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A new analytical framework for the understanding of Diophantus's *Arithmetica* I–III

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Abstract This study is the foundation of a new interpretation of the introduction and the three first books of Diophantus's *Arithmetica*, one that opens the way to a historically correct contextualization of the work. Its purpose, as indicated in the title, is to renew the traditional discussion on the methods of problem-solving used by Diophantus, through the detailed exposition of a new analytical framework that aims to give an account of the coherence and progressive nature of the material included in the three first books of the *Arithmetica*. One outcome of this new 'toolbox' is a new conspectus of the problems and solutions contained in the latter, which is presented in appendix. The first part of the article clarifies, as a necessary preliminary, the key notions and terminology underlying our analysis. Among these new concepts is the notion of "method of invention," which accounts in general for any way, by which "positions" (*hypostaseis*) are used in the *Arithmetica*. The next part proposes a complete inventory of the various methods of invention found in the three first books. Finally the last part presents the above mentioned conspectus and proposes a series of preliminary conclusions that can be drawn from it.

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In a nutshell, the issue between the traditionalists, who barge into foreign mathematical worlds through the mathematical door, and the new historians of mathematics, who insinuate themselves into ancient mathematical cultures through what one can call the historical door, is the position of the former that form and content are independent variables in the mathematical domain that can be separated arbitrarily without thereby damaging the identity and wholeness of ancient texts, while the latter question this arbitrary separation, pointing out the errors and distortions to which it necessarily leads and exposing the blatant anachronism that is its inseparable companion.

Sabetai Unguru[1].

1 Introduction

This study is the foundation of a new interpretation of the introduction and the three first books of Diophantus's *Arithmetica*, one that opens the way to a historically correct contextualization of the work. Ever since the Middle Ages and the Renaissance, determining the exact character of this treatise, its purpose, the goal of its author, and the nature of the mathematical practice involved in it, has been the subject of much speculation by both ancient and modern mathematicians, and historians of mathematics.

If we restrict ourselves to the modern period, in which the history of mathematics became a discipline, we shall see that the above discussion continues to focus on a few central questions that are still unsettled. It is not our objective here to survey these questions but to focus on one of them, which is indicated in the famous judgment passed by Hermann Hankel on Diophantus's work. Having discussed the impossibility of recognizing any "principle of grouping" of the 50 classes of problems he recognized in Diophantus, beyond the fact that simplest problems come first, he continues by discussing Diophantus's solutions: "Almost more different in kind than the problems are the solutions, and we are completely unable to give an even tolerably exhaustive review of the different terms which his procedure takes. Of more general comprehensive methods, there is in our author no trace discoverable: every question requires a quite special method, which often will not serve even for the most closely allied problems. It is on that account difficult for a modern mathematician, even after studying 100 Diophantine solutions to solve the 101st problem. . ."; after a few emphatic sentences expressed in a brilliant language, he concludes: "in this way the reader also hurries with inward unrest from problem to problem, as in a game of riddles, without being able to enjoy the individual one. Diophantus dazzles more than he delights." As he himself finally confesses, this judgment is a reflection of his own experience on reading Diophantus: "that is the general impression which I have derived from a thorough and repeated study of Diophantus's arithmetic."[2]

[1] From Unguru's introduction to Chap. 6, "Methodological Issues in the Historiography of Greek Mathematics" in (Christianidis 2004).

[2] The quotation is from Hankel's 1874 brilliant book *Zur Geschichte der Mathematik in Alterthum und Mittelalter*, and the translation—which is worth reading in full—is taken from Heath 1910 (repr. 1964), p. 54–55.

As Heath's commentary shows, this judgment, even by Hankel's time, was not shared by all scholars: Heath refers to Nesselman, who had a different view, and refers back to Euler's more favorable judgment.[3] Generally speaking, most of the subsequent historical studies on Diophantus have followed the implicit agenda that this debate laid down.

In particular, Heath, discussing Hankel's quotation, says that "it might be inferred from these remarks of Hankel that Diophantus's object was less to teach methods than to obtain a multitude of mere results" (Heath 1910/1964, 55). This comment epitomizes one basic dilemma on which the historians of the last century were called to take a position. In this respect, it is important to notice that while Heath talks about "teaching of (general) methods" as opposed to "obtaining mere results," he disregards completely the didactic dimension involved in the first option he himself makes, and he only examines the generality of Diophantus's solutions. The fact that the aforesaid phrase belongs to a long chapter of his book entitled "Diophantus' methods of solution" is quite characteristic.

This article again takes a position on this much-discussed issue of the methods of problem solving used by Diophantus. At the same time, however, we aim at renewing this discussion in a twofold sense. First, by concentrating on what we call the "methods of invention," that is, the methods through which Diophantus solves the main difficulty (or difficulties) involved in each problem, instead of considering the "methods of solution" as a whole. And second, by bringing to the fore a dimension of the Diophantine work which is often overlooked by traditional scholarship, namely the interpretation of Diophantus's work not only as a problem solving undertaking but, at the same time, as an enterprise aiming at a subtle purpose: to give the reader the means to solve problems.

The objective of this article, as indicated in the title, is to renew the earlier discussion by presenting a new analytical framework that aims to give an account of the coherence and progressive nature of the three first books of the *Arithmetica*. One outcome of this new 'toolbox' is a new conspectus of the problems and solutions contained in the latter, which is presented in the first appendix.

The choice of analyzing *only* these books is justified on two grounds. First, Diophantus's own introduction says that the search for progression implied that the easiest treatments should be put at the beginning as if there were *elements* for the rest[4]; this implies that we might expect the organization of the problems to be more easily identified at the beginning of the whole series of the problems than at the end. The second reason is that the position of these three first books survived in Greek as precisely the three first books among the thirteen announced by Diophantus, as is beyond question, whereas the subsequent books, either in Arabic or in Greek, are both more complex and, since the Arabic books plausibly come next, oblige us to take into account the bias introduced by the translation process.[5]

Beyond the interpretative problem raised by Hankel, the analysis of Diophantus's text also raises philological problem, pertaining to the question of possible inter-

[3] (Heath 1910/1964, 55–56).

[4] *Arithm*. 16.2–4.

[5] For the Arabic books attributed to Diophantus, see (Sesiano 1982; Rashed 1984).

polations or displacements in the course of the textual transmission. Our *working hypothesis*, in this respect, is to consider the introduction and three first books of the *Arithmetica* as a whole that reasonably reflects the beginning of a *coherent* work, which originally encompassed thirteen books. We take this part of the *Arithmetica in its entirety,* as edited by modern scholars (Tannery and Allard),[6] as being part of this coherent whole, voluntarily ignoring, for the moment, any decision to exclude parts of this work as 'interpolated' or 'inauthentic.' Like modern editors, we are convinced that there are such 'interpolated' passages and will indicate in due course which parts are interpolated according to our reading. But we will show that, on the basis of our toolbox, it will be possible to propose new and reasonably convincing criteria that might enable us to declare such or such a passage 'interpolated,' beyond the traditional (stylistic and philological) criteria.

This article has three parts. The first clarifies, as a necessary preliminary, the key notions and terminology underlying our analysis (part 2). Among these new concepts is the notion of "method of invention," which accounts in general for any way, by which "positions" (*hypostaseis*) are used in the *Arithmetica* (2.6). The next part proposes a complete inventory of the various methods of invention found in the three first books (part 3). Finally the last part presents the conspectus we mentioned above and proposes a series of preliminary conclusions that can be drawn from it (part 4).

2 The key notions and terminology underlying the analysis of *Arithmetica* I–III

2.1 Synoptic presentation of our 'analytical toolbox'

As was mentioned above, the purpose of this analysis is to account for the bulk of the problems treated in books I–III of the *Arithmetica*. It is not, therefore, to account only for the statements of the problems, but for the progressive nature of their treatments, taken as a whole. To investigate this set of statements-and-treatments, we need specific *analytical tools* that we list and explain in this part. The structure of the conspectus found in the first appendix and presented in part 4 is fundamentally based on these tools and therefore reflects, in a very condensed way, their basic use.

Most of these tools and notions are directly borrowed from Diophantus's own vocabulary, so that this list can be read as clarifying some key terms of Diophantus's vocabulary: they are indicated below by a parenthesis containing the corresponding Greek term. Other tools are present through linguistic patterns or specific formulas in his treatise, although they do not receive specific names in the original treatise. In such cases, we tried to invent relevant categories and names that would be as intuitively close to their object as possible.

For the sake of clarity, we distinguish between the following types of tools:

(a) The first characterizes precisely the correspondence between problem and solution and how the latter is related to the former as 'answer.' The related terminol-

[6] Our analysis is based on the two available editions of the text. The most recent is André Allard's (Allard 1980) and is not yet published. The second one is Tannery's (1893–1895). Since the second remains the most available edition, our usual references refer to Tannery's edition.

ogy includes the notion of *problem* (*problêma), statement of a problem (protasis), prescription (epitagma), to solve* a problem (*lyein*), and *solution* of a problem.

(b) The second tool is the system of qualifications, names and designations (*epônymiai*) used for *numbers*, since the treatment of problems relies on the clever use of these verbal and conceptual possibilities. The related terminology includes the generic notion of *number (arithmos*); its various qualifications with reference to the task stated in the problem (like *sought, given, found* numbers), to the algorithmic prescriptions within the statement (like *ho loipos, ho genomenos*) or to their *kinds* (like *squares* or *cubes*); its possible designations (*epônymiai*) when numbers are understood as *species (eidê*).

(c) The third constitutes, in a sense, our 'atomic' unit of analysis of the treatments, namely the notion of *position* (*hypostasis*), which is the correspondence between numbers of the statement and species formed from elements of the "arithmetical theory". With respect to position, we also introduce the term *expression* to designate the second term of the correspondence.

(d) The fourth are related to the various end-points of the chain of positions that constitute each treatment of a problem. We introduce specific terms for such end-points, namely *equations, double equations (diploisotês), dead ends*. The two first terms are related to the generic term *equality (isotês*) and *equal* (*isos*) used in Diophantus, but we shall see that the latter are used in various contexts and not only for the ending points of solutions.

(e) The last tool will enable us to characterize the use of repetition in the progression of Diophantus's problems: they are the notions of *method of invention* and *heuristic explanation*. The two expressions are not Diophantine, although they voluntarily allude to the terms *invention (heuresis*) and the verb *to invent (heurein*) as well as to the notion of *way (hodos*) all used by Diophantus.

The notion of "method of invention" is, in turn, the key basis of the following part, which proposes an inventory of such methods.

In general, one can see that most of these notions and tools directly correspond to Diophantus's own vocabulary or borrow key elements of it. In any case, they are meant to reflect *recognizable* aspects of the study. Furthermore, they are retrieved in the final conspectus, of which they are the basis.

For each of the above terms and the corresponding notions, we first propose significant examples explaining their meaning; then a generic characterization, including the specific Greek terms, expressions or formulas attached to it. Finally, we indicate how these notions are retrieved in the final conspectus.

2.2 The duality between problems and their treatments or solutions

2.2.1 The various ways by which the notion of "problem" might be understood in the Arithmetica

The notion of problem, taken in the most straightforward sense, refers to a textual unit in the manuscript tradition of the *Arithmetica*, comprehending the statement of an arithmetical task (asking for numbers verifying specific constraints) as well as the treatment or 'solution' of this task: this treatment leads to determined numbers checking the constraints. Such "problems" are classified by book and number in Tannery's or Allard's edition, following number already found in the manuscripts.[7]

Diophantus, though, does not refer to such textual units when referring to problems: even when the text happens to refer to a previously treated problem, this is never done by a precise reference to a given book or to any textual unit within it, but rather to a specific *task* or *challenge* that is, or not, fulfilled.[8] Even more important is the fact that a problem in the "textual" sense might encompass several problems in the 'functional' or 'challenging' sense of the term: this is the case, in particular, when Diophantus is led to state auxiliary problems in the course of a given solution.

A good example is problem III.10, which is among the first cases, in which such auxiliary problems are found. The problem (or task) stated at the beginning is (a) "*to find three numbers so that the <product> on any two of them, adding to itself a given number, makes a square" (Arithm*. 158.2–4). The given number is further specified as being 12 and, in the course of the treatment, Diophantus is led to make 52 *dynameis* and 12 units a square number. He then remarks that, if 13 were a square, then obtaining an equation that would bring to this end would be easy.[9] Thus, he is led to the following remark: "since it is not so, I am led to (b) *finding two numbers, so that their <product> be a square and, furthermore, so that each of them with 12 makes a square*" (*Arithm*. 158.22–24). He then remarks that if he finds squares instead of just numbers, the product of them will automatically be a square; so that "it <i.e., the aforesaid problem> becomes <a new problem, namely:> (c) *to find two squares, each of which with 12 units makes a square*" (*Arithm.* 158.26–27). Two new problems are formulated here, which are distinct from each other (the second is more specific that the first) and from the problem that led to them, which is recalled near the end of the treatment of III.10 by the expression "I come to the <problem> at the beginning." (*Arithm.* 160.4).

In general, then, the word *problem (problêma*) and the related expression *to propose a problem (proballesthai)*, when they are not specified by a proposition number, is usually used by Diophantus to designate a task or challenge containing an internal difficulty. Such a *problem* therefore calls for some activity or action to solve this difficulty and build a possible answer.[10] Stylistically, such problems are characterized by verbs like "*heurein*," "*diairein*," "*zêtein*," "*poiein*," which all refer to a task to be performed. The word problem, within the introduction of the *Arithmetica*, obviously

[7] When used in this sense, we usually append the book and proposition number, unless the context makes clear that this meaning is intended: thus "problem III.10" refers to the whole textual unit (*Arithm.* 158.2–160.11).

[8] On this point see (Netz 1998).

[9] That is, it would be easy to posit the square and thus create a solvable equation. Indeed, 52 was previously obtained as the product of 4 (which is a square) and 13 (which is not). If, then, 13 had been chosen a square (like 4), the product would have been a square: it would then have been easy to make a position for the still indeterminate square so as to obtain a 'good' equation. The process of obtaining the 'right' equation is called *isôsis* in this passage ('equalization').

[10] This meaning is akin to the original meaning, which these words have in ancient rhetoric, in which the problem is a challenge set to an orator by his audience, the action itself of "challenging" being called "*proballesthai*" (Bernard 2003; Knorr 1986). The obvious difference here is that the kind and the contents of the problems is not the same as those found in rhetoric, and furthermore the human context is not defined in Diophantus.

is used with this meaning, the second occurrence referring to the internal "weaving" of difficulties which is characteristic of arithmetical problems (*Arithm.* 4.10). When a problem calls for the "unraveling" of its intrinsic difficulty, the conclusion of the treatment is sometimes formulated as *poiousi to problêma*: "<the found numbers> make the problem."

An example of an explicit reference to the proposition of a problem, that is, the "setting" itself of the task, is found in an interesting remark within problem II.3. The statement is to find two numbers, such that their product has a given ratio to their sum or difference. One number is set 1 *arithmos*, the other 2 *arithmoi*, according to the following explanation: "indeed they <i.e., the two sought numbers> can be also proposed in given ratio" (*Arithm.* 84.16–17). This obviously alludes to the fact that one might have looked for two numbers *in given ratio*, with the same other prescriptions. This problem is stated in general and solved in the first book.

The notion of 'problem as a task' cannot be too sharply separated from the 'textual' meaning, as the example of "problem III.10" shows: although the later contains *four* different treatments, corresponding to *three* statements, clearly the first and fourth give to the whole its thematic unity, which is summarized by the recurring statement treated twice. Thus, this example shows that *it is important, for analytical purposes, to pair the statement of a problem and any of its specific treatments*. Such a notion of statement and 'elementary treatment' is somehow intermediate between 'problem as textual unit' and 'problem as task'; to designate this notion, we will also use here the term problem, eventually followed by a number identifying which problem is exactly designated (like problems III.10.1, 2a, 2b, 3 in this case).

The final conspectus, when read vertically, basically refers to this very last notion of problem: thus, each line corresponds to one 'elementary problem' *and* its corresponding treatment. Thus, problem III.10 (as a textual unit) actually corresponds to four different lines, the first being the first attempt to build a treatment, leading to a dead end; the second and third correspond to the above-mentioned auxiliary problems, the last line corresponds to the same statement than the first, but with a different treatment. Thus, problem III.10 contains four treatments and 'problems' (the third meaning of the term), but three different problems in the second meaning (problem as task). Since the conspectus, as well as our analysis, follows the third meaning, we consider that we have here four different problems. The thematic unity is classically indicated by the name of the four problems, which all begin with 'III.10."

These distinctions appear to be useful not only for the analysis of auxiliary problems, as in problem III.10, but also for the analysis of "variants" of a same problem, as in problem II.8, which can then be identified as containing two different problems II.8 and II.8 *alit.*, although the statement remains the same.[11]

That these distinctions are also useful for a detailed understanding of the structure of Diophantus's treatment might also be recognized from the ways by which the statement of a problem might be 'decomposed' into various prescriptions (*epitagmata*). Some of these prescriptions might be considered to constitute problems by themselves,

[11] This example will be studied in more detail below (2.4.2).

as we shall now see. Clarifying the notion of *epitagma* will also help us to elucidate the intimate connection between the notions of *problem* and *solution*.

2.2.2 Prescriptions (epitagmata) stated within the statement (protasis) and recalled within the treatment of a problem

The word *epitagma* is the substantive name corresponding to the verb *epitassein*, which is used in various means to designate anything that might be prescribed, that is, a *prescription*. This might be some relationship that the sought number(s) should satisfy, but it might as well designate the given number when the problem is instantiated.[12] In some cases, therefore, the *epitagma* just means "problem" or, more precisely, is used by Diophantus to emphasize the *prescriptive* aspect or aspects of a problem[13]: this is the case, in particular, in the conclusive formula "*poiousi to epitagma*," "<the found numbers> satisfy the prescription," that is synonymous of "*poiousi to problêma*."[14]

The prescriptive aspect of the statement usually consists in verbal statements *asking one to obtain* a number or numbers of a certain kind. These prescriptions are introduced, therefore, by verbs in the infinitive, like *dielein, poiein, heurein* etc., eventually completed by a formula like "*hopôs* ...," "*hina* ..." again followed by a verb. In other words, *epitagmata* cannot be dissociated from the verbal formulas by which the problem is effectively proposed, namely the *protasis* (that is, *statement* or *proposition*) or the problem. For this reason, the set of prescriptions constituting the problem is sometimes named "*ta tês protaseôs*."[15]

These "prescriptions" make a bridge between the statement and its treatment. For they are obviously contained, so to speak, within the dense formulas that constitute the statement. But these "general prescriptions" are also recalled *within the treatment of the problem*: they are then detailed into "elementary prescriptions," for which the name *epitagma* is again used. Such a use within the treatment of a given problem can be recognized through the analysis of typical expressions, in which "*epitagma"* is associated with "*lyein," to solve*, that is, when one of the prescriptions is said to be *solved*. For example, problem III.1 asks to find three numbers, such that if the square on any of them be removed from the sum of all three it remains a square. Then, having posited 5 *dynameis* for the sum and one *arithmos* for the first number, and 2 *arithmoi* for the other, Diophantus concludes that "two of the *epitagmata* are now solved" (*Arithm.* 138.11). For the sum, which was posited 5 *dynameis* wanting the

[12] We shall come back to this use in Sect. 2.3.1 and see that the distinction between *epitagma* as verbal prescription and *epitagma* as specification of a 'given' problem should not be exaggerated.

[13] This must be contrasted with the basic nuance contained in "*problêma*," in which the emphasis is laid on the *challenging* aspect *for someone*, as we have seen.

[14] The expression is used in the conclusion of five problems in book III (6, 7, 10, 12, and 15).

[15] Thus, in problem I.28 the expression "they make what is said in the proposition" is used as identical with the expression "they make the problem," the latter formula being the conclusion of a whole series of Diophantus problems. See "Index graecitatis apud Diophantum" (*Arithm.*, ii, p. 281), *s.v.* "*protasis*" (*kai poiousi* (vel *poiei*) *ta tês protaseôs*).

square on the first, namely 1 *dynamis*, or wanting the square on the second, namely 4 *dynameis*, makes a square in each case: two of the "elementary prescriptions" are thus fulfilled. Thus, the two notions of *epitagma* and *lyein* (solving) are intimately connected to each other. In book III, this connection is made clear by the many places in which prescriptions (*epitagmata*) are explicitly mentioned to be *solved*.[16]

By generalization, then, the notion of prescription (*epitagma*) might be logically applied to any case in which the 'general prescription' contained in the statement is explicitly recalled *within the treatment* and, at the same time, decomposed into its elementary components. A standard example is problem II.20, in which it is asked to find two numbers such that the square of either added to the other gives a square. In the course of the treatment, the two numbers are posited "so that the square on the first, adding to itself the second, makes a square" (*Arithm.* 114.14–15); that is, so that the first prescription, which is here explicitly stated, is fulfilled or solved. Then the other prescription included in the problem is recalled by the formula *loipon esti*: "it remains that the square on the second, adding to itself the first, <shall> make a square" (*Arithm.* 114.15–17); in other words, the second *epitagma* has to be checked. In this case as in general, these explicit reformulations enable one to identify what are the two *epitagmata* that constitute the whole of the problem. The statement of the problem might thus be said to be explicitly reformulated and 'decomposed,' within the course of the treatment, into its various "elementary prescriptions."

In almost all of the problems of books I–III, determinate numbers are found once *all* the *epitagmata* have been solved and the requested *arithmos* is determined. In some cases, though, determinate numbers are found for one of these "partial prescriptions," as if it was treated as a separate problem. In this case, therefore, we chose to represent in the conspectus such *epitagmata* as separate problems, since we can make separate treatments correspond to them. Problem III.15 *alit.*, for example, has three prescriptions,[17] the first of which is treated and "solved" separately: two determinate numbers are explicitly found, that "solves one of the *epitagmata*" (*Arithm.* 172.11–15). This decomposition of the treatment of problems into sub-problems, each one of them corresponding to a separate prescription, only appears in a few problems of the three books.[18]

As we have seen, the particular notions of problems and prescriptions that we have elucidated are hardly separable from the key notion of *treatment* or *solution* of the problem. As a synthesis of this first explanation, we shall go back to this last notion.

[16] III.1: *esti duo tôn ep. lelymena* (Arithm. 138.11); III.12: *lysomen hen tôn ep.* (*Arithm.* 166.13); III.14: *hina êi lelymena duo tôn ep.* (*Arithm.* 170.12); III.15 *alit.* and III.16: *kai lelytai hen tôn ep.* (*Arithm.* 170.2 and 178.11). In III.15 *alit.* we also find *kai menei hen tôn ep.* (*Arithm.* 174.19) and in III.17 the prospective expression *exês dei kai ta loipa duo ep. kataskeuasthai* (*Arithm.* 180.14). Many more similar examples are found in other books, like in the (Greek) book "IV."

[17] ? X, Y, Z: $XY + X + Y \rightrightarrows \square$ (E_1), $YZ + Y + Z \rightrightarrows \square$ (E_2), $ZX + X + Z \rightrightarrows \square$ (E_3). The various symbols we use in the abbreviated transcriptions of the problems are explained in the 2nd appendix.

[18] Beyond problems III.15 *alit.* and III.16 (which is very close to the latter), we have I.20, perhaps II.28 and II.29. All other cases, in which a problem is divided into "sub-problems" are cases, in which auxiliary problems are introduced, like in III.10.

2.2.3 The notion and role of solving (lyein) problems within their Diophantine treatment or solution

The previous discussion shows that we have generally used the terms 'treatment' or 'solution' to designate the *process by which a problem is solved*, that is, by which *its difficulty is unraveled*: this meaning is directly deduced from, and coherent with, the very first occurrence of *lyein* in the introduction of the *Arithmetica*. It is then opposed to the notion of weaving (*plekesthai*) applied to the statements of the problems. The latter being weaved into internal difficulties, solving (*lyein*) them just means the contrary, that is, to *unravel* the difficulty. Therefore also, this notion must be sharply distinguished from the numerical results of this process, for which Diophantus sometimes uses the expression "*heurêmenoi arithmoi*" or "*heuriskomenoi arithmoi*".[19]

Within the course of the treatments, unraveling the difficulty of the problem is essentially done by the choice of one or several relevant *positions* (on which more below) and then, after the establishment of the equation, by finding the value of the *arithmos*. For this very last process, Diophantus also uses *lyein*, to solve, but does not apply it to a problem but to an equation. The corresponding procedures used to solve equations are definitely different and distinguished from the general process of 'solution' applied to a problem.[20]

Furthermore, the verb *lyein* mostly appears within solutions themselves in association with *epitagma*. We have seen the example of problem III.1 above. Similarly, in problem III.12 we find: "if, therefore, having exposed some square <number>, we shall set some part of it for the third, and the remainder for the product of the first and second, *we shall solve one of the prescriptions*" (*lysomen hen tôn epitagmatôn*) (166.1–3, emphasis added). The meaning cannot be clearer: if we do that, then we solve one of the requests; to solve means to "do" something, it designates some specific process which is part of the entire treatment. The same meaning of "*lyein*" is found in other instances within the third book of the *Arithmetica*. For example in problem III.14 we read: "Let them be set the first 1 *arithmos*, the second 4 *arithmoi* and 4 units, the third 1 unit, so that two of the prescriptions *might be solved*" (*hina êi lelymena duo tôn epitagmatôn*) (*Arithm.* 170.1–2, emphasis added).

All this shows that the most important part of the solution of a problem, which *includes* the solution of an equation, is actually not the latter, but the first part, by which positions are found for the numbers in the statement. As we have explained elsewhere, what Diophantus takes as a theme within the *Arithmetica* is the conversion of a problem to equation and not the solution of the equation; the latter seems to be of secondary importance in his work.[21]

[19] See "Index graecitatis apud Diophantum" (*Arithm.*, ii, p. 271), *s.v.* "*heuriskein.*" For example in problem III.15 it is said that "these numbers being found" (*Arithm.* 160.4), speaking of the determinate numbers found in the first part of the solution. See also 2.3.1 below

[20] For the discussion of this second meaning of '*lyein*' (to solve), see 2.5.4.

[21] See more on this in (Christianidis 2007). Note that these notions of solution and solving of a problem are distinct from the one they usually have for a modern mathematician, namely the *outcome* of the treatment of the problem, whether numbers, or equations, or figures (for the solution) and obtaining this result (for solving). The Diophantine notion is closer to the notion of resolution, but note also that the Latin *resolutio* sometimes refers to the Greek *analysis*.

Correspondingly, therefore, we shall use here the word *solution* to designate generally the whole treatment of a problem, and write more specifically about "the core of the solution" (or any similar expression) for the most and main part of it, namely the progressive conversion from the statement to an equation. Such an understanding is all the more useful and legitimate that it helps to differentiate between two 'problems' (in the third meaning of the term) that might share the same statement, or lead to the same equation, but that might neatly differ by their 'solution' in the strict sense.[22]

2.3 The possible statuses of numbers within the problems and their solutions: qualifications and designations

We have thus seen that *epitagmata* are crucial for the core of the solution of any problem, for they represent re-statements (i.e., reformulations of the statement), within the latter. Moreover, *epitagmata* are both formulated with references to the specific *operations* that are prescribed, and to the *numbers* on which they apply, or of which they are the results. We need, therefore, to identify how these numbers are named or, better said, qualified.

Indeed, most of these numbers are generically called "number" (*arithmos*) by Diophantus, either explicitly or not. This term obviously is generic and might receive various specifications, according to the function numbers have within statements and/or treatments. These distinctions, as far as statements and *epitagmata* are concerned, are established at the level of the *qualifications* that apply to numbers (like *found, given, loipos, genomenos*, etc.), according to the roles they have either within the statement or its reformulations. Some of these qualifications (like *found, given*, etc.) might refer to the general task that defines the problem (2.3.1); others (like *loipos, tetragônos, pleura*, etc.) might refer to an operation that defines them (2.3.2). 'Qualified numbers' have generally to be grammatically understood as common names or nouns and are used within *epitagmata*. As such, they must be distinguished with other names, like *dynamis*, which are used as 'designations' (*epônymiai*) for numbers, when they are used as terms of the "arithmetical theory" and are explicitly the result of an 'artificial' process of naming (*onomasia*) (2.3.3). We might finally define the generic (and non Diophantine) notion of *constraint* (2.3.4), which includes *epitagmata* as well as qualified numbers like *tetragônoi*.

2.3.1 Numbers qualified by reference to the general task of the problem

S*ought* (alternatively, *required* or *requested*) numbers correspond to the Greek *zêtoumenoi arithmoi* in Diophantus and naturally designate the numbers, the finding of which is called for by the statement of the problem.[23] This qualification is explicitly used within the treatment, usually in formulas like "let the sought number be set ..." (*tetachtô ho zêtoumenos* ...), or similar expressions. Within the statement, they are

[22] We shall see a good example of this with problem II.8. See 2.4.2 below

[23] In the abbreviated transcription of *protaseis* that we will use throughout the article in order to avoid full-scale quotations, we will use the letters X, Y, Z ... to denote the sought numbers.

recognized by the fact that they are the object(s) of the verbs defining the main tasks and the numbers to be found (like *heurein, dielein,* etc.).

Sought numbers are most of the time qualified by ordinal or comparative adjectives like 'the first, the second' (*ho prôtos, deuteros*), 'each of' (*hekastos*) or 'the remaining one' (*ho loipos*), the greater / the lesser (*ho meizôn / elassôn*) when other sought numbers have been designated already. The latter qualifications are of course the basic means, by which *epitagmata* might be distinguished from each other; but they might also be used in the statements of problems.

The *given* or *prescribed* numbers (*epitachthentes* or *dedomenoi arithmoi*), as their name indicate, are those numbers that are liable to be *further* prescribed within a specific sentence which comes after the general statement but before the treatment proper. In problem III.10, for example, the 'given number' in the basic statement is specified by the sentence "let the <number> 12 be further prescribed" (*epitetachtô dê ton 12*). That the verb *epitassein* is here used in a coherent manner with the meaning of *epitagma* clarified above[24] is clear from several such 'further specifications,' in which not only the given number or numbers receive a determinate value, but the verb used in the statement is also recalled, as in problem II.12 (*Arithm.* 100.1–3, repeating the statement with determinate numbers). Given numbers, therefore, are inseparable from a specific form of prescription, namely, a prescription which is slightly delayed.[25]

Within the statements, numbers other than sought or given appear under explicit qualifications. In problem III.10 stated above, for example, the *product* of each pair[26] added to 12 shall make *a square*: such numbers might be designated as 'partial results' corresponding to some prescribed operation. Problem II.9 asks for the addition the same <sought> number to two given numbers, so as to make each of them *squares*: such numbers are characterized by their kinds, like "a square" (in books II and III), "a cube" (in other books). In these two cases, they are not given. We shall generally designate by *indeterminate* numbers, then, those numbers present in the statements or their reformulations and that are *not given*.[27] The term is non-Diophantine, but the notion is useful for the analysis of solutions and the numbers thus qualified are

[24] Cf. 2.2.2.

[25] Such 'differed prescriptions' are sometimes called *instantiated* problems, but this name might be misleading, since the distinction between 'general' and 'instantiated' statements or problems is not an explicit concern in Diophantus. The statement of these 'differed prescriptions' sometimes follows a condition of solvability that Diophantus calls "*prosdiorismos*." The name appears in the statement as well as within the treatment.

[26] Literally, the <number formed> under (*hypo*) the pair of numbers.

[27] Note that the Greek expression *arithmos aoristos* is used by Diophantus in a thoroughly different meaning. For the meaning of this expression, which however does not appear at all in the problems of the first three books of the *Arithmetica*, see (Klein 1968/1992, 134). In the Tannery's edition of the Greek text of the *Arithmetica* the word "*aoristos*" appears, in fact, once in the introduction, when Diophantus defines the technical term "*arithmos*" of the arithmetical theory. Thus, in Tannery's text the "*arithmos*" is defined as "having in itself an indeterminate multitude of units" (*echôn en eautôi plêthos monadôn aoriston*) (*Arithm.* 6.4). However, this word does not exist in any of the known manuscripts of the *Arithmetica*, and it is in fact inserted into the text by Tannery himself, who 'corrected' the text of the manuscripts by taking into account a lecture of the text found in a letter of the Byzantine scholar Michael Psellos (11th century). This letter is published in the second volume of Tannery's edition (*Arithm.* ii, 37–42). For an elucidating and convincing discussion of the whole issue see the corresponding comment of Rashed in his edition of the Arabic books of the *Arithmetica* (Rashed 1984, iii, 121).

easily identified. Their names refer to operations and will be explained in the next part (2.3.2).

Within the final conspectus, all these various qualifications of numbers are retrieved and enumerated within the first columns. Furthermore, in any of our abbreviated transcriptions of the statements, we will always use determinate numbers instead of the "given numbers" as such.

Found numbers (*heurêmenoi* or *heurethentes arithmoi*) is an expression used (although rarely) for the determinate numbers that satisfy the prescriptions of the problem. In problem III.10.3, for example, the opening formula is "those numbers having been found" (*toutôn heurethentôn*), referring to the determinate numbers effectively found in III.10.2b. Such numbers, by definition, can be only found after the treatment, once the *arithmos* has been determined. They are the completely determinate multitudes of units, or parts of unit, that complete the solution of a problem.

The "epitagmatic" *qualifications* listed above have in common that they designate the *role* of numbers within the statement of a problem or within its solution, when the prescriptions are reformulated. Concerning indeterminate (that is, non given) numbers, we have seen that it is important, in parallel, to regard some of these qualifications with reference to operations, be they part of an *epitagma* or not.

2.3.2 Numbers qualified by reference to operations

Diophantus's introduction makes it clear that numbers might be distinguished by specific kinds, such as "square," "cube," "side" whenever they are to be considered, within the statement, in relation to each other. The basic kinds of numbers enumerated in the introduction are the following: (a, b) square numbers (*tetragônoi*) and their sides (*pleurai*), which are respectively the numbers resulting from the *multiplication* of a number by itself, and the *squared* number itself; (c) cube numbers (*kyboi*), resulting from the *multiplication* of a square number by its side; (d) numbers resulting from the multiplication of a square number by itself, or (e) from the multiplication of a square number by the cube of the same side, or finally (f) from the multiplication of a cube by itself[28] (*Arithm.* 2.18–4.7). In all cases, these qualifications are always referred to specific operations that are here made explicit once and for all.

The following sentence of the introduction suggests that many other relations are possible within the statement and 'weaving' of a problem: "from all these <kinds of numbers>, either by addition, subtraction, multiplication or ratios, either to each other or each of them relatively to their own sides, most of the arithmetical problems are woven" (*Arithm.* 4.7–10). One might also add, when comparing with the actual statements, that the operations in question are most of the time made explicit *as well as their results*. Thus, for example, the generic and instantiated statements of problem I.7

> From the same number, <it is asked> to *remove* two given numbers and *make* the remainders *have a given ratio* to each other.

[28] As indicated already in (Christianidis 2007, 295–296), these three last categories plausibly have no specific names in this part of the Diophantine text.

> Let it be <further> prescribed *to remove* from the same number the numbers 100 and 20, and to *make* the greatest *thrice* the least.

Such a double statement not only prescribes operations to be performed, like subtraction/removal (*aphairesis*) or ratio-making[29] but also refers to the resulting numbers, which are named '*tous loipous*' (the remainders) in the generic statement, and '*ta meizona, ta elassona*' in the instantiated formulation. Similarly, a host of qualifications are used in statements or in solutions and, like the 'kinds' of numbers listed above, refer to an operation. The only difference with 'kinds' is that the operation in question is explicitly contained in the prescription, whereas for kinds it refers to the correspondence between names and operations that is made explicit in the introduction only.

2.3.3 Qualifications vs designations (epônymiai) of numbers: kinds and species

The vocabulary of 'qualifications' by reference to operations is characteristic of statements and *epitagmata*, be they stated or reformulated, and is used only for them. But Diophantus's introduction makes clear that numbers might also receive a designation (*epônymia*) and then become the terms or *elements of the "arithmetical theory"* (*stoicheia tês arithmêtikês theôrias*).[30] Most of these terms are what is later called *species* (*eidê*), like the *arithmos*, the *dynamis*, the *kybos*, the *dynamodynamis,* and so on. But a simple multitude of units (*monads*) might also be a term in the theory—more precisely, it can also *play this role*. The designations refer to some kind of relationship between species, insofar as they signify it: this is explicitly said by Diophantus when he comes to the multiplication of species: "they might by clear to you, for they are almost prefigured in their very naming (*onomasia*)" (*Arithm.* 6.24–25).

The usefulness of the distinction between kinds and species of numbers, which parallels the distinction between 'common names' (or qualifications) and *epônymiai*, is that it points out toward *different* contexts, in which they play different roles. Kinds and qualifications of numbers are for the statement or the reformulation of its prescriptive contents; species (and designations) are for the terms of the "arithmetical theory" and are used to compose what we will call below *expressions* within a *position*. For example, if we are told that an indeterminate square is set, in the course of the solution, as "1 *dynamis*," this "1 *dynamis*" is a term of the "arithmetical theory." Similarly, if we are told that an indeterminate square is set, in the process of solution, as 25 units, these 25 units just play the role of a term of the "arithmetical theory"; in this case, they obviously do not have the status of *given* numbers of the problems.[31]

[29] A ratio is not itself an operation, but it is usually translated into a multiplication by Diophantus. In the example the prescribed ratio 3 to 1 implies a multiplication by three (*Arithm.* 24.10–11).

[30] Translating *epônymia* by *designation* follows Tannery's proposal. What should be the 'right' translation of this word, and more fundamentally, how the notion itself is to be understood, is a difficult question that will not be discussed here.

[31] In our abbreviated transcriptions, we will use we use lower case letters like x, x^2, x^3, etc. to denote the species of numbers, that is the terms of the "arithmetical theory." We keep the usual number symbols for

We must finally remember that there are two terms which are formally transversal to the two contexts (statements and "arithmetical theory"), namely the terms *arithmos* and *kybos*. The latter makes its appearance from the book IV onward and will not concern us here. The former is most of the time kept implicit when understood as having a role (and the corresponding qualification) within the statement, and it is most often explicitly used when it designates the species of *alogos arithmos,* as defined in *Arithm.* 6.3–5.

Within the conspectus, kinds of numbers are, like qualifications, retrieved on the left columns, in which the statements are analyzed into their various components. The last notion, which is mentioned in the same place, is the notion of constraint, which we shall now clarify.

2.3.4 Kinds of number as condensed prescriptions: the notion of constraint

In problem III.10, which was taken as example above (cf. 2.2.1), the difficulties in problems (a) or (b) are essentially based on three prescriptions that end with an indeterminate square to be obtained. The difficulty in problem (c), by contrast, is made of two such prescriptions: each of the sought numbers, added to 12, must make a square; but, furthermore, these numbers *have to be squares*. These 'imposed kinds' therefore act as if they were also *epitagmata.* Like them, they partially express the specific difficulty of each problem and act as constraints on the solution. In general, therefore, we shall call a *constraint* (of a stated problem) *any element within the problem that calls for a treatment* or *solution*.

That kinds of numbers are close to prescriptions might easily be recognized if we artificially translate the one into the other. For example, instead of asking to divide a square into two squares numbers, as in the actual statement of problem II.8, we might as well ask to divide a square number into two numbers, *so that these parts might make squares*. In this new and highly artificial formulation, what was before expressed as a 'requested kind' becomes a verbal *epitagma*. But, of course, the artificiality of such a formulation comes from the fact that the operation called for by this artificial *epitagma* is actually contained in the very notion (and name) of 'square' (*tetragônos*), as explained in Diophantus's introduction.

The notion of a constraint has no specific name in Diophantus but, on the one hand, we have just seen that it is very close to the notion of prescription; moreover, it can be unambiguously recognized from both the statement and the treatment, the latter having to take into account, so to speak, such constraints. The recognition of the latter is important for the *analysis* of Diophantus's solutions, insofar as to each of the constraints might correspond a specific choice within the treatment that "answers" and "solves" these constraints. But we have to note that there is no explicit concern for how many constraints there are in a problem or for the whole set of *all* found numbers that might observe a given set of constraints: such a concern is typically modern.

Footnote 31 continued
determinate numbers that are used as terms of the "arithmetical theory." Kinds of numbers, especially the squares, are denoted by the sign $\square$, eventually followed by capital letters like U^2.

Within the conspectus, the total of constraints is indicated in a specific column. The abbreviations we use for the statement of problems also make clear the distinction between the various kinds of numbers and the various kinds of constraints.[32]

2.4 The analysis of the main stage of the solution into a chain of positions

2.4.1 The various stages of the solution and their relative importance

In order to introduce the next key notion of our 'analytical toolbox,' namely position (*hypostasis*), let us check the structure of a standard example, in which this meaning is made clear, namely problem I.12:

> To divide a prescribed number into two numbers twice, so that one of the <numbers obtained> from the first division should have a given ratio to one of the <numbers obtained> from the second division, and the remaining <number> of the second division should have a given ratio to the remaining <number> of the first division.

This is a standard statement, which includes four sought numbers, three given data and two prescriptions. The given number is then further prescribed as being 100 and the two given ratios as the double and triple. Here is the treatment:

> Let the least of the second division be set (*tetachthô*) 1 *arithmos*. The greatest <among the numbers obtained> from the first division, therefore, will be 2 *arithmoi*, and, consequently, the least from the first division <will be> 100 units wanting 2 *arithmoi*; and since the greatest from the second division is the triple of the latter, this will be 300 units wanting 6 *arithmoi*. There remains that the <numbers> of the second division, composed with each other, should make 100 units; but, when they are <indeed> put together, they make 300 units wanting 5 *arithmoi*. The latter are equal to 100 units, and the *arithmos* becomes 40 units.
>
> <Let us come> to the positions (*epi tas hypostaseis*). I have set (*etaxa*) the greatest of the <numbers out> of the first division 2 *arithmoi*, this will be 80 units; <I have set> the least of the same division 100 units wanting 2 *arithmoi*, this will be 20 units; <I have set> the greatest of the second division 300 units wanting 6 *arithmoi*, this will be 60 units; <I have set> the least of the second division 1 *arithmos*, this will be 40 units. And the proof is clear.

This part is the complete solution and is clearly composed of distinct, consecutive parts. The first it the "setting" of various numbers evoked in the statement (the two parts of the given numbers for each division). Then an equation is obtained, on which the various operations, which are described in a general manner in the introduction, might be performed. Unlike most of the previous problems I.1–11, but very much like

[32] Thus, the three above problems (a), (b) and (c) are abbreviated in the following manner:
(a) $?\ X, Y, Z\colon XY + \text{Gvn} \rightrightarrows \square(U^2),\ YZ + \text{Gvn} \rightrightarrows \square(V^2),\ XZ + \text{Gvn} \rightrightarrows \square(W^2)$.
(b) $?\ X, Y\colon XY \rightrightarrows \square(U^2),\ X + 12 \rightrightarrows \square(U^2),\ Y + 12 \rightrightarrows \square(V^2)$.
(c) $?\ X, Y\colon \square X, \square Y,\ X + 12 \rightrightarrows \square(U^2),\ Y + 12 \rightrightarrows \square(V^2)$.

in the huge majority of the following problems, these operations are not fully stated here.

The *arithmos* once determined, the solution continues with the standard clause "to the positions" (*epi tas hypostaseis*). What follows is, very clearly, the series of sought numbers with their corresponding expressions. Although the term *hypostasis* in most of its occurrences within the '*epi tas hypostaseis*' clause is associated with the verb *tassein* (*tetachthô, tassô, etaxa*), which means "to set," it is remarkable that the first 'matching pair' recalled has not been introduced by '*tetachthô*' before, but by '*estai'*. In general, then, the term *hypostasis*, translated here, following Tannery, as "*position*," might plausibly be interpreted as referring in general to the explicit correspondence between one number evoked in the statement and some combination of species of the "arithmetical theory." While the formula used for such 'positions' might be, like here, explicitly recalled by a formula like 'I have set' (*etaxa*), the 'values' of the sought numbers, in most cases, are directly calculated.

The solution is finally completed by a proof (*apodeixis*), which is here left implicit. It consists in checking that the found numbers indeed satisfy the prescriptions.

The case of problem I.12 is representative of many of Diophantus's treatments. Among these various parts of a standard solution, indeed, only the first is usually developed and given in full for the majority of the problems in *Arithmetica* I–III. In particular, the solution of the equation obtained (when the latter is quoted) is most often not fully worked out. There are many problems in which, as in I.12 here, it is skipped altogether and replaced by the final 'value' of the *arithmos*; the proof is absent as well, or referred to as 'clear.'

By contrast, what is present and, so to speak, prominent in most treatments is the series of *positions* (*hypostaseis*), each one being introduced by specific justifications, or calculations, or both. In other words, to each obtained *position* corresponds a 'positioning,' a specific process by which it is obtained.

2.4.2 Differentiating variants through positions: the case of problem II.8

Problem II.8 represents an interesting 'test' for the analysis by positions and positioning. For it includes one statement (to divide a given square number, be it the number 16, into two squares)[33] but proposes two variants. In the terms proposed above, there is one problem with the meaning of 'task,' but two problems (II.8 and II.8 *alit.*) if to the statement we add each of the two treatments proposed. The positions of the first of them are the following:

> Let the first (sought square) be set 1 *dynamis*; then the other <square> will be 16 units wanting 1 *dynamis*; therefore, 16 units wanting 1 *dynamis* will have to be equal to a square. I form <*plassô*> the <aforesaid> square from any number of *arithmoi* wanting as many units as there are in the side of the 16 units; be it 2 *arithmoi* wanting 4 units. The square itself will be 1 *dynamis* <and> 16 units wanting 16 *arithmoi*. (*Arithm*. 90.12–17)

33 In abbreviation: ? $X, Y{:}\,\square X, \square Y, X + Y \rightrightarrows 16$.

Here four positions are successively introduced. First the position for the first of the two sought square numbers is introduced by the formula "let it be set" (*tetachthô*).[34] Next, the position for the second sought square is deduced from the first position and from the prescription of the problem; it is therefore introduced by "*ho ara heteros estai*."[35] Then the third position for the side of the same square is introduced in two steps; first by an explanation, introduced by the verb "I form" (*plassô*), that gives the general form that should be adopted[36] and then by choosing a particular case, which is introduced by the formula "let it be <then>."[37] The last position is for the square itself (that is, the second sought number) and is directly deduced from the position for the side.[38]

The chain of positions for the variant is the following:

> Let it be set, again, the side of the first <sought square> 1 *arithmos*, and <let it be set the side> of the second, any number of *arithmoi* wanting as many units as there are in the side of the divided <square number>; be it, then, 2 *arithmoi* wanting 4 units. The squares, therefore, will be 1 *dynamis*, on the one hand, and 1 *dynamis* <and> 16 units wanting 16 *arithmoi*, on the other hand. (*Arithm.* 92.4–7)

Here there are again four positions, but they are not at all the same as in the first treatment: the two first are set for the sides of the sought squares (*and not one for the first sought square*, as before)[39] and two are deduced from these, namely the squares of these sides: the positions of the two sought numbers are thus obtained.[40] The difference between the treatments thus becomes very clear from the comparison of the positions, which can be summarized in the following table:

P #	Positions of the 1st solution	Positions of the alternative solution
1	$X := 1x^2$	side of $X := 1x$
2	$Y := 16 - 1x^2$	side of $Y := 2x - 4$
3	side of $Y := 2x - 4$	$X := 1x^2$
4	$Y := 4x^2 + 16 - 16x$	$Y := 4x^2 + 16 - 16x$

In the aforementioned examples the positions were introduced by expressions like "*tassô*," "*tetachthô*," "*plassô*," "*estô*," "*estai ara*," etc. But positions might be introduced by other expressions as, for example, the expression "that is to say" (*toutesti*) by which the last position is introduced in problem II.14 (*Arithm.* 104.10). Therefore, what is important in order to recognize *in general* when a position is introduced in the solution of a problem is not the presence of one or another of the above verbs in the

34 In abbreviation: $X := 1x^2$.

35 This position is $Y: = 16 - 1x^2$.

36 Namely, $mx - \sqrt{16}\cdot$ in modern transcription where m is "any number we wish."

37 This position can be abbreviated into s$Y := 2x - 4$, in which sY designates the side of Y.

38 In abbreviation: $Y := 4x^2 + 16 - 16x$.

39 In abbreviation: s$X := 1x$, $sY := 2x - 4$ (same convention that before, sX and sY are the sides).

40 In abbreviation: $X := 1x^2$, $Y := 4x^2 + 16 - 16x$.

text, *but the marking off when a key element of the statement of the problem is being translated into the language of the "arithmetical theory*." This brings us to a more general and synoptic characterization of 'position' and to the corresponding notion of *expression*.

2.4.3 Position and expression: general characterization

In general, then, we will use here the term *position* to translate what is called *hypostasis* in most of its occurrences in the Greek text of the *Arithmetica*.[41] Both the Greek term and its English translation share the interesting characteristic that they might evoke the corresponding process.[42] This process is the "setting" or "positioning" in the terms of the "arithmetical theory" of one of the indeterminate numbers appearing in the statement of a problem. The position, *qua* result of this setting, is the realized *correspondence* between one indeterminate number, on the one hand, and an arranged combination of some terms of the "arithmetical theory," on the other. It is announced in the text by various verbs beyond "*tetachthô*" or "*tassô*," like "*peplasthô*" or "*plassô*," "*estai*" or "*estin*" or "*estô*," "*poiêtô*," "*ginetai*," etc. and preceded by sometimes complex explanations.

Although Diophantus has no generic name for the combination of species in question,[43] is it always easily recognizable and will be called here an *expression*. From the above characterization, it is therefore clear that *positions* are not reducible to *expressions*: an expression does not express, by itself, the whole process of "setting" nor the precise correspondence with the constituents of the problem. Only the explicit and verbal description of the "setting" does this.[44]

Let us note that the word "*hypostasis*" seems to be used, within the *Arithmetica*, with a variety of meanings that is not restricted to its meaning as 'position.' But the latter is still the most frequently used meaning among all the occurrences of the term. As mentioned above in the examples, it is indeed found within the "*epi pas hypostaseis*" clauses that conclude many treatments of the problems.[45] "*Epi tas hypostaseis* <*poiêsomen*>" means to go back to the positions, in order to put into the "expressions" the numerical value of the *arithmos* that has already been found, and to find thus the numerical values of the sought numbers that the statement of the problem

41 "Position" in English is inspired by the Latin *positio* adopted by Tannery in his translation of the "*epi tas hypostaseis*" clause (*ad positiones*), which as a rule concludes the treatment of the problems.

42 To *hypostasis* corresponds *hyphistêmi*; to *position* correspond *to set, to posit.*

43 In the introduction, Diophantus uses such periphrases such as "*hyparchonta eidê kai leiponta*," *Arithm.* 14.5, 7, 8, 9.

44 In the following, we will use specific abbreviations to distinguish the two notions of position and expression. Thus, the position (as result) will be denoted by the sign :=, while the expression will be given by the simple combination of concrete numbers and lower case letters, like $1x$, $1x^2$, $2x+5$, $5x-7$, etc. For example, the positions in problem I.15 are $X := 1x + 30$ and $Y := 2x - 30$, while the expressions are respectively, $1x+30$ and $2x-30$. Note, that the signs + and − in expressions should not be understood with reference to operations, but qualify the species behind it as "forthcoming" or "existing" (*hyparchonta*), on the one hand, and "wanting" (*leiponta*), on the other. See more on this, which is true also in Arabic algebra, in the interesting (Oaks 2009).

45 97 problems within the Greek text of the *Arithmetica* have this clause.

was calling for. This meaning is confirmed by the use of the verb "*etaxa*," I have posited, following this clause in some of its occurrences, as in problem I.12 above. The most explicit occurrence of the term with this meaning is in the statement "the obtained position is ineffective" (*scholazei hê gegenêmenê hypostasis*) that appears in the alternative solution of problem III.15 (*Arithm.* 174.3–4). The meaning of this clause is that the last positioning obtained does not serve our purpose because it leads to something impracticable.[46] Problem IV.19 provides another clear-cut example in which the term *hypostasis* is undoubtedly used and made clear as bearing the meaning of position. Indeed this problem, which is solved "in the indeterminate" (*en tôi aoristôi*), has the following concluding remark: "For to seek <to solve a problem> indeterminately means that the positioning (*hypostasis*) <must be> such that, of whatever <multitude of units> someone wishes the *arithmos* to be, when he sets it to the positions (*epi tas hypostaseis poiêsas*), the prescription is preserved" (*Arithm.* 232.6–8).[47]

There seem to be other occurrences of hypostasis with a different meaning, namely "numerical value," as in the striking example of problem I.39, where the text has "the *hypostasis* of the *arithmos* is not manifest (*adêlon*)" (*Arithm.* 78.19). Another occurrence is within problem III.12 (*Arithm*. 166.17).[48] There are also a couple of other occurrences in which the interpretation of *hypostasis* as numerical value seems possible. We do not discuss here in detail these occurrences, for this is not our present purpose here, and requires more detailed analysis.[49]

Hypostasis, finally, is used once in the introduction, within the following sentence: "All this (*touto*) should be worked out with subtlety (*philotechneisthô*) within the *hypostaseis* of the statements, as far as possible, and until one species is left equal to one species" (*Arithm*. 14.21–22). This occurrence is somewhat mysterious, because this is the very first place in which this notion is used. Since the "*touto*" unambiguously refers to the kind of calculations on species and combination of species previously described (*Arithm*. 6.22–14.20) and since such calculations are performed *within the treatments of the problems,* this meaning is perfectly coherent with the meaning of *hypostasis* as "position." Indeed, the calculations in question are usually implied in the passage from one position to the other, or by the heuristic explanations that justify it, as we shall see.

[46] This same expression is further made clear in the next problem where a similar situation is described by "we will fall into the same difficulty <as in the last problem>" (*eleusomai ôsautôs eis aporon, Arithm.* 176.14).

[47] *Hypostasis* has the same meaning when it appears in the other instances of "solutions in the indeterminate."

[48] "And the lesser is 16 units, the greater 64 units; and whatsoever of them will be used in order to produce the equation I will find the numerical value (*tên hypostasin*) of the x; for, if we say the 64 units of the greater to be equal to $1x + 54$, we obtain the x <to be> 1 unit; if, again, we say the 16 units of the lesser to be equal to $1x + 54$, we obtain the x <to be> 1 unit" (*Arithm*. 166.14–20).

[49] In particular, this would lead us to discuss both Diophantus's precise meaning in the aforesaid places and the interpretation of Byzantine commentators, who have generally interpreted *hypostasis* with this second meaning.

2.4.4 The importance and usefulness of positions for our analysis

From the above examples (I.12 and II.8, II.8 *alit.*), it is clear that *positions* and the corresponding 'positioning' have a double role in Diophantus's treatment of an arithmetical problem. On the one hand, they serve to find out the equation.[50] Thus, the equation in the problem I.12 is obtained after five positions have been found and by equating the expression found for the very last with the given number.[51] On the other hand, positions give also the means for determining the sought numbers in the final part of the solution.

Thus, one could say that they play the role of "bridges" that connect the statement of the problem with its counterpart within the "arithmetical theory": indeed, through the setting of positions, the problem is transferred within the "arithmetical theory" to the equation; conversely, from the obtained positions the calculation of the sought numbers is made possible, once the numerical value of the *arithmos* has been found.

For the same reason, it is clear that the process of setting the chain of positions plays a crucial role in Diophantus's strategy for solving arithmetical problems. It would hardly be an exaggeration to say that this process constitutes the core of the *solution* or *unraveling* of each problem (see 2.2.3 above). The expressions themselves are most of the time chosen (for the first position or positions) or calculated (when they belong to a position derived from others) only because they are made to correspond, within a position, to one of the numbers considered in the problem. That is, the correspondence provided by positions is also what makes it possible to *justify* what calculations should be performed and on which expressions, as we shall see.

Only within the treatment of the finally obtained equation are operations performed on the expressions that are on each side (*merê*) of the equation: these calculations and their rationale are *specific* to *equations* and are explained once and for all in the introduction.[52] Generally speaking, therefore, these specific operations are kept distinct from the deduction-like procedures, by which positions are obtained.

In general, therefore, the leading motive of solutions is bestowed on the deductive 'chaining' of positions, which forms the core of the solution as a whole. For the same reason, it is important to identify how the chaining of positions comes to an end in general (2.5) and how positions might be justified (2.6).

2.5 The end-point of a coherent chain of positions: equation, double equation and dead end

By our own choice of definition for the notions of problem (as the couple of a statement and a particular treatment) and position (as defining our 'minimal' unit of analysis), we have to complete the picture by listing here all the possible ways in which, in the three first books of the *Arithmetica*, the coherent chaining of positions comes to

50 Equation is actually not the only possible 'end-point' of the chaining of positions; see below, 2.5.

51 On the notion of the equation on the way they might be obtained in general see below, 2.5.1.

52 On the solution *of equations* and the specific operations practiced on them, see more below (2.5.4).

an end. Only five cases arise within books I–III: three of them have been met in the above examples and directly lead to the equality between two expressions (2.5.1); one is explicitly named by Diophantus and requires a special procedure to reach an equation: this is the *double equation* (2.5.2); finally a chain of positions might arrive at a *dead end* (2.5.3). We complete the list with some remarks on the difference between equation and what we call here 'prospective equalities,' the same word *isos* being used with respect to two different notions at least.

2.5.1 Directly ending with an equality between expressions (*or equation*)

The example of problem II.8 discussed above (2.4.2) illustrates one straightforward way to stop the chain of positions in a given solution: by equating two expressions obtained for the same indeterminate number. Indeed, two positions for the indeterminate square and its side have been found. One can then equate the two expressions posited for the same number: "the latter <*tauta*, i.e., $4x^2 + 16 - 16x$> are equal to 16 units wanting 1 *dynamis* <$16 - 1x^2$>" (*Arithm.* 90.16–17). This phrase describes an equation that is the outcome of the procedures applied until then. The statement of the equation marks the beginning of a new kind of procedure, applied this time to the equation, the outcome of which will be the finding of the *arithmos*: "Let the wanting be added and let the similar terms <be subtracted> from similar terms: 5 *dynameis* <are> equal to 16 *arithmoi*, and the *arithmos* becomes 16 fifths." (*Arithm.* 90.17–19). In the column entitled "ending point" of the conspectus, this case is signalled by "ee," standing for "**e**xpression equal to **e**xpression."

A second way to end the chain of positions is illustrated by problem I.12 discussed above (2.4.1): the last position obtained (the sum of the two numbers obtained from the second division is obtained $300 - 5x$) makes it possible to equate the expression $300 - 5x$ to the given number 100. In the conspectus, such a case is signalled by "eg," standing for "**e**xpression equal to **g**iven <number>."

Finally, problem I.15 asks to find two numbers, such that, when each of them takes a given number from the other, the remainders have to each other a prescribed ratio. The first positions yield for these remainders the expressions $2x - 80$ and $1x + 80$. The concluding sentence giving the equation says that "it remains that $1x + 80$ is the triple of $2x - 80$. Thrice the lesser, henceforth, are equal to the greater <of the two numbers obtained>" (*Arithm.* 36.24–26). Thus, the final equality is obtained after *converting* the one expression ($2x - 80$) by a simple multiplication by three, obtaining $6x - 240$ and equating the result to the other expression $1x + 80$. The conversion of one of the expressions, derived from part of one of the prescriptions, has to be done before the equation is obtained.[53] In the conspectus, this case is symbolized by "ece," standing for "**e**xpression equal to **c**onverted **e**xpression."

Note, that in all three cases we will here call the obtained equality the *finally obtained equation* or, when there is no ambiguity, simply the *equation*. By this term, therefore, we mean equalities that are *exclusively* formulated in the terms of the "arithmetical theory" and, eventually, in the terms of the given numbers of the problem (when there are

[53] Note that this way of obtaining the equation could easily be replaced by a supplementary position and an equation of the "ee" type.

such given terms). "Exclusively" means that no more indeterminate number enters the equality. Furthermore, we shall also speak about a *valid equation*, when the *arithmos* might *effectively* be found from such the equation by the procedures indicated in the introduction.

As can be seen from the above examples, finally obtained equations are recognized within the text by their situation (at the end-point of a chaining of positions) and by clauses containing the word "equal" (in full or in abbreviated form), such as, "are equal" (*isoi eisin, isa eisin*), "these <are> equal" (*tauta isa*), etc. relating to each other expressions with expressions or with given numbers. Valid equations are recognized by the fact that the statement of the final equation is immediately followed by the determination of the *arithmos*.

We must note, however, that in many cases Diophantus uses the word '*isotês*' as well as the adjectives '*isos*' and '*isa*' in a broader sense than the aforesaid use with reference to 'equations' or 'valid equations' only. For example, in problem II.8, the sentence "16 units wanting 1 *dynamis* will have to be equal to a square" (*Arithm.* 90.13) does not refer to the finally obtained equation, but to what we might call a "prospective equality" or an "equation-to-be-obtained." We shall go back to this subtle but important point below (2.5.4).

2.5.2 Ending the positions with a double equation (diploisotês)

A typical example of "standard" *double equation* is given in problem III.13, which asks to find three numbers so that the product of any two of them, when the third number is subtracted, makes a square.[54] The first positions are $YZ - X := 4x^2 + 15x$ and $XZ - Y := 4x^2 - (1x + 4)$ and, for each of them, the formula used is "the <rectangle> on the second and third <numbers> wanting the first is 4 *dynameis* 15 *arithmoi, equal to a square*" (*Arithm.* 168.3–4, emphasis added). Here, as in II.13 *alit.*, this last clause must obviously be read "which *must be* made equal to a square," in accordance with the previous sentence, which recalls that the product of the 2nd and 3rd number "must produce a square" (*Arithm.* 168.5–6). Then the text reads "again the equation obtained is double" (*ginetai . . . diplê hê isôsis*, 168.10) and this formula immediately opens toward the special procedure applied in this case,[55] from which two equivalent equations are obtained,[56] each of them leading to the very same *arithmos*.[57]

In general, then, *double equations* (translating *diploisotês, diplê isotês* or *diplê isôsis*) are a special form of end-point of a chain of positions. First introduced in problem II.11, they are formed when two different expressions must be made simultaneously equal to indeterminate squares. When such a form appears, they are most of the time explicitly introduced within the text by special formulas like "*kai estai/ginetai diplê hê isotês*." This "transition" immediately introduces a special treatment, based

[54] $?\ X, Y, Z: XY - Z \rightrightarrows \square(U^2),\ YZ - X \rightrightarrows \square(V^2),\ ZX - Y \rightrightarrows \square(W^2)$.

[55] The excess of one expression on the other is $16x + 4$, which is decomposed as the product of 4 units and $4x + 1$ (*Arithm.* 168.10–12); from these factors two squares might be calculated, one on half the sum, the second on half the difference of these two factors (*ibid.* 168.13–15).

[56] Either $4x^2 + 15x = (2x + 5/2)^2$, on the one hand, or $4x^2 - (1x + 4) = (2x - 3/2)^2$, on the other.

[57] Here, the text does not make explicit the valid equations but only the *arithmos* obtained: $25/20$.

on a calculation on the difference between the two expressions, that (a) enable one to find the two indeterminate squares and (b) to obtain one of two (equivalent) *finally obtained* equations, from which the *arithmos* is found.[58] Any double equation, therefore, actually "hides" two *valid* equations that are equivalent to each other, so that the *arithmos* can be effectively deduced from either of them.

Note that it is unclear whether "*diplê hê isôsis*" designates the double treatment or the couple of obtained equalities themselves. The distinction is not easily made, because the form of double equation immediately calls for the corresponding treatment.

The double equation occurs, as far as the first three books of the *Arithmetica* are concerned, 9 times (in a total of more than 100 solutions), and its first occurrence is found in problem II.11. Any double equation actually hides a valid equation, which is the result of the "equalization" leading from the one to the other. In the conspectus, this case is symbolized by "de," standing for "**d**ouble **e**quation."

2.5.3 Ending the chaining of positions with an explicit failure: dead end

An interesting case of such 'explicit failure' or *dead end* is found in problem III.10.[59] Taking $X := 13x$, $Y := 1/x$, and $Z := 4x$ for the first positions and thus solving two of the prescriptions, one has to check the third prescription ($XZ + 12$ must produce a square): "there must also be, furthermore, that the first and third <sought number> with 12 units should make a square; but the first and third is $52x^2$; there will have to be, therefore, that $52x^2 + 12$ make a square" (*Arithm.* 158.18–20). These straightforward statements are immediately followed by the following remark: "... and if the numbers of the 13 units <from which 52 was obtained> of the first <position of> number was a square, *the equalization would be easy to handle <eucherês ên hê isôsis>*" (*Arithm.* 158.20–21). Here *isôsis* refers to the process of "equalization" that would lead to a valid equation, once the indeterminate square is found.[60] Meanwhile, an intermediary discussion is introduced that leads to this rectification: "but since it is not so, I am led to <the following problem>: to find out two numbers so that ...."[61] The formula "*moi apêktai eis to heurein*" typically marks the transition between the dead end proper and the auxiliary problem to which the "correction" of the problem leads.

Dead ends might take the *form* of apparently "normal" double equations or of prospective equalities. The important and distinctive characteristic of dead ends is that, whatever the point at which the treatment is declared "sterile" or defective, this point indicates the fact they are not amenable to a direct treatment leading either to new positions or to the *arithmos* without a *revision* being performed which usually leads

58 That, once the two squares have been found, it is indifferent to take one or the other equation to obtain the *arithmos*, is very clearly explained in problem III.12 (*Arithm.* 166.14–20).

59 The statements implied in problem III.10 are presented and discussed above (2.2.1). The abridged form of the first statement is ? $X, Y\colon XY + 12 \rightrightarrows \square$, $YZ + 12 \rightrightarrows \square$, $ZX + 12 \rightrightarrows \square$.

60 This is done a bit later in *Arithm.* 160.4–10, once the position for $XZ + 12$ has been found to be $1x^2 + 12$ and not $52x^2 + 12$.

61 This formula introduces problem (b) discussed above.

to an auxiliary problem. Therefore and in general, such "dead ends" are essentially recognizable by the fact that they are not followed by further new positions or by the determination of the *arithmos* but by a discussion of why the equation is sterile and how it might be modified.

We must note that the argument pointing out the "sterility" of the treatment does not bear, in general, on the solvability of the problem obtained, but on the fact that, *if the choice of numbers within the previous positions had been different, the problem could then have been solved*. There is no theoretical implication, therefore, about the solvability of the obtained equation or problem: a "dead end" only means that a first treatment leads to the idea that a *modified* treatment would lead to a "better," that is, more manageable problem. For the same reason, "dead ends" always appear as first steps and they are usually complemented by the solution of an auxiliary problem and to a "backward reasoning" (on which more below 3.3.5).

There are exactly four dead ends in book III.[62] In the conspectus, this kind of ending point is indicated by a cross "X."

2.5.4 Equations and prospective equalities

Problem II.13 *aliter* prescribes to subtract two given numbers (6 and 7, in the instance) from the same number so that the result is a square.[63] A first series of positions guarantees that one of the prescriptions is solved.[64] Then the text continues: "One must therefore also subtract 7 units from the 1 *dynamis* and 6 units and make <the result> a square; therefore, 1 *dynamis* wanting one unit is equal to a square" (*Arithm.* 102.13–15). The last part of this sentence is obviously not the end-point of the positions, but is an elliptical paraphrase of the beginning and must be read, therefore: "therefore $1x^2 - 1$ <must be> equal to a square." This is not yet a *finally obtained* equation[65] but only an equation-to-be-obtained: the equation *still has to be completed* and, for this purpose, one ought to obtain a position for the indeterminate square to be obtained and use the corresponding expression. This is indicated in the immediately following sentence: "I form the square from $1x - 2$; the square itself, therefore, is $1x^2 + 4 - 4x$" (*Arithm.* 102.16–17). Only then a *finally obtained* equation is formed by equating this expression to $1x^2 - 1$: "the latter <i.e., the expression for the square> are equal to 1 *dynamis* wanting 1 unit"—these last words, and only them, refer to the equation effectively obtained. In this case, as often, the procedures that lead to the *arithmos* are not detailed and the *arithmos* is immediately given: "and the *arithmos* becomes $5/4$" (*Arithm*. 102.17–18).

In general, then, we do not count here as "equations" in the strict sense defined above certain equalities, in which indeterminate numbers belonging to the statement still enter and which, therefore, require further positioning. Such equalities are recognized by

[62] Problems 10, 11, 15, and 16.

[63] ? X: $X - 6 \rightrightarrows \square(U^2)$, $X - 7 \rightrightarrows \square$.

[64] $X := 1x^2 - 6$ and $U^2 := 1x^2$ for the indeterminate square corresponding to 6.

[65] Contrarily to what Tannery's disposition of the text suggests: here as always, he misleading isolates the last part of the sentence from the rest, as if we had obtained an equation in a modern manner. Allard's edition does not artificially isolate the equation in the same manner.

the fact that they are introduced with adjectives like "*isos*," "*isê*," etc. and that they are sometimes named "*isôsis*" by Diophantus. But they are introduced in such a way that they can be replaced by "*deêsei . . . isos einai*," "*ison poiein*" or "*loipon esti*" with no change of meaning. As mentioned above, such equalities have to be understood as *equations-to-be-obtained.* We shall also designate them as "prospective equalities."

By contrast, we designate by *equation,* either alone or within the expression "double equation," the end-point of a series of positions that solves a given problem and the beginning point of special procedures leading to the final determination of the *arithmos*, without any introduction of new positions. Such equations, understood as "dividing points" between the two kinds of procedures, are recognizable within the Diophantine text (either in the introduction or in the few problems in which the equations are explicitly solved) by the fact that they indeed divide between the two sets of procedures (those enabling to make positions, and those that enable to find out the *simplified equation*, out of which the *arithmos* can be determined) and by special formulas used to introduce them.

The procedures that are applied to *equations* (not to *problems*) and that lead to the *arithmos* are part of what might be called the *solution of an equation*, corresponding to the use of the verb *lyein* in the few cases, in which it is used with reference to an equation. These procedures are (1) the manipulation of the equation through the two basic operations on equations, which are "to remove like from like" (<*aphairein*> *homoia apo homoiôn*) and "to supply the wanting species <on each side>" (*prosthei-nai ta leiponta eidê*),[66] so as to receive either the form of what we called "simplified equations" above, namely "one species equal to one species" or the form "two species equal to one species," and (2) the finding out of the numerical value of the "*arithmos*."

As mentioned before, the two concepts, solution of a problem, and solution of an equation, should not be confused. The solution of a problem comprises more than the mere solution of an equation, namely, the positioning that unravels the core of the difficulty intrinsic to the problem; and, as we already said (cf. 2.2.3 above), the positioning is really the main focus subject of the *Arithmetica*. The solution of an equation, by contrast, only constitutes a very distinct part of the whole solution of a problem and, most often, is not fully worked out in actual solutions (see 2.4.1 above).

2.6 The use of repetition within series of positions: the notion of *method of invention*

So far, we have defined and/or generalized terms that are present in Diophantus and have clarified the structure of solution. We here introduce a new notion that might help us to categorize the various ways, by which positions are introduced. We shall first discuss the use of *heurein* (to invent) with the problems and propose a definition of "invention" that is coherent with this use. Then, for the sake of a finer characterization of the various "modes of positioning," we shall introduce two important notions, that of 'heuristic explanation' (2.6.2) and that of 'method of invention' (2.6.3). We shall

[66] *Arithm*. 14.11–20. These operations respectively correspond to the *muqābala* and *jabr* operations described in al-Khwārizmī's famous treatise of algebra, but they are standard in the ancient mathematical vocabullary.

define the latter notion through the examples already given and then show that this definition is legitimate, taking into account of Diophantus's own words at the end of the introduction.

2.6.1 To find or invent (*heurein*) *numbers and positions within the solutions of problems*

The word *invention* usually translates the Greek *heuresis*, which is used by Diophantus in the introduction. The verbs *to find* or *to invent* usually translate the corresponding verb "*heuriskein*," which is used by Diophantus within the problems. This straightforward translation should not let us forget a simple, but important problem: does the (single) use of invention (*heuresis*) in the first sentence of the preface correspond to the use of the verb *heurein* within the problems?

Strictly speaking, the verb "*heuriskein*" is used for the numbers to be found in each problem. For example Diophantus writes in one of the statements that directly follow problem I.38: "Similarly, in the same <way>, can be found (*heurethêsontai*): two numbers in a given ratio such that the square of the greater has to the lesser a given ratio" (*Arithm.* 76.11–12). But the verb might also refer to the finding out of positions, as is shown by the following occurrence in problem I.21: concerning a sought number that already was posited, it is said, "but <this number> was already *found* <to be> 1 *arithmos* <and> 10 units" (*alla kai* 1 *arithmos* <and> 10 *monades heurethê*) (*Arithm.* 48.26–27). This refers to the first positioning of the number discussed in 48.13–14. There are other examples, especially in the fifth Greek book, in which Diophantus uses the verb "*heuriskein*" not for determinate numbers but for expressions.

Since finding out the determinate numbers that fulfill the prescriptions of the problem derives from their preliminary positioning, the two uses of the *verb* appear to be coherent with each other. Moreover, since invention (*heuresis*), taken in the Diophantine sense, might be interpreted as directly referring to what we have mentioned above to be the main part of the solution of a problem,[67] namely the establishment of positions, it seems also legitimate to call *here* "invention" the very process *in general*, by which positions are established for the solution of problems.

Nevertheless, we strongly emphasize that to translate the Diophantine *heuresis* (as used in the introduction) and to define invention in the above manner are distinct operations. To make them equivalent or not must still be considered to be a question that calls for more contextual elements.[68] For the sake of this analysis, though, we do not need to discuss this point: it is sufficient to understand 'invention' in the above manner, with reference to the solutions of problems. At least, this definition is justified on account of the compatibility with Diophantus's use of the verb *heurein* for the 'finding out of positions' and is sufficient for this analysis.

[67] This thesis has been proposed in (Christianidis 2007).

[68] The basic reason for this is that *heuresis* is used in the introduction in a markedly different manner, and with different connotations, than in the problems. In comparison, *this* single occurrence calls for more comparison with the tradition of ancient rhetoric, as already suggested in (Christianidis 2007). We shall come to this *separate* problem in the conclusion of this article.

2.6.2 *Comparing and distinguishing the ways, in which positions are established: heuristic explanations*

The above analysis and examples show, that positions are often (though not always) introduced with *explanations* or *justifications*. From this comes the characteristic similarity between a logical reasoning and the establishment of the chain of positions. Since we have generally called 'invention' the way in which positions are established, we will here call *heuristic explanation* any justification of a given position or set of positions. As we have seen, they are usually introduced by such words like "*ara*," "*epei*," "*epeidêper*," "*hina*," "*ôste*," or other equivalent expressions. The expression 'heuristic explanation' is purely ours and has no equivalent in Diophantus, although what this expression designates within the text is clearly defined.

In what follows, heuristic explanations will naturally play a significant role in the identification and characterization of the various ways in which positions are introduced. But many of these ways are not (or not only) signaled by these heuristic explanations. In other words, we can characterize these ways, even when they are only partly explained or when the rationale is absent. The reason is that the establishment of a position is basically expressed in the form of procedures, explaining what we shall do, whereas heuristic explanations have to do with the rationale for these procedures, explaining why we should perform them. The difference between the two, though, should not be exaggerated, because procedures might in some cases enter a heuristic explanation, as we shall see, or because heuristic explanations might become superfluous once they have been repeated several times. Procedures used to obtain positions are usually exposed through significant repetitions that are recognizable through the vocabulary used or their 'typical context.' Recognizing them calls for an active comparison between various treatments, whereas explanations tend to make the use of analogy superfluous.

We need, therefore, a new analytical artifact for the analysis of *repetition* within the ways, by which positions are introduced: this is the notion of *method of invention*.

2.6.3 *The notion of method of invention: its definition and legitimacy*

(i) Examples and definition

Let us take the four positions obtained within the solution of problem I.12 (cf. 2.4.1 above), that we might abbreviate as: ? X, Y: $100 \rightrightarrows X + Y(\mathrm{E}_1)$, $100 \rightrightarrows Z + W(\mathrm{E}_2)$, ($W < Z$ and $Y < X$), $X : W \rightrightarrows 2 : 1$ (E_3), $Z : Y \rightrightarrows 3 : 1$ (E_4)

P #	Positions	Heuristic explanation	Ref.
1	$W := 1x$	none	
2	$X := 2x$	none	P_1, E_3
3	$Y := 100 - 2x$	none	P_2, E_1
4	$Z := 300 - 6x$	*kai epei estin* + E_4	P_3, E_4
5	$Z + W := 300 - 5x$	*loipon esti* + E_2	P_1, P_4, E_2

Equation (eg): $300 - 5x = 100$ from P_5 and the given number.

The third column indicates the type of heuristic explanation and the next column refers to the implicit reference requested by the position (either one or more previous positions or one prescription). In this example, the last four positions are clearly obtained in a similar way: in each case, one previous position and one prescription serves as reference and 'guide' for the position, which consists in a straightforward deduction that follows one calculation step implied by the *epitagma*. The very first position is clearly of a different kind, since it comes first (and cannot, therefore, take into account any previous position) and has no heuristic explanation; on the other hand, it is clearly similar to the 'first position' made in problems 1–4 of book I. These five positions, therefore, must fall into two distinct categories.[69]

Taking now as example problem II.8 (complete text in 2.4.2 above), the solution of which is summarized in the following table: ? $X, Y : 4^2 \rightrightarrows X + Y$ (E_1), $\square X$ (E_2), $\square Y$ (E_3).

P #	Positions	Heuristic explanation	Remarks
1	$X := 1x^2$	None	E_2
2	$Y := 16 - 1x^2$	None	P_1, E_1
3	Side of $Y := 2x - 4$	*deêsei ara* + E_3 // 4 is the side of the given square, 2 is arbitrary	
4	$Y := 4x^2 + 16 - 16x$		

Equation (ee): $4x^2 + 16 - 16x = 16 - 1x^2$ from P_2 and P_4

In this case, the two first positions are obtained through the two methods already used in problem I.12: the first position is obtained in the same way in the two problems, and the 2nd of problem II.8, like the last four in problem I.12, is obtained through a calculation referring to the first prescription of the problem and the first position. Only the last two are characterized by the formula '*plassô*,' used in this problem for the first time and by the reference to the type of equation to be obtained. This way of obtaining a position is repeated and used several times in several problems of books II and III, whenever one needs to position an indeterminate square.

These elementary examples suggest that, whatever the position considered, it is always possible to compare the way in which it is obtained to similar cases, of which it is therefore the repetition—with the exception, of course, of the very first occurrences.

In general, then, we use here the expression *method of invention* to denote any regular procedure actually found in Diophantus's work that (a) ends up with the formation of one or more positions and (b) which is repeated in several problems in a clearly similar way.

The first clause (a) might be also expressed in the following way: a method is a procedure belonging to the core of the *solution* (cf. 2.4.1 above); the second clause (b) points out the fact that these procedures are subject to repetition and are therefore liable to be compared to each other.

[69] In the nomenclature that will be presented in part 3, the first is a 'simple, non-derivative position,' the three following are 'derivative' and 'direct reworking.'

(ii) Legitimacy

Like ‘heuristic explanation,’ the notion of ‘method of invention’ is our own invention and does not belong to Diophantus’s vocabulary. Even the notion of “double equation,” which Diophantus *does* name explicitly, as we have seen, is actually *not* a method of invention, by our own definition[70]: it is at best a useful technique of “equalization” by which a pair of prospective equalities is transformed into two equivalent equations.[71] Yet, just as a ‘heuristic explanation’ designates specific and well-defined aspects of Diophantus’s text, the notion of ‘method of invention’ corresponds to specific and recognizable regularities of the text that are announced in the last paragraph of Diophantus’s introduction. Because of the importance of this paragraph we consider it useful to give below its full translation.[72]

> From now on, we shall follow the way along the propositions (*epi tas protaseis chôrêsômen hodon*), having at our disposal much collected matter on the species themselves. Since <the things involved in the collected matter> are many in number and large in size, <and since> for this reason they are mastered only slowly by those who acquire them and <since> there are in them things that are difficult to memorize, I thought it worth to divide what in them is susceptible <of being divided>, and moreover to arrange those being at the beginning, by way of elements, from the most simple to the most complex, as it seems convenient to do. In this way, indeed, their route might become easier to follow for the beginners and their introduction will be memorized. As for their complete treatment, it will be done in thirteen books.

This passage, although essential,[73] is difficult to translate because it is very allusive. What is characteristic and puzzling is the fact that Diophantus actually does not make clear what he speaks about. He uses a number of participles (*ontôn, bebaioumenôn, epidechomena, echonta*) and pronouns (*auta, autois, autôn*), which are all indeterminate. The question is indeed whether these terms refer to one and the same thing or to several, and which ones. The crucial observation is that all of them are in the neuter gender and therefore cannot refer to the statements (*protaseis*, which is feminine). Neither can they refer to the species (*eidê*) in the way they have been introduced, because although they are in the neuter gender, they obviously cannot be said to be “many in number” and even less “large in size.”[74]

70 Cf. 2.5.2 above.

71 Indeed, it does not lead to one or several positions but only permits to obtain a valid equation. See 2.5.4 above and the explanations given in problem III.12 (*Arithm.* 166.14–20).

72 *Arithm*. 14.25–16.7 T, 378.9–17 A.

73 It is curious that modern studies on Diophantus seem to neglect this paragraph. The most that they usually see in it is a rather conventional closing sentence of the introduction. For example Heath does not consider the paragraph so important so as to give it a complete translation (Heath 1910/1964, 131).

74 The difficulty of this passage is manifest in the way it appears in Allard’s edition of the sentence translated above by “Since <the things involved in the collected matter> are many in number and large in size.” This translation is based on Tannery’s edition, which proposes for the Greek text the reading “*Pleistôn d’ontôn* tôi arithmôi *kai megistôn tôi ogkôi*” (*Arithm*. 14.27–28 T). Allard, for his part, proposes for the same text the reading “*Pleistôn d’ontôn* tôn arithmôn *kai megistôn tôi ogkôi*” (*Arithm*. 378.10–11 A). The

The most straightforward interpretation is that these "things" which are qualified as "many" (*pleista*) and "large" (*megista*) refer to the same things, to which the mysterious "collected matter" also refers, since this matter is also qualified as "much" (*pleistê*) in the previous sentence. The plausible reason why this cannot be said is that *it shall be discovered along the way*, just as the aforesaid sentence explicitly says. This interpretation is confirmed by the fact that, a few lines before, the author alludes in a hardly less mysterious way to the "positions of the statements," on which the operations and rules presented henceforth in the introduction are to be "worked out with subtlety" (*philotechneisthô*). The word and notion of "position," indeed, has not be explained nor used at this stage, although it appears to be crucial within the treatment of the problems and only then.[75] In other words, the indeterminate "things" of which Diophantus speaks of designate in all probability the whole multitude of techniques and arguments that have to be learned *within the treatments of the problems themselves*. This use of indeterminate pronouns to designate these arguments is coherent with what is found within the problems themselves, especially in the solutions in which a series of positions is skipped because it is similar to a previously given model: the similarity is simply indicated by "*dia ta auta*"[76] or "*dia tôn autôn*"[77].

Furthermore, it is important to remark that, in spite of the relative silence of Diophantus on the nature and contents of the "collected matter" he alludes to, he does say that *its arrangement is not arbitrary*, but proceeds from the most elementary to the most complex. It is therefore logical to try to specify and name what has been left unsaid, by following very carefully the way, in which the *elements* of this "matter" are exposed. In this respect, the crucial and easy observation that even a superficial reading of the treatments shows, is that there are indeed *arranged in progressive order* and obviously *follow some patterns*, almost any treatment being most often similar to others, with very few exceptions. In most cases, indeed, the *repetition* of these patterns is made *explicit* by Diophantus, with words like "*palin*," "*homoiôs*," "*hôs pro toutou*" etc. and by other features that we will detail below (part 3). On the other hand, as we have seen in the above examples, not all treatments are similar to each other, but certain "patterns" obviously differ from the others. Precisely these two features, that the "patterns of solutions" are repetitive, on the one side, and non-uniform, on the other, legitimate the project of determining, from a careful analysis of the solutions to the problems, what are the main classes of such patterns. Such classes we call here "methods of invention," introducing an expression that *is not found* in Diophantus but actually refers to patterns that *are* recognizable within the text.

The above explanation makes clear the legitimacy of the definition we proposed. As for the usefulness of it, it will be recognized from the possibility of 'categorizing' methods of invention: this is the purpose of the inventory of such methods, to which we shall now turn.

dative "*tôi arithmôi*" in the text of Tannery is changed in the genitive "*tôn arithmôn*" in Allard's text. But the obtained reading "Since the numbers are many and large in size" proposed by Allard is meaningless.

75 Cf. 2.4 above.

76 Like in I.18 (*Arithm*. 40.20) or in I.19 (*Arithm.* 44.5).

77 For example in I.25 (*Arithm.* 60.3).

3 Inventory of the methods of invention

3.1 Introduction: the leading principles of the inventory

Taking as our basis the previously defined notion of 'method of invention,' we have checked all the problems[78] within the first three books of the *Arithmetica* and found out that they can indeed all be classified in a few, recognizable categories. As we have seen, the crucial adjective, in the last sentence, is 'recognizable.' For Diophantus, on the one hand, no more introduced any specific *name* for most of the methods he effectively uses in the treatments of his problems than he used any generic name for what we called "method of invention." On the other hand, the recognizable regularities of the specific vocabulary and situation of each 'positioning' allow us to build categories among these methods.

The particular names chosen here for each method of invention, therefore, are entirely ours. But they are not arbitrary, because our definition of each method refers in any case to specific terms and/or formulas that are recurrent and recognizable. Accordingly, these names have been chosen in a way that tries to reflect the gist of the procedure. In other words, anyone using the characterization we propose for each method in terms of the *recognizable signs*, by which it is recognized for each position or series of positions, should be able to check our classification. Accordingly, we indicate, for each new method introduced in our inventory, what these 'signs of recognition' are, that enable one to identify each method.

We must note, however, that our inventory, though not arbitrary, cannot be entirely objective or faithful to the text, for two main reasons. The first and main reason is that it is not always possible to classify unambiguously each way of obtaining one (or more) positions under one or the other method. This might be due to the sometimes elliptical formulations implied by the accidents of the textual transmission. This might also come from the original wording, especially since a detailed explanation or characteristic expression given for the first occurrences of a method might naturally become afterward elliptic.[79] But, in the first place, it belongs to the natural limits of any such enterprise of 'artificial' classification that the actual occurrences do not fit exactly the proposed schema: what one can expect from this is, at best, a reasonable correspondence leaving out only few ambiguous cases.[80] In other words, some *hesitations* are allowed and inevitable in certain cases; but since, on the whole, most of the cases fall unambiguously under one heading, our classification can be considered to be reasonably faithful to the text.[81]

The second reason is that the *choice* for the 'signs of recognition' is partly guided by the text, and partly by our own appreciation of what differences should be counted

[78] 'Problem' should be understood with the precise meaning defined above, in 2.2.1: as the pair of a statement and one particular treatment.

[79] This phenomenon is characteristic of the ancient mathematical literature at large.

[80] The problem, *mutatis mutandis*, is the same as the one mentioned by Bernard Vitrac in his paper on the notion of *dynamis* (Vitrac 2008, 82), in which he also proposes a classification of the uses of this notion in the ancient mathematical literature, depending on the various possible contexts and authors.

[81] We shall come back to the possible ambiguities of our analysis and their significance in part 4.4.

as 'essential' or 'negligible' each time. This choice, therefore, could be changed and is open to more or less specifications, under certain limits. As mentioned, our general criterion for defining methods of invention was that it should enable one to distinguish one method from the other, with little 'personal' variability. But we have also introduced, within each heading, *variants* that account for possible specifications introducing, for each method, subtler distinctions in their uses: distinguishing one variant from another is more open to personal appreciations.

With these two important caveats in mind, it must be noted that our method is essentially inductive. The essentially modern term 'method' we chose should not induce anyone to think that we are looking, like some of the Renaissance mathematicians, to some *hidden art* that Diophantus would have consciously hidden away. Our purpose is both more modest and, in a sense, historically stronger: we would like to account for the structure of Diophantus's problems taken as a whole and in their actual wording. For the very same reason, our classification cannot be considered valid beyond the three books we have taken into consideration. To take into account more books, either within the remaining three kept in Greek or within the four Arabic books attributed to Diophantus, would inevitably lead to a different (and plausibly richer) classification.

From the examples given above (2.6.1), we can see that methods of invention in general fall into two distinct categories, namely those that permit one to obtain the *first* position(s) within the given treatment of a given arithmetical problem; and those that permit one to obtain *further* positions that depend on these first positions. For this reason, we call the latter "derivative" and we also begin our inventory with them (3.2); non-derivative methods, the repetitive character of which is essentially related to the fact that problems form series, will be studied in the second stage (3.3).

3.2 Derivative methods

3.2.1 Introduction: example and definition

The examples of problems I.12 and II.8, given above, have made clear the difference between 'first' and 'derived' positions. Problem I.22 will show us a more sophisticated example of the difference between the two, as well as providing an initial idea of the various kinds of possible derivations. Its statement is the following: *to find three numbers so that if the first gives the third of itself to the second, the second the fourth of itself to the third and the third the fifth of itself to the first, then, having given and received, all three numbers become equal to each other.*[82]

The statement and the seven positions constituting the core of the solution can be summarized in the following manner:

$$\begin{gathered} ?\ X, Y, Z\colon X - 1/3\ X + 1/5\ Z \rightrightarrows U(\mathrm{E}_1), \\ Y - 1/4\ Y + 1/3\ X \rightrightarrows U(\mathrm{E}_2), \\ Z - 1/5\ Z + 1/4\ Y \rightrightarrows U(\mathrm{E}_3) \end{gathered}$$

[82] This formulation is not a literal quotation, but a mixture of the generic (*Arithm.* 50.21–23) and instantiated (*ibid.* 52.1–3) version of the problem.

P #	Positions	Heuristic explanation	Ref.
1	$X := 3x$	since it gives its third	E_1
2	$Y := 4$	since it gives its fourth	E_2
3	$U := 1x + 3$		P_1, P_2, E_2
4	$\frac{1}{5} Z := 3 - 1x$	*loipon esti* + E_1 // but giving ⅓, receiving $3 - 1x$, it <X> becomes $1x + 3$	E_1, P_1, P_3
5	$Z := 15 - 5x$		P_4
6	$Z - \frac{1}{5} Z := 12 - 4x$	*deêsei ara* + E_3 // but giving ⅕, $3 - 1x$, the remainder is …	P_5, E_3
7	$U := 13 - 4x$	*labôn* from Y its fourth, 1	P_2, P_6

The equation is obtained by equating the expressions posited in P_7 and P_3 (ee): $13 - 4x = 1x + 3$. The *arithmos* is deduced to be 2 units.[83]

We can observe that these seven positions are not obtained in the same way: the two first do not depend on any previous position.[84] *All the* others, by contrast, can be called *derivative,* because they are derived from these two first. In this case, the derivation essentially relies on the intelligent observation of the required prescriptions, which are very explicitly stated or reformulated within the corresponding heuristic explanations.

In general, then, methods of invention will here be called "derivative" when they fundamentally depend on (and are derived from) positions and expressions already obtained within one and the same treatment. As such, they might be opposed to the methods used for the first positions within a given treatment. We simply call the latter "non-derivative" and shall treat them in the following section (3.3).

Derivative methods naturally enable one to obtain a significant proportion of all positions obtained in books I–III (slightly more than half of them).[85] They fall under a limited number of recognizable categories that we now enumerate. In the above example (problem I.22), the derivation depends either directly on the prescription or on its reformulation; this is not always the case, though, and checking the references according to which the derivation is justified enables us to build three distinct categories of non-derivative methods, which we call the *reworking method* (3.2.2), *quasi-simulation* (3.2.3), and *plassô-method* (3.2.4).

3.2.2 The reworking method and its two basic variants

This method is the one used in the previously given examples I.12 (for positions 2–5) and I.22 (for positions 3–7). The important point is that, in all cases, the operation *effectively done* on each expression follows a *prescribed* operation. In general, following this method, as the name indicates, just means to deduce a new position by applying more or less complex operations to one or several of the expressions obtained in previous positions, these operations being in turn indicated

83 No particular explanation is given in this case on the treatment of the equation, which is left implicit.

84 The (non derivative) method for these two will be named later the "two-at-once method" (see 3.3.2 below).

85 On a total of some 560 positions for books I–III, 311 are obtained by derivative methods.

within one of the prescriptions of the problem, either directly or in a reformulated form.

For this method, therefore, the implicit or explicit reference to one or several prescription(s) of the problem is always present and determinant. For prescriptions, by their own nature (cf. 2.2.2) have a procedural form, in which the operations to be performed, as well as the partial results obtained, are made explicit. This crucial feature of the statements makes it possible to refer to such operations. For the same reason, going through positions by reworking is equivalent to saying that the operations indicated by the prescriptions are performed on the expressions already obtained.

This special kind of reference also constitutes the basic criterion that distinguishes this derivative method from others. As the above example makes clear, the aforesaid reference to prescriptions is often made explicit within heuristic explanations by formulae like "and since" (*kai epei estin*), "it remains <to obtain>" (*loipon estin, loipon einai*), "I wish furthermore" (*loipon thelô, epei thelô*), "we must have furthermore" (*deêsei, deêsei ara*), each time followed by the explicit restatement of the prescription, usually in a slightly different form. The paraphrase might include, in particular, some expressions already found and the restatement.[86]

In our abbreviated transcriptions of the treatments, we always indicate the application of this method by the abbreviation "rw" (for "reworking").

Beyond this general definition and abbreviation, two basic variants might be introduced, depending on the way the new expressions are deduced from the preceding ones. We shall now examine some cases, showing that reworking is not necessarily as "easy" or "straightforward" as it may seem.

(i) By *direct* or *straightforward* reworking of one or more positions [rw-d]

Problem I.22 mentioned and analyzed above shows the typical way in which the fifth position (P_5) is obtained from the previous one (P_4). If we have a position for the fifth of a number, then the position for the number itself is easily obtained by multiplying the expression found for the fifth by five. Therefore, the simple operation, here, is to multiply by five; it is indicated, within the heuristic explanation, by the reference to the first prescription.

Such a kind of reworking might also imply several previous positions: in problem I.22, for example, this is the case for the way in which the 6th position is calculated by subtraction of the expressions contained in positions P_4 and P_5, or the third (P_3) from the expressions contained in the two firsts (P_1 and P_2). Similarly, when one has to posit the sum of two or three numbers and when positions have already been obtained for each one of them, this is a case of direct or straightforward reworking from several previous positions.

Direct reworking is indicated by "rw-d" whenever we need to abbreviate the reference to it.

86 Let us note that, from the point of view of the 'global' economy of the complete chain of positions, these heuristic remarks also help the reader to follow the total chain of the positions found, by following the series of prescriptions that they help to solve—especially if the number of prescriptions is high.

(ii) By the *indirect* reworking of one or more positions [rw-i]

In problem I.22 above, the fourth position is deduced from the three first positions in a particular and, so to speak, 'analytical' manner. The reasoning, indeed, is the following, when made totally explicit: we know that removing from the first number (which is posited $3x$ by P_1) 1 *arithmos* (which corresponds to the third), and receiving something, *which is precisely what we want to posit*, we must obtain the expression $1x + 3$. Then we can guess (and then posit) this "something" to be $3 - 1x$: for, added to $2x$, it indeed makes $1x + 3$.

Another example of indirect reworking is found in problems II.30 or II.31, in which the first positions[87] finally yield $12x^2$ (resp. $20x^2$ in II.31) for the product of the two sought numbers. Then the two sought numbers are set $1x$ and $12x$ (resp. $2x$ and $10x$ in II.31) so that the previous product is indeed obtained. This example shows that 'reworking' need not be "deterministic": $3x$ and $4x$, in the case of II.30, would have been possible choices.

Indirect reworking is abbreviated by "rw-i" when needed.

(iii) Complex cases of reworking

Direct reworking, by its own nature, implies in most (if not all) cases, a reasoning: this reasoning, be it explicit or not, basically consists in justifying an operation performed on expressions, by recalling a prescription. We have seen, though, that prescriptions might be reformulated so as to make clear what kind of reworking is expected. The inherently 'analytical' character of indirect reworking might make it much more complex. It implies, at least, a certain amount of "guessing," as in the examples above. In some cases, it might also imply the preliminary reworking and restatement of a prescription in the form of a "prospective equality," so as to make indeterminate numbers appear, for which we already have a position.

A good example of this way of thinking is problem I.24, the statement of which is to find three numbers such that, when each of them has taken a prescribed part of the other two (taken as one), then the three numbers obtained are equal (*Arithm.* 56.12–58.12). The two first positions with the first prescription enable one to posit, as a third position, the three numbers together as $1x+3$, and the number to be obtained as $1x+1$. Then one of the prescriptions to be observed is recalled in a slightly modified form:

> ... furthermore, it will have to be that the second number taking from the two others, taken together, their fourth, shall become $1x + 1$.[88]

Then this prescription is again transformed, with the obvious aim to obtain an indeterminate number only composed of the second sought number and the three together, since we have obtained a position for the latter:

> <Let us multiply> everything by four (*panta tetrakis*): hence, four times the second adding to itself the two <others>, is also thrice the second, adding to itself the three; thrice the second, therefore, adding to itself the three, makes $4x + 4$.

[87] They are obtained by the simulation method, on which more below (3.3.4).

[88] That is, "one and the same number" in the statement has been replaced by the expression already found by the position of this obtained number.

We can thus deduce the position for thrice the second, by removing $1x + 1$ (for the three) to $4x + 4$ (obtained through the previous operation). Then the second number is obtained by direct reworking.

A similar line of thought seems to be followed in problem III.5, in which the sum of the three sought numbers is posited $1x^2 + 2x + 1$ and the excess of two of them together over the third is posited as 1 unit.[89] This choice might be partly explained by an implicit reworking of the prescription similar to the one used in I.24, which amounts to saying that, if the two exceed the third by a square, then the three exceeds twice the third by the same square: hence twice the third might be posited $1x^2 + 2x$ (indirect reworking, left implicit) and the third obtained (and posited) $1/2x^2 + 1x$ by direct reworking.

Reworking always takes as its reference either the special form of previous positions and the operations that they might suggest, or the operations contained within the prescriptions, eventually transformed into prospective equalities. The two other derivative methods take different references: a specific algorithm for quasi-simulation (3.2.3), and the expression equal to an indeterminate square[90] for *plassô*-method (3.2.4).

3.2.3 Quasi-simulation

(i) Example and definition.

This method is used only for two special groups of problems in book I, namely problems I.16–19[91] and for the famous four problems I.27–30. The method consists in deriving *several* positions *at the same time*, by considering one and the same algorithm.

Let us take as example problem I.16, in which this method is met for the first time. The statement asks to find three numbers so that any two of them taken together make given numbers. The abbreviated form of the instantiated problem (with numbers 20, 30, and 40) is the following, together with the outline of the treatment:

? X, Y, Z: $X + Y \rightrightarrows 20\,(E_1)$, $Y + Z \rightrightarrows 30\,(E_2)$, $Z + X \rightrightarrows 40\,(E_3)$.

P #	Positions	Method	Heuristic explanation	Reference
1	$X+Y+Z := 1x$	all-together	none	
2	$Z := 1x - 20$	quasi simulation	if from $1x$ (i.e., the 3 numbers) I remove 20 (i.e., 2 of them), I obtain $1x - 20$ (i.e., the 3rd)	20 comes from E_1
3	$X := 1x - 30$			30 comes from E_2
4	$Y := 1x - 40$			40 comes from E_3
5	$X+Y+Z := 3x - 90$	rw-d	it remains to obtain the three together equal to x …	From P_2, P_3, P_4

The equation $3x - 90 = 1x$ (ee) comes from P_1 and P_5.

89 These positions are again obtained by the method of simulation.

90 This is the only case that appears in the first three books. In other books, however, the case expression equal to an indeterminate cube also appears. See for example the problems of the fourth Greek book: "IV." 26 (*Arithm.* 250.5–6) and "IV." 27 (*Arithm.* 250.19–20).

91 Excluding the alternative treatments 18 *alit*. and 19 *alit*.

The crucial explanation, which is given for the *three* positions P_2, P_3, P_4, is the following:

> Let the three be set 1 *arithmos*. And since the first and second <sought numbers> make 20 units, if from the 1 *arithmos* I remove 20 units, I will have the third, 1 *arithmos* wanting 20 units. For the same <reason>, the first will also be 1 *arithmos* wanting 30 units, and the second 1 *arithmos* wanting 40 units. (*Arithm.* 38.9–12)

The second sentence is clearly not meant to explain that, when I remove 20 units from $1x$, I obtain $1x - 20$, as a literal and de-contextualized interpretation might suggest. Just as what is obtained is "the third number," "1 *arithmos*" has to be understood as "the three numbers together," "20 units" as "the first and the second" and, more generally, as "two of these numbers"; "the third," finally, can be understood generally as meaning "the remaining number." The general heuristic explanation is therefore that, *if from three numbers I remove any two of them, I obtain the third (that is, the remaining one)*. This general explanation holds for the three positions P_2, P_3, and P_4, as the heuristic mention "for the same reason" (*dia ta auta*, 38.11) clearly indicates.

So the particularity of this method is that it is bestowed on one and the same *common* algorithm for *several* positions. The latter are all *derived* though, since they depend on the first position, which is not explained. The first position here is obtained by a method that we call "all-together" method, on which more below (3.3.3.1).

Quasi-simulation is akin, to some extent, to *indirect* reworking in the sense that the deduction of the new positions is not straightforward. But it is clearly distinct from reworking, since what serves as basic reference here is the recognition of a *specific* algorithm, which is neither the same as any of the prescriptions of the problem, nor derived from it by some kind of reformulation.

The three "general" algorithms or relationships used for these problems (that we might name "quasi-simulators") are the following:

- "If from three (resp. four) numbers together I remove two (resp. three) of them I obtain the third (resp. the fourth)": problems I.16 (3 positions) and I.17 (4 positions).
- "Three (resp. four) numbers taken together are the same as twice one of them with the excess of the two (resp. three) other numbers on this last number": problems I.18 (3 positions) and I.19 (4 positions).
- "If from half the sum of two numbers I add and remove half their difference, I obtain the two numbers"[92]: problems I.27 to I.30 (2 positions each time).

Finally, as already stated, quasi-simulation is limited to a very special group of problems that all involve the sum or difference of several numbers.

(ii) Remark on the name "quasi-simulation"

The name "quasi-simulation" adopted for this method derives from the following observations.[93] Like the method of simulation, quasi-simulation requires the explicit

[92] The lesser is obtained by removing, the greater by adding.

[93] Obviously, this part of our explanations should be read after having examined the description and analysis of the method of simulation (3.3.4).

statement of a procedure (here an algorithm in each case), from which *several* positions (and not only one) can be deduced at once. The major difference with simulation is that it does *not* justify the *first* position chosen—although it might have been used for this purpose: on the contrary, the first position is given at the outset, without particular justification and therefore by one of the other methods that we will list afterward (3.3.3). The advantage of such first positions, in the problems mentioned, is only discovered *in retrospect* and from the derived positions.

Also, the underlying procedure is not stated in full generality, as it is done in "true" simulations: it is only given through the particular case of one of the prescriptions and using the positions already set, and it is only *checked* afterward that the positions chosen then solve one of the prescriptions. The other positions are thus obtained by analogy: the procedure is not stated in full but just signalled as being similar to the first one given.

3.2.4 The plassô-method

The first example of use of this method is found in the first solution of problem II.8, as seen above.[94] In this example, the argument serves to posit the side of one indeterminate square and then the square itself, the last that must be posited. The heuristic explanation only states that the expression for the side must be taken as whatever number of *arithmoi* one wishes, wanting the side of the given square (4 in this case). The latter choice is motivated by the following purpose: in the resulting equation, the number 16 is eliminated, so that we obtain an equation of the kind "one species equal to one species." Although this purpose is not made explicit here, this is done in similar cases, like in problem II.12 (*Arithm.* 100.13–14). Moreover, here as in most similar cases, the opening word of this sentence is the word "*plassô*" (I form), which is a quasi-standard formula.

In problem II.8, this procedure is used to supply an expression for one of the requested numbers. However, this is not always the case. On the contrary, in most cases the method is used in the first three books in order to provide expressions for indeterminate numbers other than sought numbers. For example, the statement of problem II.32 asks for three numbers such that the square on each of them, added to the following one, makes a square.

? X, Y, Z: $X^2+Y \rightrightarrows \square(U^2)$ (E_1), $Y^2+Z \rightrightarrows \square(V^2)$ (E_2), $Z^2+X \rightrightarrows \square(W^2)$ (E_3).

P #	Positions	Method	Heuristic explanation	Ref.
4	$Z^2 + X := 16x^2 + 25x + 9$	rw-d	*deêsei ara* + E_3	P_1, P_3, E_3
5	$W := 4x - 4$	*plassô*	*tauta* ($16x^2 + 25x + 9$) *isa tetragônôi*	
6	$W^2 := 16x^2 + 16 - 32x$			

The equation obtained is $16x^2 + 25x + 9 = 16x^2 + 16 - 32x$.

In this problem, the *plassô*-method is used for the last two positions, in order to supply an expression for the square W^2, expression which is imperative for the formulation of the equation, after the expressions of the three sought numbers have

[94] See 2.4.2 above, third position.

been found by means of other methods (positions P_1 to P_3). No heuristic explanation is given here; a possible explanation however could be that the number of *arithmoi* (4) has been chosen so that $16x^2$ might be eliminated in the resulting equation.

In general, then, the "*plassô*-method" is used for the position of the indeterminate number to be obtained in the prescription, this number being a square as far as the first three books of the *Arithmetica* are concerned. This latter prescription is generally the last to be determined before one obtains an equation: all the others indeterminate numbers necessary for this equation have already been posited. As a consequence, the indeterminate square to be posited by this method can be made equal (in the prospective sense) to an expression already obtained. This method is introduced in most (though not all) cases by the characteristic formula "*plassô ek*" (I form from) or "*peplasthô*" (let it be formed).

The *plassô*-method is derivative, insofar as it depends fundamentally on the careful observation of the obtained expression, to which the indeterminate square to be posited must be equalized. In this sense, it derives from the previous positions by which this expression could be obtained. It is akin, therefore, to indirect reworking, though different for two major reasons. First, this method is almost always introduced by the specific formula "*plassô*"; secondly, it refers not only to the prescription, containing the indeterminate square to be obtained, but also to the expression obtained. By contrast, it introduces a position for the side of the indeterminate square, in such a way the resulting equation will be valid and easy to solve. So the two obvious references for derivation in this method, are the form of the expression on the one hand, and the kind of equation to be obtained on the other.[95]

3.3 Non-derivative methods

As their name indicates, non-derivative methods are methods of invention that enable one to obtain positions, usually the first positions in the whole chain constituting the core of the solution, which do *not* depend on any previous one. Six different categories of such methods can be distinguished: "simple" method (3.3.1), "two-at-once" (3.3.2), "all-together" and "sum-against difference" (3.3.3),[96] "simulation" (3.3.4) and "backward reasoning" (3.3.5). Among these six, the first four are mainly distinguished according to the number or nature of the first positions they provide, with little or no heuristic explanation; the last two, by contrast, are essentially characterized either by their special justification (for simulation) or by their dependence on a previous attempt to solve a problem, which therefore can be seen as a partial justification (for backward reasoning).

[95] We do not really know in which way Diophantus's found expressions like the Y's in II.8 or the W's in II.32, in other words how the "*plassô*-method" is practiced in order to obtain these or other similar expressions. Historians of mathematics have proposed since the middle of the 19th century a range of interpretations regarding this particular point of Diophantus's practice.

[96] We associate these two in the same discussion, because they have an intrinsic relationship to quasi-simulation, as we shall see.

3.3.1 The "simple" method and its variants

(i) Examples and definition.

We have seen, in problem I.12 discussed above (2.4.1 and 2.6.3) that the first position was not justified but still was similar to first positions made in the first problems. This position was simply for the sought number, the chosen species (*arithmos*) "directly" corresponding to its 'analogue' within the various 'kinds of numbers.' Similarly, in problem II.8 (*ibid.*), the first position sets one of the sought squares one *dynamis*, in direct correspondence with the requested kind.

Problem I.5 offers a variant on the same method: it asks to find two numbers, one of the prescriptions bearing on fractions of these sought numbers (the third and fifth in the chosen instance).[97] The first position, then, is to take 1 *arithmos* for the fifth of the second number. Although this is not a direct correspondence between the sought number and the corresponding *arithmos*, it comes very close to it: the basic principle remains, to take for one of the sought numbers the 'analogous' species.

The example of problem I.22 (3.2.1) follows a clearly distinct way to begin the chain of positions: while, in the previous examples, all but the first position are dependent on a previous position, in problem I.22 the second position does not derive from the first. The method used for I.12 or II.8 is thus characterized by the fact that *only* the first position is independent from any previous position.

In general, then, the "simple" method is used whenever *one* (and *only* one) of the sought numbers is posited either equal to the species corresponding to the kind of this sought number (like in I.12 or II.8), or to an expression which is very close to it (like in problem I.5). This method is not associated with any specific expression,[98] but only to the particular status of the position done among the whole chain of positions.

The "simple" method is amply used by Diophantus throughout the first book of the *Arithmetica*. It then appears very sporadically in books II and III, in many cases because the problems of the end of book II and book III call for auxiliary problems of a much simpler kind than the main problems treated in those books.

In the first book, the choice for the sought number, for which the "simple" method is used, does not seem to be entirely arbitrary in the case in which there are several possibilities (that is, more than one sought numbers). Very often, the *lesser* number is chosen (when it is clear which is the lesser, of course). This most probably is (implicitly) justified by the fact that, for the subsequent positions, more additions than subtractions will be done on the whole. This care for 'easy' calculations is sometimes made explicit in one of the variants of the "simple" method, as we shall now see.

(ii) Variants and possibility of heuristic explanations

Two main variants of the "simple" method might be distinguished: either there is an exact correspondence between kind and species, as in problems I.12 or II.8 above,

[97] In abbreviation: ? X, Y: gvn (100) $\rightrightarrows X + Y, 1/3X + 1/5Y \rightrightarrows$ gvn (30).

[98] The expression '*tetachthô*' is often used to introduce the first position, but precisely for this reason it is often associated with "simple" method, though not characteristic of it.

or the correspondence is only close, but not exact, as in the case of problem I.5. Within our conspectus, the use of the second variant is signalled by italics.

In the second case, "closeness" just means that the expression is built from the species corresponding to the kind of the sought number (the *arithmos*, in the above example) and from very simple operations (fractions, additions) that generally follow one of the prescriptions.[99] In certain cases, this choice might be made explicit in a heuristic explanation.

An interesting example for this is problem I.21, in which two alternative choices are exposed for the first position.[100] Indeed, two solutions are proposed for the problem, the difference (actually a nuance) between the two lying in the choice of the sought number first posited. In the first variant, the translation starts as follows:

> Let the lesser <sought number> be set 1 *arithmos* augmented by the 10 units by which <this number> exceeds the third part of the middle; therefore, the middle will be 3 *arithmoi*, the lesser thus having the third part of the middle and 10 units (*Arithm.* 48.13–15).

Here the first position is thus $Z := 1x + 10$, and the explanation makes it clear that this choice yields a simple expression for the middle number $(3x)$. But the text continues:

> Or, <we might start> in this way as well: let the middle be set 3 *arithmoi*; and since we want the lesser to exceed the third part of the same middle by 10 units, it will be 1 *arithmos* and 10 units. (*Arithm.* 48.16–18)

Here the choice $Y := 3x$ insures that Z will be posited with an 'easy' expression. These heuristic explanations make clear in this case what is usually left implicit elsewhere, namely, that such expressions makes easier (handier) the expression chosen for the next position, on account of the prescription that will be followed for this first derivation.

3.3.2 The "two-at-once" method

(i) Examples and definition

We have seen with problem I.22 an example of solution, in which the two first positions, not only the first, are non-derivative (3.3.1). Another, interesting example of use of this procedure is found in problem II.3, for it is probably the only one that alludes to a heuristic explanation for the use of this procedure.[101] The statement asks to find two numbers so that their product has to their sum a given

[99] In the case of problem I.5, this is the fifth, which is one of the given parts in the statement and the second prescription.

[100] Abbreviated statement: $?\, X, Y, Z(X > Y > Z) : X - Y \rightrightarrows 1/3\, Z, Y - Z \rightrightarrows 1/3\, X, Z - 10 \rightrightarrows 1/3\, Y$. The parenthesis accounts for the fact that the three numbers are named "greatest, middle and least" in the statement itself.

[101] We have already evoked this explanation when presenting Diophantus's notion of problem and the corresponding notion of "proposing a problem" (2.2.1).

ratio (6:1, in this case) and the two first positions are presented in the following way:

> Let the sought numbers be set 1 *arithmos* and 2 *arithmoi*; indeed, they can also be proposed in a given ratio (*Arithm.* 84.16–17).

This heuristic explanation obviously alludes to the following problem, which is indeed evoked in problem I.34: to find two numbers *in a given ratio* so that the product has a given ratio to their sum. In problem I.34, the "simple" method is used for one number (the lesser) and the second position is found by derivation, following the prescription that the numbers must be in given ratio (which is systematically taken to be 2:1 in this series of problems). In problem II.3, the "two-at-once" method is used, since the two numbers are set "at the same time" $1x$ and $2x$, without a derivation from one to the other. But this "double setting" is cleverly justified by a virtual problem, already treated, in which the same choice *would* represent the use of simple and a direct derivation.

In general, then, we call "two-at-once" the method in which the *two* first positions are non-derivative and are set for two indeterminate numbers that are either sought numbers or a simple combination of them. In particular, it is characteristic of this method that the second position is not dependent on the first. From a grammatical point of view, the "two-at-once" method is recognizable either by the '*ho men …ho de*' structure, or just '*ho prôtos … ho deuteros*,' or '*ho loipos de …*' instead of '*ho deuteros*' for the second number, so that the balance between the two numbers set 'simultaneously,' so to speak, is always expressed and recognizable. Nevertheless, such structures are usual in the Greek language and are not used only when the "two-at-once" method is used.

Unlike the simple method, "two-at-once" method is less elementary and appears not infrequently in book II, as well as in some problems of book III. The reason is that it is used in more sophisticated problems like II.8 *alit.*, II.9 or II.10.[102]

(ii) Variants and special cases

The example of problem I.22 (part 3.2.1) shows that the use of "two-at-once" method might include explanations bearing on the special choice for each of the positions, in a similar way than for the second variant of simple method. Problem I.22 asks to find three numbers, so that each of them giving prescribed parts of itself to the following one, having given and received the results are equal. The first positions run thus:

> Let the first <number> be a certain number of *arithmoi* having a third part, since it gives its third: let it be 3 *arithmoi*. And <let> the second <be> a certain number of units having a fourth part, since it gives its fourth: let it be 4 units. (*Arithm.* 52.4–6)

The heuristic explanation here given for each position is the same as the one sometimes given for the simple method in its 2nd variant.

[102] In problem II.9, for example, in which we are asked to divide a number, which is the sum of two squares, into two other squares ($? \; X, Y : 2^2 + 3^2 \rightrightarrows X + Y, \square X, \square Y$), Diophantus begins with the two following positions $X := 1x + 2$ and $Y := 2x - 3$.

Moreover, some cases of use of the “two-at-once” method show that a simple combination of sought numbers might be chosen instead of sought numbers. Thus, problem I.24 asks for three numbers so that each of them, adding to itself a given part of the two others taken as one, makes equal numbers.[103] The two first positions $X := 1x$, $Y + Z := 3$ are introduced at the same time, 3 being chosen because it has a simple third part, on account of the first prescription. Hence, by direct reworking, the positions $X + 1/3\ (Y + Z) := 1x + 1$ and $X + Y + Z := 1x + 3$. Here the second position is not for one sought number, but a simple combination of two of them (their sum). This variant is used only in problems I.24 and 25.

We also count as “two-at-once” method the argument used for problem II.19, which asks for three *squares* verifying one prescription.[104] The two first positions are presented in the following way: “Let them be set the least number $1x^2$, and middle one $1x^2 + 2x + 1$, obviously <calculated> from $1x + 1$” (*Arithm*. 112.18–20). Here the choice accounts for the two first constraints by a simple correspondence between two of the sought numbers and squares expressed in the terms of the “arithmetical theory.” The second position is similar to the second variant of simple method (we take $(1x + 1)^2$ instead of $1x^2$).[105]

3.3.3 The “all-together” and “sum-against-difference” methods

The two methods now examined are used in the very specific context of two groups of problems in book I (problems I.16–19 and I.27–30) and have in common that they are intrinsically associated with one and the same derivative method, namely the quasi-simulation method that is used in all eight problems of these two groups. We therefore describe the two methods separately and then discuss their relationship to quasi-simulation.

3.3.3.1 “All-together” method As we have seen (cf. 3.2.3), the solution of problem I.16 asks to find three numbers so that any pair of them is equal to a given number, the first position taken being $X + Y + Z := 1x$ and the following others using quasi-simulation. While problem I.17 calls for the use of the same position (1 *arithmos* for the sum of all four sought numbers), in problem I.18 three numbers are required, so that any two of them exceeds the third by a given number.[106] Then the first position is $X + Y + Z := 2x$, a choice which, like in the second variant of simple method, facilitates the subsequent calculations. The three following positions are obtained by quasi-simulation: $Z := 1x - 10$, $X := 1x - 15$, $Y := 1x - 20$, and the expression through direct reworking. Problem I.19 follows a similar procedure.

[103] In abbreviated form: ? X, Y, Z: $X + 1/3\ (Y + Z) \rightrightarrows Y + 1/4\ (Z + X)$, $X + 1/3\ (Y + Z) \rightrightarrows Z + 1/5(X + Y)$.

[104] ? X, Y, Z: $\square X, \square Y, \square Z$, $(X - Y) : (Y - Z) \rightrightarrows 3 : 1$.

[105] Alternatively, but in the spirit of this method, a different choice could have been $Z := 4$ and $Y := 1x^2$. Then X is derived $X := 4x^2 - 12$ and the *plassô*-method enables one to find the *arithmos*.

[106] In the instance: ? X, Y, Z: $X + Y \rightrightarrows Z + 20\ (E_1)$, $Y + Z \rightrightarrows X + 30\ (E_2)$, $Z + X \rightrightarrows Y + 40\ (E_3)$.

For these four problems I.16–19, then, we call the "all-together" method the procedure that consists in positing the sum of all sought numbers as one or several *arithmos/oi*.

3.3.3.2 The "sum-against-difference" method This method is used only in problems I.27–30, in which either the sum or the difference of two sought numbers X and Y, or the sum or difference of their squares, is given along with their product. It consists in setting for the difference $X - Y := 2x$, in case the sum $X + Y$ is given, or in setting $X + Y := 2x$ for the sum, in case the difference $X - Y$ is given.

3.3.3.3 "All-together," "sum-against-difference" methods, and quasi-simulation As noted above, the two methods previously described are restricted to two special groups of problems and are the only methods among all the others listed here to do so. Moreover, they have in common that the very same method of derivation is applied in all these problems, namely quasi-simulation. As we have seen already (3.2.3), the method of quasi-simulation serves to derive several positions at once, by reference to a common algorithm.

Interestingly, we might observe here that each quasi-simulation could easily be turned into a full-scale simulation,[107] since the algorithm in question might serve to justify the very first positions. Thus, in the case of problem I.16, the series of positions could easily be turned into the following simulation[108]:

<table>
<tr><th>P #</th><th>Positions</th><th>Method</th><th>Heuristic explanation</th><th>Remarks</th></tr>
<tr><td>1</td><td>$X+Y+Z := 1x$</td><td rowspan="4">simulation</td><td rowspan="4">if from the three numbers I remove two of them, I obtain the third</td><td></td></tr>
<tr><td>2</td><td>$Z := 1x - 20$</td><td>20 comes from E_1</td></tr>
<tr><td>3</td><td>$X := 1x - 30$</td><td>30 comes from E_2</td></tr>
<tr><td>4</td><td>$Y := 1x - 40$</td><td>40 comes from E_3</td></tr>
<tr><td>5</td><td>$X+Y+Z := 3x - 90$</td><td>rw-d</td><td>it remains to obtain the three together equal to $1x$ …</td><td>from P_2, P_3, P_4</td></tr>
</table>

This could correspond to the following, slight rephrasing of the first explanations given in problem I.16:

> And since the 1st and 2nd sought numbers make 20 units, if from the three numbers I remove 20 units, I will have the third. Hence, let the three be set 1 *arithmos* and the third be set 1 *arithmos* wanting 20 units. For the same <reason>, the first will also be 1 *arithmos* wanting 30 units, and the second 1 *arithmos* wanting 40 units.

That is, instead of setting from the outset the three numbers equal to 1 *arithmos* by a specific method, even this first position might be derived from the general observation of the algorithm that would enable to derive four positions at once.

One plausible interpretation is that such a choice was not preferred at this stage, probably because simulation had to be considered a more sophisticated method that

[107] For the notion of simulation, see the next part, 3.3.4.

[108] Compare the table given in 3.2.3.

had to be introduced progressively. Hence the prudent introduction of a similar method, but framed in a way which is more akin to the other problems found in the first book, which usually follow the pattern "the first or two first positions should be found by some method and the others by derivation." The choice for the first position is guided, in this context, by the special form of the problems, the two groups I.16–19 and I.27–30 sharing enough similarities in their structures as to make possible and easy the recognition of the method to be used in each case.

3.3.4 The method of simulation

3.3.4.1 Examples and definition

(i) The two first examples: problem II.20 and II.26

Let us take, as a first, simple example, problem II.20, which is to find two numbers, so that the square on each of the two, added to the other, makes a square.[109] The text simply begins with the following words:

> Let the first number be set $1x$, the second $1 + 2x$, so that *the square on the first <number>, adding to itself the second, might produce a square*. (*Arithm.* 114.14–15)

The positions $X := 1x$ and $Y := 2x + 1$ are here introduced on the basis of a justification, namely the italicized procedure, which is formulated in a sufficiently general manner so as to make possible the two following interpretations:

(a) It can be read as representing one of the two prescriptions, $X^2 + Y \rightrightarrows \Box$.
(b) But it can also be related to the algorithm $x^2 + (2x + 1) \to (x + 1)^2$, which is entirely expressible in the terms of the "arithmetical theory" and corresponds to a tacit knowledge concerning the possible derivation of an expression (x^2 added to $2x + 1$) into another, namely $(x + 1)^2$.

Here and in similar cases, we will transcribe this ambiguity by using the sign "/": $X^2/x^2 + Y/(2x + 1) \to sq.$

The simulation, in this case, yields the two first positions (the square obtained might have been posited as well, but this is not done in this case) and the others are obtained through usual methods:

P #	Positions	Method	Heuristic explanation
1	$X := 1x$	simulation	simulator $X^2 / x^2 + Y / (2x + 1) \to sq.$
2	$Y := 2x + 1$		
3	$Y^2 + X := 4x^2 + 5x + 1$	rw-d	*loipon esti* + E_2
4	$V := 2x - 2$	*plassô*	
5	$V^2 := 4x^2 + 4 - 8x$		

The equation (ee) is $4x^2 + 5x + 1 = 4x^2 + 4 - 8x$ obtained from P_3 and P_5.

Our second example is taken from problem II.26, which asks to find two numbers, so that their product, added to either, makes a square, and the sides on the obtained

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109 $?\ X, Y\colon X^2 + Y \rightrightarrows \Box(U^2)\ (E_1),\ Y^2 + X \rightrightarrows \Box(V^2)\ (E_2)$.

squares added together make a given number (6 in the instance).[110] The first positions are obtained through the following justification:

> Since, as soon as *there are two numbers, the greater of which is four times the lesser minus one unit, their product adding to itself the lesser makes a square*, I posit the lesser $1x$, the greater $4x - 1$ and it comes, similarly, that *their product adding to itself the lesser makes a square*. (*Arithm.* 122.9–14)

As in the previous case, there is here a procedure (formulated in the first italicized sentence).[111] In this case, the procedure is an algorithm and makes again possible the two following interpretations:

(a) As one of the prescriptions $XY + X \rightrightarrows \Box$, once the "lesser" and "greater" are identified with each of the sought numbers.
(b) As the algorithm $x \times (4x - 1) + x \rightarrow (2x)^2$, again expressible in the terms of the "arithmetical theory" and to which the reformulation strongly alludes. Later on, the text makes clear that $2x$ is the side of the obtained square (*Arithm.* 122.17).

In abbreviation, this double meaning can be written thus: $X/x \times Y/4x - 1 + X/x \rightarrow sq$.

P #	Positions	Method	Heuristic explanation
1	$X := 1x$	simulation	simulator $X / x \times Y / 4x - 1 + X / x \rightarrow sq.$
2	$Y := 4x - 1$		
3	$U := 2x$		
4	$V := 6 - 2x$	rw-i	*loipon esti* + E_2 + E_3 + P_3
5	$XY + Y := 4x^2 + 3x - 1$	rw-d	P_1, P_2
6	$V^2 := 4x^2 + 36 - 24x$	rw-d	P_4

(ii) Definition

In general, then, we call "simulation" the method by which one or more prescriptions might be solved through *the explicit statement of a procedure that can be interpreted in two ways,* one related to the prescription, the other related to an algorithm framed in the terms of the "arithmetical theory." We call "simulator" such an explicit procedure, from which several positions are set as starting points of the treatment. The simulator might be stated directly as a heuristic explanation for the positions, before or after taking them, or can itself be justified by an intermediary reasoning.

The most recognizable mark of a simulation, therefore, is the explicit statement of such a procedure. By procedure is meant a succession of several steps of calculation yielding an outcome. In all the simulations belonging to the books examined here, the outcome is only *specified in kind*, as being a square.

The second feature of simulation is the fact the simulator can be interpreted *in two ways*. This is related to the characteristic way, in which the simulator is formulated.

[110] $?\, X, Y\colon XY + X \rightrightarrows \Box(U^2)$ (E_1), $XY + Y \rightrightarrows \Box(V^2)$ (E_2), $U + V \rightrightarrows 6$ (E_3).

[111] In abbreviation, this procedures reads $l \times g + l \rightarrow sq.$, where l and g are respectively the "lesser" and "greater" number, the greater g being equal to four times the lesser, minus one ($4 \times l - 1$).

Thus, in the first example, the sentence can be read selectively as "the square on the first number, adding to itself the second, makes a square," which makes clear the relationship to one of the prescriptions; on the other hand, when one reads the whole sentence beginning with the positions and 'the first number' is mentally replaced by '1 *arithmos*' and 'the second number' by '2 *arithmoi* and 1 unit,' then we can recognize an algorithm framed in terms of the arithmetical theory $[x^2 + (2x + 1) \rightarrow (x + 1)^2]$. In the second example, the double interpretation is made possible by the clever use of the wording 'the greater / the lesser,' since each member of this pair can refer either to numbers of the statements or numbers in the "arithmetical theory." Also, the repetition of the formula "their product adding to itself the lesser makes a square" conveys the same effect.

This second feature also explains the name we have chosen for this method. This comes from the analogy with the contemporary technique of scientific modeling, for which computerized means are used to describe natural processes by artificial means. By analogy, the term 'simulation' (or 'simulator,' 'simulate') evokes the attempt to imitate an arithmetical prescription by finding out an artificial procedure that, in a sense, account for it.

In general, the formulation of simulators is often bestowed on the ambiguity of a term like 'number,' which might allude to numbers within the statement of the problems or equally, to terms of the "arithmetical theory" (cf. 2.3.3).

The method of simulation is applied in the books II and III of the *Arithmetica* in a wide range of problems: 22 problems in book II on a total of 40 approximately, 20 in book III on a total of about 35 problems. Beyond these mere figures,[112] the conspectus, which is presented in the first appendix at the end of this article, shows that they are massively used from the middle of book II onwards. These forty simulations are furthermore seen to obey an order of growing complexity, depending on the types of the simulators, the number of the simulators as compared to the numbers of simulated prescriptions, or the length and sophistication of the justifications for the simulator. While it does not appear possible to define clear-cut categories between such and such a kind of simulation, it is still possible to distinguish between four stages of simulations arranged in the order of their growing complexity. Each stage might be seen as a variant of simulation, although a formal and clear-cut definition of such variants is pointless: we will rather illustrate each of them by one typical example.

3.3.4.2 Simulators derived from the prescription Only the simulations used in problems II.11 *alit*., 12 and 13 *alit*. are of this kind. For example, the statement of II.11 reads, "to add the same (required) number to two given numbers so as to make each of them a square."[113] The simulator is here deduced at length from the first prescription, in the following way:

> I first look for some number, that adding to itself 2 units produces a square <number>; or, alternatively, some number that, adding to itself 3 units, produces a square. <So> from whatever square I remove the units, <the result>

[112] We have to remember, that the totals include variants and auxiliary problems.

[113] In the instance and in abbreviation: ? $X, Y: 2 + X \rightrightarrows \square$ and $3 + X \rightrightarrows \square$.

> will be the sought <number> . Let us suppose that <we do this> with the 2 units, and let them be removed from 1 *dynamis*; the remainder will be $1x^2 - 2$, so that it is clear that, *if it adds to itself* 2 *units, this will produce a square.*" (*Arithm.* 98.4–10)

Like in problem II.20 above, the simulator, in this case, is stated in the last words and in retrospect: X / $(x^2 - 2) + 2 \rightarrow sq$. This simulator might be read as expressed in the terms of the "arithmetical theory" ($(x^2 - 2) + 2 \rightarrow x^2$, understanding by "it" the $x^2 - 2$ previously set), but could also be read as directly related to the prescription $2 + X \rightrightarrows \square$, understanding "it" as the sought number of which it is question before the positions.

3.3.4.3 Simulator related to a familiar formula Problem II.14 is the first in which this variant appears. The problem here is to divide a given number into two parts, and to find furthermore a square, which, when added to each of the two parts, makes a square.[114]

> Take two numbers so that the squares on them are <together> less than 20 units; let them be the numbers 2 and 3. And if $1x$ is added to each of them, the squares on them will be $1x^2 + 4x + 4$ for the one, and $1x^2 + 6x + 9$ for the other. If then from each of them I remove the $1x^2$, that is, the square, *I shall have the sought <numbers> which, when adding to themselves a square, obviously make a square*. (*Arithm.* 104.2–8)

The simulators are, like in the previous example, expressed *in fine* in terms that are very close to the last two prescriptions: these are two simulators built on the same model $[(x + a)^2 - x^2] + x^2 \rightarrow sq$. The first is $X/[(x + 2)^2 - x^2] + Z/x^2 \rightarrow sq$. and the second $Y/[(x + 3)^2 - x^2] + Z/x^2 \rightarrow sq$. But the basic justification essentially relies on the recognition, from the familiarity with such expressions, that any square like $(x + a)^2$ can be regarded as a number $(2ax + a^2)$ added to a square (x^2).

The gist of the justification for the simulator, therefore, relies on a quite straightforward identification of the 'squares' mentioned in the prescriptions, and the squares present in the (seemingly standard) development of expressions like $(x + a)^2$ into $x^2 + 2ax + a^2$. Problems 15, 16 and 20–23 in the same book are very much of the same kind, but slightly more complex than the previous one, since the simulator is not deduced from one prescription, but rather from the straightforward reflection on standard algorithms, *that have, therefore, to be effectively familiar* to the reader and/or disciple. Such simulations require some "topical knowledge."

3.3.4.4 Simulator related to an arithmetical algorithm Problem II.26 commented above is a good example of this variant: the simulator, in this case, is neither derived from a prescription (3.3.4.2) nor from some familiar algorithm intrinsic to the "arithmetical theory" (3.3.4.3), but from another algorithm, which pertains in general to the knowledge of arithmetical algorithms. In II.26 this 'arithmetical algorithm' is expressed in 'generic' terms, as we saw: the 'numbers' involved might be terms of the "arithmetical theory," or 'simple numbers,' it does not actually matter.

[114] In the instance and in abbreviation: ? X, Y, Z: $20 \rightrightarrows X + Y, \square Z, Z + X \rightrightarrows \square, Z + Y \rightrightarrows \square$.

In many cases, though, the simulator is first stated with 'simple' and determinate numbers and then *completed* by what we call an "in-species" argument. For example, problem II.25 asks to find two numbers so that the square on their sum, wanting either number, makes a square.[115] The solution uses simulation and is expressed in the following manner:

> I first take some square, from which, by removing some <well-chosen> numbers, I leave out a square. Let <the number> be the <number> 16. *The latter, when it leaves out* 12 *units, becomes a square, and again when* 7 *units, it becomes a square.* Hence, I posit them again *en-dynamei*, so that the one is $12x^2$, the other $7x^2$ and the <square> on the two, $16x^2$, and it remains that *the <square> on the two, wanting each of them, makes a square*. (*Arithm.* 120.14–20)

Here a kind of generic procedure, which gives the rationale for the two simulators chosen, is given: "one square wanting one or the other number makes another square."[116] 16 being chosen for the first square, the two simulators deduced from this idea are $16 - 12 \rightarrow sq.$ and $16 - 7 \rightarrow sq.$ and are then restated "*en-dynamei*," that is, as $16x^2 - 12x^2 \rightarrow sq.$ and $16x^2 - 7x^2 \rightarrow sq$. More precisely, the last formulation makes a kind of synthesis of the various restatements: $(X+Y)^2/16x^2 - X$ or $Y/12x^2$ or $7x^2 \rightarrow sq$. The basic idea underlying the simulation, therefore, is found in simple arithmetical relations, the knowledge of which (again) relies on the arithmetical culture of the reader/disciple, and which are then restated with species in a second step.

In general, then, we call an argument "in-species" when, out of a numerical algorithm, an algorithm framed in terms of species, either *en-arithmois* or *en-dynamei*, is created. Such an argument is used, either explicitly or implicitly, when a simulator can be first formulated without species,[117] and can be afterward reformulated with the above-mentioned species. This argument thus implies a reformulation of the first simulator in term of species, from which the positions might be deduced. This argument is not, therefore, a method of invention *per se*, but must be considered as a complement to simulation, and the two taken together might be considered as a variant of simulation.

There are 15 cases of such a variant in books II and III, six of which are explicit, like the above example II.25. In several cases, though, the argument is used implicitly, like in II.24, to which II.25 is actually similar[118]:

> And since $1x^2$, *when you add to it* $3x^2$, *or* $8x^2$, *makes a square*, I posit among the sought numbers, the one $3x^2$, the other $8x^2$ and finally the square on the two $1x^2$, so that there remains that *the <square> on the two, adding each of them to itself, makes a square*. (*Arithm.* 118.22–120.1)

Here it is not stated explicitly, like in II.25, that a simple square (like 1) might be chosen, to which other numbers like 3 or 8 might be added, so as to give squares (like

[115] $?\ X, Y: (X+Y)^2 - X \rightrightarrows \square, (X+Y)^2 - Y \rightrightarrows \square$.

[116] $sq. - nb1/nb2 \rightarrow sq.$

[117] Like *arithmos, arithmêton* or *dynamis*.

[118] The problem is the same, except that the numbers are added and not removed from the square on the two.

4 and 9), and then that this arithmetical algorithm might be formulated "*en-dynamei*"; but it could be formulated in this way with no loss of meaning, and the formulation of II.25 ("I posit them *again en-dynamei*") clearly alludes to the procedure of II.24, which is thus recognized to be an application of the same method.

The pair of problems II.34 and 35, which concludes book II, is another case of two similar treatments using in-species, the one (II.34) being explicit about this use, the other (II.35) implicit. The statement of the simulator, in this case, is spectacular and worth quoting:

> And since, *whenever a number is measured by some number, and when we take the one, according to which it is measured, and when we remove the lesser from the greater among these two (the one that measures, the one according to which it is measured), then the square on half of the remainder, adding to itself the number from the beginning, makes a square,* I posit the sum of the three <sought numbers> with a certain number of *dynameis* having three measuring numbers. Let it be 12. 1 unit indeed measures the latter according to 12 <times>, as well as 2, according to 6, and 3 units, according to 4. And if I remove the measurer from the one, according to which it measures, and if I take half of the remainders, I posit the three <in the following manner>: as for the first, 5 units and $1/2$; for the second, 2 units, for the third $1/2$ unit; and it is clear that *the square on each of them, adding to itself the <number>* 12, *makes a square, one of them* 12 *and* $1/4$, *the other* 16, *the last* 42 $1/4$. I form them therefore *en-arithmois*: the first <is> $51/2\,x$, the second $2x$, the third $1/2\,x$. (134.16–136.1–4)

We have here a mixture of an 'abstract' and generic simulator (as in II.26) and the choice of a 'simple' number (as in II.24). The generic algorithm holds for three separate prescriptions and might be summarized thus: $[1/2(m-t)]^2 + nb \rightarrow sq.$ when $nb = m \times t$, when m measures nb according to t. The three separate simulators are then obtained (a) through three different choices of measures of the same number 12: 1, 3 and 2, and (b) through the explicit use of "*en-arithmois*" argument.

The simulator might be interpreted as a kind of 'arithmetical reading' of Euclid *Elements* II.5, like those that are suggested in some *scholia* to Euclid. This seems, then, to presuppose again in the reader a familiarity with such an 'arithmetical reading' of geometrical propositions.

3.3.4.5 Simulators related to a combination of prescription and formula There are a few problems in book III, namely problems III.5, 6 and 6 *alit.*, 8 and 9, which introduce sophisticated simulations that subtly combine the reflection on prescriptions *and* on algorithms expressed in the terms of the "arithmetical theory." This is the case, for example, in problem III.6, which asks to find three numbers that either taken all-together or in pairs, make a square.[119] The simulator is expressed in a very allusive way:

> Let them be set the three <sought numbers> equal to a square, $1x^2 + 2x + 1$; and the first with the second, $1x^2$. *The remaining third* will be, therefore, $2x + 1$.

119 In abbreviation: ? X, Y, Z: $X + Y + Z \rightrightarrows \square(\mathrm{E}_1)$, $X + Y \rightrightarrows \square(\mathrm{E}_2)$, $Y + Z \rightrightarrows \square(\mathrm{E}_3)$, $Z + X \rightrightarrows \square(\mathrm{E}_4)$.

And again, since we look for the second and the third to make a square, let it be made $1x^2 + 1 - 2x$, from side $1x - 1$. And the three are $1x^2 + 2x + 1$; *the remaining first*, therefore, is $4x$. (*Arithm.* 146.17–148.1)

The simulation amounts here to the recognition that, *when from three numbers taken together two of them are removed, the remainder is the remaining third*—to this alludes the expressions "the remaining [first or third]." Hence $(X + Y + Z)/x^2 + 2x + 1 - (X + Y)/x^2 \rightarrow Z/2x + 1$ simulates E_1 and E_2 at the same time; similarly, a judicious choice for $Y + Z$ simulates E_3 and yields a manageable expression for X, according to the simulator $(X+Y+Z)/x^2+2x+1-(Y+Z)/x^2+1-2x \rightarrow X/4x$. The basic idea, beyond the recognition of this relationship, is to take a square for the three (as requested by E_1) and remove a square that will yield a simple result: $1x^2$ in the one case, $1x^2+1-2x$ in the other, in order to simulate E_2 and E_3. Both the reflection on the special form of the prescriptions, and the knowledge of the potentialities of algorithms expressed in the terms of the "arithmetical theory," are here used in a subtle combination.

3.3.4.6 Other simulations introduced as a first step of more complex methods Finally, there are some problems in book III, which are included in III.10, 11, 15 *alit.* and 16, in which simulations are introduced that are by themselves less sophisticated than those examined above, but that enter a complex set of intertwined problems, for which the method of "backward reasoning" is used (on which more below 3.3.5). In other words, the sophistication of such treatments does not lie in the simulation by itself, but in its reappraisal through another method.

Problem III.10 is a good example of this: we are asked to find three numbers such that the product of any two of them, adding a given number (12) to itself, makes a square.[120] The first problem studied in this context begins with a simple simulation completed by an in-species argument:

P #	Positions	Method	Heuristic Explanation
1	$U^2 := 25$	simulation with *en-arithmois*	simulator built from E_1: *sq.* (25) – 12 → XY (13)
2	$XY := 13$		
3	$X := 13$		when set *en-arithmois*, the product remains 13
4	$Y := 1$		
5	$X := 13x$		
6	$Y := \frac{1}{x}$		
7	$V^2 := 16$	simulation with *en-arithmois*	simulator built from E_2: *sq.* (16) – 12 → YZ (4)
8	$YZ := 4$		
9	$Z := 4x$		when set *en-arithmois* + P_8 and P_6
10	$ZX := 52x^2$	rw-d	*deêsei ara* + E_3

What characterizes this treatment is the fact that it leads to a dead end, which in turns leads to the formulation of a new, auxiliary problem and finally to a "correction" of this first simulation. So the sophistication of the complete treatment lies not in the

120 ? X, Y, Z: $XY + 12 \rightrightarrows \square\,(E_1)$, $YZ + 12 \rightrightarrows \square\,(E_2)$, $ZX + 12 \rightrightarrows \square\,(E_3)$.

simulation but in this "improved repetition" of a previous treatment. This is done by "backward reasoning," which we will expound next.

3.3.5 Backward reasoning

By "backward reasoning," we mean the method of invention by which the first positions of a problem essentially depend on the treatment of two previously treated problems, one leading to a dead end and the second providing numbers. The method essentially consists in following the schema of the solution leading to the dead end, while taking as numbers those obtained in the second problem. It amounts, in other words, to bypassing the difficulty that leads to the initial dead end by using the outcomes of an auxiliary problem.

In the example of problem III.10, which contains, as we have seen, three different statements and four different solutions (cf. 2.2.1), the treatment of problem (a) leads to a dead end and to problems (b) and (c), and problem (c) has the following conclusion "and the one <found number> is 4, the second 1/4. *Each of these numbers, together with 12 units, makes a square*" (*Arithm.* 160.1–3). The treatment of problem (d) can then be summarized by the following table:

P #	Positions	Method	Heuristic Explanation	Remarks
1	$X := 4x$	backward reasoning	these numbers <4 and ¼ > being found.	ref. to positions found in attempt (a) and nbs found in (c)
2	$Y := {}^{1}\!/_{x}$			
3	$Z := \frac{1}{4}\,x$			
4	$ZX := 1x^2$	rw-d	*loipon estin …*	from P_1, P_3
5	$W := 1x + 3$	*plassô*		
6	$W^2 := 1x^2 + 6x + 9$			

What make this last treatment remarkable are its shortness and its dependence on problems (a) and (c). The solution or results of the latter constitute the basis and explicit justification for the first positions taken in (d). This explicit dependence upon previous problems, which goes beyond a mere analogy, is the distinctive mark of backward reasoning.

In book III this method is used for the four problems III.10, 11, 15 *alit.* and 16. The method used for III.13 might be interpreted as some kind of backward reasoning, but this is hard to decide it in an unambiguous manner. We shall come back in the next section to this ambiguity (see especially part 4.4).

4 Presentation of the conspectus

We now present the conspectus of the treatments of Diophantus's problems in the three first books of the *Arithmetica* that can be deduced from the previously established classification of the methods of invention (part 3) and the preliminary discussion justifying the very notion of "method of invention" (part 2).[121]

[121] The conspectus is presented in the first appendix. The original conspectus itself uses colors that can be seen by downloading the file on the website of Centre Koyré (see project 'mathématiques et histoire'). The colors are not essential to the conspectus, but they make it easier to read.

It is important to note from the outset that, by its very nature and foundation, this conspectus does not reflect merely the order of the *statements* of the problems but first and foremost the order of their *treatments*. In this sense our conspectus differs from other conspectuses that were made in the past by historians like Paul Tannery[122] or T.L. Heath[123] and which relied on the statements of the problems only. We have also established our own list of the statements, transcribed in abbreviated and convenient formalism,[124] accompanied by a detailed analysis of the various positions taken each time, as well as the method used and the heuristic explanations given. To reproduce here this detailed analysis in a dedicated appendix would have made this (already long) article reach an unreasonable length; we chose, therefore, to reproduce only examples of these detailed analyses for the sake of illustrating the methods exposed in part 3.

We claim that only tables showing the order of the *treatments* really enable one to get an *overview* of the coherent ordering of Diophantus's problems. By contrast, any synopsis of the statements is certainly useful for reference, but little order and progression emerges from it.

We first briefly expose the legend of the conspectus (4.1), then discuss what conclusions can be drawn from its examination: the limited number of methods used, their progressiveness and the increasing complexity of the treatments (4.2); we next expose the new criteria it might provide for judging whether or not such or such a problem and its treatment might be considered as interpolated or displaced (4.3). We finally discuss the limits of our analysis, and examine for this purpose the more complex cases, for which the conspectus appears useful but also somewhat limited (4.4).

4.1 How to read the table

Each line of the table corresponds to the treatment of a problem introduced by an explicit or semi-explicit statement, that is, to a problem in the Diophantine sense, as explained in 2.2.1. Thus, a problem in Tannery's ordering,[125] might comprise several problems, which are recognized by sub-numbering in the table (for example, the 10th problem of book III is divided into the four 10.1, 10.2a, 10.2b, and 10.3). In a few cases, some of these problems might also correspond to one single prescription of the statement of a "problem," as we have seen in 2.2.2.[126]

The first columns, under the heading 'statement,' refer to some of the numbers contained in the statement: sought numbers, number of constraints (2.3.4), numbers

122 See (*Arithm*. ii, p. 287–297): *Conspectus problematum Diophanti*.

123 See (Heath 1910/1964, 260–266): *Conspectus of Arithmetica*.

124 The formalism adopted to abbreviate the statements is partly inspired by Ver Eecke's French translation of the *Arithmetica* and presented in the second appendix.

125 This order corresponds to that of the manuscript tradition.

126 See also note 18.

'abbreviated' with capital letters in our synoptic writing of the statement,[127] and finally given numbers or ratios (2.3.1).

The number of positions done in each treatment is indicated in the next column. These figures are usually correlated to the numbers listed in the columns 'statement.'

The number of positions done for each problem is detailed in the next group of columns: positions are "classified" according to the method used to establish them. Hence the columns are classified under two main heading, corresponding to derivative (3.2) and non-derivative (3.3) methods. Each column is then named after the methods listed in part 3.

Some conventions have been adopted for particular methods:

- For the *plassô*-method, the number '1' indicates that only the square is set, and '2' indicates that both the square and its side are posited.
- For the "simple" method, italicized '1' correspond to the variant we called 'simple method 2' (3.3.1).
- For the "two-at-once" method, underlined '2' refer to the fact that the second number is posited a determinate number of units.
- For simulation, the number of simulators as well as of simulated constraints is indicated. A special column also indicates the "in-species" variant. In the "non-derivative" group, the columns are ordered from left to right in the order of growing complexity or sophistication.
- When we had hesitations about the interpretation of a solution, we have indicated this either by a question mark ('?') or two alternative numbers (e.g., '1/2').

The next column on the right indicates the kind of ending obtained in each treatment (cf. 2.5). The "ambiguous treatments" are indicated in the penultimate column, that is, the cases in which we have hesitated between various possible interpretations, due to an elliptic or damaged text or to the limits of our interpretation. In the last column are added remarks on the explicit dependence of a treatment on previous treatments.[128] These two columns will prove to be useful for the discussion of the most complex treatments (below section 4.4).

The dotted lines indicate treatments that we deem to be interpolated either for stylistic reasons or on the basis of the criteria explained below (4.3).

Finally, we have indicated, at the end of each book and at the end of the entire conspectus, the total number or problems and of positions obtained for each category.

4.2 The limited number of methods and their progressivity

What are the main conclusions that might be drawn from the global examination of this conspectus? The first and most obvious conclusion comes from the recognition that a *few* methods of invention are systematically used for the total number of positions

[127] These numbers are all indeterminate. However, a statement contains other indeterminate numbers, those expressed as 'partial results,' like *rest, sum,* etc. (cf. 2.3.1). Hence, only sought numbers and "squares to be obtained" are represented by our abbreviation system and are, therefore, enumerated in this column.

[128] This column has not been reproduced in the printed version of the article, for lack of space, but is available on the electronic version of the table.

held in the three books, which amount to approximately 560. Even when taking into account the problems that are difficult to interpret, and the possible variants of the methods, the whole bulk of positions fall under one of these 11 methods.[129] If one restricts oneself to the roughly 250 positions which are not derived, the proportion between this number and the 8 methods involved remains significant. Therefore, it seems justified to speak about only a *few* methods being involved and this also confirms, in retrospect, that our definition of a method as *repetitive* is well grounded in the text: they *are* really repeated on a significant number of positions each time.

The second, no less obvious conclusion stems from the recognition that the methods are not only few in number, but that they are really used in a very progressive order. Hence, Hankel's famous judgement[130] proves to be false: a reader following the propositions in their order might really and progressively be imagined to learn the specificity of each method, and how one might pass from one method to more sophisticated treatments. This progressivity is made manifest by the fact that, *grosso modo*, the table of non-derivative methods seems to exhibit a kind of diagonal beginning from the left corner of the first problem and ending up to the right corner of the final lines. Given the remarkable regularity of this progression, it is even sound to speculate that the problems were most plausibly arranged not according to their statements, but, first and foremost, *in a way, that would enable one to study one method after the other*. In any case, that the problems have been arranged in such a way, as to begin with simple treatments and going forward with more sophisticated cases, is precisely what is said at the end of the introduction.[131] What the conspectus shows, therefore, seems plainly coherent with Diophantus's account of the arrangement of problems. Most plausibly, the problems were arranged not as a series of statements, but a series of *statements-and-treatments* arranged in such a way as to provide the reader and learner a sense of the variety of ways in which this treatment might be conducted.[132] This characteristic is related to the general analysis of Diophantus's project, an issue to which we will come back in the conclusion.

Finally, what the conspectus also shows is the growing *complexification*, again announced in Diophantus's introduction, of the treatments. For example, a problem like II.8, most famous for the introduction of indeterminate squares in the statement, appears through our conspectus as the stage, at which the "*plassô*-method" is introduced for the first time.[133] This treatment is certainly 'elementary,' as compared with the many other problems in which *plassô* is freely used as a final complement to other positions. This growing complexity is typically what the synopsis of the third

[129] 3 are derivative, and 8 non-derivative, counting the "simple 2" variant, "all-together" and "sum-against-difference" as distinct, and "in-species" as a variant of simulation.

[130] Cf. the introduction of this article.

[131] Translated and commented in 2.6.3 (ii).

[132] From the few experiences of collective reading of these problems that we have tried on modern teachers, it very clearly appears that the "translation" process, that is, the series of positions held in each problem, is *the very last aspect of the treatment to which they paid attention*. A modern reader is too much focused on aspects of the statements that might be studied with *other* tools, to recognize the core of Diophantus's procedure. On this issue, see Bernard 2011.

[133] It appears along with an alternative treatment that is more in line with the previous ones. See the comparative analysis in 3.2.4 above.

book shows, with its spectacular and alternate use of several methods for inter-nested problems, like the four problems 10, 11, 15 and 16, in which backward reasoning is used. More generally, the last column shows that this third book in the form, in which we have it,[134] seems rich in cross references relating one treatment to previous ones. Hence the impression of 'horizontal explosion' that one has when reading this part of the conspectus. We come back to this phenomenon in section 4.4, because this question of complexity of the treatments also brings our interpretation to its natural limits.

4.3 The possible definition of new criteria for isolating interpolations and displacements

4.3.1 Mathematical incoherency: not always a reliable criterion

An interesting consequence of the conspectus is the possibility of building new criteria for suspecting or, on the contrary, disproving the possibility of a particular interpolation within the text. Indeed, it is well known that strictly philological criteria are not the only ones used by editors like Tannery for making a difference between what they consider to be interpolated and what is authentic to them. More or less explicit criteria about the *mathematical coherence* of the argument, as well as arguments about the alleged "style" of Diophantus as the author of the *Arithmetica*, come into play. The "style," here, does not refer to what philologist call stylistic variations, which really refer only to a certain regularity of the text as we have it, but really to the alleged "constant behavior" of the author of the text, when he is taken to be a mathematician and as if we would really *see* him at work—or, to recall Norbert Schappacher's suggestive words, "to read over his shoulders."[135] Such criteria most often appear to be unreliable, because they rely too much on presuppositions about what is or not mathematical, and who is or who is not a mathematician: such presuppositions are generally open toward more or less blatant anachronisms. The risk of such misunderstanding is always very high in the case of treatises that, like Diophantus's, are made of *series of problems*. A problem, indeed, strongly appeals for mathematical invention, whatever the period considered. Moreover, the terms used for the description of procedures generally *evoke* the personal involvement of a teacher. The risk of anachronism is even higher when this treatise bears such a prestigious name as "Diophantus," this prestige largely deriving from the rich traditions of mathematical reading and commentary that this text underwent in various contexts and periods.[136]

It is not our purpose here to confirm or disprove other interpretations of the text on the basis of a discussion or the eventual misreading of the text, due to special biases. Some of these interpretative problems, which are related to the long-lasting history of

[134] Indeed, the presence of cross references might be due to scholiasts. This hypothesis is not in contradiction with the interpretation developed here: scholiasts might have understood the spirit of the text and contributed to the text by making some transitions more explicit.

[135] See (Schappacher 1998). The article is available online in English and in a reworked form, (Schappacher 2005).

[136] The kind of anachronism related to the confusion between modern and ancient teaching situations is discussed in the introduction of (Bernard and Proust *forthcoming*).

interpretation, both mathematical and historical, of Diophantus's text, would deserve a separate discussion. For this moment, we focus only on the *positive* criteria for suspecting interpolations or displacements that might be derived from our conspectus and its underlying interpretation. The strength of this interpretation, as we have seen above, lies precisely in the fact that it might provide a firm basis for interpreting the *coherency* of a significant part of the Diophantine problems (namely those of the three first books) both with each other (that is, in their proper arrangement) and with the contents of the introduction.

What this study shows is just that the conspectus and its basis, taken as analytical tools, fit in reasonably well with the chosen portion of the text and with the contents of the introduction, taken as a whole. Since, however, this conclusion *is only approximately true*, because there are problems that seem isolated or "strange" within this coherent whole, then this strangeness might be naturally interpreted as stemming from changes made on a primitive state of the text in the course of its history—interpolated or displaced problems being two such possible changes.

If, then, we take our interpretation and analytical framework as a reasonable conclusion, confirmed by its relatively good fit to most of the text as we have it, then what should be the problems that might be suspected to have been interpolated or displaced at some stage of the history of the text, and on this sole basis?

4.3.2 First criterion: fitting in (or not) within the global progressiveness of problems

Very clearly, the major and first criterion that might be derived from our interpretation is (A) *the degree, in which a given problem fits in the recognized progressivity of the whole*. Thus, it is difficult to suspect a problem like I.33, inserted into a monotone and long series of similar problems, to be interpolated. On the contrary, the two problems II.17 and II.18 fit in quite badly within the series treatments through simulations in which they are inserted. The first problems of book II raise more interesting problems, for they seem to be very close to the last problems of book I, but are obviously different from them in the systematic use of the "two-at-once" method. This use, in turn, is coherent with what is found in problems II.8 *alit.*, II.9 and 10. It all happens as if these problems constitute a transition between the treatments of the first book and the more sophisticated statements found at the beginning of the second book. Therefore, the *repetitive* characteristic of these problems, together with the fact that they present slight but meaningful variations on the model they repeat, strongly call for their being "genuine."

Another, interesting example is provided by the alternative treatment of problem I.18. We have signaled this problem (as well as I.19 *alit.*) as ambiguous, because there is one position that either appears *not* to be done through *any* of the methods otherwise listed, or through an implicit chain of positions not stated in the text. In the first interpretation, one sought number is *directly* calculated and posited through an algorithm on given numbers, furthermore with no detailed explanation of this "direct position"[137]:

[137] Moreover, if the fact of finding alternative proofs is not unusual at other places, it most often seems to obey the pattern of providing a 'more standard' treatment than the first one, in which a new method is introduced. For example, the first treatment of II.8 introduces the *plassô*-method for the first time, whereas

Problem I.18: ? X, Y, Z: $X + Y \rightrightarrows Z + 20\,(E_1)$, $Y + Z \rightrightarrows X + 30\,(E_2)$, $Z + X \rightrightarrows Y + 40\,(E_3)$

P #	Positions	Method	Heuristic explanation	Remarks
1	$Z := 1x$	simple 1	*epei* + E_1	the 3rd is chosen
2	$X + Y := 1x + 20$	rw-d		P_1, E_1
3	$Y := ½ (20 + 30) = 25$	dir. inf.	*palin epei* E_2 (and E_1)	
4	$X := 1x – 5$	rw-d	*epei* P_2 and P_3	P_2, P_3
5	$Z + X := 2x – 5$	rw-d	*loipon dei* + E_3	P_1, P_4, E_3

Equation (ece): $2x - 5 = 65$ from P_5 and conversion of P_3 $Y + 40 \rightarrow 65$

All this encourages us to suspect an interpolated treatment that does not correspond well to the 'spirit' of the rest. This would be coherent with the philological argument of Tannery, who deemed this alternative proof, missing in some manuscripts, as interpolated.

In the second interpretation, the position is derived from the previous one by two implicit positions not stated in the text, but allusively mentioned by the adverb "*palin*" and the analogy with the previous explanation for the second position, therefore, leading to a similar but implicit position.

P #	Positions	Method	Heuristic explanation	Remarks
1	$Z := 1x$	simple 1	*epei* + E_1	The 3rd is chosen
2	$X + Y := 1x + 20$	rw-d		P_1, E_1
3	$Y := ½ (20 + 30) = 25$	rw-d	*palin epei* + E_2	Implicit $Y – X := 30 – 1x$ or $X – Y := 1x – 30$ with P_1, E_2
4	$X := 1x – 5$	rw-d	*epei* P_2 and P_4	P_2, P_4
5	$Z + X := 2x – 5$	rw-d	*loipon dei* + E_3	P_1, P_5, E_3

In this second case we might similarly suspect an interpolation, but on different grounds than in the first: here, the *elliptic* justification seems unlike the other treatments in the same book as well as the first treatment (problem I.18). In other words, we might consider it interpolated on the second criterion described below.

This example shows how dependent the decision to classify a treatment as interpolated might be on the possible interpretation of it through our analytical framework. We have, therefore, to remain very prudent with using such criteria, while recognising that it really makes it possible to be more articulate about the reasons why we might judge something to be interpolated. This example also shows that these new criteria do not contradict the classical criteria (philological or stylistic): they just enrich our "toolbox" for such kind of analysis.

4.3.3 Second criterion: leaving unsaid positions that would normally be stated

Another criterion for suspecting either interpolation or, at least, a damaged text is (B) *the extent, to which the treatment leaves unsaid positions that would normally be*

Footnote 137 continued
the second one is more in line with the previous problems. But, on the basis of this first interpretation, problem I.18 does the contrary: the first treatment is standard and the alternative one is not.

stated. A typical example is provided in problems II.28 and 29, which are strangely elliptic, when compared to the 'standard' treatments. The key point is that the impressive regularity of treatments allows us to recognise a kind of 'standard' level of detail in the exposition. When this level of detail is lacking and the exposition of positions is elliptic, we might suspect a damaged or misplaced text. If indeed we suppose that these two (in fact, six) problems are in their 'right' place in the whole order, then it is surprising that key steps, such as the recognition, in the third step of II.28, that the first prescription is solved through the use of the numbers found in the auxiliary problem and an "in-species" argument, is intriguing. Book II indeed appears as the first book in which simulations are introduced, and this is done in a very progressive manner and with all intermediary positions described quite in detail. One finds more elliptic formulations in book III, that might indeed be interpreted as being voluntarily elliptic; but, in this case, it seems 'out of place' to skip crucial steps. Moreover, the complex structure of these two problems, which brings them close to the problems like III.10 or III.11, in which "backward reasoning" is used, reinforces their similarity with the kind of problems found in III.[138] These problems, therefore, might be seen as having been displaced from an original position within book III, or having been badly damaged from a previous state, in which all the positions were explained.

4.4 The limits of our interpretation: elliptic, ambiguous and complex treatments

Having discussed some of the main conclusions that are derived from the analytical framework we proposed, we now discuss the main limits of this analysis.

The first limit is related to the very possibility to apply this analytical framework to *all* of the problems. Identifying, indeed, what is the complete chain of positions exhausting the indeterminate numbers and leading to an end-point, presupposes that the positions are effectively expressed and that either 'standard' expressions, repetitive patterns or heuristic explanations are used for their introduction. They indeed allow us to recognise, at each time, to which method of invention it refers. If, however, expected positions are lacking, either by textual accident, because they were voluntarily skipped or when heuristic explanations are absent, we might hesitate between various possible interpretations, as the above example of I.18 *alit.* shows. In such cases, we speak about *ambiguous* treatments, and such cases are systematically signaled, within the conspectus, in the very last column on the right.

The first and crucial remark, concerning these "ambiguous" problems, is that there are few of them. There are ten 'serious' ambiguities on the 117 treatments listed in the conspectus. Among these ten, the ambiguity actually bears on very few positions. Therefore also, the existence of these ambiguous cases does not put in question the basic validity of the analytical scheme. The latter allows us to interpret *the vast majority of the examined problems* in a non-ambiguous manner or with very few significant ambiguities. Even in the cases where there is an 'essential' ambiguity, the range of

[138] All the problems in book III have three requested numbers: if II.28 and 29, which have only two requested numbers, were initially part of book III, for example before problems 17 and 18, the statement of which are very similar, then someone might have considered it should be replaced at the very end of the series of problems, in book II, which have only two requested numbers.

possible interpretations is always limited to two or three at most or can be recognized to confirm the conclusions drawn from the conspectus.[139]

Our second remark is that some of these ambiguous treatments raise another issue already mentioned, namely the growing complexity of the treatments of problems. As we have seen, the conspectus visibly reveals the phenomenon, especially for book III in which problems or, more often than not, *series of problems* appear, that call for the combined use of several methods of invention, sometimes together with 'methods of equalization' (like double equation) as well as with references to other treatments (as shown in the last column of the conspectus). This system of references is typical of the third book, which either resorts heavily to explicit references to already solved problems or relies on the similarities between the treatments.[140] This system of cross reference seems to culminate in the (non-derivative) method of "backward reasoning," which is typical in this respect, for it relies, by its own definition, on the reference to two other auxiliary treatments. In other words, all happens as if the first problems of the third book had been introduced, to some extent, in order to prepare for the introduction of a new method (backward reasoning), in which the reference to other treatments is not only suggested, but becomes the substantial core of the method.

This very feature of backward reasoning calls for an important observation on our analytical framework. On the one hand, the latter appears to be sufficient for the analysis of these complex treatments, like those of the problems constituting the whole of III.10. Indeed, each "sub-problem" can be identified by a separate statement and can be analyzed from the statement to the end-point of the series of positions. All of the positions can be reasonably interpreted in terms of the methods introduced in the two first books, with the exception of course, of "backward reasoning" itself, which appears in this problem (III.10) for the first time.

On the other hand, what is special with "backward reasoning" is the following. Although it is clearly non-derivative (for it provides the first positions for a 'fresh' treatment of the problem at hand), it might legitimately be considered "derivative" in a different sense than the one defined in 3.2.1 above. Namely, the positions depend on those held in two *previous* treatments and not to previous positions of the *same* treatment. Therefore, this meaning is different indeed from the definition of derivation, but is also quite similar to it. After all, in the four problems of III.10 (or similar cases), the *statements* of the first and last problems opening toward the corresponding treatment are basically the same: the last treatment amounts to coming back to the first attempt and to change it slightly. Therefore also, the three or four steps might legitimately be considered as one and the same complex treatment of the very same problem.

Here, the crucial observation is that, precisely because such treatments are complex, their understanding requires that they be decomposed in several stages. Each one is made of a statement *and* a possible treatment and can, in turn, be analyzed according to the positions it implies. The analytical framework that we propose precisely allows us to do such decomposition successfully, precisely because the core idea of this analysis

[139] For example, the ambiguity existing between "two-at-once" and "simple" + "direct reworking" for the very last problems of the first book, only confirm that a smooth transition is established between the first and second books.

[140] This system might be partly due to the addition of scholia; see note 134 above.

is to analyze each different position separately and to examine how it is introduced and eventually justified. But then, in problems like III.10, we need to push this core idea to its very limits, for we need to take into account a new method named 'backward reasoning' that accounts for the fact that 'first time positions' might depend in some essential sense on previous treatments.

In general, then, to account for the possible deep relationship between different (that is, separate) treatments points out the natural limit of the analytical framework we developed. Unlike the problem posed by a plausibly damaged or elliptic text, the (plausibly) voluntary references to previous treatments oblige us, in principle, to account for the *system* of cross references between *different treatments*. These cross references, in particular, open toward the possibility to introduce ellipses, because a given treatment has to be understood, explicitly or not, as being analogous to others that come before it. Let us analyze, in conclusion, a few examples of this interesting kind of growing complexity within the third book.

Problem III.2[141] introduces three simulators built on the same model, that might be summarized by the following algorithm: $(X+Y+Z)^2/x^2/ + X$ or Y or $Z/(a^2 - 1) \times x^2/ \rightarrow sq$. Taking 3, 8 and 15 for the number written here a^2-1, we obtain three simulators for each of the constraints and the resulting squares are $4x^2$, $9x^2$, $16x^2$. In the same way, problem III.3[142] uses as simulator $(X+Y+Z)^2/16x^2/ - X$ or Y or $Z/7$ or 12 or $15 \times x^2/ \rightarrow sq.$, which relies on the idea that taking a square and subtracting three *gnomons* to it yield squares; in the fourth problem[143] the simulator X or Y or $Z/2$ or 5 or $10 \times x^2/ - (X+Y+Z)^2/x^2/ \rightarrow sq.$ which again follow a similar pattern. The latter is explained for the first time in problems II.24 and 25. In the problems of book III, the explanation tends to become elliptic, probably because these explanations were already given, and above all because the general pattern underlying all these simulators emerges through the comparison and repetition of particular simulators. The principle is to keep the idea (here, to add or subtract well-chosen gnomons to one or several squares, so as to obtain the same or different squares) while making slight changes.

An even more interesting example is found in the comparison between the treatments of problems III.12 and 13. The key idea behind the simulator used for problem III.12[144] is to decompose a square like $(1x+3)^2$ into two numbers, the first being a square (9) and the second a product $1x(1x+6)$: this enables him to simulate one of the constraints by positing the three numbers as $1x$, $1x+6$ and 9. Now the interesting characteristic of problem III.13,[145] the statement of which is analogous to that of III.12, is that it begins right away with two positions that yield the product of two of the sought numbers to be $1x(1x+4)$, that is, a product very similar to $1x(1x+6)$ in the previous treatment:

141 $?\,X, Y, Z\colon (X+Y+Z)^2 + X \rightrightarrows \square$, $(X+Y+Z)^2 + Y \rightrightarrows \square$, $(X+Y+Z)^2 + Z \rightrightarrows \square$.

142 $?\,X, Y, Z\colon (X+Y+Z)^2 - X \rightrightarrows \square$, $(X+Y+Z)^2 - Y \rightrightarrows \square$, $(X+Y+Z)^2 - Z \rightrightarrows \square$.

143 $?\,X, Y, Z\colon X - (X+Y+Z)^2 \rightrightarrows \square$, $Y - (X+Y+Z)^2 \rightrightarrows \square$, $Z - (X+Y+Z)^2 \rightrightarrows \square$.

144 $?\,X, Y, Z\colon XY + Z \rightrightarrows \square$, $YZ + X \rightrightarrows \square$, $ZX + Y \rightrightarrows \square$.

145 $?\,X, Y, Z\colon XY - Z \rightrightarrows \square (\mathrm{E}_1)$, $YZ - X \rightrightarrows \square (\mathrm{E}_2)$, $ZX - Y \rightrightarrows \square (\mathrm{E}_3)$.

P#	Position	Method	Heuristic explanation	Remarks
1	$X := 1x$	simulation		E_1 + implicit ref. to III.12
2	$Y := 1x + 4$			
3	$XY := 1x^2 + 4x$		*deêsei ara* + E_1 // if I set $Z :=$ $4x$, I will have E_1	
4	$Z := 4x$			
5	$YZ - X := 4x^2 + 15x$	rw-d	*deêsei ara* + E_2 and E_3	E_2
6	$XZ - Y := 4x^2 - (1x + 4)$	rw-d		E_3

The treatment ends up with a double equation with difference $6x + 4$.

What is interesting are the two correlated facts that, on the one hand, the heuristic explanation of III.13 is much shorter than the one given in III.12, but on the other hand there is a perceptible analogy between the two simulations, that partly explains the initial choice. When, indeed, we form a square like $x^2 + 4x + 4$ (just as we had formed $x^2 + 6x + 9$ previously) and we remark that $1x^2 + 4x$ is a product $1x(1x + 4)$, then we can remark that if 4 is added, the result is a square, and when $4x$ is removed, we also obtain a square. In other words, the product $1x(1x + 4)$ is "intermediary" between two squares, $1x^2$ on the one hand, $(1x + 2)^2$ on the other.

We thus retrieve with the pair of problems III.12–13 the idea of following an implicit pattern while changing the simulator. This idea is present among the first problems of book III and is of course retrieved in the method backward reasoning, which is basically about doing a variation on a chain of positions previously introduced.

This subtle system of 'patterns,' that become predominant in book III, is only partly reflected in our analysis and through the conspectus. This remark is of little consequence for the analysis, for we chose to limit the investigation to the three first books: it amounts to saying that the framework is adapted for these problems and only them. But a generalization to the remaining seven other books that are known either through the Greek manuscript tradition or through the Arabic translation of Qustā ibn Lūqā would require us to adapt the analytical framework, for some of these books go even beyond the third book in terms of complexity and cross references from one treatment to another. Such an extension or reform of the framework presented here obviously requires a separate study.

5 Conclusion

Let us come back, in conclusion, to what this analysis adds to the understanding of Diophantus's project as a whole, and what kind of further research this understanding calls for. Diophantus's general project, as announced in the very first lines of the introduction, is very slowly to get Dionysius, *the* reader of the problems, to a progressive grasp of the way or ways, in which a problem might be solved and unraveled and, by the same token, to develop his ability of invention in arithmetical problems. Our new problem, then, is to bridge the gap between the classification of treatments in progressive order proposed in this article, and the interpretation of this progressivity in terms of *mathêsis*, that is, in the terms of a project *which is conceived as a way for others to learn*. For this, we need in principle to define historically what it means, in the ancient context, to develop one's capacity for invention and through which specific learning techniques.[146] One of us had already suggested in a previous paper[147] that developing the capacity for invention (or “learning invention”) should be understood on the background of ancient rhetorical practice and of the corresponding treatises.

This comparison with rhetorical practice, for the sake of completing our reading and explaining in more detail how indeed the progressivity of problems might lead one to invention, that is, to become capable of “inventing” positions for converting problems to equations, is the objective of another study that we are now preparing. What this article essentially provides, in this respect, is a reasonably firm basis for such a complementary study. Our purpose, indeed, is to compare in some detail the progressivity of Diophantus’s treatments of problems and the necessary progressivity of rhetorical exercises; for this, we need to have a clear idea of how the problems are arranged and according to which plausible transitions. Only when this comparison will be completed shall we be in a position to re-evaluate Hankel’s judgement, which really was about learning how to solve problems and not just about their factual arrangement.

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Appendix 1: the conspectus

			STATEMENT					DERIVATIVE M.			NON DERIVATIVE METHODS									
TANNERY pg #		PB #	Nb of sought nbs	Nb of constraints	Nb of abbreviated nbs	Nb of given nbs or ratios	**Total # of positions**	Rwg method	Quasi simulation	Plassô-method	Simple method	Two-at-once method	All-together + Sum-against diff	Simulation	Simulators	Simulated constraints	in-species argument *deeper color if explicit*	Backward reasoning	END POINT OF THE CHAIN OF POSITIONS	**AMBIGUOUS** (red: serious ambiguity)
		BOOK ONE																		
16.9		1	2	2	2	2	3	2			1								eg	NO
16.24		2	2	2	2	2	3	2			1								eg	NO
18.8		3	2	2	2	2	3	2			1								eg	NO
18.26		4	2	2	2	2	3	2			1								eg	NO
20.10		5	2	2	2	4	5	4			*1*								eg	NO
22.5		6	2	2	2	4	5	4			*1*								eg	NO
24.2		7	1	1	1	3	4	3			1								ee	NO
24.21		8	1	1	1	3	4	3			1								ee	NO
26.13		9	1	1	1	3	4	3			1								ee	NO
28.6		10	1	1	1	3	4	3			1								ee	NO
30.2		11	1	1	1	3	4	3			1								ee	NO
30.22		12	4	4	4	3	5	4			1								eg	NO
32.20		13	6	6	6	4	7	6			1								eg	NO
34.25		14	2	1	2	1	4	2				2							ece	NO
36.13		15	2	2	2	4	4	3			1								ece	NO
38.2		16	3	3	3	3	5	1	3				1						ee	NO
38.19		17	4	4	4	4	6	1	4				1						ee	NO
40.10		18	3	3	3	3	6	4	1				1						ee	NO
42.2		18 alit.					5	4			1								ece	YES
42.16		19	4	4	4	4	7	5	1				1						ee	NO

[146] Indeed, arbitrary presuming that “learning” something has a straightforward meaning, which is *immediately* understandable, generally amounts to introduce an anachronism. The latter goes all the more unnoticed that it is easy to commit: we are all bent to presume that the way we learn, either by listening, writing, or reading, is universal, as if the ways by which we were taught were timeless.

[147] (Christianidis 2007, 293).

44.13		19 alit.					8	7			1								ece	YES
46.9		20	3	3	3	3														NO
46.15		20.1	2	2	2	2	3	2			1								eg	NO
46.20		20.2	2	2	2	2	3	2			1								eg	NO
46.27		21	3	3	3	4	6	5			*1*								ee	NO
46.27		21.bis	3	3	3	4	6	5			*1*								ee	NO
50.3		21 alit.	3	3	3	4	4	3			*1*								ee	NO
50.21		22	3	2	3/4	3	7	5				2							ee	NO
54.2		23	4	3	4/5	4	12	10				2							ee	NO
56.11		24	3	2	3	3	8	6				2							ece	NO
58.14		25	4	3	4	4	6	4				2							ece	NO
60.11		26	1	2	3	2	3	2			1								ece	NO
60.23		27	2	2	2	2	4	1	2				1						eg	NO
62.20		28	2	2	2	2	4	1	2				1						eg	NO
64.12		29	2	2	2	2	4	1	2				1						eg	NO
66.1		30	2	2	2	2	4	1	2				1						eg	NO
66.19		31	2	2	2	2	4	3			1								ece	NO
68.5		32	2	2	2	2	4	3			1								ece	NO
68.21		33	2	2	2	2	4	3			1								ece	NO
70.12		34	2	2	2	2	3	2			1								ece	NO
70.25		34 ii iii	2	2	2	2														NO
72.7		35	2	2	2	2	3	2			1								ece	NO
72.21		36	2	2	2	2	2	1			*1*	2?							ece	YES
74.9		37	2	2	2	2	4	3			*1*	2?							ece	YES
74.24		38	2	2	2	2	2	1			*1*	2?							ece	YES
76.11		38 i -iv	2	2	2	2														
76.26		39	1	1	1	2	14	13			1								ece/ee	NO
		44	# positions				213	147	17	0	31	10	8	0			0	0		
			# pst. der/non der				213	164			49									

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			STATEMENT					DERIVATIVE M.			NON DERIVATIVE METHODS									
TANNERY pg #		PB #	Nb of sought nbs	Nb of constraints	Nb of abbreviated nbs	Nb of given nbs or ratios	**Total # of positions**	Rwg method	Quasi simulation	Plassô-method	Simple method	Two-at-once method	All-together + Sum-against diff	Simulation	Simulators	Simulated constraints	in-species argument *deeper color if explicit*	Backward reasoning	END POINT OF THE CHAIN OF POSITIONS	**AMBIGUOUS** (red: serious ambiguity)
		BOOK TWO																		
82		1	2	1	2	1	4	2				2							ece	NO
82.15		2	2	1	2	1	4	2				2							ece	NO
84.10		3.1	2	1	2	1	4	2				2							ece	NO
84.25		3.2	2	1	2	1	4	2											ece	NO
86.4		4	2	1	2	1	4	2				2							ece	NO
86.16		5	2	1	2	1	4	2				2							ece	NO
88.1		6	2	2	2	2	4	3			1								ece	NO
88.20		7	2	1	2	2	2					2							ece	NO
90.9		8	2	3	4	1	4	1		2	1								ee	NO
92.1		8 alit.					4	2				2							ece	NO
92.16		9	2	3	2	2	5	3				2							eg	NO
94.11		10	2	3	2	1	5	3				2							eg	NO
96.5		11	1	2	3	2	3	2			1								de	NO
98.1		11 alit					5	1		2				2	1	1			ee	NO
98.24		12	1	2	3	2	5	1		2				2	1	1			ee	NO
100.21		13	1	2	3	2	3	2			1								de	NO
102.8		13b					5	1		2				2	1	1			ee	NO
102.21		14	3	4	5	3	6	1						5	2	3			eg	NO
104.14		15	3	4	5	3	4	1						3	2	3			eg	NO
106.8		16	2	3	4	2	6	2		2				2	1	1			ee	NO

108.1		17	3	3	3	6	10*	8?				2*							ee	NO
110.6		18	3	4	3	7	12*	11?			1*								ece	NO
110.15		17 alit	3	2	3	6	9*	7*				2*							ee	NO
112.13		19	3	4	3	1	4	1		1		2							ee	NO
114.11		20	2	2	4	0	5	1		2				2	1	1			*ee*	NO
114.24		21	2	2	4	0	7	1		2				4	1	1			ee	YES
116.16		22	2	2	4	0	5	1		2				2	1	1			*ee*	NO
118.6		23	2	2	4	0	4	1		1				2	1	1			ee	NO
118.20		24	2	2	4	0	5	2						3	1/2	2	3		ee	NO
120.12		25	2	2	4	0	3	0						3	1/2	2	3		ece	NO
122.4		26	2	3	4	1	5	2						3	1	1			ee	NO
124.2		27	2	3	4	1	4	2						2	1	1			ece	NO
124.19		28.1a	2	4	4	0	3	1				2							pb	YES
124.24		28.1b	1	2	2	0	3	1		1	1								ece	NO
126.8		28.2	2	4	4	0	5			2				3	1	3	3		ece	YES
126.16		29.1a	2	4	4	0	3	1				2							pb	YES
126.21		29.1b	1	2	2	0														NO
128.13		29.2	2	4	4	0	5	1		2				2	1	2	2		ece	NO
128.13		30	2	2	4	0	4	3						1/2	1/2	2	1/2		ee	NO
130.10		31	2	3	5	0	5	3						2	2	3	2		ee	NO
132.4		32	3	3	6	0	6	1		2				3	1	2			ee	NO
132.23		33	3	3	6	0	5	1		1				3	1	2			ece	NO
134.13		34	3	3	6	0	8	1						7	1	3	4		ee	NO
136.11		35	3	3	6	0	5	1						4	1	3	3		ee	NO
		40	# positions				179	60	0	26	5	26	0	62			20	0		
			# pst. der/non der				179	86			93									

			STATEMENT					DERIVATIVE M.			NON DERIVATIVE METHODS									
TANNERY pg #		PB #	Nb of sought nbs	Nb of constraints	Nb of abbreviated nbs	Nb of given nbs or ratios	**Total # of positions**	Rwg method	Quasi simulation	Plassô-method	Simple method	Two-at-once method	All-together + Sum-against diff	Simulation	Simulators	Simulated constraints	in-species argument *deeper color if explicit*	Backward reasoning	END POINT OF THE CHAIN OF POSITIONS	**AMBIGUOUS** (red: serious ambiguity)
		BOOK THREE																		
138.4		1	3	3	6	0	5	1						4	2	3	4		ee	NO
140.6		2	3	3	6	0	9	2						7	1/3	3	7		ee	YES
140.21		3	3	3	6	0	6	1						5	1/3	3	5		ee	NO
142.11		4	3	3	6	0	6	1						5	1/3	3	5		ee	NO
144.4		5	3	4	7	0	8	2		1				5	2	3			ee	YES
146.1		5 alter																		NO
146.15		6	3	4	7	0	8	2		1				5	2	3			ee	NO
148.10		6 alter	3	4	7	0	8	3		1				3	1	2			ee	YES
150.6		7	3	3	6	0														NO
150.8		7.1	3	5	3	0	5	2		1		2							ee	NO
150.21		7.2	3	3	6	0	4	3					1						*ece*	NO
154.2		8	3	4	7	1	8	4		1				3	1/3	3			*ee*	NO
156.2		9	3	4	7	1	8	4		1				3	1/3	3			*ee*	NO
158.2		10.1	3	3	6	1	10	1						9	2	2	3		X	YES
								3						4	2	2	3		X	
158.22		10.2a	2	3	5	1														NO
158.24		10.2b	2	4	4	1														NO
160.4		10.3	3	3	6	1	6	1		2								3	*ece*	NO

160.13	11.1	3	3	6	1	10	1						9	2	2	3		X	YES
							3						4	2	2	3		X	
162.8	11.2	2	4	4	1														NO
162.11	11.2'	1	2	2	1														NO
162.19	11.2''	1	2	2	1	3	1		2	1								ece	NO
164.7	11.3	3	3	6	1	6	1		2								3	*ece*	NO
164.19	12	3	3	6	0	7	4						3	1	1			de	YES
							2						5	1	1				
166.25	13	3	3	6	0	6	2						4	1	1			de	NO
168.19	14	3	3	6	0	6	1		2				3	1	2			ee	NO
170.11	15	3	3	6	0	5	2			1			2	1	1			de	NO
172.9	15al.1a	2	1	3	0	4	1		1		2							ee	NO
172.16	15al.1b	1	2	3	0	3	2			1								X	NO
174.5	15al.2a	2	2	4	0	5	1		2				2	1	1			ee	NO
174.20	15al.2b	1	2	3	0	3											3	de	NO
176.11	16.1	3	3	6	0	2					2							X	NO
176.16	16.2a	2	2	4	0	5	1		2				2	1	1			ee	NO
178.12	16.2b	1	2	3	0	6											6	de	NO
180.8	17	3	3	6	0	4	2						2	1	1			de	NO
182.4	18	3	3	6	0	4	2						2	1	1			de	NO
182.19	19	4	8	12	0	6	1						5	1	4	5		ee	NO
186	20	3	4	5	0	4	1						3	1	3			eg	NO
188.7	21	3	4	5	0	4	1						3	1	3			eg	NO
# problems	*33*	# positions				165	45	0	16	2	6	1	80			32	15		
		# pst. der/non der				165	61			104									
TOTAL	**117**	**# positions**				**557**	**252**	**17**	**42**	**38**	**42**	**9**	**142**			**52**	**15**		
		# pst. der/non der				**557**	**311**			**246**									

			Total # of positions	Rwg method	Quasi simulation	Plassô-method	Simple method (italics when variant 2)	Two-at-once method	All-together + Sum-against diff	Simulation			in-species argument *deeper color if explicit*	Backward reasoning	
# problems	*44*	# positions	213	147	17	0	31	10	8	0			0	0	
		# pst. der/non der	213	164			49								
# problems	*40*	# positions	179	60	0	26	5	26	0	62			20	0	
		# pst. der/non der	179	86			93								
# problems	*33*	# positions	165	45	0	16	2	6	1	80			32	15	
		# pst. der/non der	165	61			104								
TOTAL	**117**	**# positions**	**557**	**252**	**17**	**42**	**38**	**42**	**9**	**142**			**52**	**15**	
		# pst. der/non der	**557**	**311**			**246**								
				DERIV			NON DERIV								

Appendix 2: abbreviations used

The abbreviations used in this article are partly inspired by the ones used by Paul Ver Eecke in the comments accompanying his French translation of the *Arithmetica* (Diophante 1959). However, they are completed here by some specific signs for several crucial notions or procedures that are discussed for the first time in this article. Note, however, that this system of 'transcription' has intrinsic limits that become obvious in the case of operations (see below).

- Indeterminate numbers, which are either sought numbers or squares-to-be-obtained appearing in the statements of problems, are symbolized by capital letters. Thus, X, Y, X^2, Y^3 are used for sought numbers. The squares to be obtained are sometimes abbreviated by a special symbol $\square$ and in parentheses U^2, V^2 in order to have a special letter U, V for the sides. For example: $\square(U^2)$. Finally, when a sought number X is required to be a square, this is written $\square X$ and represents a constraint in kind.
- The terms of the "arithmetical theory" are symbolized by lower case letters like x, x^2, x^3, etc. Thus, the unknown "*arithmos*" is symbolized by x. Accordingly, lower case letters symbolize throughout this article species of numbers. An expression is symbolized by a combination of lower case letters. Examples: $2x$, $1x^2$, $5x + 1$, $1x^2 - 2x$, etc.
- The signs for operations, like +, −, etc. do not have the same meaning when used to connect the above symbols:
  - When they connect, capital letters, like $X + Y$, they denote the result of a prescribed addition between terms within the statement.
  - When they connect lower case letters, like $1x^2 + 5x - 3$, they denote a collection of species that are of different types. For a thorough discussion of this interpretation of such "expressions," that are unlike modern 'polynomials,' see (Oaks 2009).

  Paradoxically, then, these signs never denote the operations that are expressed in the Greek text, which are kept distinct from their outcomes. In particular, the sign −, when used for terms of the "arithmetical theory," denotes a wanting term (cf. *leipsis*). In such cases, it does *not* designate the operation of subtraction (cf. *aphairesis*).
- The sign := is used to introduce a position as a result.
  Examples: $X := 1x$, $Y^2 := 2x^2$, $X + Y := 2x$, etc.
- The sign = is used to denote finally obtained equations (cf. 2.5).
  Examples: $5x + 2 = 3 - 1x$, $1x + 1 = 4$, $1x^2 - 1 = 2x + 3$, $1x = 2$, etc.
- The sign $\rightrightarrows$ is used to denote the prescriptive aspect of the prescription of a problem: it might be read "*has to* be" or "*has to* make."
  Examples: $X^2 + Y^2 \rightrightarrows 16$, $Y + 12 \rightrightarrows \mathrm{K}(V^2)$, $XY + Gvn \rightrightarrows \mathrm{K}(U^2)$.
- The sign → is used to denote the outcome of a procedure. Accordingly, this sign is used only in the simulators. Example: $X^2/x^2 + Y/(2x + 1) \to sq$. The sign "/" indicates the double meaning of the algorithm, that is, as corresponding to a prescription of a problem (in which case the capital letters are taken into account), or as expressed within the "arithmetical theory" (in which case the lower case letters are taken into account).

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