Arch. Hist. Exact Sci. (2011) 65:625–658
DOI 10.1007/s00407-011-0086-1

History of the Lenz–Ising model 1965–1971: the role of a simple model in understanding critical phenomena

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Received: 24 May 2011 / Published online: 21 August 2011

Abstract This is the last in a series of three papers on the history of the Lenz–Ising model from 1920 to the early 1970s. In the first paper, I studied the invention of the model in the 1920s, while in the second paper, I documented a quite sudden change in the perception of the model in the early 1960s when it was realized that the Lenz–Ising model is actually relevant for the understanding of phase transitions. In this article, which is self-contained, I study how this realization affected attempts to understand critical phenomena, which can be understood as limiting cases of (first-order) phase transitions, in the epoch from circa 1965 to 1970, where these phenomena were recognized as a research field in its own right. I focus on two questions: What kinds of insight into critical phenomena was the employment of the Lenz–Ising model thought to give? And how could a crude model, which the Lenz–Ising model was thought to be, provide this understanding? I document that the model played several roles: At first, it played a role analogous to experimental data: hypotheses about real systems, in particular relations between critical exponents and what is now called the hypothesis of scaling, which was advanced by Benjamin Widom and others, were confronted with numerical results for the model, in particular the model's so-called critical exponents. A positive result of a confrontation was seen as positive evidence for this hypothesis. The model was also used to gain insight into specific aspects of critical phenomena, for example that diverse physical systems exhibit similar behavior close to a critical point. Later, a more systematic program of understanding critical phenomena emerged that involved an explicit formulation of what it means to understand critical phenomena, namely, the elucidation of what features of the Hamiltonian of models lead to what

Communicated by : Roger Stuewer.

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kinds of behavior close to critical points. Attempts to accomplish this program culminated with the so-called hypothesis of universality, put forward independently by Robert B. Griffiths and Leo P. Kadanoff in 1970. They divided critical phenomena into classes with similar critical behavior. I also study the crucial role of the Lenz–Ising model in the development and justification of these ideas.

1 Introduction

In a paper of 1980 titled "Model-Making in Physics," the prominent and versatile theoretical physicist Rudolf E. Peierls expressed a generally accepted view of the crucial role of models in modern physics,[1] when he wrote: "Physicists tend to use models of various kinds to aid their understanding of complicated physical situations."[2] Models "serve as aids in thinking more clearly about physical problems, by creating simpler situations, more accessible to our intuition."[3] Peierls also noted that models "serve different purposes, and they vary in their nature accordingly."[4] However, he aimed neither at describing what it means to understand a physical system, nor at analyzing systematically how different models can contribute to this understanding.

I present below a thorough study of how a model, the Lenz–Ising model, was used in the 1960s to understand a physical field, the behavior of critical phenomena, which can be understood as limiting cases of (first-order) phase transitions. This model was (and is) regarded as a crude representation of the various physical systems exhibiting critical behavior, and prior to the 1960s it was rarely used to gain insight into real systems.[5] Yet, since the early 1960s, it has been incontestably the most influential model of critical phenomena.[6] I focus on two questions: What kind of insight was the employment of this model thought to provide? And how could such a crude model lead to such understanding?

This article, which is self-contained, is the third and last in a series of three papers aimed at shedding light on modeling as an activity in physics by examining the history of the Lenz–Ising model. In the previous two papers, I studied the development that led to the view in the early 1960s that the model is relevant for understanding critical phenomena. I now examine in detail the roles played by the model in the theoretical investigations of these phenomena after it was realized that the model is physically relevant. I document the road followed by physicists in the 1960s: At first, it was used to gain insight into specific aspects of critical phenomena (such as that diverse physical systems exhibit similar behavior close to a critical point) as well as a sort of reference data for testing hypotheses about physical systems. Later, a more systematic program of understanding critical phenomena emerged that involved an explicit formulation of what it means to understand critical phenomena. Attempts to accomplish

[1] For similar views, see, for example, Ziman (1965); Fisher (1983); and Schweber and Wächter (2000).

[2] Peierls (1980, p. 3).

[3] Ibid., p. 17.

[4] Ibid., p. 3.

[5] Niss (2009).

[6] Binney et al. (1999).

this program culminated with the so-called hypothesis of universality, which classifies critical phenomena according to specific aspects of the asymptotic critical behavior; the Lenz–Ising model played a critical role in the development and justification of this hypothesis.

A complete description of the theorizing about critical phenomena in the 1960s as well as the role of the Lenz–Ising model in those years is beyond the scope of my paper. Rather, my aim is to give a general picture of the uses of the Lenz–Ising model by focusing on issues and materials representative of the field. Geographically, I restrict my study to Western physicists, thus leaving out their colleagues in the USSR, mainly to limit the scope, but also because the Cold War prevented much interaction between Soviet physicists and those in the West.[7]

2 Model and theory

The physicists I study here agreed that to explain a macroscopic phenomenon means to derive its macroscopic features from the behavior of its microscopic constituents and that nonrelativistic quantum mechanics is the underlying foundational theory for this enterprise. Since the late 1920s, however, it also had been clear that many physical systems cannot be reconstructed in terms of this foundational theory because the resulting equations are too complicated.[8] For a system to be mathematically tractable it has to be simplified and therefore models enter. In reconstructing phase-transition problems, the consensus view is that we proceed "from an assumed form of intermolecular interaction to predictions of the thermodynamic properties of an assembly by a mathematically rigorous argument."[9] The physical systems of interest consist of a huge number of atomic or molecular particles and are cooperative, i.e., their behavior can be understood only as the consequence of the cooperation between units of the system. In this article, I thus will take a *model* to be a microscopic representation of a physical system, including the interaction energy between the microscopic building blocks. These models are often called statistical–mechanical models since the procedure to obtain thermodynamic properties, such as the specific heat, from the molecular constituents and their interactions, is the formalism of statistical mechanics.[10] A *theory,* in contrast, is a more systematic account of the phenomenon in question, and they come in two varieties: foundational theories,[11] such as quantum mechanics, and more phenomenological theories, such as the theory of critical phenomena. In this

[7] The Russian physicist Anisimov (1998), now at University of Maryland, has recalled that novel ideas and concepts in the physics of critical phenomena were formulated by Soviet scientists originally and independently of Western scientists.

[8] Schweber and Wächter (2000).

[9] Temperley (1956, p. 2).

[10] Such models take much for granted: the masses of the constituents, the charge of the electrons, the origin of their spins, and the values of their spins, etc. Furthermore, the conditions within the atomic nucleus are deemed insignificant for the modeling. Since these assumptions were consensual for the physicists discussed here, they are irrelevant for the present purposes.

[11] Some would say "effective field theory," see Schweber and Wächter (2000, p. 593).

article, I use theory in the latter sense, and when the former is used, I add the qualifier "foundational."

3 Critical phenomena as a research field in the 1960s

Physicists and historians demark an epoch in the history of phase transitions from the mid-1960s to about 1970,[12] during which a particular region of the phase diagrams of substances caught the attention of physicists when the so-called critical region and the critical phenomena associated with it were recognized as a research field in itself. The end of the epoch is marked by the advent of a radical new approach, Kenneth G. Wilson's renormalization group of 1971.

The critical point of water is the prototypical example of critical phenomena: The boiling temperature of water depends on the ambient pressure (the smaller the pressure, the lower the temperature). For pressures below 218 atmospheres and temperatures lower than 374°C, it is possible to differentiate the liquid phase from the vapor phase. Above these values of pressure and temperature only a single phase of high density exists. So this pressure–temperature pair, called the critical point, marks a limiting type of phase transition with qualitatively new properties. Since the last quarter of the 19th century, it was known that some condensing gases and binary-liquid mixtures exhibit a strong increase of light scattering, particularly in the forward direction, immediately above the critical temperature of condensation. This phenomenon, now called critical opalescence, gives the near-critical liquids their characteristically milky appearance. Other examples of critical phenomena of relatively long standing are the Curie point of ferromagnets, the Neél point of antiferromagnets, and the order–disorder transition in binary alloys.

Despite its briefness, the epoch was rich with theoretical ideas and experimental results, and many of the crucial ideas for understanding the phase-transition problem were formulated, in particular, the notions of universality and scaling, which together with the renormalization group form the three pillars of our modern understanding of critical phenomena.[13] I focus on the role of the Lenz–Ising model in the formulation and justification of the hypotheses of scaling and universality, while the renormalization group is left out because, despite its great importance, because in terms of the role of the Lenz–Ising model, it is fair to say that the major developments occurred prior to the formulation of the renormalization group.

The Lenz–Ising model was (and is today) seen as a mathematical structure that can be interpreted as representing a number of physical systems exhibiting critical phenomena, including ferromagnetism, antiferromagnetism, localized adsorption or absorption, gas–liquid phenomena, the order–disorder transition in alloys, and binary solutions.[14] The mathematical structure of the Lenz–Ising model consists of a lattice in which each site can be in either of two states; the state is represented by a variable σ that can attain two values. The meaning of the two kinds of states depends on the

[12] See, for example, Kadanoff (1976); Domb (1996); and Ashrafi et al. (unpublished).

[13] Stanley (1999).

[14] Niss (2005, 2009).

particular interpretation of the model; if the model is supposed to represent a ferromagnet or antiferromagnet, for instance, a state is a direction of the spin of an electron. In the simplest versions of the model, only nearest neighbors on the lattice interact directly and the energy contribution of a nearest-neighbor pair depends on the states of the two sites.[15] For a ferromagnet, if two neighboring spins are parallel, i.e., if the variable σ has the same value for the two sites, they contribute the energy—J to the total potential energy, and if they are antiparallel they contribute J. The Hamiltonian H (for zero external field) is thus given by

$$H = -\sum_{n,m} J\sigma(n)\sigma(m), \tag{1}$$

where the sum is over all pairs of nearest neighbors n and m. The sign of J distinguishes a ferromagnet ($J > 0$) from an antiferromagnet ($J < 0$). In the so-called lattice–gas model of a gas, the state describes the occupancy or vacancy of the site by a gas atom (or molecule), while the energy of nearest neighbors is a representation of the interaction between the constituent atoms or molecules. The model is found to represent most of these systems fairly crudely; thus, for instance, the spin in the magnetic version of the model is confined to point in one of two directions disagrees with a full quantum-mechanical treatment. Likewise, the confinement of the atoms of a fluid to a lattice is also a major simplification. Consequently, the Lenz–Ising model may be said to be a *simple* model.

A good point of departure for characterizing critical phenomena as a research field in the 1960s is the conference on "Phenomena in the neighborhood of critical points" held in 1965 at the National Bureau of Standards in Washington, D.C. In the introduction to its proceedings, the prime organizer and chairman, Melville S. Green, recapitulated some recent scientific developments that "contributed to the feeling on the part of a number of scientists, and in particular, on the part of those who formed themselves into an ad hoc committee to organize it that April 1965 was an appropriate moment for a conference on critical phenomena."[16] Green mentioned several aspects of the recent development. First, a theoretical result showing that the three-dimensional Lenz–Ising model is at odds with fundamental assumptions of two classical approaches, the so-called mean-field model and a theory of phase transitions advanced by the Russian physicist Lev Landau in 1937.[17] Green's second point was that experiments had revealed that physically disparate systems actually share important features: The singularities of the specific heat of argon and oxygen are very much like the one found for the λ transition of helium-4 (at which helium goes from a normal liquid to a superfluid) and that of the curve of magnetization as a function

[15] Nearest neighbors are defined as two sites connected by a bond on the lattice, for instance, for a square lattice each site has four nearest neighbors.

[16] Green (1966, p. xi).

[17] The mean-field approach replaces the field acting on a particle situated at a given lattice site by a mean value rather than the correct fluctuating field arising from the particle's interaction with its neighbors. A description of Landau's theory would lead us to too far astray (some information can be found in Niss (2009), pp. 252–253), so I will only say that the Lenz–Ising model showed that a crucial assumption, which the free energy can be expanded in the so-called order parameter, is wrong.

of temperature for ferromagnets and antiferromagnets, which has a shape similar to the coexistence curve of the liquid–gas. Third, Green mentioned the realization that quantities estimated numerically for the three-dimensional Lenz–Ising model in its various guises closely follow these curves. His final point was that both theoretical and experimental objections to the classical theory of light scattering near the critical point proposed by Leonard Ornstein and Frits Zernike in 1914 had been made. He concluded that the time was ripe for a general understanding of critical phenomena,[18] and that the pertinent questions at the conference could be summarized in two: are the phenomena really analogous, and what are their shared features that can account for the singular behavior of their analogous properties?[19]

A problem of central interest in this enterprise, both experimentally and theoretically, was the determination of the asymptotic laws governing the approach to a critical point. Much of the discussion concerned the way in which various physical quantities (specific heats, susceptibilities, etc.) diverge to infinity or converge to zero as the temperature or other variable approaches its critical-point value. This behavior is captured by critical *exponents* (sometimes called indices) that had become the central variable characterizing critical phenomena. For instance, the zero-field magnetization, also called the spontaneous magnetization M, disappears when the temperature tends to the critical temperature T_c as

$$M(T) \propto (T_c - T)^{\beta} \quad \text{as } T \to T_c. \tag{2}$$

Here β is the critical exponent characterizing the critical behavior of the zero-field magnetization close to the critical point.

The exponents are a convenient way of describing the singularities of the system, but they contain considerably less information than the complete form of the function because a range of parameters that are specific to each system are deemed of lesser significance. For instance, the critical temperature can vary from one system to the next by up to six orders of magnitude,[20] but this is irrelevant for the value of the critical exponent; what matters are the overall features of the behavior, and this is epitomized in the critical exponents. The justification for focusing on the critical exponent rather than the entire function was that it was often found experimentally that the corresponding term dominates near the critical point.[21]

It was generally accepted in the 1960s that fluids and ferromagnets show analogous behavior if we let magnetization of a ferromagnet correspond to the density of a fluid and the magnetic field to chemical potential. Consequently, the same critical exponents are used for the two types of systems (see Table 1), which shows the definition of the exponents in two notations, one by Michael E. Fisher,[22] which is the one in use today, and one by Benjamin Widom.[23] The table also shows what was known, and,

18 Green (1966).

19 Ibid., p. xi.

20 Stanley (1999, p. S364).

21 Stanley (1971, p. 40).

22 Given in Essam and Fisher (1963).

23 Widom (1962, 1964, 1965a, b).

Table 1 The definition and values of critical-point exponents

Physical quantity	Exponent		Experimental value		Value for two-dimensional Lenz–Ising model[a]	Value for three-dimensional Lenz–Ising model
	Fisher's notation for fluids and magnets	Widom's notation for fluids	Fluids	Magnets		
Specific heat at constant magnetic field/volume below T_c	α'	α^*	≥ 0 (Logarithmic singularity)	≥ 0?	0 (Logarithmic singularity)	$\geq$0?
Spontaneous magnetization/ shape of coexistence curve	β	$1/d$	0.33–0.36	0.33 ± 0.015	1/8	$0.312 \cong 5/16$ 0.303 to 0.308
Zero-field isothermal susceptibility/ compressibility below T_c	γ'	f	$\geq$1.2?	?	7/4	1.23 to 1.32
Shape of the critical isotherm	δ	g	4.2 ± 0.1	4.22 ± 0.05	15	5.20 ± 0.15
Critical-point decay of correlation	η		$>$0?	$>$0?	1/4	$0.059 \cong 1/18 \pm 0.006$
Specific heat above T_c at constant magnetic field/volume	α	α	$\geq$0? $\leq$0.2	$>$0?	0 (Logarithmic singularity)	$\geq$0 $\leq$0.2

Table 1 continued

Physical quantity	Exponent		Experimental value		Value for two-dimensional Lenz–Ising model[a]	Value for three-dimensional Lenz–Ising model
	Fisher's notation for fluids and magnets	Widom's notation for fluids	Fluids	Magnets		
Zero-field isothermal susceptibility/ compressibility above T_c	γ		$>1.1?$	1.35 ± 0.02	7/4	1 1/4
Correlation length above T_c	ν		$>0.55?$	≥ 0.66	1	0.644 ± 0.002

The values are taken from Fisher (1966a) and represent the view ca. 1965. The queries indicate significant uncertainties or lack of values

[a] See Niss (2009) for the sources of these values

perhaps more important, not known about critical exponents for liquid, magnets, and the three-dimensional Lenz–Ising model.

4 Use of model results as "experimental" results

One of the first functions of the Lenz–Ising model in the 1960s was to provide a series of reference data for testing theoretical ideas; model results thus played a role analogous to the role of experimental data in other areas of physics. Since the late 1940s, the Lenz–Ising model had played an important *negative* role with respect to the classical theories of phase transitions. In particular, observed discrepancies between the behavior of the model and the predictions of these theories were seen as major problems for the latter,[24] as reflected in Melville Green's introduction above. In the 1960s, the model also played a *positive* role: when a theoretical idea agreed with model results, this was seen as positive evidence for the validity of the latter. The model played this role in two contexts: (1) critical scattering, i.e., the scattering of light in the critical region, and (2) proposed relations between the various critical exponents.

4.1 Critical scattering

The spectacular phenomenon of critical opalescence was discovered by the British scientist Thomas Andrews in 1869 and was quickly confirmed in a host of other experiments. In 1908, the Polish physicist Marian Smoluchowski traced critical opalescence to abnormally large density fluctuations arising when the critical point is approached. Two years later, Albert Einstein showed that a quantitative theory based on this idea does indeed result in an enormous increase of light scattering near the critical point.[25] The approach of Smoluchowski and Einstein assumes that density fluctuations in different volume elements are statistically independent of each other, but as pointed out by the Dutch physicists Leonard S. Ornstein and Frits Zernike in 1914, this assumption is not valid close to the critical point. Ornstein and Zernike were able to show that critical opalescence can be explained by a rapid increase of the correlation range of local density fluctuations, a range that tends to infinity at the critical point. To treat the correlation between different volume elements, Ornstein and Zernike introduced a new function, now called the pair distribution function, $g^{(2)}(r)$, which is a measure of the probability that if an atom is placed at the origin, another will be found at a distance r.[26] The net correlation function is defined as

$$G(r) = g^{(2)}(r) - 1.$$

In a fluid, $G(r)$ falls quite rapidly to zero as r increases.

[24] Niss (2009).

[25] Pais (1982, pp. 100–104), Münster (1965, p. 205).

[26] An analogous function can be defined for magnetic systems describing the correlations between spins at lattice sites.

Ornstein and Zernike expressed the fluctuation in the number of particles in a macroscopic volume of a fluid as an integral over the correlation function. This enabled them to relate the large forward scattering of light near the critical point to the correlation function. In a subsequent paper, Zernike went further and showed that the correlation function $G(r)$ of a fluid decays exponentially at large distances with a characteristic correlation length $1/\kappa$—that is,

$$G(r) \sim \frac{e^{-\kappa r}}{r} \quad \text{for } r \to \infty.$$

Here κ is a function of temperature, and κ vanishes at the critical point, i.e., the correlation length becomes infinite at the critical point. This means that the critical point correlation function is no longer exponentially damped but is predicted to follow the law

$$G(r) \sim \frac{1}{r} \quad \text{for } T = T_c \text{ and } r \to \infty.$$

The validity of the Ornstein–Zernike theory was a major issue at the turn of the 1950s. Several alternative approaches had appeared, which are essentially equivalent as regards their consequences but are based on somewhat different basic assumptions.[27] However, the experimental tests of the theories were inconclusive.[28] In 1960, Melville S. Green criticized the Ornstein–Zernike theory based on a density expansion developed by Joseph Mayer. Green conjectured, deliberately neglecting terms in the expansion that the correlation function at the critical point behaves like r^{-2} rather than r^{-1} as predicted by the Ornstein–Zernike theory. The next year, Stillinger and Frisch (1961) tested Green's method by applying it to the two-dimensional lattice-gas with the result that $G(r) \sim r^{-4/3}$. This result could be compared to an exact result implicit in the work of Kaufman and Onsager (1949) (which Fisher (1959) made explicit) that

$$G(r) \sim r^{-1/4} \; (T = T_c, d = 2).$$

From this comparison, Stillinger and Frisch concluded that Green's method gives a result that, while not unreasonable, is too great for the lattice-gas model, and thus the neglected terms in Green's method should be considered "in obtaining a proper description of at least the critical point."[29] Stillinger and Frisch's use of the Lenz–Ising model to test the validity of the method is analogous to the use of the model in testing the validity of mean-field theories in the 1940s.

What should replace the "suspect" Ornstein–Zernike theory?[30] Theorists turned to the Lenz–Ising model. Stillinger and Frisch conjectured that in three dimensions the true result at the critical point is

[27] Fisher (1966b) mentions theories by Rocard, Klein and Tisza, Debye, Fixman and Hart, Yvon, and Brout.

[28] In 1966, Green could write that "an obvious experimental problem" (Green 1966, p. xi) is to confirm experimental deviations from the Ornstein–Zernike theory.

[29] Stillinger and Frisch (1961, p. 752).

[30] Fisher (1964, p. 958).

$$G(r) \sim r^{-\alpha},$$

with an exponent α that satisfies $1 < \alpha < 2$.[31] To "provide support for this conjecture,"[32] Fisher (1962) discussed the possible behavior of the correlation functions at and above the critical point and showed that for the three-dimensional lattice-gas there are strong indications that $\alpha = 7/4$.[33] Despite that a lattice-gas with nearest-neighbor interaction is "only a very crude approximation to a real (continuum) gas," the properties of this model "should provide a reasonably accurate guide to those of more realistic models,"[34] because experience suggests that refinements of the lattice-gas model that will make it approach a real continuum gas do not lead to significant changes in the asymptotic behavior of the correlation function. So the Lenz–Ising model not only helped to discredit the Ornstein–Zernike theory, it also provided a rough estimate of the behavior of the correlation function of real systems. For the correlation function at the critical point, for instance, Fisher could conclude in 1964, based on analyses of the Lenz–Ising model that for a three-dimensional system "one should evidently expect that $G(r)$ behaves asymptotically as $1/r^{1+\eta}$ with $0 < \eta < 1$."[35] That the value of the exponent η, which measures the deviation from the Ornstein–Zernike theory (which corresponds to $\eta = 0$), is larger than zero for the three-dimensional Lenz–Ising model was gradually established throughout the 1960s. This was particularly significant as this exponent proved to be a consistent challenge to experimentalists, in particular, to establish that $\eta > 0$.[36]

The story is similar for the temperature behavior of the correlation length κ, but with a twist. The three-dimensional Lenz–Ising model was used to establish that the correlation length behaves as

$$\kappa \sim (T - T_{\mathrm{c}})^{\nu}.$$

Moreover, Fisher and Burford (1967) found a value of $\nu = 0.643 \pm 0.003$ for the model. When they compared this to the value of $\nu = 0.647 \pm 0.0022$ obtained by Jens Als-Nielsen and Ove W. Dietrich for experiments on neutron scattering in beta-brass,[37] they noted the remarkable agreement and concluded that:

> The very close agreement of these results with our analyses of the Ising model is most satisfying. While it may be premature to draw physical conclusions (e.g., about effective higher-neighbor interactions, etc.) from observed deviations in amplitudes, it is clear that the Ising model provides an excellent detailed account

31 This α is not the same exponent as the α in Table 1.

32 Fisher (1962, p. 172).

33 As we shall see below, the value of this exponent turned out to be significantly above unity. Fisher (1962) implicitly supposed that the correlation length exponent assumed the value 1.

34 Ibid., p. 173.

35 Fisher (1964, p. 958).

36 I am grateful to Jan Sengers for pointing this out to me. As late as 1975, Tracy and McCoy could state that "no experiments to date clearly and unambiguously establish that $\eta > 0$" (Tracy and McCoy 1975, p. 369). Sengers and Shanks (2009) describe the history of various critical exponents.

37 Als-Nielsen and Dietrich (1967a, b); Dietrich and Als-Nielsen (1967).

> of the order-disorder process in beta-brass. One may in the future hope to see it tested as precisely in other real physical systems thereby increasing and deepening our insight into critical phenomena.[38]

Thus, there was feedback on the perception of the Lenz–Ising model.

4.2 Exponent relations

The relations between exponents were the results of two largely independent research agendas. The major actor in the first agenda was Benjamin Widom (b. 1927), who trained as a physical chemist at Columbia and Cornell, with a theoretical dissertation at the latter institution on energy transfer in molecular collisions supervised by Simon Bauer.[39] Most of Widom's doctoral coursework was in physics and mathematics, which was a "little unusual" in physical chemistry at Cornell, "but not unprecedented." Upon finishing his doctoral studies in 1952,[40] he became a research associate in the Department of Chemistry of the University of North Carolina before joining the Cornell chemistry faculty two years later. At North Carolina he worked with the physical chemist Oscar K. Rice, who was preoccupied with the discrepancy between experiments and the predictions of van der Waals theory. Rice helped spark Widom's research interests in the critical behavior of liquids, and from 1955 he published a series of papers focused on the construction of an equation of state that could incorporate nonclassical critical-point exponents. The shape of the so-called coexistence curve that divides the $\rho - T$ plane into two regions was particularly in focus. Below the coexistence curve is the inhomogeneous or two-phase region where both liquid and vapor phases are present in equilibrium, and ρ denotes the average density of fluid in the container. In 1945, the English chemist Edward A. Guggenheim plotted the existing experimental data for a number of gases and argued that the data could be well fitted by the formula[41]

$$\frac{\rho_{\mathrm{L}} - \rho_{\mathrm{G}}}{\rho_{\mathrm{c}}} = \frac{7}{2}\left(1 - \frac{T}{T_{\mathrm{c}}}\right)^{\frac{1}{3}}. \tag{3}$$

Here ρ_{L} and ρ_{G} are the densities in the liquid and gas phases, respectively, and ρ_{c} is the density at the critical point. Guggenheim's plot gave an exponent of $\frac{1}{3}$ in contradistinction to the $\frac{1}{2}$ result of classical theories of the van der Waals type.

The Lenz–Ising model played a major role in Widom's thinking. Rice asked him to review for an informal seminar the two Yang and Lee papers on the Lenz–Ising model

38 Fisher and Burford (1967, p. 619).

39 The following is based mainly on Widom's recollections in Widom (2011) as well as in an interview with the Physics of Scales group in Widom et al. (2003).

40 Widom's degree was awarded in 1953.

41 In his 1945 paper, Guggenheim only claimed that the data for argon could be fitted to this curve, but in Guggenheim (1950) he gave the wider claim.

shortly after they were published in 1952,[42] and "this was an important part of [his] education."[43] Moreover, as he related in an interview:

> I never myself analyzed a microscopic model to determine its critical-point exponents other than models that I was able to transcribe into the Ising model and make use of what had already been known about the Ising model. I worked a lot with lattice-gas models, and with lattice liquid mixtures, and so on, so I've done a lot with those, but always making the transcription to the Ising model and making use of what others had found for the values of critical-point exponents.[44]

Widom proposed various hypotheses to take the nonclassical results into account from which he derived relations between critical exponents like the ones above. I will mainly deal with his justification of the relations and only describe briefly the hypotheses (later I will discuss one of them, his famous homogeneity (now scaling) hypothesis)

Widom followed Rice, who had published a relation between critical exponents for liquids in 1955, when he published his first critical-exponent equation in 1962, namely,

$$f = 2\left(1 - \frac{1}{d}\right) \quad [\gamma' = 2(1 - \beta) \text{ in Fisher's notation}]. \tag{4}$$

He derived this relation from thermodynamical considerations and a conjecture that two features of classical systems approaching the critical point are true for fluids in general. This led to several other relations: Two years later, Widom derived the following relation from another conjecture about critical behavior of fluids[45]:

$$g = 1 + fd \quad \left[\text{i.e., } \delta = 1 + \frac{\gamma'}{\beta}\right]. \tag{5}$$

He put forward another two relations in a 1965 paper on the surface tension of liquids,[46] where he studied the so-called square-gradient theory of 1958 for the interfacial tension of a fluid in the neighborhood of the critical point.[47] The two equations related critical exponents characterizing the temperature dependence of the thickness of the interface between the liquid and gas phases ν and the surface tension μ, respectively, to the other exponents:

$$\nu = \frac{f}{2} \text{ and } \mu = \frac{f}{2} + \frac{2}{d}. \tag{6}$$

[42] Yang and Lee (1952), Lee and Yang (1952).

[43] Widom (2011, p. 3).

[44] Widom (2003).

[45] This was a generalization of a result by Widom's mentor, Rice (1955), who had implicitly assumed that $f = 1$.

[46] Widom (1965b).

[47] The theory was originally proposed by van der Waals and elaborated by Cahn and Hilliard (1958).

How were all these conjectures tested? A major problem throughout most of the 1960s was that experimental results for a sufficient number of critical exponents for a single physical system to test the relations were lacking.[48] Widom's answer to this challenge in his papers of 1962 and 1964 was to examine the validity of the relations for exponent data for the Lenz–Ising model as well as the mean-field model. Information about the values of the exponents for the classical model and the two-dimensional Lenz–Ising model had been available since the late 1950s, but values for all of the exponents for the three-dimensional Lenz–Ising model gradually appeared only in the first half of the 1960s. After noting that Eq. 4 is satisfied for classical systems, Widom (1962) wrote that, "It is significant that, however, the relation is also satisfied by $f = 7/4$, $d = 8$, which are the values that characterize two-dimensional systems [the Lenz–Ising model], because here the validity of the hypotheses might be in doubt."[49] In other words, the lattice-gas model is discussed on a par with experimental results.

This role was even more pronounced for the square-gradient theory, where conclusive results for the three-dimensional Lenz–Ising model were available. In 1965 Widom tested the relations predicted by this theory against three sets of data: (1) real experimental results for inert gases; (2) numerical values of the three-dimensional lattice-gas model; and (3) exact values for the two-dimensional lattice-gas model. The confrontation led him to conclude that had the results "for the three-dimensional cases [sets 1 and 2] been previously known they would have been counted outstanding successes of the square-gradient theory."[50] So not only experimental results for real three-dimensional fluids, but also results for the three-dimensional lattice-gas model would have counted in favor of the square-gradient theory. Widom also used the lattice-gas model to argue that the square-gradient theory cannot be true, despite its apparent success, because the values of $\nu = 1$ and $\mu = 1$ for the two-dimensional lattice-gas model contradicts the predictions $\nu = 7/8$ and $\mu = 9/8$ of the square-gradient theory, and the latter therefore must "be judged fundamentally incorrect."[51] In this case, he relied only on the lattice-gas model without invoking experimental results, even though he expressed uncertainty about the agreement between the lattice-gas model and real fluids.[52] Moreover, he used the discrepancy to justify a rejection of the square-gradient theory and to propose a new theory. As a proof of the value he ascribed to the lattice-gas model, he once again placed the model on a par with experiments: "The resulting theory is in accord with all the facts which are rigorously known analytically, numerically, or experimentally about interfacial tensions and correlation lengths in fluid systems of two or three dimensions."[53] In short, the lattice-gas model was used to falsify some theories and verify others. In an e-mail answer to my question whether his use of the lattice-gas model as experimental results was controversial, he responded that the milder word "questioned" is more accurate. Moreover:

48 Fisher (1967a).

49 Widom (1962, p. 2704).

50 Widom (1965a, p. 3895).

51 Widom (1965a, p. 3896).

52 Nothing in the square-gradient theory restricted its (proposed) validity to three dimensions. Thus, the theory should be valid for two dimensions. I am grateful to Michael E. Fisher for pointing this out to me.

53 Widom (1965a, p. 3893).

> At the time it was not as obvious as it is now that the critical-point behavior of the lattice gas would be the same as experiment, and that the differences between them is due mostly to inaccuracies in the experimental results. That was sometimes questioned but was gradually accepted.[54]

The motivation behind the second agenda was somewhat different as it aimed at determining the number of independent exponents for various physical systems by examining whether any simple relationships between the different exponents are generally valid. This effort "can perhaps best be described as enlightened empiricism,"[55] according to Levelt Sengers et al. (1977), who emphasized its phenomenological, rather than its theoretical basis.

The first relation in this agenda was proposed in 1963 by John W. Essam and Michael E. Fisher. Fisher (b. 1931) will play a major role in what follows.[56] He received his undergraduate and graduate education in the Physics Department at King's College, London, enrolling there in 1948 and graduating in 1951. After two years of national service he returned to the department and began doctoral studies under the supervision of Donald MacKay,[57] a pioneer in information theory. His Ph.D. degree was awarded in 1957 for a thesis entitled "The Solution of Problems in Theoretical Physics by Electronic Analogue Methods."[58] In parallel with his preoccupation with the mathematical aspects of theoretical physics, Fisher became interested in statistical mechanics. According to Cyril Domb, Domb's inaugural lecture at King's College in 1955 on problems of statistical mechanics combined with Fisher's somewhat pessimistic view on the prospects of analogue computing helped persuade Fisher to shift from this subject to statistical mechanics.[59] Domb and Fisher collaborated on several subjects, in particular on the Lenz–Ising model but also on polyelectrolyte molecules in solutions and on random walks. Domb suspects "that it was the challenge of exact methods which Michael found particularly attractive,"[60] and it is fair to say that Fisher was more interested in the mathematical aspects of the modeling than was the average theoretical physicist.

The main purpose of Essam and Fisher's paper was the determination of critical exponents of the Lenz–Ising model by following G. A. Baker's pioneering use of the so-called Padé Approximant procedure.[61] For our purposes, the significant part

54 E-mail to the author by B. Widom, August 19, 2009.

55 Levelt Sengers et al. (1977, p. 44).

56 The following is based on Cyril Domb's recollections of Fisher's early career in Domb (1991).

57 Domb (1991).

58 Fisher published several papers on the solutions of various differential and integral equations arising in mathematical physics by such methods in the 1950s.

59 Domb (1991).

60 Domb (2003, p. 491). After getting his degree, Fisher continued to work at King's College, where he became full professor in 1964. Two years later he left for the United States to take up a position at Cornell where he was first Professor of Chemistry and Mathematics and then Professor of Chemistry, Physics and Mathematics. In 1987, he transferred to the University of Maryland.

61 For details of this method, see Domb (1996).

of the paper is the final section where they raised the question of possible relations that might exist between critical exponents. Consideration of a heuristic model, the so-called droplet model, led them to suggest, "with due caution,"[62] the validity of the following identity for both liquids and magnets

$$\alpha + 2\beta + \gamma = 2. \tag{7}$$

Here the exponents characterize the following quantities: α is the specific heat (for both liquids and magnets); β is the spontaneous magnetization (magnets) or the shape of the coexistence curve (liquids); and γ is the susceptibility (magnets) or the compressibility (liquids). A priori, these quantities need not all be related, but this was exactly what Essam and Fisher conjectured nonrigorously. Their relation resembles Widom's relation in Eq. 3, except that the latter assumes that α is zero, corresponding to a finite or logarithmic specific heat C_v, while the former allows for the possibility that C_v becomes infinite at the critical point.[63]

Essam and Fisher's conjecture stimulated the British physicist G. Stanley Rushbrooke (1915–1996) to investigate whether anything could be said rigorously based purely on thermodynamics, which would then apply to all models,[64] He showed rigorously that there was indeed a thermodynamic relation for magnets[65] and that the equality (6) holds as an inequality (= is replaced by $\geq$). Rushbrooke's inequality was soon followed by several other inequalities.[66]

The next year, Fisher (1964) put forth other, nonrigorous relations, namely

$$\gamma = (2 - \eta)\nu \tag{8}$$

and

$$\delta = (4 - \eta)\nu. \tag{9}$$

Fisher faced the same challenge of lack of experimental results as Widom, and he responded in the same way. He tested the relations against results for the two-dimensional Lenz–Ising model as well as results obtained by the classical approaches,[67] while a test with the three-dimensional results had to wait until 1967 when such results became available.[68]

62 Essam and Fisher (1963, p. 809).

63 Widom noted this point already in his paper.

64 Rushbrooke (1963).

65 Shortly thereafter, Fisher (1964) proved that the inequality also holds for liquids.

66 **?**.

67 They also tested the relation for results obtained for a special construct, the so-called Bethe lattice.

68 Fisher (1967b). His is based on a lecture given in March 1965, but it was not published until 1967.

4.3 The scaling hypothesis for thermodynamics

The work on relations among critical exponents culminated with Widom's famous hypothesis of homogeneity, or what is now called the scaling hypothesis.[69] The hypothesis of homogeneity assumes that the singular part of the free energy is a homogeneous function of its variables.[70] From this hypothesis, he could derive many of the important nonclassical properties of a fluid. In particular, he obtained the relation in Eq. 5 as well as Essam and Fisher's relation in Eq. 7.

In an interview, Widom has related how his ideas were shaped and how he took the Lenz–Ising model into account on a par with experimental results: "So I took that classical equation of state, and that form of it near the critical point, and I asked myself, what are the least changes that one would make in this in order that these exponents come out with what we know to be from experiment and Ising model calculations, what we know to be their non-classical values."[71] His attempt to make the least radical change led him to a modified classical equation of state and thermodynamics that enabled him to calculate the heat capacity, which he found to diverge logarithmically. He related once again the role of the Lenz–Ising model: "That was pretty astonishing, because I already knew from Onsager that the heat capacity of the two-dimensional Ising model diverged logarithmically,"[72] referring to a crucial theoretical result by the Norwegian chemist Lars Onsager in 1944.[73] Widom continued that he then asked himself,

> what was it about this particular equation of state, what were the features of this particular equation of state that allowed me to do that calculation and that led to that answer? And I saw that the particular features of that equation of state, the very special one that I was working with, a highly specific one, those of its features that allowed me to incorporate the non-classical critical-point exponents, and allowed me to do that calculation for specific heat were a certain homogeneity of form that is now called scaling. And so I said that if one imagines that instead of the highly specific one that I was working with, which I had no reason to think was correct, if I said that I'll just abstract that crucial feature of it, homogeneity, and imagine that that's what does it, then I again calculated the heat capacity and again found a logarithmic divergence.[74]

69 Widom (1965b). Two other papers are usually seen as independent origins of the scaling hypothesis: Domb and Hunter (1965) and Patashinskii and Pokrovskii (1966). However, Widom (1965b) is the most explicit, and I have chosen his paper as the exemplar of these three papers. Patashinskii and Pokrovskii (1966) should be credited with having added substantially to the subject that culminated in the development of the renormalization group theory of Kenneth Wilson. I am grateful to Leo Kadanoff for pointing this out to me. Their non-Western paper, however, is beyond the scope of my paper.

70 For further discussion of the hypotheses of homogeneity and scaling, see Stanley (1999).

71 Widom et al. (2003).

72 Ibid.

73 Onsager (1944).

74 Widom et al. (2003). But (to a high degree of conviction) we now know that in three dimensions, the divergence is stronger than logarithmic. Both for the three-dimensional Lenz–Ising model and for real fluids $\alpha \cong 0.1$.

The Lenz–Ising model played a role not only in Widom's development of the hypothesis of homogeneity, but also in the discussions of others of the validity of the hypothesis. As remarked by Kadanoff et al. (1967), the homogeneity assumption "is not given very strong justification beyond that it appears to work" in Widom's paper,[75] since Widom was mainly preoccupied with deriving consequences from the assumption. Kadanoff and his group at the University of Illinois were thus led to confront the scaling hypothesis with results for the two-dimensional and three-dimensional variants of the Lenz–Ising model. Their conclusion was a testimony to the model's status as being equivalent to experimental results for real systems (which Kadanoff et al. found wanting): "The close equality among these in both the two-dimensional case and three-dimensional one serves to support the scaling law hypothesis."[76]

5 Uses of the Lenz–Ising model to obtain understanding

The Lenz–Ising model also was used in constructive ways for understanding the workings of real systems. Cyril Domb, a pioneer in the study of the Lenz–Ising model in the 1950s, started this application. While mainly preoccupied with analyzing the model mathematically, he also used the model a few times to understand the most general features of phase transitions. He argued that, for instance, it is possible to account for experimental results assuming only short-range interatomic forces based on model results.[77] In the 1960s, however, the model was used to gain deeper and deeper understanding of critical phenomena, at first for the solution of specific tasks, but gradually in a more systematically way, culminating with the formulation and fulfillment of a research program in the later half of the decade.

The first examples of its constructive use for specific tasks appear in a paper by the Chinese–American physicists (and brothers) Yang and Yang (1964) and lecture notes by Fisher (1965) given at University of Colorado. Both present a similar constructive use of the model, but since Fisher's, at more than 150 pages, is the more elaborate, I will concentrate on his work.[78]

[75] Kadanoff et al. (1967, p. 263).

[76] Others, including Fisher (1967b) and Domb and Hunter (1965), however, did not look as favorably on the discrepancies between the three-dimensional model and the relation as Kadanoff et al., and they expressed reservations about the correctness of the relations. This is, however, again a testimony to the model's status as on par with experimental results.

[77] Niss (2009, pp. 270–272).

[78] The Yang brothers compared the lattice-gas model with the results of Voronel and co-workers (Niss 2009). They connected the behavior of real physical systems to that of the model:

> Now the critical phenomenon of a lattice gas originates from a rapidly changing balance in the competition between the occupied and unoccupied sites. One is thus led to the suggestion that in real gases, the critical phenomenon originates from a rapidly changing balance in the competition between holes and occupied volume. (Yang and Yang 1964, p. 304).

In other words, they thought that the lattice-gas model might explain the origin of critical behavior qualitatively, i.e., a feature found in the model might be transferable to the real gas. Only if the model is believed to capture the essential feature of the physical phenomenon are conclusions about it transferable to conclusions about the real system. Then, such a transfer requires fundamental confidence in the validity of the model.

Fisher used the Lenz–Ising model to say something about the analogies between disparate physical systems such as ferromagnets and liquids, in particular the conclusion that these systems have the same values of critical exponents to a high degree of experimental precision. For instance, in a number of systems an exponent β of roughly 1/3 appears: Guggenheim's plot for liquids gave an exponent of 1/3, and his equation is analogous to Eq. 2 for magnetic systems. The same exponent is found for the coexistence curve for a binary-fluid system,[79] and for the magnetization of a ferromagnet (EuS).[80] Such agreements are not only a numerological coincidence, Fisher argued, because the Lenz–Ising model in its various guises can "show theoretically that this analogy, between what are at first sight very different physical systems, is not merely superficial, but can be made quite precise."[81] Based on the mathematical analogy between the Lenz–Ising model as a representation of a gas and of a ferromagnet, Fisher was able to explain the coincidence of the values for these two physical systems. He continued:

> Of course, these relations are only exact for an Ising ferromagnet and a lattice gas. However, in as far as we believe that these models are at all "realistic," we may now draw the theoretical conclusion that we should *expect* the critical behavior for gases and ferromagnets to be very similar.[82]

In this way, the Lenz–Ising model could help explain why we find the same critical behavior in real systems.

Next, Fisher used a comparison of the Lenz–Ising model with experiments to draw some equally important conclusions regarding the nature of critical points. He again discussed the critical exponent β. Based on numerical studies of the three-dimensional Lenz–Ising model,[83] he concluded that β is about 5/16 = 0.3125. It is surprising, he wrote, "that such a simplified model of a magnet or a gas could lead to a result for the exponent β so close to the experimentally observed one third laws."[84] From this "the conclusion is forced on us that the detailed properties of the Hamiltonian become relatively unimportant in the critical region, whereas the dimensionality becomes a prime factor."[85] He argued for this statement by appealing to a graph by Burley (1960) depicting the magnetization of the two and three-dimensional Lenz–Ising model as a function of temperature for various types of lattices (simple quadratic, honeycomb, simple cubic, face-centered cubic, etc.), in both two and three spatial dimensions. Burley's figure clearly shows that the dimensionality is the decisive feature for the

Footnote 79 continued
Yang and Yang, presumably, had this confidence in the lattice-gas model to the degree that they advanced this suggestion.

79 For example, in the mixture with the impressive name Perfluoromethylcyclohexane in carbon tetrachloride.

80 Fisher (1965, pp. 16–18).

81 Ibid., p. 18.

82 Ibid., p. 34, emphasis in the original.

83 These numerical studies are so-called series expansions. For a description, see Niss (2009), pp. 267–269.

84 Fisher (1965, p. 106).

85 Ibid.

shape of the magnetization curves: these curves fall into two groups, one for each dimension, and the differences between the members in each group are much smaller than the differences between the two groups.

The agreement between the theoretical and experimental values of β was good, but it was not perfect. "The artificial nature of the Ising model does therefore make itself felt," Fisher found, "but ... to a much smaller extent than might have been guessed."[86] This led him to write that "[an] outstanding theoretical task is to characterize just which *relevant* features of real systems are oversimplified by the model."[87] One way to examine this question for the Lenz–Ising model, he continued, would be to compare it with the more reliable Heisenberg model for magnetism.[88] However, since several of the relevant critical exponents could not be obtained for the Heisenberg model, such an examination was not feasible.[89]

6 The role of the model in explaining scaling

Benjamin Widom described the most important scaling properties, but he did not discuss how they might arise. In 1966, Leo Kadanoff at the University of Illinois derived Widom's scaling law by applying certain transformations to the Lenz–Ising model. Kadanoff (b. 1937) got both his undergraduate degree and doctorate in physics at Harvard. His Ph.D. thesis of 1960 was entitled "Theory of many-particle systems: superconductivity; and the acceleration of a charged particle by a quantized electric field." In his own words:

> My thesis advisers, Paul Martin and Roy Glauber, continually directed my attention to the relation between a microscopic description of reality and a macroscopic description. Thus, a gas is composed of molecules, but it also obeys the laws of fluid mechanics. A microwave cavity contains not only photons but also an electric field. Or again, a fluid near its critical point is a bunch of molecules, but they [can] also be described as a scale-invariant field theory.[90]

Kadanoff's interest in second-order phase transitions was aroused by Kurt Gottfried and Paul Martin at Harvard, who pointed out that this problem was interesting and not at all understood.

86 Ibid.

87 Ibid., emphasis in the original.

88 See Niss (2009, footnote 69) for a description of this model.

89 In fact, Fisher was pessimistic about the prospects of determining the value of β for the Heisenberg model:

> Unfortunately, there seems no way at present in which one might seek to estimate β for the Heisenberg model. The low-temperature behavior in that case is given by the spin wave expansion and its correction terms which have proved exceedingly difficult to calculate. There are, however, good reasons for believing that the spin wave approach yields only an asymptotic series (terms like $e^{-J/kT}$ are neglected) so that even the complete series might not describe the critical-point behavior. (Fisher 1965, p. 106).

90 Kadanoff (1999, p. 7).

> I entered the problem by studying Lars Onsager's solution of the two-dimensional Ising model. The exact solution of this model of two-dimensional magnetism had been announced in the 1940s, but it had never been fully analyzed. Onsager and C. N. Yang had calculated some of the thermodynamic properties, but there was really no explanation of what physics might be demonstrated by Onsager's solution. Here was a tiny little world, just waiting to be explored and perhaps even captured.[91]

Moreover, having previously studied the correlations in a number of contexts, including the connection between hydrodynamics and space-time correlations, he was "pushed" toward looking at the spatial correlations. His main tool for studying such correlations was the two-dimensional Lenz–Ising model; at first he calculated the spin-spin correlations of this model,[92] and he then used the model to derive Widom's scaling law.[93] The basic idea of his derivation was that the singularity in the free-energy is caused by the presence of large-scale fluctuations, so that the correlation length ξ is much greater than the lattice spacing a. This means that it is possible to find a number L satisfying $1 << L << \xi/a$. We imagine dividing the lattice into cells of side length aL. Kadanoff then formed a new Lenz–Ising model by perceiving each cell of area $(aL)^2$ as a single spin site with a spin variable with a value equal to the average of the local magnetization in the block. This seems reasonable because of the strong correlation over short distances. The new spin variable should behave in an essentially identical fashion to the original variable, i.e., interactions among the cells should produce correlations identical in structure to the correlations in the original Lenz–Ising model. As the correlation length gets larger and larger as we approach the critical point, this means that the new system is further away from its critical point than the original system. We assume the state of the cell system to be described in terms of the effective magnetic field and temperature, which measure the distance of the system from critical. They depend on L as well as on the original magnetic field and temperature. Based on these ideas, Kadanoff obtained a relation between the free energies of what is effectively the same system at different distances from its critical point. This relationship turns out to be identical with Widom's homogeneity assumption.

Thus, by operating on the Lenz–Ising model in this way, Kadanoff was able to obtain the scaling relations. The procedure involves postulates that, in the words of Widom (1974), were "figuratively, but not literally, correct."[94] However, no one seems to question Kadanoff's use of the Lenz–Ising model to justify the scaling result.

6.1 Critical fluctuations

The Lenz–Ising model also played a crucial role in establishing what the Dutch-American experimentalist Johanna Levelt Sengers, who started to work on fluids in

91 Ibid., p. 157.

92 Kadanoff (1966b).

93 Kadanoff (1966a).

94 Widom (1974, p. 118).

the 1950s, called “the most important achievement of the 20^{th} century with respect to the understanding of the fluid critical region,” namely, “the insight that Van der Waals’ equation, although it gives a qualitatively correct description of the liquid-vapor transition, is subtly but fundamentally wrong near a critical point.” The reason is that critical behavior is governed by *fluctuations* and these fluctuations are ignored in descriptions of the mean-field type such as Van der Waals’ equation. She dated these important insights to the second half of the 20th century.[95]

As we saw above, critical fluctuations had been well known since the turn of the century, and in its first two decades it also became clear that at the critical point the length that characterizes the decay of the density correlations at different points in the system, the correlation length, becomes very large compared with all microscopic length scales. However, the crucial significance of the divergence of the correlation length and fluctuations for understanding critical phenomena does not seem to have been appreciated before the 1960s. Indeed, many earlier authors seem rather to have been preoccupied with solving the problem of why the fluctuations are infinite in finite systems. Moreover, the realization that the solid state exhibits phenomena analogous to critical fluctuations had to wait to the 1950s.[96] In 1953, Palevsky and Hughes found that the intensity of scattering of neutrons on magnetic materials, such as iron, exhibits a strong maximum at the Curie point with an abrupt change of slope. Drawing an analogy to the Ornstein–Zernike theory, the Belgian theorist Léon Van Hove showed that this can be understood by assuming that the correlation range for the fluctuations of local magnetization increases strongly as the Curie point is approached; consequently, this effect is just a new kind of critical opalescence. Subsequently, other systems exhibiting critical opalescence (with scattering of neutrons, light, and X-rays) were discovered.

I have documented that the Lenz–Ising model played a role in discrediting the Van der Waals’ equation in particular and the mean-field theories in general in Niss (2009). Here I will focus on the crucial role played by the model in establishing the importance of fluctuations for phase transitions.

Several authors played a role in realizing the role of fluctuations for critical phenomena. Benjamin Widom pointed out the importance of the correlation length for critical phenomena.[97] Following E. W. Hart, he identified the correlation length ξ with the thickness of the interface between the liquid phase and the vapor phase. Moreover, he made the understanding of the role of fluctuations more quantitative. It had long been known that when a fluid (say, vapor) is in equilibrium with its conjugate phase (liquid) fluctuations give rise to small domains of the conjugate phase within the bulk of the first phase. Widom conjectured that the volume of the conjugate region is equal to ξ^d, with d the dimensionality. Moreover, by associating a free energy of CkT with this fluctuation (C is a constant and k is Boltzmann’s constant), he derived an exponent relation.

95 Levelt Sengers (1979, p. 395).

96 Münster (1965).

97 Widom (1965a) and Widom (1965a, b).

Kadanoff used the Lenz–Ising model extensively to gain insight into how fluctuations govern critical phenomena.[98] In 1967, he and his co-workers argued that the infinities in thermodynamic derivatives are caused by a correlation length that diverges. First, they described the connection between correlation functions and thermodynamics in Landau's theory of fluctuations, in particular that the "large-scale fluctuations in the order parameter are the source of the singularities in thermodynamic derivatives near the critical point."[99] This statement, however, is perhaps only "qualitatively correct,"[100] so to investigate the correctness of Landau's theory, they quoted results from theoretical investigations of the Lenz–Ising model. From an examination of the spin-spin, energy density-energy density, and energy–density–spin correlation functions for the two-dimensional Lenz–Ising model, they concluded that "we see again how the divergences in thermodynamic derivatives are connected with the very long range of the correlation functions,"[101] and they derived the logarithmic divergence of the specific heat, i.e., Onsager's important result. They apparently judged that an appeal to the Lenz–Ising model would be sufficient to persuade the reader. Moreover, they related the long range of the correlations to the universal properties:

> We expect that the long-ranged correlations are relatively insensitive to the details of the interactions between spins. If a correlation extends over a hundred or a million lattice constants, this correlation should be sensitive only to the grossest features of the interaction and should not be affected by a change from a bcc to a fcc lattice or to the introduction of some next-nearest-neighbor interaction.[102]

Once again, the Lenz–Ising model played a critical role in endeavoring to understand the importance of fluctuations and correlation functions for critical phenomena:

> The Ising model solutions do, in fact, bear this out. The correlations in the two-dimensional case are basically the same for square and triangular lattices. In the three-dimensional case, so far as we can tell, $\gamma = 1.25$ equally well for bcc, fcc, and simple cubic lattices.[103]

7 The Lenz–Ising model and a theory of critical phenomena

In the second half of the 1960s, the Lenz–Ising model, along with similar simplified models, was put to more systematic use to gain insight into critical phenomena in

98 Valery Pokrovsky, a student of Lev Landau, has recalled that Landau was entertaining similar ideas: "Around 1960 Landau formulated the general problem of fluctuation-driven phase transitions via a calculation of the path integral over all configurations of the order parameter (unpublished)." (Pokrovsky and Valery 1998, p. 15).

99 Kadanoff et al. (1967, p. 400).

100 Ibid.

101 Ibid., p. 402.

102 Ibid.

103 Ibid.

general. Borrowing a term from the title of a review paper by Michael E. Fisher,[104] a *theory* of critical phenomena appeared in the 1960s.

7.1 The aim of a theory of critical phenomena

In his review paper, Fisher found it necessary to give an explicit exposition of the main aim of theory,[105] which he contrasted to the traditional approach "sometimes held (implicitly or explicitly) to be the calculation of the observable properties of a system from first principles using the full microscopic quantum-mechanical description of the constituent electrons, protons and neutrons."[106] This approach, which is often the one taken in quantum chemistry, is rarely feasible for the complicated systems of condensed-matter physics. More importantly, Fisher claimed, "it is not even a very sensible one!"[107] He elaborated in lectures given in South Africa in 1983: "If one had a large enough computer to solve Schrödinger's equation and the answers came out that way, one would still have *no understanding* of why this was the case!"[108] Numerical answers do not necessarily provide understanding. Instead, as he had written in this review:

> [The] aim of the theory of a complex phenomenon should be to elucidate which features of the Hamiltonian of the system would lead to the most characteristic and typical observed properties. Initially one should aim at a broad qualitative understanding, successively refining one's quantitative grasp of the problem when it becomes clear that the main features have been found.[109]

The physicists I have studied in the present paper subscribed to this "modern attitude"[110] toward the task of the theorist.[111]

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[104] Fisher (1967a).

[105] In fact, Fisher did so in his three reviews papers: Fisher (1965, 1967a, 1983). They all express roughly the same view; consequently, I will not treat them individually.

[106] Fisher (1967a, p. 619). He wrote in 1983: "The traditional approach of theoreticians, going back to the foundation of quantum mechanics, is to run Schrödinger's equation when confronted by a problem in atomic, molecular or solid state physics! One establishes the Hamiltonian, makes some (hopefully) sensible approximations and then proceeds to attempt to solve for the energy levels, eigenstates and so on." (Fisher, 1983, p. 46).

[107] Fisher (1983, p. 46).

[108] Ibid., emphasis in the original.

[109] Fisher (1967a, p. 619).

[110] Fisher (1983, p. 46).

[111] Widom stated in a letter to me that he subscribed to Fisher's view at the time and that he knows of no one who would have disagreed. Leo P. Kadanoff and his group at the University of Illinois stated in a review paper of 1967 what they considered to be the central question of critical phenomena: "This simplicity and similarity among phase transitions is not fully elucidated theoretically. Some of the qualitative features of this behavior are reasonably well understood; others remain a complete mystery." (Kadanoff et al. (1967), p. 395). The particular subject of their paper was "what can be learned by comparing different phase transitions with each other and with the existing theories. How are different phase transitions alike? In what ways do they differ? Why should we expect these similarities and differences?" (Kadanoff et al. (1967), p. 395). Stanley, Hankey, and Lee wrote: "In recent years considerable experimental and theoretical attention has been directed toward the study of critical-point exponents. Very recently increasing attention has been focused on

This perception of understanding has consequences for the methodology of investigation since the fulfillment of this aim requires extensive use of simplified models like the Lenz–Ising model, which "has been increasingly rewarding,"[112] but to distinguish relevant features from irrelevant ones, the theorist should employ several simple models and not concentrate on a single one.[113] Fisher, ascribing this philosophy to the Soviet physicist Yakov I. Frenkel and his 1946 argument for the use of caricature models,[114] described the role of such models:

> We may well try to simplify the nature of a model to the point where it represents a "mere caricature" of reality. But notice that when one looks at a good political cartoon one can recognize the various characters even though the artist has portrayed them with but a few strokes. Those well chosen strokes tell one all one really needs to known about the individual, his expression, his intentions and his character. So, accepting Frenkel's guidance … a good theoretical model of a complex system should be like a good caricature: It should emphasize those features which are most important and should downplay the inessential details.

For instance, if the system in question involves spins, physicists should focus only on the features of the spins needed for reproducing the essential features of the real system. Moreover, whether a full quantum-mechanical description is necessary or a classical one will do, is decided only by a test of the latter's ability to reproduce the essential traits. If a model is to give more insight than can be obtained directly from experiments, Fisher wrote in 1967, the model better be tractable mathematically. As a reality check, "one should always attempt to refine a model in order to test how far its defects as a true microscopic description affect the conclusion drawn."[115] The analysis of the models of critical phenomena had reached a stage where "[one] can be confident that the deviations now observed between theory and experiment are consequences of oversimplifications of the models (rather than deficiencies of calculation)."[116]

Footnote 111 continued
the question of precisely which features of a physical system are relevant for determining the critical-point exponents and which are not relevant." Stanley et al. (1971, p. 246).

112 Fisher (1967a, p. 619).

113 Fisher (1983, p. 47).

114 The term "minimal model" is sometimes used instead, for instance in Goldenfeld (1992) and Batterman (2002). The latter subscribed to the use of the former, which defined minimal model as the "model which most economically caricatures the essential physics." (Goldenfeld 1992, p. 22). Strictly speaking, a caricature model need not be the *most* economical model, but I think the notions of caricature model and minimal model essentially try to capture the same aspect of models, namely, a focus on a few features.

115 Fisher (1967a, p. 619). Fisher argued that the preceding development in critical phenomena, in fact, had followed this route: "The recent history of the study of critical phenomena has, in the main, followed the course of simplifying the physical models while improving and strengthening the mathematical techniques to the stage where, at last, fairly accurate theoretical treatments can be given for models which, while gross oversimplifications of reality in many respects, do certainly embody a number of the vital features of the particles and interactions leading to phase transitions and critical points." (Fisher 1967a, pp. 619–620).

116 Ibid., p. 718.

7.2 The hypothesis of universality

In 1970, this program for a theory of critical phenomena was fulfilled by the hypothesis for which Leo P. Kadanoff in 1971 coined the term "universality."[117] The hypothesis, although implicit in much earlier work, was formulated independently by Kadanoff, then at Brown University, and by Robert B. Griffiths at Carnegie-Mellon University. Both had introduced the idea (but not the term "universality") at the 1970 Midwinter Solid-State Research Conference at Newport Beach, California, January 18–23, 1970.[118] In his paper,[119] Griffiths stressed that theoretical estimates of critical exponents (indices) "coming largely from analyses of various lattice systems (Ising, Heisenberg, etc.),"[120] indicate that the indices are independent of the details of the Hamiltonian. He continued:

> The indices do, however, depend on (a) the lattice dimensionality d, (b) the "symmetry of the order parameter" in the sense that, for example, Heisenberg and Ising model indices are unequal, and (c) the range of interaction, provided it decreases sufficiently slowly.[121]

In short, the hypothesis of universality is analogous to the Mendeleev periodic table[122] and fulfills the program advanced by Fisher: the aspects of the system that matters for the type of phase transition are items (a–c). The order parameter is used for quantifying the amount and kind of ordering that is built up in the neighborhood of the critical point and is (normally) chosen to be zero above the critical point and non-zero below the critical point. For example, in a ferromagnetic crystal the zero-field magnetization M is typically used as the order parameter because M is a measure of the degree of alignment of the magnetic moments throughout the crystal. This parameter is non-zero below the Curie point and vanishes as this point is approached (in zero external magnetic field).[123] The symmetry of the order parameter varies for different systems. For example, at $T = 0$ the magnetization of the isotropic Heisenberg ferromagnet is free to point in any direction, while the magnetization of the Ising model has only an up–down degree of freedom.

The idea that apparently dissimilar liquid systems present considerable similarities near the critical point was already inherent in van der Waals's work and fairly well known thereafter. We have seen that Guggenheim presented a plot of a host of

[117] Kadanoff (1971). I have made no attempt to give a complete account of the complex development that led to this hypothesis.

[118] Laramore (1970). They published their ideas in Griffiths (1970) and Kadanoff (1971). Their papers form the basis of my present description.

[119] Kadanoff's version of the hypothesis of universality reads: "All phase transition problems can be divided into a small number of different classes depending upon the dimensionality of the system and the symmetries of the order [*sic*] state." See Kadanoff (1971), p. 103.

[120] Griffiths (1970, p. 1479).

[121] Ibid.

[122] Stanley (1999).

[123] For the liquid-gas transition the difference between the densities of the liquid and gas is a suitable order parameter, since this quantity vanishes as the critical point is approached from below.

different liquids falling on roughly the same curve. The relation between different areas of physics was noted in 1895 by the French physicist Pierre Curie who, when measuring the magnetic equation of state of nickel, was struck by the great resemblance between his plot of the magnetization versus temperature and the plot of others of the density–temperature isobars of carbon dioxide near the critical point. In 1907 another French scientist, Pierre Weiss, fashioned his mean-field theory describing the equation of state of nickel after van der Waals's equations of fluids. A more systematic point of view was provided by Lev Landau in 1937, who pointed to the universality among different phase-transition problems by providing a single formulation that in principle should contain all phase transitions.

In his history of critical phenomena, Domb (1996) noted that the papers of Griffiths (1970) and of Kadanoff (1971) were

> two key publications ... which converted the background assumptions into a coherent formal hypothesis. The individual ideas in these publications were not new, but their synthesis provided stimulus to further theoretical and experimental investigation to check whether the hypothesis was indeed widely satisfied. More importantly they summarized clearly and precisely results which had been derived empirically, and which any *theory* of critical phenomena must explain.[124]

The experimental background to their publications was the realization, mentioned earlier that details of the system undergoing the phase transition might be irrelevant and that many different phase problems might be essentially identical, "are very old ones in the theory of phase transitions."[125] How to classify such phase transitions, however, was new in Griffiths's and Kadanoff's work. The theoretical background was formed by a host of results for various models that gradually emerged, not least in the 1960s. Cyril Domb, one of the protagonists in this endeavor, mentioned three steps toward the hypothesis of universality in his book on the history of the subject, probably identifying the essential, if not all of the steps:

1. The realization that the critical exponents for a given model depend on dimension and not on lattice structure appeared first for the Lenz–Ising model when Domb and Sykes (1957) conjectured that for this model the exponent for susceptibility, γ, is equal to 5/4 for *all* three-dimensional lattices.
2. Domb and Sykes (1962) suggested that for the Lenz–Ising model with general spin γ is independent of spin.
3. The paper by Jasnow and Wortis (1968) discussed below.

One might add another step, as Fisher (1998) does, namely, the realization that exponents of the Heisenberg model seem to differ from those for the Lenz–Ising model (Rushbrooke and Wood 1958). These results, however, were much less conclusive as it was much more difficult to analyze the Heisenberg model mathematically.

In their paper, Jasnow and Wortis (1968) of the University of Illinois studied two questions echoing Fisher's aim of a theory: "How many different types of critical

124 Domb (1996, p. 244); emphasis in the original.

125 Kadanoff (1971, p. 103).

behavior are there, as classified by the critical exponents?"[126] And: "Which features of the dynamics and kinematics of a given system serve to determine its critical exponents and which are irrelevant?"[127] The Lenz–Ising model played an important role in their exploration of these questions as they based their exploration on the examination of a general Hamiltonian H, the so-called spin-infinity (classical) anisotropic Heisenberg model that had the Lenz–Ising model as a special instance. In the notation of my paper:

$$H = -\sum_{n,m,\alpha} \eta_\alpha J \sigma_\alpha(n)\sigma_\alpha(m). \tag{10}$$

Here the Latin index stands for lattice sites that are summed over all nearest-neighbor pairs, while the Greek index stands for the Cartesian components x, y, z. The classical spin vector of site n, namely, $\sigma(n) = (\sigma_x(n), \sigma_y(n), \sigma_z(n))$ has unit magnitude. The exchange interaction J measures the coupling strength between sites 1 and 2. The pure numbers η_α are called anisotropy parameters as they describe the anisotropy of the interaction.

Equation 10 encompasses several different models of those important for critical phenomena, in particular:

(a) $\eta_x = \eta_y = \eta_z = 1$: Classical isotropic Heisenberg model.
(b) $\eta_x = \eta_y = 0, \eta_z = 1$: Lenz–Ising model given by Eq. 1.
(c) $\eta_x = \eta_y = 1, \eta_z = 0$: The so-called classical *x*–*y* model.

However, several other models are *not* covered by Eq. 10, in particular quantum-mechanical magnetic models as well as continuum models of gases.

Jasnow and Wortis studied numerically how the values of the exponents γ (for the susceptibility) and ν_1 (for the correlation range, i.e., a measure of the characteristic decay of disturbances in the system) change on varying the anisotropy parameters.[128] First, they changed the Lenz–Ising model (b) to the classical isotropic Heisenberg model (a) by varying $\lambda = \eta_x = \eta_y$ from 0 to 1 in increments of 0.2 (keeping η_z constant at 1) and determining γ and ν_1 for each step. Second, they applied a similar procedure to study the change in interpolating between the classical isotropic Heisenberg model (a) and the classical $x - y$ model (c). They proposed that three sets of critical exponents are sufficient to characterize the singularities in their study:

Ising like: $\gamma = 1.23, 2\nu_1 = 1.25$
$x - y$-like: $\gamma = 1.32, 2\nu_1 = 1.34$
Heisenberg-like: $\gamma = 1.38, 2\nu_1 = 1.40$
Concerning the questions posed, they concluded:

> We find three sets of critical indices. The anisotropy parameters can be varied, altering the dynamical situation, without changing the critical indices, as long as

[126] Jasnow and Wortis (1968, p. 740).

[127] Ibid.

[128] Both the susceptibility and the correlation length were studied at temperatures above the critical temperature. Jasnow and Wortis also studied the specific heat, but were much less confident about these results.

> the ground-state manifold does not change. That is to say, the cases which could be treated indicated that the interactions matter only insofar as they determine the symmetry of the order parameter in the ground state.[129]

Both Griffiths and Kadanoff had published several papers on the Lenz–Ising model in the 1960s,[130] and it is clear from the above that while the Lenz–Ising model was not the only model among the models that formed the background to the universality hypothesis, it was a crucial element in the few pieces of the puzzle that led to one classification of systems exhibiting critical behavior.

What role did the Lenz–Ising model play in Griffiths's and Kadanoff's justification of hypothesis of universality? Kadanoff cited critical data for testing universality as well as theoretical arguments. Data concerning the equation of state for CO_2, xenon, and helium by Vicentini-Missoni, Levelt-Sengers, Green, Schofield, Lister and Ho show that for even these diverse materials the parameters are approximately constant. The theoretical argument was mainly the one from Jasnow and Wortis's paper, from which Kadanoff drew the following conclusion:

> The symmetries of the ordered state fall into three classes depending upon the value of λ. Apparently, as the symmetry changes so does the value of the critical indices, but within each symmetry group, the indices seem to remain constant, as required by universality.[131]

Griffiths, in his discussion of the hypothesis, appealed only to theoretical analyses of lattice models. He mentioned several models and papers, including that of Jasnow and Wortis. The Lenz–Ising model (and variations of it) took center stage in Griffiths's paper: He considered results (mainly of other authors) for the Lenz–Ising model with interaction energy J that differs in different directions, for the Lenz–Ising model with next-nearest-neighbor interactions, for the Lenz–Ising model with $S > 1/2$, that is, $S = 1, 3/2, 2$, and for the Lenz–Ising model with interactions not only restricted to nearest neighbors, but of long range.[132]

In short, the Lenz–Ising model played a major role in both Kadanoff's and Griffiths's justification of the hypothesis of universality as well as in the establishment of this hypothesis. Its role was constructive in that it helped to establish a common point of view that could classify phase transitions. However, the explanation of why such a classification exists had to wait for the renormalization group technique. The acceptance of the notion of universality provided feedback on the perception of the Lenz–Ising model. Kadanoff (2009) answer to his own question, "why study a simplified model like the Ising model?" is typical: The strategy of studying physical questions by using highly simplified models

[129] Jasnow and Wortis (1968, p. 749).

[130] These include Griffiths (1964, 1966, 1967a, b) and Kadanoff (1966a, b).

[131] Kadanoff (1971, p. 107).

[132] In the later case, Griffiths generalized arguments put forward by Nagle and Bonner (1970), who had studied an one-dimensional Lenz–Ising model with long-range ferromagnetic forces decreasing as $r^{-1-\varepsilon}$, with r the distance between spins and ε a positive parameter.

> is made rewarding by a characteristic of physical systems called “universality”, in that many systems may show the very same qualitative features, and sometimes even the same quantitative ones. To study a given qualitative feature, it often pays to look for the simplest possible example.[133]

8 Conclusion

A remarkable feature of the development I have described is that the Lenz–Ising model was put to many disparate uses during less than a decade: the model went from being applied mainly in a destructive way, i.e., to discredit theories of a function as positive as well as negative evidence, to more constructive, but hesitant uses to solve specific tasks, to the culmination in is crucial role in the theory of critical phenomena. The theory was based on the idea that to understand critical phenomena meant to be able to elucidate what features of the Hamiltonian led to what kinds of critical behavior.

More remarkably, the theory meant a change in the epistemology of the phenomena, i.e., in the perception of what a physical theory of phenomena is and should be based upon. I have noted that Fisher explicitly contrasted the aim of the modern theory of critical phenomena to the aims of more traditional earlier approaches, and the former imply a much more systematic approach to critical phenomena. However, the modern approach also marked a shift of focus away from a discussion of the physical aspects of the model. Prior to the modern approach, i.e., until the mid-1960s, researchers focused on the model’s ability to capture the features of empirical data, for instance by comparing the value of the exponents with real materials, and the extent to which the models lack of realism was able to explain possible discrepancies. While such discussions were not completely abandoned in the second half of the decade, they no longer took center stage. Instead, the focus was on obtaining a more abstract understanding of how the overall features of models, for example, its dimensionality, are responsible for critical behavior.

Furthermore, the modern approach diverged from previous approaches with respect to what phenomenological features we try to understand. This becomes clear if we compare the view expressed by Michael Fisher with the attitude of Cyril Domb, Fisher’s mentor and a pioneer in the use of the Lenz–Ising model, in his inaugural lecture about ten years prior to Fisher’s statement of the modern approach. Domb had given an idea of what we should attempt to obtain by describing the goal of his endeavors noting how we can use the knowledge that the theoretical chemist can give us about a water molecule and how two such molecules interact:

> If we could therefore apply the statistical technique to an assembly consisting of a vast number of water molecules we should be able to account for all the physical properties of water; we should be able to show that at sufficiently low temperatures water consists of a crystalline solid at 0°C. it melts into a liquid, and that at 100°C. it vaporises; we should be able to understand all the peculiarities of water, why the solid is lighter than the liquid, and why the liquid has its

133 Kadanoff (2009, p. 784).

> maximum density at a temperature of about 4°C., and what specific characteristic of the water molecule gives rise to these almost unique properties. We should be able to predict the properties of water or ice in regions of pressure and temperature so far inaccessible in the laboratory, but which may be of considerable astrophysical or geophysical interest and importance.[134]

This view – that we should aim at detailed predictions – is in marked contrast to the approaches by Fisher noted above. While Fisher tried to extract the general features of a range of physical phenomena, Domb had advocated an attitude toward the phenomena where we should look for much more particular aspects of phenomena. We should, for instance, predict the values of the phase-transition temperatures. The modern program is indifferent toward these values; rather, it focuses on the overall features described by the critical exponents. This, of course, is because the model is somewhat removed from real systems: a more systematic use of the model necessarily must be somewhat abstract, because the merits of the model cannot be in close relation to real systems. The history of the Lenz–Ising model from 1920 to 1970 is thus the story of a model that went from relative obscurity to a prominent position in understanding critical phenomena, and how this became possible by a profound change in its epistemology.

Acknowledgment I thank Michael E. Fisher, Leo P. Kadanoff, and Jan V. Stengers for their valuable comments on a draft of my paper, and Roger H. Stuewer for his careful editorial work on it.

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[134] Domb (1955, p. 403).

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