbert (1902b, p. 234). Torretti (1978, p. 186) has rightly pointed out that Hilbert understood a closed Jordan curve to be a continuous function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}^2$, such that $f$ is injective on $(a, b)$ and $f(a) = f(b)$.

[17] The Jordan curve theorem was established by Camille Jordan in his 'Cours d'analyse' in 1893. Although the theorem seems trivially true, the proof Jordan provided was not simple. Already at the beginning of the twentieth century, there were several mathematicians who found Jordan's proof unsatisfactory and offered their own proofs, Veblen (1905) and Brouwer (1910) among others. Following Hales (2007), however, who recently verified Jordan's original proof, the credit for the first correct proof can go to Jordan. I thank Jeremy Gray for pointing me to this paper of Hales. See Guggenheimer (1977) for a recount of further attempts to prove the Jordan curve theorem prior to Brouwer's proof from 1910. Guggenheimer (1977) also reported on a manuscript found in Max Dehn's Archives containing a proof of the Jordan curve theorem that Dehn worked out around 1900. In that year, Dehn, who was a student of Hilbert's, finished his thesis on foundations of geometry. Dehn never published his results. Guggenheimer seemed unaware of the importance of Dehn's proof for Hilbert's *memoir*. Thus, their connection remains to be investigated.

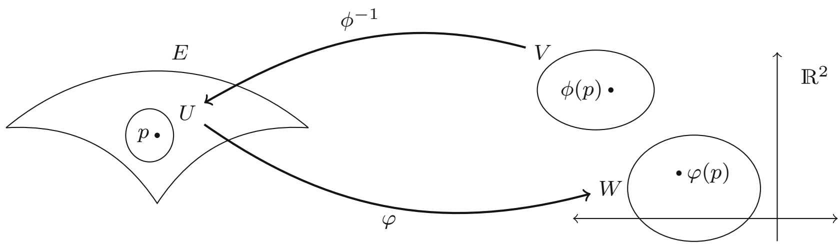


**Fig. 1** Axiom 3 states that the coordinate change $\varphi \cdot \phi^{-1} : V \to W$ is a bijective continuous mapping from $V \subset \mathbb{R}^2$ onto $W \subset \mathbb{R}^2$

4. A neighbourhood $U$ of $p$ containing the point $q$ is also a neighbourhood of $q$.
5. For every two neighbourhoods $U$ and $V$ of $p$, there exists a neighbourhood $W$ of $p$ such that $W \subset U \cap V$.
6. For any two points $p$, $q$ of the plane, there exists a common neighbourhood.[18]

Clearly, these axioms do not include any differentiability assumptions. Besides, as Scholz (1999) rightly pointed out, given that the system of Jordan domains determines what today is called a neighbourhood basis for the standard topology of $\mathbb{R}^2$, this system of axioms provides a sufficiently rich system of neighbourhoods so that, in modern terms, the plane is topologised as a $\mathcal{C}^0$-manifold, i.e. the system of neighbourhoods defines a topology with respect to which all coordinate functions are continuous.[19]

Hilbert's general concept of the plane comprises several aspects of the concept of continuity. The first axiom states that locally (i.e. inside every neighbourhood) the plane has as many points as any Jordan domain in $\mathbb{R}^2$. Hence, Hilbert regards cardinality as one criterion for continuity. From today's perspective, it can be said that axiom (2) together with axioms (4)–(6) stand for the characterisation of the local topology, since they guarantee that each neighbourhood is provided with a sufficiently rich system of neighbourhoods via the coordinate functions and the standard topology of $\mathbb{R}^2$. Axiom (3) combines the topological with the analytical aspect of continuity, i.e. it combines the structure determined by the system of neighbourhoods with the analytical notion of continuous function. This is expressed by the simple fact that two images of a neighbourhood are topologically related to each other in such a way that the coordinate change is a continuous function from $\mathbb{R}^2$ to $\mathbb{R}^2$ (see Fig. 1).

In his *Annalennote*, Hilbert's 'less general' definition of the plane, as he called it, does not include axiom (3) but assumes instead the existence of a global coordinate function $k : E \to B \subset \mathbb{R}^2$ (see Fig. 2) that maps the whole plane $E$ onto a bounded

[18] Hilbert (1902b), 234-235. The condition (2) is stated following Scholz (1999) who rightly translated Hilbert's unclear formulation.

[19] According to Scholz (1999), Hilbert's definition of the plane was the first attempt at an axiomatic definition of the concept of manifold. In Purkert et al. (2002), Scholz discusses Hilbert's *memoir* focusing on the historical relation between Hilbert's system of neighbourhoods and the concept of topological space. I take a different approach and focus on Hilbert's notion of accumulation point for the plane and consequently on the role played by his neighbourhoods axioms.

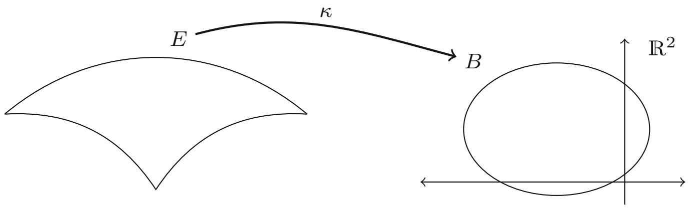


**Fig. 2** In his *Annalennote*, Hilbert assumes the existence of a global coordinate function $k$ that maps the whole plane $E$ onto a bounded open domain $B \subset \mathbb{R}^2$

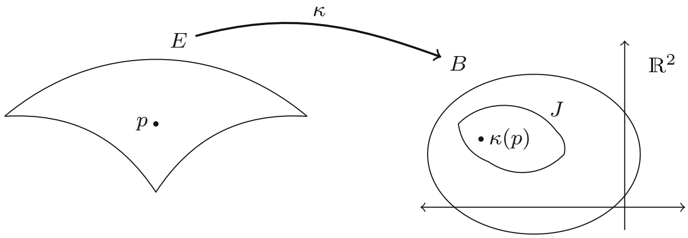


**Fig. 3** In Hilbert's *Annalennote*, a neighbourhood of a point $p \in E$ is no longer a set in $E$ but a Jordan domain $J \subset B$ containing the image under $k$ of $p$ in $\mathbb{R}^2$

open domain $B \subset \mathbb{R}^2$. This implies a restriction of the possible spaces on which he intended to establish geometry. It excludes the possibility of elliptic geometry.

Furthermore, the *Annalennote* contains another very interesting redefinition of the concept of neighbourhood that not only shows a change in Hilbert's epistemic conception of this notion but is also an important clue for understanding of Hilbert's technical approach. While in his *memoir*, a neighbourhood of $p$ is a set $U \subset E$ containing $p$ that can be mapped one-to-one onto a Jordan domain in $\mathbb{R}^2$, in his *Annalennote* it is any Jordan domain $J$ in $B \subset \mathbb{R}^2$ containing the image under $k$ of $p$ (see Fig. 3 and Hilbert (1903, p. 383)). That means that a neighbourhood of a point $p \in E$ is no longer a subset of $E$ but of $\mathbb{R}^2$.

This identification of the system of neighbourhoods in the plane $E$ with the system of Jordan domains in $\mathbb{R}^2$ went hand in hand with the restriction to hyperbolic and Euclidean geometry mentioned above. Technically, it is legitimated by the assumption of a global chart in conjunction with the other axioms. With this identification, Hilbert wanted to achieve a way of working only on $B$, the image under $k$ of the plane $E$ in $\mathbb{R}^2$. Why? Because Cantor's point-set topology of $\mathbb{R}^2$ works in $B$ and so does classical analysis.

## 2.5 Motion and continuity

That Hilbert wanted to reduce the analysis and the point-set topology of $E$ to the analysis and the point-set topology of $\mathbb{R}^2$ seems evident in his definition of motion as a continuous function.

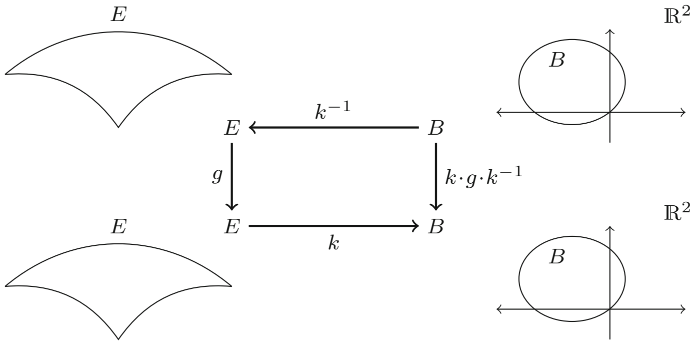


**Fig. 4** Hilbert defined motion as a bijective mapping $g : E \to E$ such that the corresponding mapping $k \cdot g \cdot k^{-1} : B \to B$ is a continuous mapping from $B$ onto $B$, where $B$, the image under the global chart $k$, is a Jordan domain in $\mathbb{R}^2$. The continuity of a plane transformation is to be checked in $\mathbb{R}^2$

A motion is a bijective continuous mapping from $E$ onto itself preserving the orientation of any Jordan curve.[20] The challenge consisted of defining the notion of 'continuous mapping' from $E$ onto $E$, because there was no theoretical framework available at that time that would enable one to define a 'continuous mapping' on the abstract set $E$; theories about abstract topological and metrical spaces did not yet exist. Hilbert mastered this challenge by focusing on the combination of coordinate functions as mappings from $\mathbb{R}^2$ onto $\mathbb{R}^2$.

In his *memoir*, a motion is as a bijective function from $E$ onto $E$ that maps neighbourhoods 'continuously', meaning that the corresponding mapping via the coordinate functions is a continuous mapping from a Jordan domain onto another Jordan domain in $\mathbb{R}^2$ (Hilbert 1902b, p. 236). In his *Annalennote*, once the existence of a global coordinate function $k$ has been assumed, the concept of motion is defined as a bijective mapping $g : E \to E$ such that the corresponding mapping $k{\cdot}g{\cdot}k^{-1} : B \to B$, $B \subset \mathbb{R}^2$, is a continuous mapping preserving the orientation of any Jordan curve (see Fig. 4 and Hilbert (1903, p. 383). This means, the continuity of a transformation is then to be checked in $\mathbb{R}^2$—just as the continuity of the coordinate change (Fig. 1).

Hilbert added then the following three axioms characterising the set of motions.

| | |
|---|---|
| Axiom I | The set of motions constitutes a group. |
| Axiom II | A 'true circle' through the point $y$ centred at the point $x$ is the set of all rotations of $y$ around $x$. A true circle is an infinite set. A rotation around $x$ is a motion $g$ such that $g(x) = x$. |
| Axiom III | The set of motions constitutes a closed system. |

Incidentally, axiom III is one of the most important and original of Hilbert's contributions to the Riemann–Helmholtz–Lie space problem because this condition characterises the continuity of the group of motions. However, my concern is how Hilbert handled this condition technically: he resorted to Cantor's point-set topological methods on $\mathbb{R}^2$.

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[20] Hilbert's general concept of motion appeared in both his *memoir* and his *Annalennote*. See Hilbert (1902b, p. 236) and Hilbert (1903, p. 383).

## 2.6 Cantor's point-set topology on the plane

By taking full advantage of the coordinate functions, and more specifically of the global chart $k$, Hilbert introduced point-set topological methods in the plane $E$. The general idea is that a sequence is said to converge on $E$ if its image under $k$ converges in $\mathbb{R}^2$ and a point $p$ in $E$ is an accumulation point of $F \subset E$ if its image under $k$ is an accumulation point of the image under $k$ of $F$ in $\mathbb{R}^2$.

The central notion is again that of a Jordan domain in $\mathbb{R}^2$ because these domains are the images under $k$ of the neighbourhoods in $E$ and because the metric topology of $\mathbb{R}^2$ allows a notion of closeness as follows: let $a$ be a point in $E$ and let $k(U)$ be an arbitrarily small Jordan domain enclosing $k(a)$, then all those points $p$ such that $k(p) \in k(U)$ are said to be arbitrarily close to $a$. This kind of topological identification of $E$ with $\mathbb{R}^2$ is emphasised by the fact that Hilbert opted for assigning the same letter to both, the points and subsets of $E$ and their respective images under $k$ in $\mathbb{R}^2$ (Hilbert 1903, p. 384).

Using this idea of closeness, Hilbert explained axiom III as follows: Let $A, B, C, A', B', C'$ be points of the plane. If there exist any motions that map points arbitrarily close to $A, B, C$ to points arbitrarily close to $A', B', C'$, then there exists a motion that maps the points $A, B, C$ to $A', B', C'$. That means:[21] If $\forall \varepsilon > 0$, there exists $\delta > 0$ and a motion $g_\varepsilon$ such that $\forall\ x, y, z$ with $\mid x - A \mid < \delta$, $\mid y - B \mid < \delta$ and $\mid z - B \mid < \delta$ their images satisfy $\mid g_\varepsilon(x) - A' \mid < \varepsilon$, $\mid g_\varepsilon(y) - B' \mid < \varepsilon$ and $\mid g_\varepsilon(z) - C' \mid < \varepsilon$, then there exists a motion $g$ such that $g(A) = A'$, $g(B) = B'$ and $g(B) = B'$.

Once he had established this idea of closeness, he then easily introduced further point-set topological concepts in $E$. His notation, however, makes it difficult for the reader to perceive whether he is referring to point-sets of $E$ or of $\mathbb{R}^2$. Nowhere did he explicitly define either the notion of convergence of a sequence of points in $E$ or the concept of accumulation point of a subset $A \subset E$. However, in his demonstrations he assumed that a sequence of points in $E$ is convergent if its image under $k$ in $\mathbb{R}^2$ is convergent. He regarded $p$ as an accumulation point of a subset $A \subset E$, if $k(p)$ is an accumulation point of $k(A) \subset k(E)$ in $\mathbb{R}^2$, i.e. if every arbitrarily small neighbourhood of $p$ contains an infinity of points of $A$ in the sense explained above.

He reduced the continuity of a mapping from $E$ to $E$ or from $E$ to $\mathbb{R}^2$ to the continuity of the corresponding mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$. This is already evident in the introduction of his *Annalennote*. There he announced some of the main results, e.g. (1) the true circle is a closed set that is dense in itself, i.e. a perfect set; (2) in order to prove that the true circle is a Jordan curve, he shows the existence of a continuous bijective mapping from the true circle to a standard circle and (3) he uses the notion of accumulation point in his definition of the 'true straight line'.[22]

This implicit notion of an accumulation point in $E$ implies that all topological properties (open, close, dense) of subsets of $E$ depend on the point-set topology of

[21] Following Hilbert, I formulate axiom III for motions as transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$. The given formulation in modern terms is my own and is equivalent to both the one given by Scholz (Purkert et al. 2002, p. 709) and the one by Toretti (1978, p. 187).

[22] See (Hilbert, 1903, pp. 386–387).

$\mathbb{R}^2$. Hilbert's concept of an accumulation point works because he made sure that the system of neighbourhoods is sufficiently rich.[23] His system of neighbourhoods generates what today is called a basis for the weak topology induced by the coordinate functions. Convergence of a sequence, as understood by Hilbert, is nothing other than the weak convergence of the sequence with respect to the coordinate functions. Hence, the system of neighbourhoods and correspondingly the system of coordinate functions made it possible to define the concept of an accumulation point in terms of Jordan domains and, consequently, to ensure the applicability of Cantor's point-set topology of $\mathbb{R}^2$ for topological investigations on the plane $E$.

And yet, the development of mathematics at that time allowed, as Riesz was to show, a different, truly point-set topological approach to the notion of continuity of space. Hilbert, however, was not interested in developing point-set theory. In fact, his *Annalennote* and his *memoir* are the only works in which Hilbert actually applied Cantor's point-set theory.

## 3 Riesz's early works and point-set theory

In his early research career, i.e. before he focused on functional analysis, Frigyes Riesz's worked on geometry, analysis situs, point-set topology and the theory of order types.

In 1902, Riesz finished his doctorate in Budapest with a thesis on geometry after spending one year in Göttingen as a student, sometime between 1899 and 1902, the precise date is unknown. During this period, Hilbert had been working on the foundations of geometry: he had lectured on geometry in the winter term of 1898/1899, his *Festschrift* appeared in 1899, his *memoir* was submitted in November 1901, his *Annalennote* in Mai 1902, and in the summer term of 1902, he lectured again on geometry. Judging by his early publications, Riesz probably attended one of these lectures. In any case, he knew Hilbert's work on the foundations of geometry very well. If he missed the lectures, he still had the chance to study them from notes that were freely accessible to students in the library of the mathematical institute.[24] Riesz came back as a young researcher and a guest of the Göttingen Mathematical Society in the winter term of 1903.

At the end of 1904, Riesz was in scientific correspondence with Hilbert. In the meantime, he published a paper on the converse of the Jordan curve theorem (1904c). This was followed by a long paper on multiple order types (1905c) and two short notes on topological questions, one concerning the theorem now known as the 'Heine–Borel' or 'Borel–Lebesgue' theorem (1905b) and the other on the concept of dimension

[23] Technically, if $X$ and $Y$ are first countable spaces, then it is valid to reduce continuity to sequential continuity as follows: $f : X \to Y$ is continuous iff whenever $x_n \to x$ in $X$, then $f(x_n) \to f(x)$ in $Y$ (Willard 1970, p. 71).

[24] The production of such lecture notes was one of those peculiarities of Hilbert's that was highly appreciated by students. Hallett and Majer (2004) edited Hilbert's lectures and notes on foundations of geometry, including the lecture of 1902. In the introduction, the editors explain how these *Ausarbeitungen* came about under Hilbert's close supervision.

(1905a). At a meeting of the Göttingen Mathematical Society in July 1905, he reported on 'his set-theoretical research and its connection to the space problem'.[25]

Most of what he covered in these four papers was building up to a long article published in 1906 entitled 'The Origin of the Concept of Space'. Its German version, 'Die Genesis des Raumbegriffs' (1906b), was supposed to appear that same year but the corresponding volume of the journal was actually printed in 1909. In the *Genesis*, Riesz attempted to contribute to Hilbert's continuity of space problem by extending Hilbert's concept of two-dimensional manifold to higher dimensions. Not only did Riesz himself point to the connection between the *Genesis* and Hilbert's foundation of geometry but he also discussed this connection in his correspondence with Hilbert (see Sect. 3.4). This connection explains why, in the papers mentioned above, Riesz investigated concepts and results that Hilbert had applied or established: e.g. the concepts of neighbourhood, accumulation point, connectedness, the theorems concerning Jordan curves, the Heine–Borel and Bolzano–Weierstrass theorems.

Riesz's early works show how intellectually close he was to both Hilbert's mathematics and the French developments in analysis which were exploring the field of point-set topology beyond Cantor's theory. Riesz's position allowed him to synthesise these mathematical cultures in his research. The best known of his synthesising ideas is undoubtedly his work, starting in 1906, on the space of functions $L^2$. His first synthesis, however, occurred in the *Genesis* and it concerns his theory of mathematical continua. This is the subject of the present article.

### 3.1 On analysis situs

The paper 'On a theorem from analysis situs' (1904c) shows that Riesz had developed a deep insight into diverse aspects of Hilbert's work on the foundations of geometry. This is apparent not only because of the multiple references to Hilbert's *Annalennote* but also because Riesz approached the converse of the Jordan curve theorem by adapting a method presented by Hilbert.[26]

In this paper, Riesz worked out an alternative proof for a theorem that had been formulated and proven by Arthur Schoenflies in 1902.[27] Sharpening Schoenflies's theorem Riesz proved that if a perfect bounded subset $P$ of the plane separates the points of the plane that are not contained in $P$ into two domains: an inside set $I$ and an outside set $O$ such that (1) every arbitrary pair points of $I$ resp. of $O$ can be joined by a Jordan curve containing only points of $I$ resp. of $O$, (2) every point $p$ of $P$ can be joined to every point of $I$ resp. of $O$ by a Jordan curve that apart from $p$ contains only points of $I$ resp. of $O$ and (3) every Jordan curve joining a point of $I$ with a point of $O$ contains at least one point of $P$, then the set $P$ can be mapped bijectively and continuously to the points of a circle.

Riesz's proof is based on the method Hilbert used to prove that the 'true circle' is a closed Jordan curve—an assertion that Schoenflies himself saw as a particular

25 See protocol in *Jahresbericht der DMV* **14** (1905), 458.

26 Riesz refers to Hilbert (1903) and to its reprint in the Appendix of Hilbert's *Festschrift*.

27 See Crilly and Johnson (1999, p. 11) and Schoenflies (1902).

case of the converse of the Jordan curve theorem for the 'true circle'.[28] The method consisted first in arranging the points of the set $P$ periodically, then in mapping the set $P$ into the circle bijectively, and finally in showing that the mapping is continuous.

Thus, already in 1904 Hilbert's work on foundations of geometry had become a source of subjects and methods for Riesz's own research.

### 3.2 On point-set topology

Riesz was not only very interested in potential applications of Cantor's point-set theory but he also explored this theory as a scientific research area. Consequently, he closely followed its further development, which was taking place primarily in France.

In January 1905, Riesz submitted the article 'On a theorem by Borel' to the *Comptes Rendus de la Academie de Sciences* in Paris (1905b). In this short note, he sketched a generalisation of Borel's theorem, which is the theorem now commonly known as the 'Heine–Borel' or 'Borel–Lebesgue' theorem. Borel's version states that for every countable open cover of a bounded and closed interval $[a, b] \subset \mathbb{R}$, there exists a finite subcover. Riesz rightly asserted that this also holds for uncountable covers and for arbitrary, bounded closed subsets of $\mathbb{R}^n$. However, Riesz lost the priority of this generalisation to Lebesgue, who in 1904 had already proved the same result.[29] Riesz's interest in the Heine–Borel theorem persisted and in the same year he succeeded in producing another generalisation to multiple order types.

Later in October 1905, Riesz published a short note 'On discontinuous sets' (1905a). There he briefly discussed the dimension problem and suggested an axiomatic definition of dimension for subsets of Euclidean spaces. According to Johnson (1981), this note represents 'a novel approach to dimension at an early date', even if Riesz's axioms do not hold within the dimension theory initiated a few years later by Brouwer, Paul Uryshon and Karl Menger.[30] In this note, Riesz also presented a theorem asserting that for every discontinuous set contained in an $n$-dimensional Euclidean space, $n \geq 2$, there exists a Jordan curve containing it. The assertion is true but Riesz's sketch-proof turned out to be wrong.

Still, the note includes two other interesting contributions. First, he called a set discontinuous if it does not contain any connected subsets. Then, he proposed the following point-set theoretical definition of the concept of connected set: a point-set $A$ is said to be connected if it cannot be split into two subsets $B$ and $C$ such that $(B \cup B') \cap C' = \emptyset$ and $B' \cap (C \cup C') = \emptyset$, where $B'$ and $C'$ are the sets of all accumulation points of $B$ and $C$, respectively. Although Riesz did not say so explicitly, this definition was meant for non-empty point-sets of $\mathbb{R}^n$. It presents some similarities to the definition introduced by Camille Jordan in his *Course d'Analyse*

28 See Schoenflies (1904).

29 A survey of the history of the concept of compactness is given by Pier (1980).

30 Johnson (1981) discusses this note by Riesz within the realm of dimension theory.

(1893) for perfect and bounded sets.[31] However, what Riesz here called connected is not necessarily connected in today's sense of the term.[32] Nevertheless, it is remarkable that Riesz did not resort to the notion of distance, as Cantor had done, but used only the concept of an accumulation point.[33]

Later in 1904, Riesz formulated a more general concept of connectedness for multiple order types and two years later he offered an absolutely accurate definition of connectedness (absolute connected) for mathematical continua that corresponds to today's notion for topological spaces.

### 3.3 On multiple order types

Riesz's article 'On multiple order types I' (1905c) was an outstanding attempt to develop a point-set topology for multiple order types, i.e. for sets provided with more than one order relation. His aim was to transfer those concepts that are based on the notion of an accumulation point from Cantor's point-set theory for linear order types (e.g. $\mathbb{R}$) to multiple ones (e.g. $\mathbb{R}^n$). He was especially interested in perfect types and perfect and connected order types, whose study he regarded as an extension of the research area of analysis situs.

Riesz called $M$ an $n$-fold ordered set if $M$ was provided with $n$ different orders. The i-th order is denoted as $<_i$, $(i = 1, \ldots, n)$. For instance, $\mathbb{R}^n$ can be regarded as an $n$-fold ordered set if for any pair $a = (\alpha_1, \ldots, \alpha_n)$, $b = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$, the i-th order relation is defined as $a <_i b$ whenever $\alpha_i < \beta_i$ $(i = 1, \ldots, n)$. Two sets are called similarly ordered if there exists a bijective mapping preserving the rank relations between the elements of the sets. An $n$-fold order type is the class of all $n$-fold order sets that are similarly ordered.

He then introduced the concept of 'neighbourhood' of an element $a$ of an $n$-fold order type $M$. A neighbourhood of $a$ is a set $\{u \in M \mid b_i <_i u <_i c_i\}$ determined for $n$ pairs of elements $b_1, c_1, \ldots, b_n, c_n$ satisfying $b_i <_i a_i <_i c_i$ $(i = 1, \ldots, n)$. In the example mentioned above, the $n$-dimensional intervals are the neighbourhoods of elements of $\mathbb{R}^n$.

Following Cantor and Hilbert, Riesz defined 'accumulation point' in terms of neighbourhoods. However, Riesz's approach is more general because it uses only pure set-theoretical notions: let $M$ be an $n$-fold order then, for $a \in M$ and $N \subset M$, $a$ is an accumulation point of $N$ if every neighbourhood of $a$ contains elements of $N$ others than $a$. In contrast, Hilbert insisted on 'arbitrarily small neighbourhoods', giving away that he had in mind the notions of distance and limit for $\mathbb{R}^n$.

31 Jordan's (but not Riesz's) concept is shortly discussed by Herrlich et al. (2002a). Johnson's observation that 'Riesz implicitly assumes that the sets under discussion are bounded' applies for Riesz's quotation of Zoretti's theorem but not to his concept of a connected set, as wrongly assumed by Johnson (1981, p. 112).

32 This was rightly pointed out by Taylor (1982, p. 269). Johnson (1981, p. 112) gives the following examples of sets being connected in Riesz's but not in today's sense: the set $(0, 1) \cup (1, 2)$ and the set of rational points in the closed interval $[0, 1]$.

33 Cantor considered the set $T$ to be connected if for every two points of the set and for every $\varepsilon > 0$, there exists a finite subset of $T$ of points $t_1, \ldots, t_n$, such that the distances $d(t_1, t_2)$, $d(t_2, t_3)$, etc., are all smaller than $\varepsilon$.

Riesz even succeeded in generalising Cantor's concept of fundamental sequence as a well-ordered sequence with arbitrary cardinality.[34] He called a 'Cantor fundamental sequence' those fundamental sequences with ordinal number $\omega$ of $\mathbb{N}$. For an element $a$ that is an accumulation point of some subset, Riesz considered all well-ordered sequences of nested neighbourhoods of $a$ that happen to have $a$ as the only common element. He called the smallest cardinality of such a sequence the 'accessibility' (Zugänglichkeit) of $a$. If the accessibility of $a$ is $\aleph_0$, then $a$ is said to be 'simply accessible' (einfach zugänglich).

From Riesz's generalised notion of fundamental sequence, it is evident that in his mind, he had envisaged very large sets, sets with large cardinality, larger than $\aleph_1$, the cardinality of $\mathbb{R}^n$. However, admitting fundamental sequences with arbitrary cardinality implies that the $n$-fold order type in question has at least the same cardinality. This degree of generality is an important prerequisite for the next section, because, according to Riesz, Hilbert restricted the cardinality of space. This is one of the things Riesz aimed to improve.

Coming back to his paper, there $T'$ designates the 'derived set' of a subset $T$, i.e the set of all accumulation points of $T$. Extending some elementary properties of sets of an $n$-fold order type $M$, he defined: (1) $T$ is dense in itself if $T \subset T'$, (2) $T$ is perfect if $T$ is closed and dense in itself, and (3) $T$ is closed if $T' \subset T$ and if every infinite subset of $T$ has an accumulation point in $T$. Riesz's concept of closed subset is more restrictive than Cantor's since the second condition in (3) is the Bolzano–Weierstrass property for closed bounded subsets of $\mathbb{R}$. Therefore, Riesz's concept of closed set is related to the concept of compactness.

Besides, Riesz defined 'connected set', as already mentioned, generalising the concept presented in (1905a): $T$ is connected if $T$ is not the union $U \cup V$ of two subsets $U$ and $V$ satisfying $(U \cup U') \cap (V \cup V') = \emptyset$. [35]

For perfect and connected 'simple order types', Riesz stated the Heine–Borel theorem.[36] He sketched an inductive proof which is similar to the one he provided in (1905b). It is also based on the idea of Dedekind cuts. Furthermore, Riesz generalised the notion of interval to $n$-fold order types calling it 'elemental domain' (Elementargebiet).

His correspondence with Hilbert provides valuable insight into the purposes behind the research Riesz had accomplished in the articles discussed above.

34 See Riesz (1905c, p. 410). Purkert (2002, pp. 688–689) has already pointed to the relevance of Riesz's concept of fundamental sequence.

35 Although the formulation in Riesz (1905c) is not identical with the one given in Riesz (1905a), the contents are equivalent. Notice that neither here nor in Riesz (1905a) the subsets $U$ and $V$ are supposed to be disjoint. That the two definitions are equivalent follows easily from the fact that for $T$ connected and $T = U \cup V$, if $U \cap V \neq \emptyset$, then at least one of the following inequalities holds true: $U' \cap V \neq \emptyset$, $U \cap V' \neq \emptyset$, $U' \cap V' \neq \emptyset$.

36 'Simple order type' is my translation of 'einfacher Ordnungstypus', where 'einfach' means $n = 1$, i.e. the set has only one order relation (see Riesz 1905c, p. 415).

## 3.4 Letters from Hungary

On 18 November 1904, Riesz sent Hilbert a report (1904a) on his latest research asking him to submit the enclosed note to the Göttingen Mathematical Society in support of its publication.[37] He wished to present his results in a short note first before publishing them in a longer and more detailed paper. The attached 'short note' is not preserved in Hilbert's Archive but from Riesz's description it is clear that it was an early version of his article 'On multiple order types I' (1905c) containing some additional results which were not included in the final version of this article but appeared later in the *Genesis* (Riesz 1906b).

What were the findings that he communicated to Hilbert? Riesz's starting point was Cantor's conviction that it is possible to generalise the so-called geometric properties of point-sets to order types by using the concept of fundamental sequences. Such investigations had been done for linear order types, but he, Riesz, now attempted to do so for multiple order types. The lack of a notion of metric or of distance represents a challenge, because the concept of a uniform continuous function cannot be defined in the old way. Fortunately, the metric approach to the concept of uniform continuity can be replaced by the Heine–Borel theorem. Riesz claimed to have generalised both the Heine–Borel and the Bolzano–Weierstrass theorem to linear perfect and connected order types.

As a generalisation of the geometric concept of 'domain' (Bereich), he considered perfect and connected twofold order types that, after removing their boundary elements, remain everywhere dense. He therefore focused on these order types, investigating their connectedness and separation properties. He referred to this work as investigations in the field of analysis situs. He was forced to give up the notion of straight line. He defined only particular line segments and polygons. Nevertheless, he managed to find a way to generalise certain results on Jordan curves to this specific kind of order types so that it was possible both to define the two orientations of a Jordan curve and to prove that a Jordan curve separates a domain into two regions.

Most of the details discussed above appear in the introduction to 'On multiple order types I' except for the assertions concerning Jordan curves. In fact, in this article Riesz does not mention either the notion of a Jordan curve or the Jordan curve theorem at all.[38] Bearing in mind the fundamental role that the notion of a Jordan curve played in Hilbert's *Annalennote*, the letter gives the impression that Riesz wanted Hilbert to know that he was also working on a generalisation of Hilbert's approach and that

37 A transcription of this letter is included in Appendix A and (Rodríguez 2006).

38 The generalisation of the Jordan curve theorem to dimension three was proven by James Alexander in 1924 once the topology had been sufficiently developed. Two important properties of a closed Jordan curve are (1) the curve separates the plane in two regions, the interior and the exterior, such that any continuous curve joining the two regions intersects necessarily the Jordan curve; (2) the interior of a closed Jordan curve is topologically equivalent to an open disc while the exterior is topologically equivalent to an open disc from which the middle point has been removed. Similar properties of spheres in three-dimensional space were much more difficult to prove. The continuous image of a sphere that does not intersect itself happens indeed to separate the space into two regions, the interior and the exterior. But it is not true that the one region is topologically equivalent to an open ball while the other one is topologically equivalent to an open ball from which the middle point has been removed. Alexander showed that the horned sphere is a counter example (see Gray 2000, p. 112).

he had began to extend one of the main tools used by Hilbert: the notion of Jordan domain. There are also other reasons to think that this was a likely scenario.

Coming back to his letter, Riesz discusses the application of his investigations on multiple order types to the foundations of geometry and the space problem. He wrote:

> Now my investigations come very close to the foundations of geometry. More precisely, to those works that, regarding the continuity as the primary property of space, base the notion of the $n$-dimensional manifold on the concept of number. Thereby they put upper bounds to the power of space, bounds that actually appear to be artificial.[39]

This passage bears some resemblance to the introduction of the *Genesis*. There, however, Riesz refers explicitly to Lie's and Hilbert's notions of an $n$-dimensional manifold. In the *Genesis*, Riesz criticises that, by defining this concept in terms of the Cartesian space $\mathbb{R}^n$, Lie and Hilbert made this notion dependent on the set of real numbers. Consequently, the power of an $n$-dimensional manifold cannot be bigger than the power of $\mathbb{R}^n$.

This remark is impressive because it indicates that Riesz was approaching the notion of continuity of space from a quite general point of view: his idea of continuous space was far beyond $\mathbb{R}^n$. From the beginning, he did not want to exclude the possibility that a continuous space could be defined that has a greater power than $\mathbb{R}^n$.

What sets did Riesz definitely not want to exclude? For sure, multiple order types with greater power than $\mathbb{R}^n$. He may already also have had sets of functions in mind. There are several reasons to think that this was likely: for one, Riesz was thoroughly familiar with sets of functions. As shown by his earlier publications, he had studied Baire's theory of real functions. There, the set of all real functions $f : \mathbb{R} \to \mathbb{R}$ plays a central role. This set has greater power than $\mathbb{R}$. Besides, Riesz referred explicitly to some sets of functions in the *Genesis*.[40] Furthermore, the research he started just after concluding the *Genesis* shows that he was ready to envisage function spaces; indeed, he was actually one of the first mathematicians to use the term 'function space'. He did so in a letter to Fréchet from 1907.[41] Besides, that same year Riesz (1907c) discussed the possibility of an analytic geometry on the function space $L^2$, the set of square-integrable real functions in Lebesgue's sense.

The rest of the letter contains remarks concerning multiple order types that appear in the *Genesis* but not in 'On multiple order types I'. Riesz continued his letter as follows:

> In my opinion the only essential features of $\mathbb{R}^n$ are: it has an $n$-fold ordering, it is dense everywhere and it is continuous (i.e it is perfect and connected), as clearly

[39] My translation of Riesz (1904a). See the equivalent passage in Riesz (1906b, p. 310).

[40] The idea was not completely new. Already fifty years earlier in his habilitation lecture, Bernhard Riemann had mentioned the idea of a spatial manifold whose elements were functions (see Scholz 1999, p. 27).

[41] A transcription of this letter (Riesz 1907b) is included in Appendix B and (Rodríguez 2006). On the development of function spaces, see Bernkopf (1966). Apparently, Bernkopf did not know of this letter from Riesz. Taylor (1982) quotes the letter but, in contrast to Bernkopf, he does not discuss the priority of the term 'function spaces'.

> appears in your definition of the plane although with the mentioned limitation. According to this, $\mathbb{R}^n$ can be regarded as a set of $n$-fold order type (Riesz 1904a).

The 'mentioned' limitation consists in putting an upper bound to the cardinality of space by basing the notion of two-dimensional manifold on the concept of number.

Thus, by regarding $\mathbb{R}^n$ as an $n$-fold order type Riesz was highlighting those properties he considered relevant for a precise concept of continuous space. Remarkably, he chose abstract point-set topological as well as order-theoretical properties. Following Cantor, he suggested that a set should regarded as continuous if it is perfect and connected, but with these properties understood according to Riesz's own theory of multiple order types.

Why was Riesz searching for such an abstract characterisation of $\mathbb{R}^n$? Because, as he announced in 'On multiple order types I', once the so-called geometric properties of sets can be expressed in terms of fundamental sequences for multiple order types, then it suffices to have the structure of a multiple order type on a set to be able to study its geometric properties—said more broadly: one can do topology on it.

In his letter, Riesz was suggesting to replace $\mathbb{R}^2$ with an abstract twofold order type that is dense everywhere, perfect and connected. Thus, Riesz aimed at defining a topology on the $n$-dimensional manifold using the topology of the $n$-fold order type instead of the standard topology of $\mathbb{R}^n$ (as Hilbert did for the plane). The convenience of Riesz's approach relied on achieving a concept of an $n$-dimensional manifold that is independent of both the system of number coordinates $\mathbb{R}^n$ and the standard point-set topology of $\mathbb{R}^n$. This procedure avoids the artificial limitation Riesz addressed above.

That Riesz's true concern was indeed the topologisation of space independently of $\mathbb{R}^n$ follows from the last part of his letter.

> I find your Annalennote in this respect well thought and if I am not mistaken (that is possible because I have not worked that thing out), then my conception of $\mathbb{R}^n$ and respectively of the plane is enough to arrive at either to Euclidean or to Bolyai–Lobachevsky geometry as a consequence of your motion axioms. That means, no manifold with bigger power than that of the continuum allows a motion in Lie's sense, respectively in yours (Riesz 1904a).

According to Riesz, if an abstract $n$-fold order type that is dense everywhere, perfect and connected is used for the foundations of geometry, then it will follow from Hilbert's axiomatic definition of motion that its power is not bigger than that of the continuum. That means that the upper bound on the cardinality of space is a necessary consequence and not an artificial assumption. This consequence is what justifies Riesz's abstract approach. It is an issue of mathematical formalism concerning the mutual independence of the elementary assumptions.

Thus, Riesz's first attempt to extend Hilbert's concept of a two-dimensional manifold consisted in an application of his theory of multiply order types. But then he got interested in a still more general approach, one based on purely abstract point-set topological methods without any ordering assumptions.

Just before the end of the year, Riesz wrote a second letter to Hilbert asking him to hold back the publication of his short note because he needed to make some corrections.[42] He announced a change of mind and explained that he had attempted to generalise the analysis situs of the theory of functions by applying his theory of multiple order types. During these investigations, he had realised that the use of fundamental sequences, or at least of subsets with a unique accumulation point, leads to a better end than the approach he had first tried.

What then happened? Riesz worked fast, and in January 1905, he replaced his note from November 1904. The original short note became his long paper 'On multiple order types I' (1905c) which appeared that same year in the *Mathematischen Annalen*. Furthermore, he got the opportunity to present his results at the meeting of the Göttingen Mathematical Society on 25 July 1905. According to the records of that day's meeting, Riesz reported on 'The connection between his soon-to-appear investigations on set theory and questions related to the space problem'.[43]

'On multiple order types I' appeared in December 1905 free from any connection to the foundations of geometry and the space problem. The title carries an 'I' indicating that Riesz planed to publish a second part of this paper. He reconsidered the connection to the space problem in the *Genesis*.

## 4 'Die Genesis des Raumbegriffs'

A second part of 'On multiple order types I' did never appear as such. However, on 22 January 1906, Gustav Rados presented a paper by Riesz to the Hungarian Academy of Sciences, that, judging by its contents, must be the second part of the paper Riesz once offered to Hilbert, or at least what had become of it.[44]

The German version of this paper was to appear in 1906 as 'Die Genesis des Raumbegriffs' (Riesz 1906b) in the journal *Mathematische und Naturwissenschaftliche Berichte aus Ungarn*. The journal itself, although Hungarian, had a German name and used to publish only articles written in German. It had been established in 1883 to provide greatly improved chances for the diffusion of Hungarian works. The Hungarian original version of the *Genesis* was published in two parts (Riesz 1906a, 1907a) in another Hungarian journal.[45]

'Die Genesis des Raumbegriffs' is an long article of 35 pages that deals with the role of experience in the development of the concept of continuous space in geometry. Here, Riesz approached Hilbert's continuity of space problem from a broad perspective that included, apart from a modern mathematical treatment, current philosophical and psychological views. Thematically it consists of two parts: the first one has an introductory and theoretical (mathematical) character (pp. 309–322), and the second one focuses on the construction of a concept of space starting from a specific empirical notion of continuity in real space (pp. 323–345).

[42] The letter is undated (Riesz 1904b), but judging by its contents, it concerns the same note and it must have been written shortly after the one from November 1904 because Riesz wishes Hilbert a happy New Year's Eve. A transcription of this letter can be found in Appendix A and (Rodríguez 2006).

[43] See report of this meeting in *Jahresbericht der DMV* 14 (1905) p. 458.

[44] See report of the Academy meetings in *Mathematische und Naturwissenschaftliche Berichte aus Ungarn* 24 (1906) p. 365.

[45] On these Hungarian journals, see Szénássy (1992) p.221.

In the *Genesis*, Riesz distinguished between geometry as an exact science and geometry as a natural science, asking how continuous space could be characterised best in each of them. In other words: what does it mean that the geometrical space is continuous? What does it mean that the real space is empirically continuous? While the first question strictly concerns a purely mathematical issue, the second unavoidably involves views about the philosophy of geometry.

Hilbert's works on the foundations of geometry keenly motivated Riesz's interest in the concept of continuous space in geometry. But Riesz's interest in the empirical continuity of the real space arose from those contributions to philosophy of geometry in which Poincaré discussed the relation of the axioms of geometry to real space. Riesz explicitly refers to Poincaré's (1905) 'La valeur de la science'.[46]

Riesz approached these two different notions of continuity by redefining the concepts of mathematical and physical continuum. These two concepts were discussed by Poincaré in 'La valeur de la science' and taken up by Riesz. Poincaré's conception of empirical continuity was contained within his notion of a physical continuum. Riesz redefined this concept in a more general and abstract way by formulating it axiomatically and in terms of point-set theory. Furthermore, he developed a rudimentary theory of physical continua that he then applied to construct a three-dimensional space as an infinite numerable sequence of physical continua (pp. 323–335). From now on, I will refer to this space as Riesz's space $\mathcal{R}$. However, I will not enter into a more detailed discussion on its construction because it would go far beyond the scope of this article.[47] The purpose of this note is to draw attention to Riesz's characterisation of continuous space in geometry.

This characterisation was accomplished in two steps: first Riesz provided the space $\mathcal{R}$ with a specific concept of an accumulation point and then he showed that $\mathcal{R}$ defines a mathematical continuum that satisfies certain additional properties.

What exactly these properties are, I discuss in Sect.4.7 where I summarise Riesz's characterisation of continuous space in geometry. In preparation, I introduce Riesz's concept of mathematical continuum in Sect.4.1. In Sect.4.2, I discuss the epistemological meaning of Riesz's approach. Then, I proceed to present some important concepts of Riesz's rudimentary theory of mathematical continua in Sects.4.3, 4.4 and 4.5. In Sect.4.6, I explain how Riesz approached the issue of dimension.

## 4.1 Mathematical continuum and its epistemological meaning

Around 1900, the only known general concept of a continuum was due to Cantor who had called a continuum of points (Punktkontinuum) any subset of $\mathbb{R}^n$ that happened to be perfect and connected. That is the reason why the term mathematical continuum was at that time commonly related to the set of all real numbers, so for instance by Poincaré (1905).

46 See Riesz (1906b, pp. 312–315). On the philosophy of geometry of that time, see Torretti (1978). On the influence of Poincaré's work on Riesz, see my thesis (Rodríguez 2006).

47 On Poincaré's idea of empirical continuity of space and Riesz's constructive notion of space, see Rodríguez (2006).

Taking an abstract point-set theoretical approach, Riesz suggested a more general concept of a mathematical continuum: an abstract set provided with a concept of accumulation point that satisfies four axioms.[48] Let $M'$ denote the set of all accumulation points of the set $M$. In modern terms, Riesz's definition reads as follow (Riesz 1906b, p. 318):

**Definition 1** (*Mathematical continuum*) $X$ is a mathematical continuum if for every element $x \in X$ and for every subset $M \subset X$ either $x \notin M'$ or $x \in M'$ and then the following conditions hold true:

1. $M = \{x_1, \ldots, x_n\}$ $(n \in \mathbb{N}) \Rightarrow M' = \emptyset$;
2. $M \subset N \Rightarrow M' \subset N'$;
3. $M = P \cup Q \Rightarrow M' \subset P' \cup Q'$;
4. $x \in M'$ and $y \neq x \Rightarrow \exists\, P \subset M$ such that $x \in P'$ but $y \notin P'$.

It is obvious that Riesz's concept of mathematical continuum is not a continuum in Cantor's sense as it does not need to be either perfect or connected.

Riesz did not just disengage the term 'continuum' from its traditional meaning; he also assigned a broader epistemological meaning to the mathematical concept of continuous space. For one, he called a mathematical continuum any arbitrary abstract set that is provided with a concrete concept of accumulation point satisfying the four axioms mentioned above. Consequently, he acknowledged that one and the same set can be regarded as continuous in different ways, each way depending on the concrete concept of accumulation point in consideration.

Given an arbitrary set provided with a concept of an accumulation point, the challenging question is no longer whether it is continuous (because according to Riesz it is) but to find out further continuity properties.

Mathematical continua in Riesz's sense are by definition provided with a continuity structure that is determined by the specific concept of accumulation point in consideration. This continuity structure results from associating to each element $x$ of the mathematical continuum a family of subsets, namely the family consisting of exactly those subsets for which the element $x$ is an accumulation point. The structure of $X$ as mathematical continuum can be described as the system of pairs $(x, A_x) \in X \times \mathcal{P}(\mathcal{P}(X))$ where $x$ is an element of the mathematical continuum $X$ and $A_x$ is the family of those subsets of $X$ of which $x$ is an accumulation point, i.e. $A_x = \{M \subset X | x \in M'\}$.

There may be elements of $X$ that are not accumulation point of any subset of $X$. Riesz called $x \in X$ 'main element' if there exists a subset $M$ of $X$ such that $x$ is an accumulation point of $M$. In his Rome lecture, he explained the fourth (separation) axiom as a condition that guaranties that to each main element $x$, there corresponds a unique family $A_x$.

Coming back to the *Genesis*, Riesz explained that he had anticipated interesting applications of his rudimentary theory of mathematical continua to analysis and the calculus of variations. As examples of mathematical continua, he cited the Cartesian

[48] Instead of accumulation point, Riesz (just as Hilbert) used the term 'Verdichtungsstelle' which literally means 'condensation point'. However, as rightly pointed by Taylor (1982, p. 268), this term should not be translated as 'condensation point' in the special sense given to it by Ernst Lindelöf in 1905.

spaces $\mathbb{R}^n$ with their standard concept of accumulation point, multiple order types with the concept of accumulation point as defined in Sect. 3.3, as well as sets of functions with the concept of accumulation point defined by a metric or by different kinds of convergence principles: point-wise, uniform, weak and strong convergence. He stressed the importance of the selected continuity structure by pointing to fundamental differences between the concepts of strong and weak extreme values in the calculus of variations (Riesz 1906b, p. 318).

Although it was already a common practice to distinguish between different kinds of convergence (weak, point-wise, etc.), it still represents a major conceptual shift to define different concepts of accumulation point on a set, thereby regarding the set as continuous in different ways. That is what Riesz did.

Thus, the first condition that ought to be assumed of an abstract set, in order to be able to investigate its continuity properties, is that it may be regarded as a mathematical continuum. This applies in particular to the set upon which geometric space may be defined.

From the viewpoint of geometry, it is not evident why Riesz thought it necessary to tackle the continuity of space problem using such an abstract approach. The first reason is one of the practical nature and goes back to those philosophical motivations that had led him to work on a concept of space whose elements are ‘groups of sensations’. To avoid any kind of philosophical and psychological controversy, he conceived those ‘groups of sensations’ as elements of an abstract set that satisfy a certain system of axioms.

Besides, this general approach certainly allowed him to overcome those limitations he had criticised in his letter to Hilbert. In his Rome lecture, Riesz made the connection to geometry acknowledging Hilbert’s approach to the foundations of geometry from 1902 as the main motivation behind his concept of mathematical continuum:

> It is in Hilbert’s work that for the first time it seems evident that the definition of this concept [the concept of continuity, LR] concerns primarily a definition of limit point, or more generally, a definition of accumulation point, whereas the concept of accumulation point is determined by the postulate asserting that the considered domains can be mapped on certain number manifolds and that the mappings fulfil certain conditions. The concept of an accumulation point for those number manifolds is, however, already determined.[49]

Riesz asserted that it is in Hilbert’s *memoir* where it seems evident for the first time that the issue of continuity is primarily about defining a concept of accumulation point. However, Hilbert solved this by using mappings from the plane to a number manifold. Not only did Hilbert assume that those mappings satisfy certain conditions but he also took advantage of the fact that the number manifold is already provided with a concept of accumulation point. In Riesz’s approach, on the contrary, the space needs only to be a mathematical continuum. What is more, the concept of accumulation point defined on the mathematical continuum is independent of any number manifold.

49 My translation of Riesz (1909, p. 18).

But most importantly, Riesz aimed to contribute a generalisation of Cantor's point-set theory to abstract sets in the sense suggested by Jacques Hadamard at the first International Congress of Mathematicians in Zurich in 1897. Riesz explicitly said so in his Rome lecture (1909). Eventually, he succeeded in his purpose as the reception of his Rome lecture played a significant role in the emergence of general topology (see Sect. 6). That is the reason why Riesz's concept of a mathematical continuum is not just mathematically but also historically closely related to the modern concept of topological space.

## 4.2 Classifying mathematical continua

Riesz paid special attention to the issue of classifying mathematical continua. He tackled this issue by defining a variety of continuity properties that a mathematical continuum may possess or not. These properties were defined in terms of elementary topological concepts.

Still, Riesz went a step further beyond Cantor's point-set topology. Having studied Cantor's theory of order types and having contributed to the theory of multiple order types, Riesz was familiar with the idea of classifying abstract sets by means of isomorphisms, the technique used in Cantor's theory to define order types. Riesz adapted this technique to classify mathematical continua according to isomorphisms that preserve their continuity structure. He called the isomorphic continua 'similarly condensed' and he named the equivalence class consisting of all mathematical continua that are similarly condensed to each other a 'condensation type' (Verdichtungstypus). A 'condensation type' is a notion comparable to the concept of homeomorphism class which is used in general topology to classify topologically equivalent spaces:

**Definition 2** (*similarly condensed—ähnlich verdichtet*) Let $X$ and $Y$ be two mathematical continua. $X$ and $Y$ are similarly condensed if there exists a bijective mapping $f : X \rightarrow Y$ such that $\forall x \in X$ and $\forall M \subset X$ it holds $x \in M'$ if and only if $f(x) \in f(M)'$ (Riesz 1906b, p. 319).

In Riesz's approach to Hilbert's continuity of space problem, the concept of similarly condensed played a key role because it provided a method for comparing the continuity structure of the space as a mathematical continuum with the continuity structure of $\mathbb{R}^3$.

## 4.3 Neighbourhoods

Riesz proceeded to generalise the well-established elementary concepts of Cantor's point-set topology in $\mathbb{R}^n$ to abstract sets, e.g. open set, connected, dense.[50] He started with the concept of neighbourhood:

[50] Some of Riesz's concepts and theorems have been already accurately described by Taylor (1982) but in a different context. My description focuses on Riesz's topological approach to Hilbert's space problem.

**Definition 3** (*Neighbourhood*) Let $X$ be a mathematical continuum. A subset $U \subset X$ is a neighbourhood of an element $x \in X$ if $x \in U$ and $x \notin (X \backslash U)'$ (Riesz 1906b, p. 319).

This concept is much more general than the one used by Hilbert (1902b; 1903). It does not depend either on $\mathbb{R}^n$ or on the notion of Jordan curve but only on the continuity structure of the mathematical continuum.[51]

Furthermore, Riesz asserted that the concept of accumulation point can be characterised in terms of neighbourhoods: let $X$ be a mathematical continuum. The element $x \in X$ is an accumulation point of the subset $M \subset X$ if and only if every neighbourhood of $x$ contains infinitely many elements of $M$.

Riesz proved only the proposition: if $x \in M'$ then every neighbourhood of $x$ contains infinitely many elements of $M$. He claimed that the converse proposition holds true and he pointed to the fact that the intersection of a finite number of neighbourhoods of $x$ is also a neighbourhood of $x$. Both assertions can be proven easily.[52]

The proof that the intersection of a finite number of neighbourhoods of $x$ is a neighbourhood of $x$ follows, as Riesz pointed out, from axioms two and three and via induction: let $U_1$ and $U_2$ be two neighbourhoods of $x$, then $x \in U := U_1 \cap U_2$. Let $U^c$ denote the complement of $U$. $U^c$ can be written as the union of two disjoint sets $U^c = U_1^c \cup (U_1 \cap U_2^c)$. By axiom three, if $x$ is an accumulation point of $U^c$ then $x$ is an accumulation point of $U_1^c$ or $U_1 \cap U_2^c$ but $x$ is not an accumulation point of $U_1^c$ and by axiom two it is neither of $U_1 \cap U_2^c \subset U_2^c$. Since $U = \cap_{i=1}^{n} U_i = \cap_{i=1}^{n-1} U_i \cap U_n$, then induction on $n$ shows that $U$ is a neighbourhood of $x$.

Although this theorem reads just as today's definition of accumulation point in a topological space, the fact is that they use different notions of 'neighbourhood'. Therefore, they do not mean the same. The neighbourhoods of an element $x$ in Riesz's sense are in general different from the neighbourhoods of $x$ in a topological space, since the former do not need to contain an open set containing $x$. This difference is the reason why the concepts of mathematical continuum and topological space are not equivalent.

With his characterisation of accumulation point in terms of neighbourhoods Riesz found a way to define a given set as a mathematical continuum starting from neighbourhoods. This approach is not altogether surprising because up to that time it was the traditional custom to define the notion of accumulation point or limit point in terms of neighbourhoods: Karl Weierstrass had used $\varepsilon$-neighbourhoods to define uniform convergence in classical analysis; Cantor developed his point-set topology for $\mathbb{R}$ starting with $\varepsilon$-neighbourhoods (implicitly, for he did not define the concept of neighbourhood at all), and Hilbert introduced a concept of limit point for the plane in

[51] As Taylor (1982, p. 268) rightly pointed out, Riesz 'says nothing about the status of an empty set or of the entire class'. This was typical at that time.

[52] Let $x$ be an element of the mathematical continuum $X$ such that every neighbourhood of $x$ contains infinitely many elements of $M$. As it happens, $x \in M$ or $x \notin M$. Case one $x \notin M$: if $x \notin M'$, then $U_x := X \setminus M$ is a neighbourhood of $x$ that contains infinitely many elements of $M$. This is a contradiction and so $x \in M'$. Case two $x \in M$: if $x \notin M'$ then by axiom two of the definition of mathematical continuum $U_x := (X \setminus M) \cup \{x\}$ is a neighbourhood of $x$ and so $U_x$ contains infinitely many elements of $M$ but $U_x \cap X = \{x\}$. This is a contradiction and so $x \in M'$.

terms of neighbourhoods (Sect. 2.6). Another important example was given by Baire (1899a, b) who built up a rudimentary point-set topology for $\mathbb{R}$ that is different from the standard one.

Baire's papers had a strong influence on Riesz for two reasons. First, because Riesz attributed the notion of what he called a 'sufficient system of special neighbourhoods' to Baire (Riesz 1906b, p. 320). Actually, Baire did not define any concept of neighbourhood at all. What he did was to characterise the concept of an accumulation point in terms of certain specific subsets which Riesz then identified as neighbourhoods.[53] Riesz succeeded in synthesising the essential properties of both Baire's subsets and Hilbert's neighbourhoods into what he called a 'sufficient system of special neighbourhoods': let $X$ be a mathematical continuum and let $x \in X$ be a main element; that is, let $x$ be an accumulation point of some subset $M \subset X$. A system of neighbourhoods $\mathcal{U}$ at $x$ is called sufficient if for every neighbourhood $V_x$ of the arbitrary main element $x$ there exists a special neighbourhood $U_x \in \mathcal{U}$ of $x$ such that $U_x \subset V_x$.

In a topological space, such a system of special neighbourhoods is called a neighbourhood base. Clearly, the advantage of special neighbourhoods is that it suffices to define accumulation point in terms of special neighbourhoods. That is what Baire did. It is one of the technical issues that made Baire's papers so influential in Riesz's *Genesis*. Not only did Riesz identify those subsets used by Baire to define accumulation point as special neighbourhoods, but he also applied most of Baire's basic concepts and procedures to the construction of his space $\mathcal{R}$ as an infinite numerable sequence of physical continua.[54] Riesz also adapted Baire's notion of an accumulation point, thereby achieving a characterisation of his space $\mathcal{R}$ as a mathematical continuum.

### 4.4 Elementary concepts of an abstract point-set topology

Riesz then generalised some elementary concepts of Cantor's point-set topology for subsets of $\mathbb{R}^n$ to subsets of a mathematical continuum: let $X$ be a mathematical continuum. An element $x \in X$ is to be called 'internal element' of $A \subset X$ if $A$ is a neighbourhood for $x$. A subset $A \subset X$ is to be regarded as 'open' if all its elements are internal elements of $A$. The element $x \in A$ is a 'boundary element' of $A$ if $x \in (X \backslash A)'$. The 'frontier' of $A$ consists of all the boundary elements of $A$ and all the boundary elements of $X \backslash A$. Two subsets $A$ and $B$ are called 'separated' from each other if $A \cap B = \emptyset$. Two subsets $A$ and $B$ are called 'isolated' from each other if $(A \cup A') \cap (B \cup B') = \emptyset$, i.e. if $A \cup A'$ and $B \cup B'$ are separated from each other (Riesz 1906b, p. 320).

However, the concept of a 'perfect set' presented some difficulties. The problem is that in Riesz's theoretical framework of mathematical continua, the relation $A'' \subset A'$ does not hold in general, and consequently the set $A \cup A'$ is not necessarily a closed set, as it is in Cantor's theory. Riesz himself pointed out this flaw, regretting that the relation $A'' \subset A'$, so important for a true generalisation of Cantor's point-set topology

53 See sections 3.2.4 and 7.1 of my thesis for a detailed account of the influence of Baire's papers on Riesz's *Genesis* (Rodríguez 2006, pp. 189–201).

54 On the space $\mathcal{R}$, see the beginning of Sect. 4.

for $\mathbb{R}^n$, was missing in his rudimentary theory of mathematical continua (Riesz 1906b, p. 320). He therefore restrained himself from giving any tentative definition of ‘closed’ and ‘perfect set’.

Why did not Riesz change the axioms in his definition of mathematical continuum so that the relation $A'' \subset A'$ would necessarily hold true? Probably, as has been already suggested by Purkert et al. (2002), he did not want his concept to become so narrow that his theory would then fail to apply on Baire’s classification of real functions.[55]

Baire’s classification of real functions $\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R}\}$ reads as follows: the class $\mathcal{C}^0$ is defined as the set consisting of all continuous functions, $\mathcal{C}^1$ is the set of discontinuous functions that are the point-wise limit of a sequence of continuous functions, and $\mathcal{C}^\alpha$ is the set of functions $f$ that are the point-wise limit of a sequence of functions belonging to the classes $\mathcal{C}^\beta$ but with $f \notin \mathcal{C}^\beta$ for all $\beta < \alpha$, with $\alpha$, $\beta$ ordinal numbers of the second class (i.e. $\alpha$ and $\beta$ are the order type of a well-ordered countable set).

Baire’s classification first appeared in (1898) and very soon became a research object in French analysis. In 1904, Henri Lebesgue showed that $\mathcal{C}^\alpha \neq \emptyset$ for $\alpha$ any countable ordinal number.[56] In 1905, Maurice Fréchet, who at that time was developing his theory of ‘$L$-’, ‘$V$-’, and ‘$E$-classes’, discussed the Baire classes within the framework of his theory of $L$-classes.[57]

Riesz definitely knew Fréchet’s paper from 1905. He referred to it in the *Genesis*, rightly acknowledging Fréchet as a pioneer of the research area to which his own theory of mathematical continua belonged (Riesz 1906b, p. 322). Most interestingly, not only had Fréchet (1905) pointed to the fact that the statement ‘every derived set is closed’ does not hold for $L$-classes in general but he also argued against the option of adding this statement as a further condition defining an ‘$L$-class’. He pointed to Baire’s classification as an example of a ‘classical’ concept of limit for which the derived set of a set is not closed.

From Fréchet’s perspective, $\mathcal{F}$, provided with the limit concept of point-wise convergence, constitutes an $L$-class. From Riesz’s perspective, $\mathcal{F}$ may be regarded as a mathematical continuum by considering $f \in \mathcal{F}$ as an accumulation point of $G \subset \mathcal{F}$ whenever there exists an infinite sequence $\{f_n\} \subset G$ that converges point-wise to $f$. In $\mathcal{F}$ as a mathematical continuum, the Baire classes are examples of sets that do not contain all their accumulation points. This is a consequence of Lebesgue’s proof that none of the classes $\mathcal{C}^\alpha$ are empty, for $\alpha$ any countable ordinal number.

This suggests that, following Fréchet, Riesz did indeed decide against a modification of his system of axioms because of the Baire classes.

55 This explanation was suggested by Purkert et al. (2002), but the author did not elaborate.

56 See Dugac (1976, pp. 312–327).

57 An ‘$L$-class’ in Fréchet’s sense is an abstract set provided with a concept of limit point that satisfies the following conditions: (1) if $a$ is limit of the sequence $\{a_n\}_{n\in\mathbb{N}}$, then it is unique; (2) if the elements of the sequence $\{a_n\}_{n\in\mathbb{N}}$ are all identical to the same element $a$, then the sequence has $a$ as limit; (3) if the sequence $\{a_n\}_{n\in\mathbb{N}}$ has $a$ as limit, then every subsequence of $\{a_n\}_{n\in\mathbb{N}}$ has $a$ as limit. Fréchet summarised his theory of $L$-, $V$-, and $E$-classes in his PhD thesis in (1906). For an accurate discussion of Fréchet’s PhD thesis and his earlier works on abstract point-set theory, see Taylor (1982).

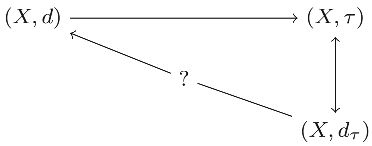


**Fig. 5** If in the mathematical continuum $(X, d)$ the relation $d(d(A)) \subset d(A)$ does not hold true for every subset $A$ of $X$, then $(X, d_\tau)$ is not similarly condensed to $(X, d)$

From today's perspective, on the other hand, the lack of the absolute validity of the relation $A'' \subset A'$ is crucial in order to understand how close Riesz came to today's concept of topological space. In order to clarify the matter, let call $d$ the derivative operator that associates $A \subset X$ its derived set $A' := d(A))$ and let $(X, d)$ be an arbitrary mathematical continuum. Finally, let $\tau$ be the system of open sets $A \subset X$ in Riesz's sense. It can easily be proven that $(X, \tau)$ defines a topological space. Therefore, one can associate with any mathematical continuum $(X, d)$ a topological space $(X, \tau)$. Furthermore, using the standard definition of accumulation point in a topological space, one can associate to every topological space $(X, \tau)$ its 'standard' mathematical continuum $(X, d_\tau)$. However, the passage from the mathematical continuum $(X, d)$ to its associated topological space $(X, \tau)$, and then from there to the mathematical continuum $(X, d_\tau)$, may not return us to $(X, d)$, i.e. $(X, d)$ and $(X, d_\tau)$ may not be similarly condensed (see Fig. 5). This is certainly what happens whenever in the mathematical continuum $(X, d)$ the relation $d(d(A)) \subset d(A)$ does not hold true for every subset $A$ of $X$.[58]

This is presumably what Hausdorff meant when he wrote in an unpublished note from 1938: "The foundation of topology on $A'$ (instead of on $\overline{A}$) seems unlucky to me because different functions $A'$ can determine the same topological space".[59] As illustrated in Fig. 5, the topological space $(X, \tau)$ can be determined by both $(X, d)$ and $(X, d_\tau)$.

## 4.5 Connectednes

Riesz considered connectedness as an additional property of a mathematical continuum. As shown in Sects. 3.2 and 3.3, he had been working for years on sharpening this notion both for point-sets of $\mathbb{R}^n$ and for multiple order types. Within the framework of mathematical continua, he formulated three different concepts of connectedness: the first one for the space itself as a mathematical continuum, the other two for its subsets.

58 That means: if $d(d(A)) \subset d(A)$ does not hold true for every subset $A$ of $X$, then $(X, d)$ and $(X, d_\tau)$ are not similarly condensed. This can be demonstrates as follows: first, notice that $(X, d_\tau)$ is both a topological space and a mathematical continuum. Therefore, $d_\tau(d_\tau(A)) \subset d_\tau(A)$ holds true for every subset $A$ of $X$. The assertion follows as consequence of this lemma: let $(X, d_1)$ and $(X, d_2)$ be any two similarly condensed mathematical continua and let $i$ be the isomorphism between them, then $d_1(d_1(A)) \subset d_1(A)$ if and only if $d_2(d_2(i(A))) \subset d_2(i(A))$ for every subset $A$ of $X$.

59 My translation of the quote appearing in Purkert et al. (2002, p. 728). Herrlich et al. (2002a, p. 753) described Riesz's mathematical continuum as a pre-topological space with a separation axiom weaker than $T^2$ and stronger that $T^1$. This characterisation is not helpful, because the concept of pre-topological space is very specialised and rather difficult to find in a standard book on general topology.

His definition of a connected mathematical continuum coincides with today's concept of connected topological space and reads as follows:

**Definition 4** (*connected mathematical continuum*) The mathematical continuum $X$ is said to be connected if it is not possible to split it into two complementary open sets $M, N$ such that $X = M \cup N$ (Riesz 1906b, p. 320).

For subsets, he defined 'connectedness' and 'absolutely connectedness'. His definition of a connected subset is identical to the definition he gave in his paper 'On discontinuous sets' (1905a). But while Riesz's connected sets may be not connected in modern terms, his concept of absolutely connected is sharper and coincides with today's notion of connectedness: the subset $P \subset X$ of the mathematical continuum $X$ is said to be absolutely connected if for every decomposition of $P$ into two subsets $M, N$ there exists at least one element that belongs at the same time to one of the subsets and to the derived set of the other, i.e. either $M \cap N' \neq \emptyset$ or $M' \cap N \neq \emptyset$ (Riesz 1906b, p. 321).[60]

However, due to the poor reception of the *Genesis* (see Sect. 6), the modern notion of connected set was not established by Riesz but by N.J. Lennes (1911) and Hausdorff (1914), who developed it a few years later independently of each other and of Riesz.[61] A simple example illustrates the difference between Riesz's notions of connected and absolutely connected set: with respect to the standard topology in $\mathbb{R}$, the set $(-\infty, 0) \cup (0, \infty)$ is connected but not absolutely connected, while the set $(-\infty, 0) \cup [0, \infty)$ is indeed both connected and absolutely connected.

Riesz pointed out that an absolutely connected subset $A$, considered in itself as a mathematical continuum, is a connected mathematical continuum. Furthermore, he stated that a mathematical continuum is connected if and only if for every pair of elements there exists an absolutely connected subset containing them. The proof of this last statement consists of two parts. Riesz observed that if the mathematical continuum is connected, then it can be regarded as an absolutely connected subset of itself (Riesz 1906b, p. 321). Thus, it follows that for every pair of elements, there exists an absolutely connected subset containing them. He omitted the second part of the proof, namely that if for every pair of elements there exists an absolutely connected subset containing them, then the mathematical continuum is connected. This can be easily proven.[62]

In his Rome lecture, Riesz discussed some deficiencies in Cantor's concept of connectedness. He presented the following two homeomorphic sets $M = R \backslash 0$ and $N = R \backslash [0, 1]$ from which $M$ is connected in Cantor's sense but $N$ is not (neither is absolutely connected in Riesz's sense). The fact that $M$ and $N$ are homeomorphic demonstrates, as Riesz claimed, that the concept of connectedness is independent of

[60] Riesz again missed assuming that the sets are non-empty (Taylor 1982, p. 269).

[61] See Herrlich et al. (2002a). The authors do not refer to Riesz's early contributions from 1905.

[62] Let $X = M \cup N$ be an arbitrary splitting of the mathematical continuum $X$ into the two disjoint sets $M$ and $N$ and let $x \in M$ and $y \in N$ be a pair of elements of $X$. By hypothesis, there exists an absolutely connected set $C \subset X$ such that $x, y \in C$. Since $C \cap M$ and $C \cap N$ are a splitting of $C$, then there exists $z \in (C \cap M) \cap (C \cap N)'$ (or $z \in (C \cap M)' \cap (C \cap N)$). The second axiom in the definition of mathematical continuum implies that $z \in M \cap N'$ (respectively, $z \in M' \cap N$), and therefore, M cannot be opened and $X$ cannot be splitted into two disjoint open sets.

the structure determined by the notion of accumulation point. Since connectedness is inherent to the notion of continuity, this means that continuity cannot be completely characterised by the notion of accumulation point (Riesz 1909, p. 21).

### 4.6 Ordering the space

Riesz resorted to his work on multiple order types in order to approach the issue of the dimension of space. He intended to introduce an $n$-fold order relation for the space.

He incorporated some of the most important concepts and results of his theory of multiple order types, pointing to the fact that the order relation provides a natural concept of neighbourhood. Using the characterisation of an accumulation point in terms of neighbourhoods he concluded that any multiple order type defines a mathematical continuum.

However, the ordering of the mathematical continuum $X$ induces a second condensation type on $X$ that is determined by the order relation. Therefore, Riesz defined: a mathematical continuum $X$ can be continuously ordered if its condensation type coincides with the condensation type of $X$ as a multiple order type (Riesz 1906b, p. 345).

### 4.7 Riesz's approach to Hilbert's continuity of space problem

Now that his theoretical framework had reached this stage Riesz felt ready for a first approach to Hilbert's continuity of space problem. He started, as mentioned at the beginning of Sect. 4, with a concept of space defined by a system of axioms. These axioms determine the relationships between what he called groups of sensations, physical points and physical continua. The space was first defined as a numerable ordered sequence of physical continua (Riesz 1906b, pp. 323–330). This space, which I refer to as the space $\mathcal{R}$, constitutes the domain on which Riesz then formulated a notion of continuous space for geometry.

According to Riesz, the notion of continuous space for geometry is to be founded on the concept of mathematical continuum. Therefore, the first condition imposed on $\mathcal{R}$ is that it can be regarded as a mathematical continuum. For this, he defined 'mathematical points' on $\mathcal{R}$ in terms of physical points. Furthermore, he introduced a rule determining when a mathematical point is an accumulation point of a set of mathematical points. Finally, he showed that $\mathcal{R}$, as the space of mathematical points, defines a mathematical continuum (Riesz 1906b, pp. 331–334).

Thus, $\mathcal{R}$ is provided with a double structure: it is defined as an infinite sequence of physical continua and as a mathematical continuum. Riesz then proved the following continuity properties of $\mathcal{R}$ which depend on this double structure:

1. $\mathcal{R}$ has a non-denumerable infinity of mathematical points.
2. $\mathcal{R}$ is a mathematical continuum.
3. The Bolzano–Weierstrass Theorem is valid on 'bounded' subsets of $\mathcal{R}$.
4. For every element of $\mathcal{R}$ as mathematical continuum there exists a countable, sufficient system of special neighbourhoods ($\mathcal{R}$ is first countable).
5. The Borel theorem is valid for 'closed' subsets of $\mathcal{R}$.

6. $\mathcal{R}$ is connected.
7. $\mathcal{R}$ can be continuously threefold ordered so that it is locally ‘similarly condensed’ to $\mathbb{R}^3$.

Riesz’s versions of the Bolzano–Weierstrass and the Borel theorems do not apply to subsets of any arbitrary abstract mathematical continuum but only to subsets of $\mathcal{R}$. The reason is that Riesz characterised both ‘bounded’ and ‘closed’ sets only in terms of special properties of $\mathcal{R}$. As already mentioned, Riesz did not give a general definition of ‘closed’ subsets of a mathematical continuum.[63]

For the proof of the first six properties, no extra assumptions were needed. But in the case of the property seven, which he needed to approach the issue of dimension, he additionally assumed that for every mathematical point $x$ of $\mathcal{R}$, there exists a neighbourhood $U$ such that $U$ can be continuously threefold ordered. The continuity of the ordering means that the condensation type induced on $U$ by the threefold order does not differ from the original condensation type of $U$. Riesz then claimed, without proper proof, that the space $\mathcal{R}$ is locally similarly condensed to $\mathbb{R}^3$, i.e. for every mathematical point $x$ of $\mathcal{R}$, there exists a neighbourhood $U$, such that $U$ is similarly condensed to $\mathbb{R}^3$.

Riesz general characterisation of continuous geometrical space reads as follows:

> In this way, one achieves a characterisation of space as a connected mathematical continuum such that every element has a neighbourhood that, considered as an independent continuum, has the same condensation type as the three-dimensional space of numbers. So, one arrives at Hilbert’s more general conception of continuous space.[64]

That means that continuous space, understood as the set upon which geometry can be established, is a connected mathematical continuum locally similarly condensed to $\mathbb{R}^3$. This notion of three-dimensional space resembles the modern notion of a three-dimensional topological manifold.

Riesz was well aware of the difference between the local and the global topology of space. He pointed to the fact that such a local characterisation leaves the question unanswered as to what the condensation type of the global space is. He therefore concluded that the space $\mathcal{R}$ was still not ready for establishing geometry upon it (Riesz 1906b, p. 350).

## 5 Continuous space as a *n*-dimensional manifold

Both Hilbert’s and Riesz’s conception of continuous space was that of an $n$-dimensional manifold (with $n = 2$ in Hilbert’s case, $n = 3$ in Riesz’s). However, they approached the challenge of defining this concept with different degrees

[63] I translate as ‘bounded’ what Riesz called ‘im Endlich gelegen’, i.e. ‘located in a finite region’ (Riesz 1906b, pp. 335–336). Riesz’s notion of ‘bounded’ is connected to the concept of compactness, see next section.

[64] Riesz (1906b, p. 359), my translation.

of generality. This was due not only to their different motivations, but also to the differences in their mathematical practices.

The continuity of space problem set by Hilbert involved the challenge not only of finding an accurate definition of a continuous space but also of formulating an adequate characterisation of the continuity of the group of motions, as specified in Hilbert's fifth Paris problem.

Riesz experienced a broader spectrum of motivations which included Hilbert's axiomatic foundation of geometry, recent contributions to the philosophical discussion of space, Cantor's theory of order types, as well as recent developments of analysis and point-set theory in France. Riesz was particularly interested in Fréchet's early works on abstract point-set topology, because Fréchet's topological approach to the study of functions whose domain are sets of functions generalised both the classic analysis as well as Cantor's point-set theory.

The methods Hilbert used to define the plane as a two-dimensional manifold consisted in mapping Jordan domains in $\mathbb{R}^2$ onto neighbourhoods and in working with the point-set topology of $\mathbb{R}^2$. Thus, his procedure was determined by those mathematical methods common to the classical analysis of Weierstrass, Cantor and Lie. These methods used basic concepts such as the limit of sequences, the continuous function on $\mathbb{R}^2$ (continuous function as a mapping that preserves the convergence of sequences) and the Cantor's point-set topology of $\mathbb{R}^2$.

One original aspect of Hilbert's approach that is important for the present discussion, is that he induced a weak topology on the plane using the standard topology of $\mathbb{R}^2$. It is precisely this technical solution that struck Riesz and eventually led him to develop his theory of mathematical continua.

Riesz regarded the space primarily as a mathematical continuum, i.e. he demanded that the space is a set provided with its own notion of an accumulation point and, consequently, with its own point-set topology. He additionally assumed that each element of the space possesses a neighbourhood whose continuity structure as a mathematical continuum happened to be similar to that of the three-dimensional space $\mathbb{R}^3$. The fundamental difference to Hilbert's definition is that the continuity structure did not need to be induced but could be defined from the start independently of $\mathbb{R}^3$. It is the difference between having a strong instead of a weak topology.

Riesz's general approach proves that the modern practices in the French analysis influenced him profoundly, above all Baire's exploration of other topologies for $\mathbb{R}$ and Fréchet's attempts to develop an abstract point-set topology. Nevertheless, it was definitely Hilbert's technical solution of inducing a topology on the plane that gave Riesz the key idea to provide the space with its own concept of accumulation point, as Riesz explicitly acknowledged in his Rome lecture (see quotation in Sect. 4.1).

Thus, the basic concepts that played an important role in Hilbert's and Riesz's characterisations of continuous space were cardinality, neighbourhood, accumulation point, condensation type, the Bolzano–Weierstrass theorem, the Borel theorem, connectedness, dimension, motion and continuous function. However, they may have been understood and applied differently by both authors. These differences will be discussed in the following paragraphs.

### 5.1 Cardinality

In Hilbert's work, the cardinality of the space was determined by the axioms defining the plane as a two-dimensional manifold. From the perspective of an axiomatic foundation of geometry, Riesz found this implication unsatisfactory because it means that the definition of the space is unnecessarily restrictive (see Sect. 3.4). According to him, a foundation of geometry based on Hilbert's notion of motion has the same implication on the cardinality of the space. All the same, Riesz proceeded just as Hilbert had done. The characterisation of the space $\mathcal{R}$ as locally similarly condensed to $\mathbb{R}^3$ implies as well that the cardinality of $\mathcal{R}$ is $2^{\aleph_0}$.

The issue of the cardinality of space remained under scrutiny in the *Genesis*, because Riesz was particularly interested in certain sets of functions. This is revealed in his article 'Sur une espèce de gèométrie analytique des systèmes de fonctions sommables' (1907c) in which he discussed the development of an analytical geometry on the set of functions $L^2$ (the square-integrable functions in Lebesgue's sense).

### 5.2 Neighbourhood

This was the basic concept in Hilbert's definition of the plane. It also played an important role in Riesz's characterisation of the space. There is, however, a fundamental difference between Hilbert's and Riesz's concepts of neighbourhood. Hilbert defined the system of neighbourhoods using the images, under the coordinate functions, of Jordan domains in $\mathbb{R}^2$. Instead, Riesz defined the system of neighbourhoods independently of $\mathbb{R}^3$ and of any system of subsets of $\mathbb{R}^3$. Riesz's neighbourhoods are determined only by the continuity structure of the space as a mathematical continuum.

### 5.3 Accumulation point

Hilbert regarded a point $x$ as an accumulation point of a subset $M$ of the plane if the images of $x$ and $M$ under the coordinate functions on $\mathbb{R}^2$ are related in the same way. In contrast, Riesz defined the concept of accumulation point for an abstract set axiomatically via his definition of mathematical continuum.

### 5.4 Condensation type

Riesz introduced this concept to classify mathematical continua according to their continuity structures (condensation types). Hilbert did not have such a concept. Hilbert regarded $\mathbb{R}^2$ as the prototype of the continuous two-dimensional space and, accordingly, as the only imaginable continuity type. However, using Riesz's theoretical framework to describe what Hilbert did, it is clear that Hilbert's plane has either locally or globally (assuming the existence of a global coordinate function) the condensation type of $\mathbb{R}^2$. Riesz noticed this. That is the reason why he generalised Hilbert's notion of two-dimensional manifold by defining three-dimensional continuous space as a mathematical continuum locally similarly condensed to $\mathbb{R}^3$.

## 5.5 Bolzano–Weierstrass property

A set is said to satisfy the Bolzano–Weierstrass property if it holds true that any infinite subset has at least one accumulation point. In 1894, when he was searching for an adequate formulation of the continuity axiom in geometry, Hilbert resorted to the property that every infinite, monotone increasing and bounded sequence has a least upper bound.[65] In the *Genesis*, Riesz characterised the sets that satisfy the Bolzano–Weierstrass property only for the special case of the space $\mathcal{R}$. In his Rome lecture, Riesz discussed the importance of the Bolzano–Weierstrass property within the framework of an abstract point-set topology.

## 5.6 Borel theorem

This theorem, still used to characterise compact sets, belongs to the point-set topology that, after Cantor, was chiefly developed by a group of young French mathematicians at the beginning of the twentieth century, Borel among them. Hilbert never learnt to take full advantage of this branch of modern mathematics—his work on integral equations gives proof of that. His mathematical practices remained close to algebraic structural methods and to the classical analysis of Weierstrass. Even though he certainly valued Cantor's work highly and never lost an opportunity to express his support for it, the fact remains that the *Annalennote* and the *memoir* were the only occasions in which Hilbert came to apply Cantor's point-set topological methods. Therefore, it is not surprising that Hilbert did not grasp the analytical significance of the Borel theorem.

Riesz managed to characterise those subsets of the space $\mathcal{R}$ for which the Borel theorem is valid, i.e. for which each cover consisting of neighbourhoods has a finite subcover. Furthermore, he suggested in his Rome lecture another three characterisations of compactness which he discussed in connection with the Bolzano–Weierstrass property.[66] Within the framework of a general theory of mathematical continua, he characterised $E$ as compact if for every sequence $\{E_n\}$ of nonempty, infinite subsets of $E$ such that $E_{n+1}$ is a subset of $E_n$ for each $n$, there is at least one element that belongs to all the subsets $E_n$.[67]

## 5.7 Connectedness

Hilbert did not discuss the concept of connectedness explicitly. Nevertheless, from Riesz's point of view, it is clear that the last axiom in Hilbert's definition of the plane

[65] For a detailed account of Hilbert's lecture on the foundations of geometry in 1894, see Toepell (1986, p. 74) and Hallett and Majer (2004, p. 68).

[66] One of them is acknowledged by Willard (1970, p. 318) as the first attempt to describe compactness using the finite intersection property. Instead of 'compact' Riesz spoke of 'bounded set'.

[67] Fréchet also used nested sequences of subsets in 1904. Fréchet did assume that the subsets $E_n$ are closed. On Fréchet's the notion of 'compactness', see Taylor (1982), and on its further development, see Pier (1980).

ensures that the plane is a connected mathematical continuum.[68] Riesz, on the other hand, had worked on sharpening the concept of connectedness for years. He considered it of great importance for the characterisation of continuous space. At the same time, he was well aware that this property is independent of the notion of 'condensation type'.

## 5.8 Dimension

In Hilbert's work, the two-dimensionality of the plane was determined by the axioms defining the coordinate functions as bijective mappings onto $\mathbb{R}^2$ with continuous coordinate changes. In Riesz's treatment, the three-dimensionality of space was determined in a similar way, except that here the corresponding coordinate functions were supposed to be continuous (Riesz does not need coordinate changes). It is interesting how Riesz approached the issue of dimension in the *Genesis*. He knew from his own earlier work how difficult it was to determine the dimension of an arbitrary set, even for subsets of Euclidean spaces. Therefore, he gave up such a general approach and, resorting instead to his own theory of multiple order types, he introduced the hypothesis that the space $\mathcal{R}$ can be continuously threefold ordered.[69]

## 5.9 Motion

Hilbert defined this concept as a bijective transformation of the space to itself whose corresponding mapping from $\mathbb{R}^2$ onto $\mathbb{R}^2$ was continuous, preserved the orientation of any closed Jordan curve and satisfied three further axioms. Riesz regarded a motion as a bijective continuous transformation of space in itself, i.e. a transformation that preserves the condensation type of the corresponding subsets.[70] He discussed the notion of motion only superficially because, as he said (1906b, p. 350), no geometry could yet be established on his space $\mathcal{R}$.

## 5.10 Continuous function

Hilbert used extensively the concept of continuous function from $\mathbb{R}^n$ to $\mathbb{R}^n$, $n = 1, 2$. Continuity played a fundamental role in his definition of the plane as a property of the coordinate functions. He implicitly extended the concept of a continuous function to mappings from the plane onto the plane via the coordinate functions; for instance, his concept of motion is defined as a continuous mapping from the plane onto the plane in

68 Hilbert's neighbourhoods are absolutely connected in Riesz's sense, and for every two points of the plane, there exists a neighbourhood containing them, i.e. the plane fulfils Riesz's characterisation of connectedness in terms of absolute connectedness; see Sect. 4.5.

69 See Sect. 3.2. Referring to his own article (1905a), Riesz said, 'Die größte Schwierigkeit liegt darin, daß der Dimensionsbegriff für mathematische Kontinua schwer zu fassen ist' (Riesz 1906b, p. 313).

70 'Die Bewegungen sind eindeutig umkehrbare stetige Transformationen des Raumes, d.h. solche eindeutig umkehrbare Abbildungen des Raumes auf sich selbst, für welche die einander entsprechenden Punktmengen ähnlich verdichtet sind' (Riesz 1906b, p. 350).

the sense that the corresponding mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ is continuous. Riesz did not explicitly define the concept of continuous function between different mathematical continua. However, such a concept is suggested through his definition of a 'continuously ordered mathematical continuum' and his concept of motion. A function is continuous if it preserves the condensation type of the corresponding subsets.[71]

## 6 Reception

Riesz's participation in the International Congress of Mathematicians at Rome in 1908 was crucial for the reception of his concept of mathematical continuum. If he had not attended it, then there would have been no reception at all. Not only because the *Genesis* went largely unnoticed but also because his Rome lecture was the only occasion in which Riesz summarised his theory of mathematical continua. Nevertheless, his lecture was successful for it eventually reached and inspired several future founders of general topology.

His contribution was entitled 'Continuity and abstract point-set theory' (1909). Continuity means the notion enclosed within the concept of space as a continuous manifold. Riesz made the connection with abstract point-set theory when, reading Hilbert's *memoir*, he understood that the issue of characterising continuity depends mainly on the concept of an accumulation point. This realisation, combined with the ideas he had learnt from Poincaré's philosophy about how basic geometric ideas develop in our intuition, led him eventually to conceive the space as a mathematical continuum. He did not enter this discussion but referred to Poincaré's 'La Valeur de la Science' (1905) and to his own 'Die Genesis des Raumbegriffs'.

He then presented his concept of a 'mathematical continuum'. However, the definition included only the first three axioms used in the *Genesis*. The fourth axiom is replaced by the assumption that each accumulation point $x$ of a given set $M$ is uniquely determined by the totality of those subsets of $M$ of which $x$ is an accumulation point. Addressing the issue of how a theory of mathematical continua should be further developed, he made some suggestions, mentioning the work of Fréchet as a starting point, and presenting two theorems of his own related to the Bolzano–Weierstrass property. But not every continuity property can be reduced to the concept of an accumulation point. That is the case of 'connectedness' as he illustrated with examples. Therefore, he proposed the new concept of 'chaining' which defines a relation between two sets, namely if they are near or far from each other (i.e. chained or not).

The rest of the lecture deals with the relation between the spatial structures of 'chaining type' and 'condensation type'. The first is determined by the sets being chained with each other, and the second by the accumulation point relation between the elements and the subsets of the space.[72]

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[71] See Footnote 70 and Sect. 4.6. Since sequential convergence describes the topology of any first countable space and since the space $\mathcal{R}$ is first countable, then $f : \mathcal{R} \to \mathcal{R}$ is continuous if and only if whenever $x_n \to x$ in $\mathcal{R}$, then $f(x_n) \to f(x)$ in $\mathcal{R}$.

[72] In modern terminology, 'chaining type' is called a proximity structure, see Thron (1997). According to Bognár and Császár (2006), the proximity spaces of modern topology (introduced by V.A. Efremovis in

**Table 1** List of authors that referred to Riesz' topological ideas. The references appear in works published between 1907 and 1933. The authors are ordered according to their country and the year their paper or book was published

| Year | USA | France | Austria | Poland | Germany | Switzerland |
|---|---|---|---|---|---|---|
| 1908 | | | | | Schoenflies | Fehr |
| 1910 | EH Moore | Fréchet | | | | |
| 1911 | Root | | | | | |
| 1914 | Root (2); Pitcher | | | | | |
| 1915 | RL Moore; Chittenden | | | | | |
| 1917 | | Fréchet | | | | |
| 1918 | | Fréchet | | | | |
| 1919 | Chittenden | | | | | |
| 1921 | | Fréchet | Vietoris | | | |
| 1922 | | | | Kuratowski | | |
| 1923 | | | Tietze | | | |
| 1926 | Hildebrandt | | | | | |
| 1927 | | | | Sierpinski | | |
| 1928 | | Bouligand, Fréchet (2) | | | | |
| 1929 | Putnam, Chittenden | | | | Baer | |
| 1930 | Frink | | | | | |
| 1932 | Stephans | | | | | |
| 1933 | | | | Kuratowski | | |

In my opinion, the Rome lecture cannot be fairly regarded as a short description of the ideas contained in the *Genesis*.[73] In Rome, Riesz focused only on the concept of mathematical continuum, leaving aside all the other important concepts he had developed in *Genesis*, e.g. connectedness and neighbourhood, not to mention their application to the space problem. Nevertheless, for the reception of Riesz's concept of mathematical continuum, the print version of the Rome lecture is undoubtedly the more important one of the two papers.

The reception of Riesz's topological ideas can be traced as follows: a systematic research on publications appearing between 1907 and 1933 resulted in 24 articles in 14 mathematical journals and four books containing references to Riesz. The search covered a total of 26 journals of that period. The complete list of journals searched is shown in Table 2. The references were made by 18 different authors from six countries (see Table 1). Their works were published in the USA, France, Italy, Austria, Poland, Germany and Switzerland. Among the authors, there are some of the most prominent

Footnote 72 continued
1951 independently of Riesz) are in some respect similar to the chaining types of Riesz. This similarity was responsible for the late reception of Riesz's topological ideas, e.g. Thron (1973).

[73] That is how the Rome lecture is commonly referred to, e.g. by Taylor (1982) and Bognár and Császár (2006).

**Table 2** Complete list of journals searched for references to Riesz' *Genesis* and his Rome lecture. The columns contain the country in which the journal was published, the volumes and years searched and the number of references founded in the corresponding period

| Journal | Country | Searched | Ref. |
|---|---|---|---|
| Abhand. Math. Sem. Hamburg. Universität | Germany | 1.1922–9.1933 | 0 |
| Acta Mathematica | Sweden | 30.1906–62.1933 | 0 |
| Acta Scientiarium Mathematicarum | Hungary | 1.1922–6.1934 | 0 |
| American Journal of Mathematics | USA | 29.1907–55.1933 | 4 |
| Ann. Mat. Pura appl. ser. iii and iv | Italy | 15.1907–31.1922; 1.1924–11.1933 | 0 |
| Ann. scientifiques de l'École normale supérieure | France | 37.1920–47.1930 | 1 |
| Annali della Scuola Normale Superiore di Pisa - Classe di Scienze | Italy | 9.1907–16.1930; 1.1932–2.1933 | 0 |
| Annals of Math. | USA | 8.1907–34.1933 | 1 |
| Bull. Am. Math. Soc. | USA | 10.1904–39.1933 | 6 |
| Bull. de la Soc math de France | France | 35.1907–61.1933 | 1 |
| Bull. des sciences mathématiques | France | 32.1908–57.1933 | 1 |
| Comptes Rendu Ac. Sci. Paris | France | 144.1907–151.1910 | 1 |
| Fundamenta Mathematicae | Poland | 1.1920–21.1933 | 1 |
| Göttinger Nachrichten | Germany | 1907–1933 | 0 |
| Journal für reine und angewandte Math. | Germany | 132.1907–169.1933 | 0 |
| Jahresbericht der DMV | Germany | 17.1908–29.1920 | 0 |
| L'Enseignement mathématique | Switzerland | 9.1907–32.1933 | 1 |
| Math. Ann. | Germany | 67.1909–108.1933 | 2 |
| Monatshefte für Math. und Phys. | Austria | 18.1907–40.1933 | 1 |
| Nouvelles Annales de Mathematiques | France | 7.1907–20.1920; 1.1922–3.1924; 1.1925–2.1927 | 0 |
| Proceedings of the London Math. Society | UK | 4.1907–35.1933 | 0 |
| Proceedings of the National Acad. of Sci. | USA | 1.1915–19.1933 | 0 |
| Rend. Accad. Lincei ser. 5 | Italy | 16.1907–33.1924; 1.1925–18.1933 | 0 |
| Rend. Circ. Mat. Palermo | Italy | 23.1907–57.1933 | 1 |
| Sitzungsberichte der Heidelberger Akademie der Wissenschaften. Math.-naturwis. Klasse | Germany | 15.1929 | 1 |
| Transactions of the AMS | USA | 9.1908–35.1933 | 3 |

founders of general topology: Kuratowski, Sierpinski, Tietze, Vietoris, Fréchet, Robert Lee Moore and Chittenden. All references, except the one by Schoenflies in 1908, concern the Rome lecture.

The two references made in 1908 are not relevant. Schoenflies (1908) is the only one in the list who actually referred to the *Genesis* in his report on the development of point-set theory. That could have been a good start for Riesz but unfortunately Schoenflies was apparently interested only in Riesz's theory of multiple order types and missed all the other topological work contained in the *Genesis* completely. The second reference was made by Fehr (1908) who wrote a report on the International Congress of Mathematicians for the journal *L'Enseignement Mathematique*. The report includes a brief summary of Riesz's lecture.

Table 1 shows an accurate picture of how the reception of Riesz's topological ideas proceeded chronologically and geographically. The relevant reception started in France in 1910 and on the other side of the Atlantic, in the USA. Very slowly Riesz's ideas move on towards Austria and Poland to arrive finally in Germany. The table also shows clearly that Riesz's ideas achieved their widest reception in the USA.

For the reception in the USA, E.H. Moore played a key role. E.H. Moore actually attended the International Congress of Mathematicians in Rome and was most probably present at Riesz's talk. At that time, E.H. Moore was working in what he called 'general analysis'. He had therefore an open interest in the development of abstract point-set theory.[74] E.H. Moore knew by then Fréchet's pioneer work on abstract point-set topology and must have been very interested in what Riesz had to say. In any case, two years after the congress, E.H. Moore's book on general analysis (1910) appeared. The book contains references to Riesz's and Fréchet's topological work as examples of different principles that can be applied in order to develop an abstract point-set theory. In 1910, Fréchet referred to Riesz in that same way.

Furthermore E.H. Moore propagated Riesz's approach in Chicago among five of his students successfully: Ralph E. Root, A. Pitcher, Robert Lee Moore, Edward Wilson Chittenden and Theophil H. Hildebrandt.[75] Apart from R.L. Moore, they were all followers of E.H. Moore's general analysis.

In spite of dramatic events, the reception of Riesz's Rome lecture went on as follows. First, came Root's papers on iterated limits (1911, 1914a, 1914b) and a paper by Pitcher (1914) on general analysis. Then, came the First World War with the effect that even the publication of Felix Hausdorff's book 'Grundzüge der Mengenlehre' in 1914 passed almost unnoticed.[76] During the war, two papers appeared by Chittenden (1915), Chittenden (1919), one by Moore (1915) and two by Fréchet (1917, 1918). Until 1928, Fréchet remained in France quite isolated as receptor of Riesz's ideas.

74 On Moore's general analysis, see Siegmund-Schultze (1998).

75 On E.H. Moore's school and the beginning of Topology in the USA, see Jones (1997). The author discusses the reception of Fréchet's work within E.H. Moore's school, but Riesz is not mentioned at all. The reception of Riesz's ideas by Root and R.L. Moore is shortly mentioned by Purkert et al. (2002, p. 707). A detailed discussion of the reception of Riesz's topological ideas by Fréchet gives Taylor (1985).

76 Purkert (2002) gives a detailed account of the reception of Hausdorff's 'Grundzüge der Mengenlehre'. This reception suffered the effects of the war too.

After the war, Europe was ready to catch up. Vietoris (1921) and Fréchet (1921) published papers on investigations they had started to pursue during their days at the front. After a second reference in Austria by Tietze (1923), Riesz's ideas managed to infiltrate Poland and achieved in the hands of Kuratowski (1922) and Sierpinski (1927), their most important contribution to general topology.[77]

Before 1919, the authors who referred to Riesz seemed not to have noticed Hausdorff's approach for he is not mentioned by any of them.[78] This changed radically after the war. Almost all of the papers listed in Table 1 deal with principles defining a continuity structure in an abstract set, i.e. principles on which to develop an abstract point-set topology. After the war, once Hausdorff's *Grundzüge* began to be acknowledged, his neighbourhoods axioms provided a way to topologize a set that was different to those of Riesz and Fréchet. Which approach is better? Or are they equivalent? That is in general the issue of most post-war papers: to find out which concept (accumulation point, distance, neighbourhood, closure) is the most convenient one as the basic concept for the foundation of abstract point-set topology. For instance, Tietze stated:

> A 'general topology' is arising as a theory concerning the properties that, under topological mappings, remain invariant between so-called topological spaces. These investigations are related to the study of the different systems of axioms defining the mentioned relations of limes, neighbourhoods or distance, their own relevance and the connection between them. It is in this respect that the present paper makes some contributions.[79]

Practically all post-war, authors discussed both Riesz's and Hausdorff's approach, evaluating their axioms and choosing those that seemed best adapted for their purposes. And although Hausdorff's axioms were eventually accepted as a satisfactory basis, this happened only after several efforts to improve Riesz's system of axioms (by modifying them or adding new axioms) so that it became a system equivalent or comparable to Hausdorff's.

Most importantly, this procedure can be found in precisely those papers by Vietoris, Tietze, Fréchet, Kuratowski and Sierpinski that were so fundamental for the establishment of general topology. Vietoris (1921), for instance, investigated the relation between two systems of axioms defining the notions of neighbourhood and of accumulation point. The interesting point is that his second system consisted of the three axioms of Riesz and a fourth axiom proposed by his teacher W. Gross.[80]

Another example is Sierpinski who said:

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[77] The papers mentioned in Table 1 from 1926 on are Hildebrandt (1926), Bouligand (1928), Fréchet (1928b), Fréchet (1928a), Chittenden (1929), Baer (1929), Frink (1930), Stephens (1932) and Kuratowski (1933). Putnam's reference appeared in Dresden (1929), a report of a meeting of the American Mathematical Society.

[78] Hausdorff is first mentioned along with Riesz by Chittenden (1919). Hausdorff was acquainted only with Fréchet's work. In his *Grundzüge*, he therefore does not refer either to Riesz or to any member of the Moore's school (see Purkert et al. 2002, pp. 699–713).

[79] Tietze (1923), my translation.

[80] Reitberger (1997) provides a brief discussion of Vietoris' contribution to the foundation of general topology.

> Since Fréchet's thesis we have seen various attempts to base topology (analysis situs) on one or another primitive concept, e.g. on that of limit (Fréchet), accumulation point (F. Riesz), distance or neighbourhood (Hausdorff, Fréchet), and closure (Kuratowski). The purpose of this article is to show how topology could be developed by taking as primitive the concept of derived set.[81]

This means that the accumulation point axioms of Riesz were at some point competing with the neighbourhoods axioms of Hausdorff, the closure axioms of Kuratowski, and the derived set axioms of Sierpinski. In fact, Riesz's axioms played a key role in the formulation of the axiom systems of Kuratowski and Sierpienski. This was Riesz's main contribution to the foundation of general topology.

As for the reception of Riesz's *Genesis*, apparently the only section that has been studied until now, even by historians, is the one in which the notion of mathematical continuum is introduced.[82] The question posed by Riesz about the role of experience in the emergence of the concept of space, and no mention his mathematical treatment of the issue, made the text difficult to read. Riesz wrote the *Genesis* for mathematicians who, like Poincaré, were interested in both the foundations of geometry and the philosophical discussion about space. Such mathematicians were rare. Besides, the fact that Riesz's paper did not appear in an important German or French journal reduced the chances of its reception significantly. Sadly, the efforts made by the journal *Mathematische und naturwissenschaftliche Berichte aus Ungarn* to promote the work of its authors by publishing in German were not enough to prevent this.

All the time historians of general topology have repeatedly overlooked Riesz's genuine interest in shaping the concept of continuous space for geometry. Therefore, they missed the chance of discovering that Riesz's topological work has indeed its roots in geometry, especially in Hilbert's foundations of geometry.

## Appendix A Letters to David Hilbert

### A.1 Letter by F. Riesz to David Hilbert from 18th November 1904[83]

Löcse, den 18. November 1904

Hochgeehrter Herr Professor!

[81] Sierpinski (1927) according to the translation by Moore (2008, 233). Moore (2008, 221) concludes, just I did in my thesis (Rodríguez 2006, 240), that 'Hausdorff's neighbourhood spaces did not immediately succeed in displacing their competitors', Riesz's mathematical continuum among them, but that 'there was a period of evolution during which it was not clear what concept should be taken as primitive and what axioms should be assumed for such a space.' As late as (1982), Appert, a direct pupil of Fréchet's, occupied himself with this same question.

[82] An interesting exception is Willard (1970, p. 312), who refers to the *Genesis* acknowledging Riesz's contribution to the notion of connectedness.

[83] Riesz (1904a)

Sie entschuldigen, wenn ich Sie störe. Es handelt sich um meine letzten Untersuchungen, über die ich vor ausführlicher Publikation gerne eine kürzere Note abgeben möchte und ich wende mich Ihnen mit der Bitte, dieselbe der Gesellschaft der Wissenschaften in Göttingen gütigst vorzulegen. Die Untersuchungen knüpfen an die Cantor'sche Idee an, dass die sogenannten geometrischen Eigenschaften der Punktmengen sich mit dem Begriffe der Fundamentalreihe auf Ordnungstypen übertragen lassen. Dies bezügliche Untersuchungen sind erst für lineare Ordnungstypen durchgeführt, ich unternahm es, dieselben für mehrfach geordnete transfinite Mengen weiterzuführen. Die Schwierigkeiten, die zu überwinden waren, bestanden darin, dass der Massbegriff und damit die gleichmässige Stetigkeit zu entbehren waren. Als Ersatz dafür diente die Übertragung des sogen. Heine–Borelschen Satzes – dass nämlich aus jeder unendlichen Intervallmenge, die alle Punkte eines Intervalles im Inneren enthält, eine endliche Anzahl von Intervallen ähnlicher Eigenschaft sich auswählen lässt – auf lineare perfekte zusammenhängende Ordnungstypen. Damit liess sich auch der Bolzano–Weierstrass'schen Satz übertragen. Nach allgemeinen Definitionen gehe ich auf zweifache Ordnungstypen von perfektem, zusammenhängendem Charakter ein, speziell auf solche, die nach Ausscheidung von Randelementen in einen überall dichten zusammenhängenden Ordnungstypus –Verallgemeinerung des Bereiches der Geometrie – übergehen, und beschäftige mich mit den Zusammenhangs-, resp. Teilungseigenschaften derselben, d.h. mit der Analysis Situs der Ordnungstypen. Von den gebräuchlichen Instrumenten der Mengenlehre muss ich hier teilweise die gerade Strecke entbehren, und kann nur spezielle Strecken und Poligone definieren, die mir aber genügen, um die Untersuchungen über Jordan-Kurven zu übertragen, die Teilung des Bereiches durch dieselben nachzuweisen, wie auch die beiden Sinne für die Jordan-Kurven zu definieren. Auf weitere Teilungsfragen möchte ich in der Note nicht eingehen.

Nun treten aber auch meine Untersuchungen in innigste Fühlung mit jenen über die Grundlagen der Geometrie. Jene Untersuchungen nämlich, die als primäre Eigenschaft des Raumes die Stetigkeit voraussetzen, bauen den Begriff der n-fach ausgedehnten Mannigfaltigkeit auf den Zahlenbegriff und stellen somit der Mächtigkeit des Raumes Schranken, die gewissermassen für gekünstelt erscheinen. Meines Erachtens nach sollten die einzigen wesentlichen Merkmale der $\mathbb{R}^n$ die n-fache Anordnung, das Überalldichte und die Stetigkeit (Perfektheit u. Zusammenhang), wie diese in Ihrem Ebenenbegriff, jedoch mit der obigen Beschränkung, klar hervortreten, bilden. Demgemäss wäre der $\mathbb{R}^n$ als Bereich von n-fachem Ordnungstypus aufzufassen. Nun halte ich Ihre Annalennote aus diesem Gesichtspunkte durchdacht und wenn ich mich nicht geirrt habe – was möglich ist, da ich die Sache nicht näher ausgearbeitet habe, so genügt schon meine Auffassung des $\mathbb{R}^n$, resp. der Ebene, um auf Grund Ihrer Bewegungsaxiome zur Euklidischen oder zur Bolyai–Lobatschewskyschen Geometrie zu gelangen. Also liesse keine Mannigfaltigkeit, welche von höherer Mächtigkeit als das Continuumm ist, eine Beweglichkeit im Lie'schen, resp. Ihrem Sinne zu.

Wenn Ihnen eine Note über diese Untersuchungen willkommen ist, so bitte ich Sie, mir mittels einer Postkarte die Erlaubnis zur Einsendung derselben zu ertheilen.

Zugleich erlaube ich mir, Ihnen einen Abzug meiner Annalennote mit innigstem Dank für die Publikation einzusenden. Herr Schoenflies hatte mich brieflich auf einen

Irrtum in Anmmerkung S. 410 aufmerksam gemacht, das aber auf die Sache nicht von Belang ist.

Hochachtungsvoll Ihr ganz ergebener
Riesz
Adresse: F. Riesz, Löcse /:Ungarn:/

A.2 Letter by F. Riesz to David Hilbert, undated [probably December 1904, LR][84]

Hochgeehrter Herr Professor!

Die eingehende Ausführung meiner Untersuchungen über mehrfache Ordnungstypen, namentlich jener, die die Verallgemeinerung der Analysis Situs des Bereiches der Funktionentheorie zum Ziele haben, zeigte mir, dass eine Verwendung der Fundamentalreihen oder wenigstens von Teilmengen mit einer einzigen Häufungsstelle eher zum Ziel führt, als der eingeschlagene Weg. Demgemäss wären auch die einleitenden Untersuchungen in einigen Punkten abzuändern. Ich bitte Sie deshalb, von der eingesandten Note vorläufig nicht Gebrauch zu machen. Ich werde sie demnächst ersetzen.

Ich benütze zugleich die Gelegenheit, Ihnen ein glückliches Neujahr zu wünschen.
Hochachtungsvoll Ihr ergebener
Riesz

## Appendix B A letter to Maurice Fréchet

B.1 Letter by F. Riesz to M. Fréchet from 21 May 1907[85]

Paris 21 mai 1907

Monsieur et cher Collègue!

En quelques jours je quitte la France pour aller à l'Allemagne. Je serais été ravi de faire votre connaissance; mais, étant en mission, ma route est déterminée, et je ne peux pas passer à Besançon. Ce que je regrette bien.

Je vous remercie de votre aimable lettre. En ce qué concerne les applications de mes résultats à la théorie des opérations fonctionelles, non seulement des opérations ordonnant à chaque fonction un nombre, mais aussi aux transformations linéaires (ou alors distributives) de l'espace des fonctions, moi je pensais bien que vot. résultats et ceux de M. Hadamard peuvent étre completés et généralisés; en mème temps, M. Hadamard m'avaits communiqué le mème espoir. J'aimerais bien de pouvoir lire prochainement une publication sur ce sujets. Naturellement, pour pouvoir appliquer la méthode, il faudra augmenter le domaine des fonctions et en meme temps, élargir

84 Riesz (1904b)

85 Riesz (1907b)

le sens de la notion de „opération continue“. Alors, les problèmes reviendront à la résolution des systèmes d’équations linéaires d’une infinité d’inconnues.

Je veux vous communiquer un théorème qui suit inmédiatement de mes recherches mais que je n’avais pas encore communiqué. Pour la classe des fonctions sommables, de carré sommable et pour la notion de fonction limite que j’ai introduite dans ma publication autérieure sur les ensembles de fonctions (à l’aide de la notion de distance), pour qu’une série de fonctions $f_n$ converge (au sens donné) vers une fonction limite il faut et il suffits que la distance de deux fonctions de rang assez elevé dévienne tel petite que l’on veux. Donc l’ensemble consideré rentrera dans votre classification de „classe normale“.

Je vous remercie encore de votre observation grammatique. N’etant pas français, votre belle langue me fait bien de difficultés.

Votre tout dévoué

Riesz

11.6. L’année dernière, Vous aviez la bonté de m’envoyer votre thèse. Je vous en remercie. Mais possédat quelque chose, on souhaite toujours à augmenter la fortune. Le qui vent dire que je serais heureux d’ètre en possession de vos autres travaux. Pendant mon voyage je reçois mes lettres par l’adresse de mon frère: M. Marcel Riesz, Göttingen, Gorslerstr. 6.

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