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From measuring tool to geometrical object: Minkowski's development of the concept of convex bodies

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1 Introduction

The present paper is concerned with the emergence of the modern theory of convex sets. Whereas the special instances of what we today understand by convex sets, such as the circle or regular polygons, have been studied throughout the history of mathematics, the modern theory understood as the systematic study of sets characterised exclusively by the property of convexity began only towards the end of the nineteenth century from where it developed into one of the many new disciplines of twentieth century mathematics. Today the theory of convexity is considered a central theory not least due to its expansion into almost all important areas of mathematics such as geometry, analysis, and applied mathematics.[1]

According to the history[2] presented in textbooks on the theory of convexity the German mathematician Karl Hermann Brunn (1862–1939) was the first to engage in such systematic studies. His studies were then followed by the work of another German mathematician, Hermann Minkowski (1864–1909), who developed the theory further and explored some of its many applications. Examining this history one realises that Minkowski did not know about Brunn's work until after he himself had begun his own investigations of what led to the theory of convexity. And even though Bonnesen

[1] For the significance of the theory of convexity in mathematical programming and the theory of linear inequalities see (Kjeldsen, 2000, 2001, 2002, 2003, 2006).
[2] Often presented in introductions to textbooks on the theory of convexity, see for example (Bonnesen and Fenchel, 1934) and (Klee, 1963). There are a few historical studies on the concept of convexity (Fenchel, 1983), (Gruber, 1990, 1993).

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and Fenchel in the preface to their famous monograph *Theorie der konvexen körper*[3] use the term "Brunn–Minkowski theory" and explicitly characterised the material in their book as a generalisation of this Brunn–Minkowski theory, the modern theory of convex sets primarily grew out of Minkowski's work.[4]

This prompts interesting historical questions such as: How and why did this theory of convexity emerge? Which kinds of objects did Minkowski study? Why did he initiate such studies? Did his perception of the objects change? How did he carry out his investigations? How and why was the theory of convex sets developed through his work? The study presented in this paper has been guided by these questions in order to explain:

Why and how the concept of convex bodies emerged, took form, and led to the beginning of a theory of convexity in Minkowski's mathematical practise.

The historical analysis presented in this paper reveals that three phases can be identified in Minkowski's mathematical practise leading to a theory of convex bodies. These phases will be described below, and it will be argued that the interfaces between them are distinguished by shifts in Minkowski's focus of research. It will be argued that the dynamics of knowledge production in Minkowski's mathematical practise leading to the theory of convex bodies can be characterised as an interplay between the research strategies of (1) answering known questions in new ways and (2) posing and answering new questions.[5]

The story of the origin of the theory of convex bodies in Minkowski's work is interwoven with the emergence of the geometry of numbers, another new discipline in the twentieth century created by Minkowski, and even though this is not the focus of this paper the investigations presented here will also partly explain how the geometry of numbers emerged from Minkowski's mathematical practice.

2 The first phase: geometrical treatment of the minimum problem for positive definite quadratic forms

The first traces of ideas that led Minkowski to form a concept of convex sets and to investigate them mathematically are found in his work on the so-called minimum problem for positive definite quadratic forms in n variables:

$$f(x_1, \ldots, x_n) = \sum_{h,k=1}^{n} a_{h,k} x_h x_k, \quad a_{h,k} \in \Re, \quad a_{h,k} = a_{k,h}$$

[3] (Bonnesen and Fenchel, 1934).

[4] A comparison of Brunn's and Minkowski's mathematical practise can provide answers to the question why the theory did not emerge from Brunn's work. The results of such a historical investigation will be presented in a forthcoming study, see (Kjeldsen, forthcoming).

[5] Rheinberger has used this distinction to describe the dynamics of knowledge production in experimental systems; see (Rheinberger, 1997). For the adaptation of Rheinberger's ideas to the history of mathematics, see (Epple, 2004).

The form is said to be positive definite if $f(x_1, \ldots, x_n) > 0$ for all $(x_1, \ldots, x_n) \neq (0, \ldots, 0)$. The minimum problem is the question of finding the smallest number N that can be represented by f for integer values (not all zero) of the variables $x_1, \ldots, x_n$. This minimum is essential in the reduction theory of positive definite quadratic forms.

The minimum problem was not new at Minkowski's time. Lagrange had already addressed the problem for forms in two variables in the eighteenth century. Gauss had realised its connection to the question of so-called reduced forms at the turn of the century, and Hermite had studied the minimum problem and its connection to reduced forms of an arbitrary number of variables.[6] The new contribution of Minkowski's work on the minimum problem was not the statement of the problem itself but the approach he used in its solution—the way in which he investigated and answered the problem.

As we shall see in this section, Minkowski introduced geometrical methods for dealing with the minimum problem in n variables and used a geometrical framework both to discuss the minimum problem and to come up with plausible and convincing arguments for his results. He relied on geometrical intuition not only to generate ideas and mathematical questions but also to construct proofs. Minkowski's thoughts and work with the minimum problem in this geometrical context are what characterises what I have called "the first phase" of the line of thinking that eventually led Minkowski to define the mathematical object of a convex body.

In the following I will analyse Minkowski's use of geometrical intuition in his number-theoretical work on the minimum problem so as to understand how this particular mathematical practise laid the foundation for Minkowski's introduction of convex bodies and the following theory of convexity. We will see that the merging, in the hands or mind of Minkowski, of these two mathematical frameworks—number theoretical inquires, and geometrical intuition and proof-technique—constituted the melting pot from which Minkowski changed the ways of thinking about the minimum problem and introduced new concepts, changes and concepts that, in phase three, led him to define the mathematical object we today think of as a convex set.

2.1 The minimum problem in a number theoretical framework

The number-theoretical part of Minkowski's thinking can be traced back to his teenage years as a student in the gymnasium in Königsberg. From a letter that Weber wrote to Dedekind we know that at the age of 15 Minkowski had already studied the number-theoretical work of Gauss's *Disquisitiones Arithmeticae* and Dirichlet's *Vorlesungenüber Zahlentheorie*. The letter also shows that Weber had high expectations for Minkowski's future mathematical achievements, not least because of two papers on the reduction theory of quadratic forms written by the 15-year old Minkowski:

> On this occasion I will write to you about a very promising mathematical and especially number theoretical genius who has appeared here. It is a *Primaner* in one of the local gymnasiums who will not enter university until next year

[6] For historical work on the theory of quadratic forms see (Schwermer, 1991; Goldman, 1998; Scharlau, 1977; Scharlau and Opolka, 1985).

> and on his own initiative has worked himself into higher analysis and number theory which he has studied through the first edition of your Dirichlet-lectures. Now he works on the Disquisitiones. That he studies with understanding follows from two papers he gave me in which he as far as I can see has formulated the problem himself and worked it out very cleverly. [...] In one of the papers he determined for a negative determinant the number of reduced forms for one of the three forms $(a, 0, c)(a, {}^1\!/\!_2 a, c)(a, b, a)$, [...]
> The other paper treats positive determinants [...] For a Primaner this is certainly a commendable achievement. [...].[7]

Two years later, in 1882, Minkowski submitted a long paper, which in essence is a foundation for a general theory of quadratic forms with integral coefficients, to the *Grand Prix des Sciences Mathématiques* at the French *Académie*. The occasion was the prize-question of representing an integer as the sum of five squares posed by the *Académie* the year before. Minkowski's work was so original that the prize was given to the young Minkowski even though he—contrary to the rules—had written his paper in German instead of French.[8]

Minkowski's further work on quadratic forms was inspired by Hermite's work on reduced forms, in particular the letters "sur différents objects de la théorie des nombres" addressed to Jacobi and published in Crelle's *Journal*.[9] According to Minkowski, Hermite proved the fundamental theorem of reduction of positive definite quadratic forms in those letters:

> The investigations of Mr. Hermite relate to forms with an arbitrary number of variables; they start with the statement of the fundamental theorem of reduction according to which the dimensionless ratio of the smallest number different from zero that can be represented by a positive quadratic form in n variables by means of integers to the nth root of the determinant of the form never exceeds a certain amount that only depends on n, and they represent themselves as a never-ending witness of the fruitfulness of this theorem in almost every section of number theory.[10]

Hermite himself also acknowledged that this theorem was of fundamental importance in number theory: "De nombreuses questions me semblent dépendre des resultants precedents."[11] The result is the one Hermite presented in a letter to Jacobi dated August 6, 1845 stating that for a positive definite quadratic form $f(x_0, x_1, \ldots, x_n)$ in

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[7] The letter is not dated and Minkowski is not named in the letter, but Walter Strobl (1985, pp. 145–146) convincingly argued that the student that Weber wrote about is in fact Minkowski and that the letter was written at some point between March 1879 and December 1880. For further information on Minkowski's mathematical work while he was still a student see (Strobl, 1985).

[8] There were some debates about the prize which was awarded to both Minkowski and the British mathematician Henry John Stephen Smith. See also (Strobl, 1985), (Hilbert, 1909), and (Reid, 1970). Minkowski's paper is published in his collected works (Minkowski, 1911, vol. I, pp. 3–144, (1884)).

[9] See Minkowski's note in Comptes rendus de l'Académie des Sciences, Paris 1883, t. 96, pp. 1205–1210, also published in his collected works (Minkowski, 1911, vol. I, p. 145 (1883)).

[10] (Minkowski, 1911, vol. I, p. 245 (1891a)).

[11] (Hermite, 1850, p. 263).

$n+1$ variables it is "always possible to find $n+1$ integer numbers $\alpha, \beta, \gamma, \ldots, \lambda$ such that

$$f(\alpha, \beta, \gamma, \ldots, \lambda) < \left(\frac{4}{3}\right)^{\frac{1}{2}n} \sqrt[n+1]{D}\text{"}$$

Here D denotes the determinant of the form.[12]

As mentioned above, this minimum problem plays a fundamental role in the reduction theory for quadratic forms, which is concerned with the question of finding among a class of equivalent positive definite quadratic forms a form—the so-called reduced form—whose coefficients have as simple a form as possible, which basically means having the smallest coefficients. In the reduced form the leading coefficient must be taken to be the minimum of the quadratic form for integer values (not all zero) of the variables. Reduction theory was an important part of the theory of quadratic forms, and as a consequence the minimum problem was an essential question, and—as will become clear in the following—it was the pivotal centre for the work of Minkowski that eventually led him to introduce a concept of convex sets.

2.2 Minkowski's probationary lecture: geometrical intuition and interpretation of quadratic forms

The geometrical part of Minkowski's thinking, which came to lie at the core of his ideas, was present at least from 1887 on, as can be seen from his so-called probationary lecture "Über einige Anwendungen der Aritmetik in der Analysis" for the German *Habilitation*, given on the 15th of March 1887 at the Friedrich-Wilhelm University in Bonn.[13]

In the probationary lecture Minkowski gave a new proof of the minimum problem positive definite for quadratic forms in three variables using a geometrical interpretation of quadratic forms. The idea of interpreting positive definite quadratic forms geometrically was not Minkowski's own. Gauss[14] in 1831 had given an outline of such a geometrical interpretation in a review of a book by Seeber, and Dirichlet had shown in a very clear and transparent way how to represent positive definite quadratic forms in three variables geometrically.[15] Dirichlet's method had been published in the same issue of Crelle's *Journal* as Hermite's letters to Jacobi, and even though Minkowski did not credit Dirichlet in the manuscript for the probationary lecture he did so in nearly all of his published papers concerning geometrical methods in number theory, and it is quite clear that he was inspired by Dirichlet's paper.

To understand how a positive definite quadratic form $f(x, y) = ax^2 + 2bxy + cy^2$ can be interpreted geometrically I will briefly illustrate the idea behind Gauss' outline in modern terms using vector notation. If $\mathbf{x}$ denotes the column vector (x, y), $\mathbf{x}^{\mathrm{T}}$ its

[12] (Hermite, 1850, p. 263).

[13] The manuscript for the probationary lecture is published in (Schwermer, 1991).

[14] (Gauss, 1863, vol. II, p. 188–196).

[15] (Dirichlet, 1850).

transpose and Q is the matrix

$$Q = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

then the quadratic form can be written as $\mathbf{x}^{\mathrm{T}} Q\mathbf{x}$. In the rectangular (x, y)-coordinate system the level curves $f(x, y) = \lambda$ form ellipses. We are looking for new coordinates (u, v) that will reduce the quadratic form to a sum of squares $(u^2 + v^2)$ and in this new coordinate system the level curves will form circles. To do this, we let $\mathbf{u}$ denote the column vector (u, v) and set $\mathbf{u} = A\mathbf{x}$. We need to find a matrix A such that f reduces to $u^2 + v^2$. We get

$$f(x, y) = \mathbf{x}^{\mathrm{T}} Q\mathbf{x} = \mathbf{u}^{\mathrm{T}}(A^{-1})^{\mathrm{T}} Q A^{-1}\mathbf{u}$$

If $Q = A^{\mathrm{T}} A$ then

$$\mathbf{u}^{\mathrm{T}}(A^{-1})^{\mathrm{T}} Q A^{-1}\mathbf{u} = \mathbf{u}^{\mathrm{T}}\mathbf{u} = u^2 + v^2$$

as required.

To understand this geometrically, note that the points (1, 0) and (0, 1) in the (x, y) coordinate plane are represented in the (u, v) coordinate plan by the first and second columns of the matrix A. A calculation shows that these are the points (a_1, a_2) and (b_1, b_2) in the (u, v) plane where $a_1^2 + a_2^2 = a$, $b_1^2 + b_2^2 = c$ and the angle, φ, between them is given by the formula

$$\cos \varphi = \frac{b}{\sqrt{ac}}$$

So the unit squares in the (x, y) plane are mapped to the 'Elementar-Parallelograms', as Gauss called them, or standard parallelograms in the (u, v) plane. The square of the area of each of theses is $ac - b^2$ the determinant of Q. The simpler geometric object to study is the quadratic form in the (u, v) plane with respect to the parallelogram lattice that is the transform of the square grid in the (x, y) plane. In the skew (u, v) coordinate system f represents the square of the distance from lattice points to the origin. (See Fig. 1.)

The problem Minkowski considered in his probationary lecture was to find the best approximation in integers not all zero to a solution of a system of equations $\xi = 0, \eta = 0, \zeta = 0$ where ξ, η, ζ are independent linear forms in the three variables x, y, and z. As Minkowski pointed out, if one proceeds according to the rule of probability one will pick as the best solution the values for which the sum of the squares of the errors is a minimum, that is the integer values (not all zero) for x, y, and z for which $\xi^2 + \eta^2 + \zeta^2$ is a minimum. Expressed in x, y, and z, $f(x, y, z) = \xi^2 + \eta^2 + \zeta^2$ is a positive definite quadratic form for which the minimum for integer values (not all zero) of the variables are sought. In this way Minkowski transformed the problem of finding the best solution into the minimum problem for positive definite quadratic forms in three variables.

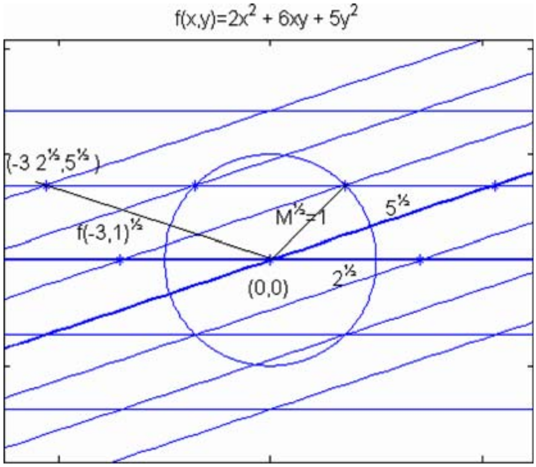

Fig. 1 Geometrical interpretation of the quadratic form f. The smallest distance in the lattice is $M^{1/2} = 1$. The *bold lines* represent the two coordinate axes

As Minkowski explained in the probationary lecture this problem can be interpreted geometrically:

> Perhaps the most natural way to encounter this square sum is through a geometrical interpretation. It is obvious to interpret the magnitudes ξ, η, ζ, which appear equally, as orthogonal coordinates in our space. To every system ξ, η, ζ there corresponds a point, to the system 0, 0, 0 the origin. There is no doubt about which points are to be considered as the ones closest to the origin: the ones with the smallest distance to this point. This distance will again be measured through the root of $\xi^2 + \eta^2 + \zeta^2$.[16]

Formulated in this way the problem became to find the minimum distance from the origin to points (ξ, η, ζ) with integer values (not all zero) of x, y, and z in the rectangular (ξ, η, ζ)-coordinate system. To examine the points in this coordinate system with integer values of x, y, and z Minkowski considered the three skew axes (x, y, z) determined by the three linear forms ξ, η, ζ. The points (x, y, z) with integer values form a lattice built up by (standard) parallelotopes, and the problem was thereby transformed into the problem of finding such a lattice point closest to the origin, that is to find the smallest distance between two lattice points. As we saw above the square of the distance from the origin to a lattice point (x_0, y_0, z_0) is measured by the quadratic form

$$f(x_0, y_0, z_0) = \xi(x_0, y_0, z_0)^2 + \eta(x_0, y_0, z_0)^2 + \zeta(x_0, y_0, z_0)^2.$$

That Minkowski's construction of the lattice is the same as Gauss's can be seen by the following illustration of Minkowski's procedure for two linear independent forms ξ

16 (Minkowski, 1887) in (Schwermer, 1991, p. 86).

and η in two variables x and y:

$$\xi = a_1 x + b_1 y$$
$$\eta = a_2 x + b_2 y$$

In the (ξ, η)-coordinate system the point $(x, y) = (1, 0)$ have the coordinates $(\xi, \eta) = (a_1, a_2)$ and the point $(x, y) = (0, 1)$ have the coordinates $(\xi, \eta) = (b_1, b_2)$. This means that the unit of the x-axes and y-axes respectively, is

$$\sqrt{a_1^2 + a_2^2} \quad \text{and} \quad \sqrt{b_1^2 + b_2^2}$$

and the angle φ between the axis is determined by

$$\cos\varphi = \frac{a_1 b_1 + a_2 b_2}{\sqrt{a_1^2 + a_2^2}\sqrt{b_1^2 + b_2^2}}$$

In the rectangular (ξ, η)-coordinate system the distance from the origin to a point (ξ, η) equals the root of $\xi^2 + \eta^2$. Expressed in (x, y) we have

$$f(x, y) = \xi^2 + \eta^2 = (a_1 x + b_1 y)^2 + (a_2 x + b_2 y)^2$$

That is, the distance from the origin to a lattice point is measured by the root of the quadratic form:

$$f(x, y) = \left(a_1^2 + a_2^2\right) x^2 + 2(a_1 b_1 + a_2 b_2)xy + \left(b_1^2 + b_2^2\right) y^2$$

Comparing the coefficients of the quadratic forms we can see that Minkowski's construction of the lattice representing the quadratic form is the same as the one described by Gauss.

In the probationary lecture Minkowski pointed out that when the lattice is given by its points, the standard parallelotope, that is the parallelotope whose corners are determined by the eight points for which the coordinates are either 0 or 1, can be chosen in a variety of ways. Every linear transformation with integer coefficients and determinant ± 1 will transform the coordinate system x, y, z into a new coordinate system x', y', z' and in this system the points with coordinates 0 or 1 will determine a parallelotope which would also represent the lattice. This means that every positive definite quadratic form that is equivalent to a given positive definite quadratic form can be used to construct the lattice. Another feature of the lattice representing a positive definite quadratic form is that the square of the volume of a standard paralllelotope is equal to the determinant of the form.

The notion of lattices became very important in Minkowski's thinking about number theory because every theorem about the lattice can be translated into a theorem about numbers. So here we witness his first step into a mathematical practice of dealing geometrically with problems in number theory, a technique he developed further in the years to come, and by means of which he laid the foundation for the mathematical discipline called the geometry of numbers.

The interesting problem now formulated by Minkowski in the probationary lecture is to determine "the points closest to the origin, or … the smallest distance between two points in the lattice."[17] Remembering that distances in the lattice are measured by the square root of the quadratic form f, the problem of the smallest distance in the lattice is in fact the minimum problem for f.

To find an upper bound for the smallest distance between points in the lattice Minkowski imagined a sphere, with the smallest distance as diameter, placed around each lattice point. The spheres around two lattice points, the smallest distance apart, will touch each other on the line connecting the two points, he argued, but they will not intersect and they will leave some free space. Minkowski accordingly concluded that the volume of such a sphere must be less than the volume of a standard parallelotope. By this intuitive geometrical reasoning Minkowski derived the following upper bound for the smallest distance between two lattice points:

> The smallest distance [between two lattice points] is smaller than the product of a constant and the third root of the volume of a standard parallelotope.[18]

Why is that? If we let M denote the smallest distance between two lattice points, and V the volume of the standard parallelotope the inequality between the two volumes is:

$$\frac{4}{3}\pi\left(\frac{M}{2}\right)^3 < V$$

leading to the above cited upper bound for M:

$$M < c\sqrt[3]{V}$$

where c is a constant. Minkowski added that if the lattice is changed through some kind of continuous process in such a way that the volume of the standard parallelotope keeps decreasing, then the smallest distance between two lattice points will then also decrease below every limit, and—as he continued—that result is not limited to the three dimensional case he investigated in the probationary lecture:

> The same applies of course to lattices of any dimension and expresses an important property of positive quadratic forms which in general was first proved by Hermite but in a much more complicated way.[19]

Minkowski did not give any further arguments for this generalisation from three to n-dimensional space. If we rewrite Minkowski's inequality above for the n-dimensional case remembering (1) that if M is the smallest distance between two lattice points then M^2 is the minimum of the quadratic form for integer values not all zero of the variables

17 (Minkowski, 1887) in (Schwermer, 1991, p. 86).

18 (Minkowski, 1887) in (Schwermer, 1991, p. 87).

19 (Minkowski, 1887) in (Schwermer, 1991, p. 87).

and (2) that the volume of the parallelotopes equals the square root of the determinant of the quadratic form associated with the lattice, then the inequality reads:

$$M^2 < \left(c\sqrt[n]{D^{1/2}}\right)^2 = kD^{1/n}$$

where k is a constant and D is the determinant of the form, expressing the property found by Hermite.

Immediately following the text quoted above there is a passage in Minkowski's manuscript that Minkowski had crossed out saying that:

> More important though, is it that the bound reached here is far more natural and much smaller than the one given by the method of Hermite, even though it is of course not (cannot be) precise since it contains the transcendental number π.[20]

Here again we see Minkowski referring to his reasoning as being much more natural than Hermite's. Besides the numerical difference between the two limits the main difference is the method by which they are reached and according to Minkowski among the different methods one can apply to reach a result some are more natural than others.

This notion of a method "more natural than others" also appears in a letter Minkowski wrote to Hilbert two years later in November 1889:

> Now I have come much further in the theory of positive quadratic forms [...]. Perhaps the following theorem (which I can prove in half a page) will interest you and Hurwitz: In a positive quadratic form in $n(\geq 2)$ variables and determinant D one can always assign integer values to the variables such that the form becomes $< nD^{1/n}$. For the coefficient n Hermite had $(4/3)^{1/2(n-1)}$, which obviously, in general, is a much larger limit.[21]

This is the same theme as the one found in his probationary lecture for the Habilitation—and again Minkowski drew attention to two aspects: (1) the proof he claimed to have can be given in just half a page, and (2) the upper bound for the minimum of a quadratic form for integer values (not all zero) of the variables is lower than the one given by Hermite. The fact that Minkowski emphasised that he can prove the theorem in half a page suggests that there was some kind of more or less tacit mutual understanding here between Minkowski, Hilbert, and Hurwitz to what constitute "good" mathematics. The fact that Minkowski's proof is only half a page is a quality in itself.

2.3 Minkowski's geometrical proof of the minimum theorem for n variables

In the probationary lecture Minkowski claimed that his result could be generalised to n dimensions, but he did not present a proof. He did not present the proof in the letter to Hilbert either. In fact two more years passed by before he published the first

20 (Minkowski, 1887) in (Schwermer, 1991, p. 87–88).

21 (Minkowski, 1889, p. 38).

geometrical proof of the minimum theorem in 1891 in the paper "Über die positiven quadratischen Formen und über kettenbruchähnliche Algorithmen".

In the introduction to this paper Minkowski first gave a brief historical summary of the development of the theory of positive definite quadratic forms as well as an outline of the most important achievements up to that date, before he explicitly stated that the present study should be seen as an attempt to fill some of the holes in the theory. The main focuses of attention were positive definite quadratic forms and their application in number theory, indicated by the title of the paper, which gives no hint about any use of geometrical intuition. Only at the very end of the introduction did he stress the advantage of extending the geometrical intuition from three to n dimensions in the proof for the minimum problem.

The idea of the proof is the same as the one he outlined in his probationary lecture. He associated a positive definite quadratic form f in n variables with a n-dimensional lattice built up of standard paralellotopes. Again, what is of interest is the smallest distance between two points in the lattice, because if $\sqrt{M}$ represents the smallest distance then its square $\left(\sqrt{M}\right)^2$ represents the smallest number different from zero that can be represented by f with integer values for the variables. In the paper Minkowski gave two upper bounds for this minimum. In the first one, which gives rise to the bound he mentioned in the letter to Hilbert quoted above, he constructed a n-dimensional cube with side length

$$\frac{1}{\sqrt{n}}\sqrt{M}$$

around each lattice point and imagined the cubes organised in a parallel pattern. The points of such a cube which have the largest distance to the centre will be corner points and, as explained by Minkowski, due to the Pythagorean theorem this distance equals $1/2\sqrt{M}$. He then proceeded as in the probationary lecture arguing that since $\sqrt{M}$ is the smallest distance between two lattice points the cubes will have no inner points in common, and they will not fill out the whole space, so a comparison of the volumes of a cube and a standard parallelotope gives rise to the following inequality:

$$\left(\frac{1}{\sqrt{n}}\sqrt{M}\right)^n < \sqrt{\Delta}$$

Since Δ is the determinant of the quadratic form, the inequality expresses that the volume of a cube is less than the volume of the standard parallelotope. Rearranging the terms Minkowski reached the bound for the minimum he had announced in the letter to Hilbert:

$$M < n\sqrt[n]{\Delta} \quad \left(= nD^{1/n}\right)$$

Minkowski immediately obtained an even lower upper bound (for large n) by extending the method used in his probationary lecture from three-dimensional spheres to n-dimensional spheres of radius $1/2\sqrt{M}$. The replacement of the cubes with the spheres does of course not change the argument because the spheres—just as the cubes—do not overlap and they do not fill the whole space either, the only thing that changes is the upper bound:

$$M < \frac{2n}{\pi e}\sqrt[n]{n\pi e^{\frac{1}{3n}}}\sqrt[n]{\Delta}$$

which, as explained by Minkowski, for large n is approximately $2/(\pi e) = 0.234\ldots$ times the other bound.[22]

As in the letter to Hilbert Minkowski again emphasised the nature of the proof:

> With the help of a geometrical expression that can be transferred to forms with more than three variables it was achieved, not only to make the fundamental theorem of Hermite regarding the minimum of a positive quadratic form appear obvious in a certain sense, but also to narrow considerably the limit required in this theorem and in its extensions. For this reason new important applications of this theorem are made possible.[23]

And he finished the argument by announcing that

> We have then put the successful theorem of Hermite about the minimum of a positive definite quadratic form in its natural light …[24]

If we consider for a moment the result that Minkowski derived detached from his mathematical practise the only difference between his and Hermite's result is that Minkowski managed to obtain a smaller upper bound. If instead we examine Minkowski's result within the context of his proof another significant difference surfaces, namely the method of reasoning used. Minkowski moved this number theoretical enterprise of positive definite quadratic forms in n variables into a completely different mathematical context—an entirely different way of thinking that—as we shall see in the next section—led to a new type of inquiry that I have identified as phase 2 of Minkowski's mathematical practice, which in time led him to single out and investigate convex sets for their own sake.

3 The second phase: investigations of the method and the construction of "measure" bodies

Until now we have seen how Minkowski used geometrical "Anschauung" in his treatment of quadratic forms to reach the fundamental theorem through geometrical intuition. The key step in his line of thought so far was the idea to construct a hypersphere (cube) around each lattice point enabling him to reach the minimum result simply by comparing the volume of the hypersphere (cube) with the volume of the standard parallelotope of the lattice associated with the quadratic form in question.

22 For more detailed calculations see (Minkowski, 1911, vol. I, p. 255–256, (1891a)).

23 (Minkowski, 1911, vol. I, p. 246, (1891a)).

24 (Minkowski, 1911, vol. I, p. 255, (1891a)).

In what follows we shall see how he further developed this line of thinking, elaborating his geometrical approach and turning the "sphere (cube)-trick" into a more general method, which led him to introduce the notions of nowhere concave bodies with middle point, "Eichkörpers" or gauge bodies, which functioned as measuring tools, and to generalize the basic fundamental geometrical idea of the length of a line segment. It will be discussed in what sense this step was an important precursor for the emergence of the concept of a general convex set and the beginning of the development of the modern theory of convexity.

3.1 The lattice point theorem and nowhere concave bodies with middle point

The first sign of Minkowski's shift in focus can be detected in a written report of a talk titled "Über Geometrie der Zahlen" he gave in Halle, Germany, in 1891.[25] He gave this talk the same year as the publication of his proof of the minimum theorem for positive definite quadratic forms in *n* variables. The demarcation line between what I have called phase 1 and phase 2 of Minkowski's mathematical practise leading up to his introduction of the concept of a convex body can be seen to cut right through these two presentations. In the 1891 publication Minkowski dealt with the minimum problem of positive definite quadratic forms treated geometrically—this paper belongs to phase 1. But as we will see, in the talk he gave in Halle the focus had shifted from the minimum problem to a scrutiny of the geometrical proof-method—the talk belongs to phase 2.

The report of the talk relates that he introduced the three dimensional lattice—not as a geometrical representation of a positive definite quadratic form—but as the collection of points with integer coordinates in the coordinate system with perpendicular axes. He defined the term "Geometrie der Zahlen" to mean geometrical investigations of the lattice and associated bodies, as well as the extension to manifolds of arbitrary dimension. He explicitly stated that the object under investigation was the lattice and associated bodies, but immediately pointed out that every statement about the lattice has an arithmetic core, indicating that this inspection of the lattice was justified not in its own right but because of its relation to number theory. It was an investigation of a *method* useful in number theory—not the building of a new theory detached from number theory. This is further supported by the fact that Minkowski felt the need to argue for the appropriateness of the use of the term "Geometry":

> Every statement about the number grid [lattice] has of course a purely arithmetic core. But the word "Geometry" appears to be quite appropriate with regard to questions that rise from geometrical intuition and to methods of investigation that are constantly guided through geometrical concepts.[26]

At this point Minkowski was working and thinking in the mathematical context of number theory, but the quote also shows that he now viewed the geometrical method in a broader sense—he no longer associated the lattice with a quadratic form. The

25 (Minkowski, 1911, vol. I, pp. 264–265, (1891b)).

26 (Minkowski, 1911, vol. I, p. 264, (1891b)).

ideas Minkowski presented in this talk laid the foundation for his upcoming book *Geometrie der Zahlen*, and for a new mathematical discipline in the 20th century with the same name.

In the talk Minkowski drew attention to a special category of bodies, which—as he phrased it—were constructed in such a way that they circumscribe the origin of the lattice in a certain manner. For these bodies one can—he claimed—obtain a relation between the magnitude of their volumes and a property of the lattice. The special category of bodies was bodies that have the origin as a middle point and whose boundary towards the outside is nowhere concave. For those bodies the asserted property with respect to the lattice is the following:

> If the volume of a body from this category is $\geq 2^3$ then this body necessarily contains additional lattice points besides the origin.[27]

This theorem is nowadays known as Minkowski's lattice point theorem (here for three dimensions). But how did that theorem come out of Minkowski's mathematical practise? And how can we explain the emergence of the bodies of this category, the *nowhere concave bodies with middle point*, in the context of Minkowski's mathematical activities? If we go back for a moment and reconsider his geometrical argument for the upper bound of the minimum of a positive definite quadratic form f we can come up with a plausible explanation for the appearance of the lattice point theorem and the nowhere concave bodies with middle point in the mathematical framework of quadratic forms. Minkowski reached this upper bound by comparing the volume of a sphere/cube with a certain radius/side length with the volume of the standard parallelotope in the corresponding lattice. In this lattice the ordinary Euclidean distance from the origin to a lattice point is given by the square root of f taken for the given lattice point. The question whether there exist integer values (not all zero) of the variables, for which a given number N can be represented by f transforms into the question whether a sphere with radius $\sqrt{N}$ and centre in the origin passes through a lattice point or not. In the talk at Halle, Minkowski considered the lattice in the rectangular coordinate system, and in such a coordinate system the points (x, y, z) for which the quadratic form equals a certain number, say $f(x, y, z) = N$, form an ellipsoid. The question of the minimum value representable by f with integer values not all zero can of course again be transformed into a question of the volume of this ellipsoid: The minimum is reached when the corresponding ellipsoid is so big that it contains a lattice point different from the origin, that is the answer to the minimum problem depends on the volume of the ellipsoid. But in the talk at Halle, Minkowski was not talking about ellipsoids; he was talking about a "very general" kind of nowhere concave bodies with the origin as middle point. If we analyse the geometrical proof he gave in the paper "Über die positiven quadratischen Form und über kettenbruchänliche Algorithmen", which was outlined in the previous section, we can see, that the crucial step in Minkowski's proof is the construction of the bodies around each point of the lattice. In the lattice corresponding to a positive definite quadratic form he first gave the argument for n-dimensional squares and then for n-dimensional spheres. The essential property of these two different kinds of geometrical bodies is that the bodies

[27] (Minkowski, 1911, vol. I, p. 265, (1891b)).

constructed around neighbouring lattice points have no inner points in common. The reason for this is exactly what we today call their convexity property. In fact Minkowski could have used a body of any shape to circumscribe the lattice points as long as the bodies did not overlap, and the only requirement needed to be put on the shape of the bodies for them to fulfil that property is that their boundary is nowhere concave and that they are symmetric around the lattice point—a property Minkowsky probably singled out somewhere around 1891 and described as *nowhere concave bodies with middle point*.

By leaving the skew coordinate system Minkowski in a certain sense also detached the method used to find the upper bound for the minimum from positive definite quadratic forms. The minimum result for such a form f can still be obtained because the level curves for a positive definite quadratic form are ellipsoids in the rectangular coordinate system, and since ellipsoids have the required property, they can be used as the bodies circumscribed around each lattice point. If the ellipsoid corresponding to the level curve $f = N$, has a volume greater than or equal to 2^3, it is large enough to contain a lattice point other than zero, meaning that N will give an upper bound for the minimum problem.

3.2 The "Eichkörper": gauge bodies and radial distances

The report of the 1891-talk that Minkowski gave at Halle is the first published written source we have that gives us information about his new geometric number theory. As we saw above, the theorem he presented in the talk was no longer about the minimum of a positive definite quadratic form, but about how big the volume of a certain body has to be in order for this body to contain a lattice point in the ordinary rectangular Euclidean coordinate system. Minkowski gave the proof of the theorem in the paper "Über Eigenschaften von ganzen Zahlen, die durch räumliche Anschauung erschlossen sind", which he read at an international mathematics conference in Chicago in 1893.[28]

In that paper Minkowski gave an outline of his book *Geometrie der Zahlen* on which he was working at the time. The main ideas along with the key result—the lattice point theorem—are presented. Since this paper gives very clear insights into Minkowski's mathematical practice I will quote from it at some length. Just as was the case with the talk given at Halle two years earlier he also began this paper by justifying its geometrical aspects:

> In number theory as in all other fields of analysis the inspiration often comes from geometrical considerations even though at the end maybe only the analytical verification is shown. Therefore, I will not be able to exhaust my theme and this is also not my intention. Here I will only talk about the geometrical figure that has the simplest relation to integer numbers, the *number grid* [*lattice*].[29]

As before, the lattice he is considering consists of all the points in three dimensional space with integer coordinates and the usual Euclidean rectangular coordinate axes. He mentioned, that what he was going to present would be included for n-dimensions

28 (Minkowski, 1911, vol. I, pp. 271–276, (1893b)).

29 (Minkowski, 1911, vol. I, p. 271, (1893b)).

in his forthcoming book *Geometrie der Zahlen*. In the following quote we shall see how he introduced an abstract notion of a metric even though he did not call it such. Instead he talked about radial distance functions, which he characterised as "generalisations of the concept of the length of a straight line". He also introduced the associated "Eichkörper"—a "measuring" or gauge body—which we today would think of as a kind of unit ball associated with the radial distance function:

> The deeper properties of the lattice are connected with a generalization of the concept of the length of a straight line by which only the theorem, that the sum of two of the sides in a triangle is never less than the third side, is maintained.
> Consider a function $S(ab)$ of two arbitrary variable points a and b, at first only with the following properties: (1) $S(ab)$ is positive when b is not equal to a, and equal to zero when b is equal to a; (2) if a, b, c, and d are four points with b different from a, and if the relationship $d - c = t(b - a)$ holds for t positive then $S(cd) = tS(ab)$. The relationship should be understood in the sense of the Barycentric Calculus and means that cd and ab are line segments in the same direction and with length (in the usual sense) in the proportion t:1. In contrast to the usual length, $S(ab)$ is called the radial distance from a to b.
> Let 0 be the origin; obviously, all the values of $S(ab)$ are determined when the set of points u for which $S(0u) \leq 1$ is given. This set of points is called the *Eichkörper* of the radial distance. In any given direction from 0 there exists a line segment from 0 in this direction with non-vanishing length and belonging to the "Eichkörper".
> If moreover $S(ac) \leq S(ab) + S(bc)$ for arbitrary points a, b, and c the radial distance is called *einhellig*. Its "Eichkörper" then has the property that whenever two points u and v belong to the "Eichkörper" then the whole line segment uv will also belong to the "Eichkörper". On the other hand every *nowhere concave body*, which has the origin as an inner point, is the "Eichkörper" of a certain "einhellig" radial distance.
> [...]
> $S(ab)$ is called *reciprocal* if $S(ba) = S(ab)$ without exceptions. This is the case when and only when the "Eichkörper" as the origin as *middle point*.[30]

Today we would call an "einhellig" reciprocal radial distance function a metric that also induces a norm, and we would think of the associated "Eichkörper" as the unit ball.

In *Geometrie der Zahlen* Minkowski proved what he had claimed in the talk, that the "Eichkörper" associated with an "einhellig" radial distance function S has the property that whenever two points u and v belong to the "Eichkörper" then the whole line segment uv will also belong to the "Eichkörper" as well as the other statement that every nowhere concave body, which has the origin as an inner point, is the "Eichkörper" of a certain "einhellig" radial distance function.[31] The first part of the proof is straight forward:

30 (Minkowski, 1911, vol. I pp. 272–73, (1893b)).

31 (Minkowski, 1953, pp. 11–13, (1896)).

Minkowski considered two points u and v belonging to the "Eichkörper" as well as a point $w = (1-t)u + tv$, $0 \le t \le 1$, on the line segment between u and v. He let u' be the point for which $u' - o = (1-t)(u-o)$ then

$$w - u' = t(v-o), \quad S(ou') = (1-t)S(ou) \quad \text{and} \quad S(u'w) = tS(ov).$$

Since the radial distance function is "einhellig",

$$S(ow) \le S(ou') + S(u'w),$$

but then

$$S(ow) \le S(ou') + S(u'w) = (1-t)S(ou) + tS(ov) \le (1-t)\cdot 1 + t \cdot 1 = 1$$

which means that w belong to the "Eichkörper", and thereby—as we would say today—that the "Eichkörper" is convex.

Minkowski proved the second part by contradiction. He assumed the existence of three points a, c, b for which

$$S(ac) > S(ab) + S(bc)$$

and determined τ such that

$$S(ac) > \tau > S(ab) + S(bc)$$

He let v' and w' denote the points for which

$$v' - o = \frac{1}{\tau}(b-a) \quad \text{and} \quad w' - v' = \frac{1}{\tau}(c-b)$$

then

$$S(ow') = \frac{1}{\tau}S(ac) > \frac{1}{\tau}\cdot\tau = 1 \quad \text{and}$$
$$S(ov') + S(v'w') = \frac{1}{\tau}S(ab) + \frac{1}{\tau}S(bc) < \frac{1}{\tau}\cdot\tau = 1$$

Minkowski then chose

$$t = \frac{S(v'w')}{S(ov') + S(v'w')}, \quad \text{and} \quad \text{thereby } 0 < t < 1$$

He then let u and v be determined by

$$u - o = \frac{v' - o}{1-t} \quad \text{and} \quad v - o = \frac{w' - v'}{t}$$

For those points

$$S(ou) = S(ov') + S(v'w') < 1 \quad \text{and} \quad S(ov) = S(ov') + S(v'w') < 1$$

This means that u and v belong to the "Eichkörper", and then also the whole line segment uv, especially the point $(1-t)u + tv$. Since $w' = (1-t)u + tv$ Minkowski had reached a contradiction because w', belonging to the line segment between u and v, also belongs to the "Eichkörper", but in contradiction to this it was also shown above that $S(ow') > 1$, which finished the second part.

As we have seen also in the previous section, at some point probably around 1891 Minkowski realized that the crucial property of the bodies constructed around each lattice point, besides the symmetry around the lattice point, is the property of convexity – as we would say today – or as Minkowski phrased it that they are nowhere concave bodies with middle points. A positive definite quadratic form gives rise to such a body, but that is the only property of the ellipsoid/sphere in the rectangular/skew lattice that Minkowski used. At some point he drew the conclusion that it does not have to be a positive definite quadratic form that is used to measure the distance in the lattice. It can be any function as long as it has the properties of an "einhellig" reciprocal radial distance function, as he sketched in the above quote from the talk.

The "Eichkörper" or measuring/gauge body of an "einhellig" reciprocal radial distance function is, as mentioned above, what we today would call the unit ball. Such a radial distance function (also called a gauge function) defines the "Eichkörper", and it measures the distance from the origin, o, to a point (x, y, z) using the length in the ordinary sense from o to (x^e, y^e, z^e) as the unit, where (x^e, y^e, z^e) is the intersection of the ray from o to (x, y, z) with the boundary of the "Eichkörper". This way of measuring the length of a straight line was indeed a "generalisation of the concept of the length of a straight line" and it gives rise to a geometry in which – as phrased by Thompson (1996):

> [...] to someone peering in from outside, it appears that the unit for measuring length is different in different directions and hence unit "circles" and "spheres" are not the familiar round objects from Euclidean geometry but are some other convex shape.[32]

Minkowski used the "Eichkörper" to prove the lattice point theorem. He considered an "einhellig" radial distance function S, and argued (for the fist time in print) for the existence of a smallest radial distance in the rectangular lattice. He called this minimum distance M. Again he imagined the bodies $S(au) \leq 1/2M$ and $S(uc) \leq 1/2M$ constructed around two different lattice points a and c, respectively. Because of the property

$$S(ac) \leq S(ab) + S(bc)$$

which ensures the convexity property of the bodies $S(au) \leq 1/2M$ and $S(uc) \leq 1/2M$, these two bodies will have no inner points in common. If the radial distance is also

[32] (Thompson, 1996, p. x).

reciprocal then $S(uc) \leq {}^1\!/_2 M$ is identical to $S(cu) \leq {}^1\!/_2 M$ and the individual bodies around the lattice points will touch each other at the boundary. Using the same line of argument, as he did in the probationary lecture and in the 1891 paper in Crelle's *Journal*, Minkowski constructed a system of non-overlapping nowhere concave bodies with a lattice point as middle point. This system will not fill out the whole space, so again – by comparing volumes – he reached the inequality

$$1 \geq \left(\frac{M}{2}\right)^3 J$$

Here J denotes the volume of the "Eichkörper" $S(0u) \leq 1$. The volumes of the bodies $S(au) \leq {}^1\!/_2 M$ are then equal to $({}^1\!/_2 M)^3\, J$.[33]

The conclusion Minkowski reached from this inequality was, that since M is the smallest distance in the lattice and $M \leq 2J^{-1/3}$ there exists at least one lattice point q for which $S(0q) \leq 2J^{-1/3}$. That this theorem is in fact the lattice point theorem he stated in the talk at Halle in 1891 is not mentioned in this paper, but in the book *Geometrie der Zahlen* Minkowski interpreted the inequality (for n dimensions $1 \geq (M/2)^n J$) as follows: If $J = 2^n$ then $M = 1$, and if $J > 2^n$ then $M < 1$ meaning that, since M is the smallest distance between lattice points, there exists a lattice point b for which $S(0b) = M \leq 1$, that is b is contained in the "Eichkörper" associated with S.[34] Remembering that Minkowski in this paper had restricted himself to three dimensions the inequality expresses that if the volume of the "Eichkörper" is greater than or equal to 2^3 then the "Eichkörper" must contain more lattice points than the origin.

Minkowski finished the discussion of the inequality with the following praise:

> The hereby gained theorem about nowhere concave bodies with middle point seems to me to belong to the most fruitful in the whole of number theory.[35]

Clearly, Minkowski at this point conceived of the nowhere concave bodies with middle point as a tool in number theory. The connection to positive definite quadratic forms and the minimum problem was not presented in the paper, but in a letter Minkowski wrote to Hermite the same year in which Minkowski also gave what he called a "quick resumé" of his book *Geometrie der Zahlen* Minkowski focused on the analytical aspects of the theory:

> The biggest part of the book treats functions φ in n variables $x_1, x_2, \ldots, x_n$, which, as the square root of a positive definite quadratic form, satisfy the conditions

[33] Contrary to the Probationary lecture and the 1891 paper Minkowski actually gave a rigorous proof of the inequality $1 \geq \left({}^1\!/_2 M\right)^3 J$, as well as an explanation for what was to be understood by the volume of a body in terms of C. Jordan's results (Jordan, 1892).

[34] (Minkowski, 1953, p. 76, (1896)).

[35] (Minkowski, 1911, vol. I, p. 274, (1893b)).

$$\varphi(x_1, x_2, \ldots, x_n) > 0, \quad \text{if not } x_1 = 0, x_2 = 0, \ldots, x_n = 0,$$
$$\varphi(0, 0, \ldots, 0) = 0,$$
$$\varphi(tx_1, tx_2, \ldots, tx_n) = t\varphi(x_1, x_2, \ldots, x_n), \quad \text{if } t > 0,$$
$$\varphi(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \leq \varphi(x_1, x_2, \ldots, x_n) + \varphi(y_1, y_2, \ldots, y_n),$$
$$\varphi(-x_1, -x_2, \ldots, -x_n) = \varphi(x_1, x_2, \ldots, x_n).$$ [36]

And the main result is that one can find integers $x_1, x_2, \ldots, x_n$ for which

$$0 < \varphi(x_1, x_2, \ldots, x_n) \leq \frac{2}{\sqrt[n]{J}}$$

where J denotes the volume of the domain $\varphi(x_1, x_2, \ldots, x_n) \leq 1$. This is the minimum result for positive definite quadratic forms if, as Minkowski indicated in the beginning of the letter, φ is the square root of a positive definite quadratic form. The "Eichkörper" is then the ellipsoid $\varphi(x_1, x_2, \ldots, x_n) \leq 1$. That is the quadratic form measures the distances in the lattice. Since the volume, J, of this ellipsoid is a constant, k_n, that depends on n, multiplied by $1/\sqrt{D}$, where D is the determinant of the quadratic form, the usual minimum result appears

$$f(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n)^2 \leq 4k_n^{-2/n} D^{1/n}$$

In the letter to Hermite, Minkowski only gave the analytical version of the theorem, but in the book *Geometrie der Zahlen* both versions are presented as well as the above application of the theorem to the minimum problem for positive definite quadratic forms, which again suggests that even though Minkowski in the sense discussed above had detached the investigation of the lattice from positive definite quadratic forms, he still considered his work on the lattice and its associated bodies as a method useful for solving problems in number theory – not least in the theory of positive definite quadratic forms.[37]

In the monograph *Geometrie der Zahlen* Minkowski devoted the first chapter to nowhere concave surfaces. He defined the "Aichkörper"[38] of a radial distance function as before and he gave the proof, shown above, of the fundamental property, which we recognise as convexity:

> The "Aichkörper" of an "einhellig" radial distance has the following characteristics:
> If two points belong to the "Aichkörper" so does every point on the line segment between them. [...]
> Vice versa, if the "Aichkörper" of the radial distance $S(a, b)$ has the property that for any two of its points η and ξ, which do not belong to its boundary, any

[36] (Minkowski, 1911, vol. 1, p. 266, (1893)).

[37] (Minkowski, 1953, pp. 76–77, and p. 196, (1896)).

[38] In *Geometrie der Zahlen* Minkowski spelled Eichkörper with an A instead of an E.

> point on the line segment between η and ξ also belong to the "Aichkörper", then the radial distance is "einhellig".[39]

But he was not considering them as "convex" sets, they were measuring bodies (gauge bodies) defined not through their "convexity" property but through a radial distance function.

In this second phase, Minkowski coined the term "nowhere concave bodies with middle point" and investigated them as a tool through which he could derive number theoretical inequalities. The constituent element is the "Eichkörper" associated with a radial distance function. The connection between these two concepts follows from the fact that the "Eichkörper" corresponding to an "einhellig" radial distance function form a nowhere concave body with the origin as an inner point, and every nowhere concave body, which has the origin as an inner point, is the "Eichkörper" of some "einhellig" radial distance function. As we have seen, Minkowski's study of his method of proof for the minimum theorem for positive definite quadratic forms led him to realize the essential property of the ellipsoid, which – through his geometrical interpretation of positive definite quadratic forms – measured the distance in the associated lattice. The only thing needed to carry out the proof for the minimum result was the convexity property of the ellipsoid; it was not associated with the ellipsoid itself. This led Minkowski to generalise the proof method and single out the general notions of nowhere concave bodies and "Eichkörpers", which he then used to measure distances in space, creating an abstract general metric space.

So how should we understand Minkowski's perception of these nowhere concave bodies at this point? The first time he used the term "nowhere concave" in print was in the report of the talk from Halle in 1891. Here he did not give a formal definition of the notion of a nowhere concave body. Rather he talked about bodies of a certain kind surrounding the origin in a certain way, bodies whose boundary toward the outside is nowhere concave. The first object he introduced in this respect was the "Eichkörper" of a radial distance function, defined in the paper he read at the international mathematics conference in Chicago in 1893. Here he also talked about nowhere concave bodies – again without giving a definition, probably due to the fact that he thought of the name as self-explanatory. That he did not give a formal definition at that point indicates that the significant element at this time in his mathematical practise, the "Eichkörper" – the measuring (gauge) body – was not interesting as a geometrical object in itself, but as a tool, a technique, through which the concept of distance in space could be generalised and measured with the purpose of solving problems in number theory.

It was not until the last chapter of the book *Geometrie der Zahlen* that Minkowski gave a formal definition of nowhere concave bodies:

> The manifold of all points $x_1, \ldots, x_n$ will be denoted by H. A given set of points B is characterised as a nowhere concave body already through the following two properties:
>
> 1. that a straight line has either no points, one point or a line of points in common with the set,

[39] (Minkowski, 1953, p. 12, (1896)).

2. that in the set some $n+1$ points, not in the same [hyper]plane, exist.

.....

In 1. is also the following property inherent:

3. when two points belong to B so does every point on the line segment between the two.[40]

Here we recognise 3. as a definition of a convex set that is often used today. The title of this last chapter is "Eine weitere analytisch-arithmetische Ungleichung" in which Minkowski used the tools he had developed in the first chapters to derive results for positive definite quadratic forms and ellipsoids, and prove an inequality involving volumes of different bodies constructed through a radial distance function. The definition of the nowhere concave body occurs in the subsection "Berechnung eines Volumens durch successive Integrationen" the content of which goes into the proof of the inequality. We can see here that, even though the importance of these bodies is stressed with regard to their use in number theory, he slowly began to think of them as mathematical concepts in their own right.

4 The Third Phase: the concept of a convex body—the beginning of the theory of convexity

The third phase is characterised by Minkowski's submersion into the geometry of nowhere concave bodies – or convex bodies as he soon came to name them – for their own sake. This also marks the beginning of the systematic study of convex sets that developed into the modern theory, as it is known today. Minkowski published four papers in which he treated different aspects of the study of convex sets.[41] A fifth (unfinished) paper was published posthumously in volume II of his *Collected Works*.[42]

The shift of Minkowski's focus is seen very clearly both from the title and the content of those papers. In the first one "Allgemeine Lehrsätze über die konvexen Polyeder" from 1897 Minkowski opened the paper with a definition of a convex body:

> A *convex body* is completely characterized by the properties that it is a closed set of points, has inner points, and that every straight line that takes up some of its inner points always has two points in common with its boundary.[43]

In a footnote he directed the reader towards page 200 in his *Geometrie der Zahlen*

> There [page 200 in *Geometrie der Zahlen*] I called these objects *nowhere concave* bodies; here I will use the shorter term *convex*.[44]

While in "*Geometrie der Zahlen*" he used the nowhere concave bodies as a tool, this first paper in what I call the third phase of Minkowski's practise is dedicated to the

40 (Minkowski, 1953, p. 200, (1896)).

41 (Minkowski, 1911, vol. II, (1897), (1901a), (1901b), (1903)).

42 (Minkowski, 1911, vol. II, pp. 131–229).

43 (Minkowski, 1911, vol. II, p. 103, (1897)).

44 (Minkowski, 1911, vol. II, p. 103, (1897)).

study of convex bodies. Minkowski argued that investigations of such bodies were mathematically interesting not only because of their applicability in other areas of mathematics, notably number theory, but also because

> The theorems about convex bodies have a special appeal because they as a rule are valid for the whole category of objects without any exceptions.[45]

Besides this general interest in convex bodies Minkowski gave a more particular motivation behind the paper. He wanted to prove a theorem that he

> had expected for a long time […]: A *convex body* that is build up by a finite number of sheer *bodies with middle point* that only touch each other at the boundaries has a middle point as well.[46]

The resemblance between this problem and the proof-method Minkowski first used in his work on the minimum problem and later, in *Geometri der Zahlen*, to prove the lattice point theorem is striking and suggests that Minkowski was led to consider this theorem through his work in phase 1 and phase 2 on these problems and the associated convex bodies—or nowhere concave bodies with middle point—as he called them at that time.

Minkowski proved the theorem not for arbitrary convex bodies but for convex polytopes in three dimensional space, but, as he claimed in a finishing remark, the theorem can be extended to manifolds of arbitrary dimensions.

4.1 Hermann Brunn's influence on Minkowski's theory of convex bodies

In this section I will discuss the question of Brunn's possible influence on Minkowski's work. I claimed in the introduction that the story, as it is presented in various textbooks, leaves the reader with the impression that Minkowski's work on convex sets was a continuation of Hermann Brunn's work. If one reads Minkowski's first paper solely devoted to the theory of convex sets without taking Minkowski's mathematical practise in phase 1 and phase 2 into account, one can indeed get this impression. In the second section of that paper, entitled "Die Grundlagen der Untersuchung", Minkowski referred to Brunn's thesis, in which Brunn, according to Minkowski, developed the following theorem:

> If a convex body is intersected by three parallel planes A', B', C' of which the middle one B' divides the distance between A' and C' in the ratio $t : 1 - t$ and if the cross sections have the area A, B, C, then the inequality
>
> $$\sqrt{B} \geq (1-t)\sqrt{A} + t\sqrt{C}$$
>
> is true.[47]

45 (Minkowski, 1911, vol. II, p. 103, (1897)).

46 (Minkowski, 1911, vol. II, p. 103, (1897)).

47 (Minkowski, 1911, vol. II, p. 108, (1897)).

The theorem Minkowski was referring to is Theorem 5 on page 23 in Brunn's dissertation "Ueber Ovale und Eiflächen" from 1887. In his thesis Brunn did not use the term "convex body", he worked on geometrical objects, which he named "Oval" (oval), "volles Oval" (full oval), "Eifläche" (egg-surface) and "volles Eifläche" (full egg-surface). The ovals are plane figures whereas the egg-forms are the corresponding figures in three dimensional space. Brunn defined an oval as:

> a closed curve that has two and only two points in common with every intersecting straight line in its plane.[48]

By the term "volles" oval Brunn understood an oval together with the inner points enclosed by the oval. He defined egg-surface and "volles" egg-surfaces in an analogous way for three dimensions. Brunn was working with what we today would call convex sets in the plane and in space and their boundaries. The theorem, which Minkowski claimed to be at the foundation of his own investigation in his first paper entirely on convex sets, was phrased differently by Brunn, who wrote:

> Among a family of parallel cross sections in an egg-surface there exists one and only one with maximum area realised through *one* cross section or through *one* continuum of congruent cross sections forming part of a cylinder.[49]

This is certainly not the same formulation as the one given by Minkowski cited above, so first of all why was Minkowski crediting the theorem to Brunn, and second, what role – if any – did Brunn's work play in Minkowski's investigations?

Regarding the first question: In the proof Brunn first considered two parallel planes each containing a rectangle with equal area, and parallel sides. He then compared a cross section, parallel to the two planes, of the obelisk formed by the two rectangles. This cross section lies between the two planes and Brunn argued that the area of the cross section of the obelisk is at least the size of the rectangle.[50] In the argument he presented the inequality

$$F' \geq F(\lambda + \lambda')^2$$

where F' is the area of the cross section of the obelisk, F the area of the two original rectangles, and $\lambda + \lambda' = 1$. With the help of this result Brunn was able to give an argument for the first part of the theorem, whereas his proof for the second part, as we shall see below, was deemed flawed according to the standards of the time. This inequality by Brunn is contained in the one presented by Minkowski in 1897 that was quoted above, which is probably why Minkowski credited the inequality result to Brunn. These types of inequalities are nowadays in textbooks and mathematical papers known and referred to as Brunn–Minkowski inequality or Brunn–Minkowski theorem.

Regarding the second question on Brunn's influence on Minkowski: As is evident from all his publications, Minkowski was very careful to acknowledge the papers by

48 (Brunn, 1887, p. 1).

49 (Brunn, 1887, p. 23).

50 (Brunn, 1887, p. 23).

other mathematicians that he had drawn on for inspiration. Brunn's inaugural thesis is from 1887 and his *Habilitation* dissertation, which Brunn saw as a continuation of the inaugural thesis, is from 1889,[51] but Minkowski's first reference to Brunn did not occur in print until 1896 where it can be found on page 209 in *Geometrie der Zahlen*, almost at the end of the book. Here Minkowski stated the inequality for n dimensions, writing that the theorem "is due to Mr. Brunn" and referred the reader to the last section of the book, Sect. 57, for the proof.[52] In a footnote on page 237 there is a second reference to Brunn. As I have also argued in (Kjeldsen, forthcoming) there are several circumstances that indicate that these references probably were added in the proofreading process. First of all, Minkowski introduced the "Eichkörper" and the nowhere concave surfaces already in the first chapter of the book and on page 200, as mentioned above, he defined the notion of a nowhere concave body. Considering Minkowski's practise of crediting his sources of inspiration, if he had been inspired by Brunn's work he most likely would have referred to Brunn at those particular places in the book. Secondly, in 1894 Brunn published what is basically a revision of parts of his inaugural thesis especially regarding Theorem 5 on page 23, the one to which Minkowski was referring. The title of this revision is "Referat über eine Arbeit: Exacte Grundlagen für eine Theorie der Ovale", and Brunn gave the following explanation for the occurrence of the work:

> The occasion to return to this subject [the inaugural thesis] the revision of which for a long time has appeared ungrateful for the author is the knowledge of similar work by Mr. Minkowski in Bonn (in the future Könningsberg). At Teubner Minkowski has published a preannouncement of a book in print entitled "Geometrie der Zahlen" in which an unexpected and fruitful connection between number theory and the geometry of bodies whose boundaries are nowhere concave is established and thereby also in analytical terms treats the theory of the latter. Thus also from other sides than a geometrical point of view a certain importance is attached to egg-forms and this has encouraged the author to supplement his doctoral thesis in the manner indicated above.[53]

About Theorem 5, Brunn wrote in the revision:

> The proof of the first part of the theorem can be more sharply clarified without essentially changing it by means of the investigations in the preceding report. Regarding the last claim in the theorem, that part of a cylinder is formed, Minkowski has brought to my attention that the indicated proof in my doctoral thesis (III, 9, 10) is too superficial regarding the occurrence of a difficulty. The author may here be permitted to close this gap.[54]

The quotes suggest that Brunn, after having seen the announcement from Teubner, probably presented his thesis to Minkowski, who then apparently pointed out some

51 (Brunn, 1889).

52 (Minkowski, 1953, p. 209, (1896)). Minkowski used the letter m instead of n to indicate the dimension.

53 (Brunn, 1897, p. 94).

54 (Brunn, 1897, p. 94).

weaknesses. Thirdly; in the very last paragraph of *Geometrie der Zahlen*, which probably was written in 1896 but apparently not published until the 1910-edition, Minkowski came back to Brunn with a reference to the revision from 1894, so that part was certainly added after 1893, that is during the proof reading.[55] Here Minkowski wrote:

> The merits of being the first one to set up the important inequality (4) [$\sqrt[n-1]{T'} \geq (1-t)\sqrt[n-1]{B'} + t\sqrt[n-1]{C'}$] goes to Mr. Brunn. (see his paper Ueber Ovale und Eiflächen, p. 23, Art. 5). Mr. Brunn's exposition refers primarily to the case $n = 3$ and is held more in geometrical terms whereas here a pure analytical presentation is given. … Because of a remark from me about the necessity of a sharper argumentation for some additional theorems as they appear through the remarks mentioned at the site (pp. 24–25, Art. 9 and 10) Mr. Brunn has returned to the case of the limit in question in the paper "Exacte Grundlagen für eine Theorie der Ovale".[56]

Finally; Brunn explicitly stated that his theorem could not be used to prove extremal properties of the sphere,[57] which was, as will be shown below, exactly what Minkowski was going to use it for and the reason why it is counted as a fundamental theorem in the modern theory of convexity. So, even though Minkowski credited the result to Brunn it is almost certain that Minkowski worked independently of Brunn, and one can say for sure that Brunn was not able to use the theorem to prove extremal properties.

4.2 Minkowski's further work on the theory of convexity

Minkowski's first paper exclusively committed to investigations of convex bodies was followed three years later, in 1901, by two further papers on convex bodies. One of them is a summary of a talk published in *Jahresbericht der Deutschen Mathematikervereinigung* where Minkowski talked about the problems of justifying the notions of the length of curves and the surface area of curved surfaces as limits of the length of polygons and surface areas of polyhedrons respectively.[58] He argued for the significance of convex bodies in the generalisation of the notion of surface area. The main issue in Minkowski's talk was to use these considerations to give a

> new and more rigorous proof of the theorem *that among all convex bodies with equal volume the sphere has the smallest surface* and *at the same time to trace this theorem back to a more substantial and analytically simpler one*.[59]

55 There are some indications that this very last part of the book was not published until 1910 with the second edition of the book. Apparently the book was published in two parts: the first part in 1896 consisting of the first 240 pages and the last part consisting of the last 14 pages was added to the 1910-edition. See also (Kjledsen, forthcoming).

56 (Minkowski, 1953, p. 254, (1896)).

57 (Brunn, 1887, p. 31).

58 (Minkowski, 1911, vol. II, p. 122, (1901a)).

59 (Minkowski, 1911, vol. II, p. 125, (1901a)).

In order to accomplish that, Minkowski used the theorem that he again ascribed to Brunn now in the following formulation:

> Let J_0 and J_1 be two arbitrary non-similar convex bodies with volumes W_0 and W_1, respectively. If one combines each point of J_0 with every one from J_1 and divides every connecting line segment in the constant ratio $t : 1 - t$, where $0 < t < 1$, then the set of all the different dividing points form again a convex body J_t and for the volume W_t of this the following inequality holds:

$$\sqrt[3]{W_t} > (1-t)\sqrt[3]{W_0} + t\sqrt[3]{W_1}$$

In the reference to Brunn's thesis Minkowski cited Brunn:

> At the mentioned cite Mr. Brunn has indeed uttered the opinion that: "This theorem cannot be used to prove the maximum property of the sphere."[60]

Here again we see Minkowski carefully crediting Brunn for being the first one to consider matters leading to the above inequality but – at the same time – emphasising that the interpretation of these matters and their significance in a broader theory of convex sets cannot be ascribed to Brunn.

In the second 1901 paper, Minkowski continued to pursue questions and results about extremal properties of the sphere, considerations that led him to introduce the notion of mixed volumes of convex bodies, which has become a key notion in parts of convexity theory.[61]

He published only one longer paper on his theory of the geometry of convex bodies before he died but a second longer work, found after his death, was published in volume II of Minkowski's *Collected Works*.[62] Together these papers can be seen as the beginning of a foundation for a systematic development and presentation of what soon became established as the theory of convexity. In these two papers Minkowski introduced many of the now-standard notions such as supporting and separating hyperplanes, mixed volumes and centre of gravity, to name some of them. In these papers he developed a geometric theory of convex sets in three dimensional space, but for example the existence of supporting hyperplanes he had introduced and proved for n-dimensional "Eichkörpers" in *Geometrie der Zahlen*.

5 Conclusion

The modern theory of convexity entered the world of mathematics at the turn of the twentieth century. It constituted a new research field in which new mathematical knowledge was created. As described in the introduction the historical research presented above has been directed towards answering the question: Why and how did the concept of convex bodies emerge, take form, and lead to the beginning of a

[60] (Minkowski, 1911, vol. II, p. 125, (1901a)).

[61] (Minkowski, 1911, vol. II, p. 129, (1901b)).

[62] (Minkowski, 1911, vol. II, pp. 230, (1903)), (Minkowski, 1911, vol. II, pp. 131).

theory of convexity in Minkowski's mathematical practise? This analysis has brought forward three phases in Minkowski's mathematical practice that led him to introduce the study of convex bodies as a new research field. The shift from one phase to the next is characterised by a change in Minkowski's focus of research, showing that the dynamics of Minkowski's production of the knowledge that led to the theory of convex bodies can be described as an interplay between two ways of generating knowledge: (1) by answering new previously unimagined questions, and (2) by answering known questions using new methods, following new roads and ways of argumentation.

The first phase is characterised by Minkowski's effort to answer a well known, important question in the reduction theory of positive definite quadratic forms, namely the minimum question. By using geometrical methods Minkowski was able to solve the minimum problem in a new way. The geometrical method he created was so powerful that it turned into a new mathematical discipline of the twentieth century, called the geometry of numbers. This is a clear manifestation of new knowledge generated by answering old questions in new and previously unknown ways. Even though Minkowski was not the first one to interpret positive definite quadratic forms geometrically, he took the method much further. Through his geometrical intuition and the way he put it to use, he moved number theory into a new epistemic place, which opened up new kinds of questions, and new pathways to explore, leading to the formation of the discipline known today as the geometry of numbers, which is an upshot of the method, the technique, and the context of argumentation more than the theorem of the minimum problem in itself.

Minkowski did not stop at the minimum problem. Instead he went on and turned his attention towards researching the geometric method itself, and that is the part of Minkowski's mathematical practise I have called the second phase. In this phase he investigated the proof-method used in his work on the minimum problem, research that led him to introduce radial distance functions, to construct the associated "Eichkörper" as well as to introduce the notions of nowhere concave bodies with and without middle point. Along with the notion of the lattice these new mathematical concepts functioned as tools in his number theoretical investigations, tools that allowed him to prove the minimum theorem through geometrical intuition without cumbersome calculations, a proof-method Hilbert later in his commemorative speech characterised as "a pearl of the Minkowskian art of inventions",[63] and indeed, as we have seen, this second phase was extremely fruitful. This phase also shows Minkowski as a mathematician of the new trend of abstraction and axiomatization that became the hallmark of twentieth century mathematics. Minkowski's radial distance functions are highly abstract and n-dimensional. They are detached from the meaning of distance that we know from our empirical, physical world and as such they have another ontological status than mathematical objects had in the beginning of the nineteenth century where they usually were thought of as abstractions of familiar objects. The associated "Eichkörpers" as well as Minkowski's nowhere concave bodies are abstract mathematical entities in n-dimensional mathematical spaces not abstractions from figures in our "real" world. Jeremy Gray has characterised this ontological change as a revolution in the

63 (Hilbert, 1909, p. XI) in (Minkowski, 1911, vol. I).

ontological status of mathematical objects,[64] a change that was notably apparent in Hilbert's *Grundlagen der Geometrie* from 1899 where the modern axiomatic method of the twentieth century mathematics served as the foundation for geometry. Minkowski was in this sense a very modern mathematician.

This new geometrical way of argumentation Minkowski developed for solving number theoretical problems also led him to recognise the essential properties—that we today call convexity—of the objects used in his geometrical line of reasoning for the minimum problem and the lattice point theorem. This led Minkowski to introduce, in phase three, the concept of a convex body as a geometrical object independent of number theory, the minimum problem, and the lattice point theorem. The convex bodies were interesting geometrical research objects in themselves, partly because of their applicability to those areas of mathematics, but also because statements about them had a certain appeal, as Minkowski wrote in his first paper exclusively committed to the study of convex bodies.

This marked the beginning of the third phase of Minkowski's mathematical practise leading to his introduction of the modern theory of convexity. This third and final phase is characterised by the generation of new mathematical knowledge obtained by asking new questions, questions about the geometry of these very general bodies, which are only required to satisfy the property of convexity.

The identification and analysis of the three phases show that the concept of a convex body as well as the modern theory of convexity emerged through an interplay between the two ways of generating new knowledge: the answering of an old question with new methods created new knowledge that opened the possibility of introducing new mathematical objects and asking new questions, together laying the foundation of the modern theory of convexity.

Analysed with hindsight we can see that Minkowski changed his perception of what a mathematician of today would call a convex set from an "Eichkörper", which he conceived as a tool, to an independent geometrical object detached from the process of measuring. This change in perception is also supported by Minkowski's change of dimension: The "Eichkörper" was a tool to handle number theoretical problems, and in order to function as such it had to "live" in n-dimensional manifolds. In accordance with that Minkowski developed in his *Geometrie der Zahlen* a theory of n-dimensional "Eichkörper". When he changed his research perspective in phase three, defined the geometrical concept of a convex body, and began to develop a theory of convexity he focused on three dimensions. Even though he claimed at several places that the results could be extended to n-dimensions he only derived results for three-dimensional convex bodies, indicating again a shift in focus from number theory to geometry in three dimensional space. This change in perception made it possible for Minkowski to ask geometrical questions, and gain new mathematical knowledge about the properties of convex bodies.

When he suddenly died from a ruptured appendix, Minkowski left a new mathematical research field where he had laid the foundation for the new discipline of convexity in twentieth century's mathematics. His work inspired others, in the beginning

64 (Gray, 1992)

notably Carathéodory[65] and Steinitz[66], who took up and extended Minkowski's work in different directions and for different purposes.

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References

Bonnesen, T and Fenchel, W.: *Theorie der konvexen körper*. Berlin: Julius Springer Verlag. 1934.

Brunn, H.: *Ueber Ovale und Eiflächen*. Inaugural-Dissertation, Munich: Akademische Buchdruckerei von F. Straub, 1887.

Brunn, H.: *Ueber Curven ohne Wendepunkte*. Habilitationsschrift, Munich: Theodor Ackermann, 1889.

Brunn, H.: "Referat über eine Arbeit: Exacte Grundlagen für eine Theorie der Ovale", *Sitzungsberichte der Mathematik und Physik*, **XXIV** (1894) 93–111.

Corry, L.: "Introduction: The History of Modern Mathematics – Writing and Rewriting", *Science in Context*, **17** (2004), 1–21.

Dirichlet, G. L.: "Über die Reduction der positiven quadratischen Formen mit drei unbestimten ganzen Zahlen." *Journal für die reine und angewandte Mathematik*, **40** (1850) 209–227.

Epple, M.: "Knot Invariants in Vienna and Princeton during the 1920s: Epistemic Configurations of Mathematical Research", *Science in Context*, **17** (2004), 131–164.

Fenchel, W.: "Convexity through the Ages." In Peter M. Gruber and Jörg M. Wills (eds.) *Convexity and Its Applications*. Basel-Boston-Stuttgart: Birkhäuser, 1983, 120–130.

Gauss, C. F.: *Collected Works*, Göttingen 1863, 188–196.

Goldman, J. R.: *The Queen of Mathematics*. Wellesley, Massachusetts: A K Peters, Ltd. 1998.

Gray, J.: "The nineteenth-century revolution in mathematical ontology". In Donald Gillies (ed.) *Revolutions in Mathematics*. Oxford: Clarendon Press, 1992, 226–248.

Gruber, P.: "Zur Geschicthe der Konvexgeometrie und der Geometrie der Zahlen". In Gerd Fischer. Friedrich Hirzebruch, Winfried Scharlau and Willi Törnig (eds.) *Ein Jahrhundert Mathematik 1890–1990*. Festschrift zum Jubiläum der DMV. Barunschweig-Wiesbaden: Deutsche Mathematiker-Vereinigung, Friedr. Vieweg & Sohn, 1990, 421–455.

Gruber, P.: "History of Convexity". In M. Gruber and Jörg M. Wills (eds.) *Handbook of Convex Geometry*. Elsevier Science Publishers, 1993, 3–15.

Hermite, C.: "Extraits de letters d. M. Ch. Hermite a M. Jacobi sur différents objects de la théorie des nombres". *Journal für die reine und angewandte Mathematik*, **40** (1850) 261–315.

Hilbert, D.: "Hermann Minkowski. Gedächtnisrede" 1909, V-XXXI in (Minkowski, 1911, vol. I).

Jordan, C.: "Remarques sur les integrals définies," *Journal de Mathématiques*, 4. series, **8** (1892) 77.

Kjeldsen, T. H.: "A Contextualized Historical Analysis of the Kuhn-Tucker Theorem in Nonlinear Programming: The Impact of World War II" *Historia Mathematica*, vol. 27 (2000) pp. 331–361.

Kjeldsen, T. H.: "John von Neumann's Conception of the Minimax Theorem: A Journey Through Different Mathematical Contexts" *Archive for History of Exact Sciences*, 56, 2001, pp. 39–68.

Kjeldsen, T. H.: "Different Motivations and Goals in the Historical Development of the Theory of Systems of Linear Inequalities" *Archive for History of Exact Sciences*, 56, 2002, 469–538.

Kjeldsen, T. H.: "New Mathematical Disciplines and Research in the Wake of World War II". In Booss-Bavnbek and Høyrup (eds.): *Mathematics and War*. Birkhäuser Verlag: Basel-Boston-Berlin, 2003, 126–152.

Kjeldsen, T. H.: "The Development of Nonlinear Programming in Post War USA: Origin, Motivation, and Expansion." In Andersen, H.B., Christiansen, F.V., Jørgensen, K.F. and Hendricks, V. (eds): *The Way Through Science and Philosophy: Essays in Honour of Stig Andur Pedersen*. College Publications, London, 2006. pp. 31–50.

Kjeldsen, T. H.: "Ovals, Egg-forms, and "Measure" bodies: Different Mathematical Practices in the Early History of the Development of the Modern Theory of Convexity." Forthcoming.

Klee, V. (ed.): "Convexity", *Proceedings of Symposia in Pure Mathematics*, volume VII, Providence, Rhode Island: American Mathematical Society. 1963.

[65] (Carathéodory, 1907, 1911).

[66] (Steinitz, 1913, 1914).

Minkowski, H.: "Sur la réduction des formes quadratiques positives quaternaires" (1883), 145–148 in (Minkowski, 1911, vol. I).
Minkowski, H.: "Grundlagen für eine Theorie der quadratischen Formen mit ganzzahligen Koeffizienten" (1884), 3–144 in (Minkowski, 1911, vol. I).
Minkowski, H.: "Über einige Anwendungen der Arithmetik in der Analysis", probationary lecture, (1887) 85–88 in (Schwermer, 1991).
Minkowski, H.: "Über die positiven quadratischen Formen und über kettenbruchähnliche Algoritmen" (1891a), 243–260 in (Minkowski, 1911, vol. I).
Minkowski, H.: "Über Geometrie der Zahlen" (1891b), 264–265 in (Minkowski, 1911, vol. I).
Minkowski, H.: "Extrait d'une lettre adressée à M. Hermite" (1893a), 266–270 in (Minkowski, 1911, vol. I).
Minkowski, H.: "Über Eigenschaften von ganzen Zahlen, die durch räumliche Anschauung erschlossen sind" (1893b), 271–277 in (Minkowski, 1911, vol. I).
Minkowski, H.: *Geometrie der Zahlen*. B. G. Teubner, 1896. (All page references is to the 1953 edition).
Minkowski, H.: "Allgemeine Lehrsätze über die konvexen Polyeder" (1897), 103–121 in (Minkowski, 1911, vol.II).
Minkowski, H.: "Über die Begriffe Länge, Oberfläche und Volumen" (1901a), 122–127 in (Minkowski, 1911, vol.II).
Minkowski, H.: "Über die geschlossenen konvexen Flächen" (1901b), 128–130 in (Minkowski, 1911, vol.II).
Minkowski, H.: "Theorie der konvexen Körper, insbesondere Begründung ihres Oberflächenbegriffs" 131–229 in (Minkowski, 1911, vol.II).
Minkowski, H.: "Volumen und Oberfläche" (1903), 230–276 in (Minkowski, 1911, vol.II).
Minkowski, H.: *Gesammelte Abhandlungen*. Leipzig, Berlin: B. G. Teubner. 1911.
Minkowski, H.: *Briefe an David Hilbert*. Berlin-Heidelberg-New York: L. Rüdenberg, H. Zassenhaus. 1973.
Reid, C.: *Hilbert*. Berlin, Heidelberg, New York: Springer-Verlag. 1970.
Schwermer, J.: "Räumliche Anschauung und Minima positive definiter quadratischer Formen", *Jahresbericht der Deutschen Mathematiker-Vereinigung*, **93** (1991), 49–105.
Scharlau, W.: "A Historical Introduction to the Theory of Integral Quadratic Forms", *Conference on Quadratic Forms*. Edited by G. Crzech. Queen's University, Kingston, Ontario, Canada. 1977.
Scharlau, W. and Opolka, H.: *From Fermat to Minkowski*. New York: Springer Verlag. 1985.
Strobl, W.: "Aus den wissenschaftlichen Anfängen Hermann Minkowskis", *Historia Mathematica*, **12** (1985), 142–156.
Thompson, A. C.: *Minkowski Geometry*. Cambridge, MA: Cambridge University Press, 1996.