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Piola's contribution to continuum mechanics

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Abstract This paper examines the contribution of Gabrio Piola to continuum mechanics. Though he was undoubtably a skilled mathematician and a good mechanician, little is commonly known about his papers within the international scientific community, principally because a large part of the Italian school of mechanics was isolated in the first half of the XIXth century. We examine and comment on Piola's most important papers, and compare them with those of his contemporaries Cauchy, Poisson and Kirchhoff.

1 Introduction

At the beginning of the XIXth century the state of theoretical mechanics and science in general did not seem remarkable in Italy. An overview of the scientific production of the time shows a deep lack of creativity and a certain cultural isolation, with the exception of some contacts of northern Italian scientists with the nearby French school; even the best works reveal a remarkable cultural gap.[1] The Italian situation in some way reflected the international one, which had reached a point of stagnation after Lagrange. The models of material point

[1] For a portrait of mathematics and mechanics at the beginning of the nineteenth century in Italy, see (Bottazzini 1994).

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and rigid body, used by the mechanicians of the XVIIIth century, had exhausted their powers: the problems which could be solved by them were either too difficult, such as the problem of the *n* bodies, or of little importance. Hydraulics was in a different situation, since much could still be obtained from these models.

New studies of mechanics emerged from this point of stagnation in two directions. In one, the rigid body model was abandoned and the idea of energy and dissipation gained importance. Indeed, at the beginning of the 1820s the mechanics of elastic continua originated in the works of Navier and Cauchy; a little later scientists focused on the idea of work, and mechanics became linked to the new discipline of thermodynamics. In the other, the theory was perfected thanks to the introduction of a general formulation and a renewed form of geometrical language, which turned out to be a powerful tool of rationalisation. The contributions of Hamilton and Jacobi in the first half of the nineteenth century were fundamental here.[2]

A lively and extremely interesting discussion of the fundamentals began in France and the rest of Europe, inspired by the publication of Lagrange's *Méchanique analitique* in 1788. In this book the founding principle is that of virtual velocities as stated by Bernoulli, suitably generalised by the calculus of variations. Italians made only a marginal contribution to this discussion, as is shown by the papers of the major Italian scientists in the most important Italian journals, the *Memorie di matematica e fisica della Società italiana delle scienze*, the *Memorie dell'Istituto nazionale italiano* and the *Memorie dell'Istituto lombardo*. Vittorio Fossombroni wrote a book[3] which was well received even in France; for example, Prony[4] recommended it to his students at the École polytechnique. Michele Araldi[5] and Girolamo Saladini[6] tried to prove rigorously the principle of virtual work, unaware of the results of the French school of mechanics. Gregorio Fontana produced studies on mechanics which were still anchored in the previous century.[7] In 1790 and 1794 Antonio Maria Lorgna[8] and in 1809 and 1811 Paolo Delanges presented papers on elasticity which are interesting from the point of view of applications but of modest theoretical content. Pietro Ferroni presented his view of the principles of mechanics.[9] In the absence of an original and creative vein, most of the Italian mechanicians simply provided critical comments in the spirit of an eighteenth century tradition.

To many Italian mathematicians and mechanicians, modernity was represented by Lagrange. This was partly because Lagrange, even after leaving Turin

2 For an overview of these aspects, see (Dugas 1950).

3 (Fossombroni 1796).

4 (Prony 1797), p. 204.

5 (Araldi 1806).

6 (Saladini 1808).

7 For instance in (Fontana 1802).

8 In 1782 this scientist had promoted the foundation of the Società italiana, which edited *Memorie di matematica e fisica*. The founders being forty in number, the society was also called Accademia dei XL and still operates with this name.

9 (Ferroni 1803).

in 1766, had remained in contact with the Italian world of science, and partly because Italians considered him Italian and this was a period of rising nationalistic feelings. Vincenzo Brunacci (1768–1818), professor of "Matematica sublime" (Calculus) in Pavia, was one of the main supporters of Lagrange's ideas. Along with the fashionable purism of the time, he accepted Lagrange's reduction of differential calculus to algebraic procedures[10] and rejected the eighteenth century concept of infinitesimal in both calculus and mechanics.[11] Brunacci transmitted these ideas to his pupils, including Ottaviano Fabrizio Mossotti (1791–1863), Antonio Bordoni (1788–1860) and Gabrio Piola (1794–1850), the brightest Italian mathematicians of the first half of the nineteenth century. As an example of the spirit of that school, one may consider the notes[12] to Giuseppe Venturoli's book on hydraulics.[13] In these notes Venturoli's proofs, which were originally obtained by using infinitesimals, were re-drawn by Bordoni and Piola using Lagrange's method of derivative functions. Lagrange's ideas were so deeply rooted in Brunacci's pupils that in due course they found it difficult to accept the "progressive" ideas of Cauchy, with whom they came in contact during his voluntary exile in Italy from 1830 to 1833. Only after some time did Piola start to appreciate Cauchy's new mathematical conceptions, and he was never to accept them completely.[14]

Though undoubtably one of the most brilliant Italian mechanicians of the XIXth century, little is commonly known about Gabrio Piola's life and work within the scientific community.[15] His name, however, is well known because in most textbooks on continuum mechanics it is associated with two tensors describing stress at a point of a body undergoing large deformations. With this paper we aim to illustrate in some depth the role played by Piola in continuum mechanics.

Count Gabrio Piola was born in Milan on July 15th, 1794, into a rich and noble family. At first he studied at home and later attended a local lyceum. He soon showed his skills in mathematics and physics, and studied mathematics at the University of Pavia, where he was a pupil of Vincenzo Brunacci and gained his doctorate on June 24th, 1816. He did not follow an academic career, though he was offered the chair of Applied Mathematics at the University of Rome. Instead, he did private teaching—one of his pupils was Francesco Brioschi, later professor of Rational Mechanics in Pavia and president of the Accademia dei Lincei. Piola began as a researcher in mathematics and mechanics in 1824, winning a prize competition from the Istituto Lombardo in Milan with a long

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10 In (Lagrange 1797, 1813b), the derivative of a function of a real variable is defined as the factor multiplying the increment of the variable in the first term of the Taylor series expansion of the function itself.

11 (Brunacci 1804).

12 (Bordoni 1833).

13 (Venturoli 1826).

14 For some of the Italian mathematicians' view of Cauchy see (Bottazzini 1989).

15 For Piola's biography and an analytical list of his papers, see (Masotti 1950).

article on Lagrange's analytical mechanics.[16] His contributions to mathematics are on finite differences and integral calculus, while in mechanics he was interested basically in continuum mechanics and hydraulics. He was the editor of a journal, *Opuscoli matematici e fisici di diversi autori*, of which only two volumes appeared. Amongst other things, the journal was the medium which presented Cauchy's mathematical theories in Italy: in fact, it contained some of Cauchy's fundamental works, which were translated from French into Italian.[17]

Piola was a highly cultured person and dedicated himself also to history and philosophy, most notably writing a paper on Bonaventura Cavalieri.[18] He was a member of numerous societies, among which the Società italiana delle scienze, and from 1825 he belonged also to the Accademia romana di religione cattolica. In fact, Piola was very attached to traditions and an extremely strong Catholic, a faith he shared with Cauchy. This is why the latter kept Piola as as a first point of reference in his stay in Italy from 1830 to 1833.[19] Piola also taught religion for twenty-four years in a parish in Milan, and was a friend of Antonio Rosmini, the most important exponent of Italian Catholic spiritualism. He died in 1850 in Giussano della Brianza, near Milan.

2 Piola's principles of mechanics

Among Brunacci's pupils, Piola was the most interested in mechanical aspects; in his works he devoted much effort to the task of eliminating infinitesimals, re-formulating the principle of virtual velocities as proposed by Lagrange. For reasons which are still not clarified by historians, there was in Italy a certain reluctance to accept the idea of force as a primitive concept, as Newton and Euler had proposed. Rather, the preferred approach was that of d'Alembert, according to whom force is a derived concept and $f = ma$ is simply a definition. According to this point of view, dynamics precedes statics. This is the belief of, among others, Giovanbattista Magistrini, to whom Piola refers:[20]

> Elements of the former [statics] cannot be but a particular determination of the elements of the latter [dynamics], and its [of dynamics] equations could not be good and general unless they did not include equilibrium with all its accidents. Putting into practice the ideas derived from putting statics prior to dynamics makes us notice this truth by means of irregularity and

16 (Piola 1825).

17 (Bottazzini 1989), pp. 28–29.

18 (Piola 1844).

19 For a short excerpt on the religious view shared by Cauchy and Piola, see (Bottazzini 1989), footnote (40), p. 29.

20 (Piola 1825), pp. IX-X.

> contradiction […]. Indeed, [this practice] is obliged to use the expedient of a certain infinitesimal mechanical motion.[21]

Piola's epistemological views are set out in the paper which won a prize from the Reale Istituto Lombardo di Scienze in Milan in 1824 and was published in 1825. These views remained unaltered in practice in Piola's later works. The metaphysics in his work is the same as that found in Lagrange: all mechanics can be expressed in terms of differential calculus and there is no need, nor is it convenient, to use other branches of mathematics which use intuition, such as geometry for example, and which can therefore lead to errors. Piola called the key tool of his treatment '*equazione generalissima*', that is, the most general and "indubitable" equation of mechanics. This equation coincides with what we now call the equation of virtual work, formulated according to Lagrange's approach using the calculus of variations. Yet the equation of virtual work may not be considered as evident *per se*; even Lagrange had doubts on the subject,

> […] it is necessary to admit that it is not sufficiently evident to be regarded as a primitive principle.[22]

In line with the Aristotelian view of science of his time, Piola could not assume the equation of virtual work explicitly as a true principle and felt compelled to derive it from first principles, which had to be absolutely evident, at least in a purely empirical sense, i.e., experienced in everyday life. In so doing, Piola abandoned d'Alembert's position,[23] where mechanics is a purely rational science just like geometry, and linked himself to the "empiric" epistemology of Newton, even though he did not accept Newton's fundamental concept of force:

> It is therefore necessary to give up all our pretences and, following the great precept of Newton, to seek in the nature those principles which can explain other natural phenomena […]. These ideas persuade us that it would be a bad philosopher who would persist in wanting to know the truth about the fundamental principle of mechanics in the same way as the axioms are evident to us. […] But, if the fundamental principle of

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[21] Gli elementi della prima non possono essere che una particolare determinazione degli elementi della seconda, e le formole di questa non si potrebbero aver per buone e generali se il caso non comprendessero dell'equilibrio con tutti gli accidenti che ad esso appartengono. La pratica stessa dei ragionamenti che impiegasi nel premettere la statica alla dinamica ci fa sentire questa verità coll'irregolarità e con la contraddizione […]. Perciocché vedesi costretta a mettere in campo il ripiego di certo meccanico movimento infinitesimale (Magistrini 1816, p. 450).

[22] […] il faut convenir qu'il n'est pas assez evident par lui-même pour être érigé en principle primitif (Lagrange 1788, p. 23).

[23] (d'Alembert 1758), p. XXIX.

> mechanics cannot be evident in itself, at least it should be a truth easy to understand and to be persuaded of.[24]

The empirical first principle introduced by Piola is the superposition of motions: the motion due to the action of two causes is the sum, in the modern sense of vector sum, of the motions due to each single cause.[25] Along with d'Alembert's definition of force, this principle leads to the property of superposition of forces. These two assumptions are not sufficient to study the mechanics of extended bodies, and the idea of mass must be introduced. Piola followed the norm of his time, by identifying mass with the quantity of matter: he believed that the substance of a given material can be considered to be formed by very small atoms which are all equal. These can be arranged in space in various different ways and constitute bodies with apparently different densities; the mechanical behaviour of a body depends only on the number of atoms it contains. In a scholion, Piola expressed clearly his ideas about atoms, or infinitesimal components in mathematics and in physics, refusing their existence in the former and accepting them in the latter:

> I, educated by Brunacci in the school of Lagrange, have always avoided the metaphysical infinitesimal, by assuming that in analysis and geometry (if we want to have clear ideas) we must always replace it with the undeterminate, as small as we need: but I accept what could be called physical infinitesimal, of which we have a clear idea. It is not an absolute zero, rather it is any such quantity that could be noticeable by other beings, but it is a zero relative to our senses.[26]

Piola then "proved" the equation of virtual work using of these ingredients, and so he believed he had eliminated all the mechanical and mathematical uncertainties he found in Lagrange's formulation. Indeed, Piola had no need to use the somewhat obscure concept of XVIIIth century infinitesimal, and he used the calculus of variations established rigorously by Lagrange.[27] The equation of virtual work for a system of constrained material points is provided by Piola in the following form:

$$\delta L + \lambda\, \delta C = 0\,, \tag{1}$$

24 È dunque necessario abbandonare alquanto le nostre pretese, e, seguendo il gran precetto di Newton, cercare nella natura que' principi con che spiegare gli altri fenomeni naturali […]. Queste riflessioni persuadono che sarebbe un cattivo filosofo chi si ostinasse a volere conoscere la verità del principio fondamentale della meccanica in quella maniera che gli riesce manifesta l'evidenza degli assiomi. […] Ma se il principio fondamentale della meccanica non può essere evidente, dovrà essere non di meno una verità facile a intendersi e a persuadersi (Piola 1825, p. XVI).

25 The same principle, using a similar vocabulary, is assumed in (Mossotti 1868).

26 Io, educato da Brunacci alla scuola di Lagrange, ho sempre impugnato l'infinitesimo metafisico, ritenendo che per l'analisi e la geometria (se si vogliono conseguire idee chiare) vi si deve sempre sostituire l'indeterminato piccolo quanto fa bisogno: ma ammetto ciò che potrebbe chiamarsi l'infinitesimo fisico, di cui è chiarissima l'idea. Non è uno zero assoluto, è anzi tal grandezza che per altri esseri potrebbe riuscire apprezzabile, ma è uno zero relativamente alla portata dei nostri sensi (Piola 1848, p. 14).

27 (Lagrange 1797).

where δL is the first-order variation of the work of all the active forces (including inertia), δC represents the first-order variation of the constraint equations and λ is a Lagrange multiplier. Hence, the virtual displacements that must be taken into account are free from any constraint and do not need to be infinitesimal. There is a weak point in Piola's proof of the equation of virtual work, i.e., the vanishing of the work of constraint reactions, which is implicitly assumed but not proved.[28] However, even if Piola had been conscious of the weakness of his reasoning, he would probably not have been severely worried. He had no doubt that the equation of virtual work was right and its rigorous proof was only a question of style, which did not modify the development of the mechanical theory.

By means of the '*equazione generalissima*', the undisputed general equation of motion, Piola's empiric and positivist strategy could be applied in a convincing and interesting way to the mechanics of extended bodies. In his papers, Piola questioned the need to introduce uncertain hypotheses on the constitution of matter by adopting a model of corpuscles and forces among them, that the French mechanicians did. Piola stated that it is sufficient to refer to evident and certain phenomena: for instance, in rigid bodies, the shape of the body remains unaltered. Then, one may use the undisputed equation of balance of virtual work; only after one has found a model and equations based exclusively on phenomena, said Piola, is it reasonable to look for deeper analyses:

> This is the great advantage of Analytical Mechanics. It can put into equations facts about which we have clear ideas, without requiring us to consider their causes, of which we have obscure ideas [...]. The action of forces, whether active or passive (according to a known distinction by Lagrange) are such that we can sometimes have some ideas about, but more often the full doubt remains [...] that nature is different [...]. But in the A. M. what is contemplated is the effect of internal forces and not the forces themselves, namely, the conditional equations which must be satisfied [...] and in such a way all difficulties that belong to the action of forces are bypassed, and we have the same equations, secure and exact, like those which would result from the thorough knowledge of the said actions.[29]

[28] (Capecchi 2003).

[29] Ecco il maggiore vantaggio del sistema della Meccanica Analitica. Esso ci fa mettere in equazione i fatti di cui abbiamo le idee chiare senza obbligarci a considerare le cagioni di cui abbiamo idee oscure [...]. L'azione delle forze attive o passive (secondo una nota distinzione di Lagrange) è qualche volta tale che possiamo farcene un concetto, ma il più sovente rimane [...] tutto il dubbio che il magistero della natura sia ben diverso [...]. Ma nella M. A. si contemplano gli effetti delle forze interne e non le forze stesse, vale a dire le equazioni di condizione che devono essere soddisfatte [...] e in tal modo, saltate tutte le difficoltà intorno alle azioni delle forze, si hanno le stesse equazioni sicure ed esatte che si avrebbero da una perspicua cognizione di dette azioni (Piola 1833, pp. 203–204).

Piola approaches mechanics in a modern way which can be found unchanged in many modern handbooks on rational mechanics. Lanczos writes the following on the subject:

> It frequently happens that certain kinematical conditions exist between the particles of a moving system which can be stated *a priori*. For example, the particles of a solid body may move as if the body were "rigid" [...]. Such kinematical conditions do not actually exist on *a priori* grounds. They are maintained by strong forces. It is of great advantage, however, that the analytical treatment does not require the knowledge of these forces, but can take the given kinematical conditions for granted. We can develop the dynamical equations of a rigid body without knowing what forces produce the rigidity of the body.[30]

3 Piola's papers on continuum mechanics

Piola's contributions to continuum mechanics are developed in a number of papers listed in the first part of the References. Many of these papers deal with hydraulics; they usually focus on particular aspects and are worth reading mainly to appreciate the author's mathematical skill. A few papers deal with continuum mechanics in general, implicitly assuming that the solid state is the preferred one. These papers were published in 1833, 1836 and 1848 and will be commented upon separately and in some detail in this section. A further paper was published posthumously by Piola's former pupil Francesco Brioschi in 1856: it may be considered a mature revision and a rewriting of the article of 1848 and for this reason will not be commented upon separately here. With the exception of the paper from 1833, all the rest consist of more than 200 pages that systematically expound a full approach to the statics and dynamics of one-, two- and three-dimensional continua. Though they are very interesting in every aspect, we believe the parts concerning balance equations to be the most interesting and will therefore focus on those.

As become usual in mathematical physics in the nineteenth century, Piola's papers contain a lot of mathematics, most of which is quite extensive. To report his equations efficiently and compactly we have decided to give them a different form by using the language of linear algebra, which the Italian mechanician could not know.[31] We have tried, however, not to corrupt the original spirit

[30] (Lanczos 1970), pp. 4–5.

[31] The precise concept of matrix and the related modern symbolism are found in (Cayley 1858). Disposition of data in matrix form stems however from the theory of determinants, well known to Cauchy, who in 1811 developed a two-index notation (Kline 1972, ch. 33).

of Piola's work avoiding any modern interpretation of the symbols, which are introduced only for the sake of brevity. Piola's actual equations can be obtained by performing matrix operations and expressing the results in scalar form. Table 1 of Sect. 3.1 will be useful for this purpose and compares some of the modern notations with Piola's.[32]

The first paper will be analysed in depth: it is short in comparison with the other two, but contains all the notable results from a mechanical point of view which are also present in the others. However, it has also many drawbacks and cannot be fully understood without analysing the others.

3.1 Piola's *Meccanica de' corpi naturalmente estesi*

Piola's first paper on continuum mechanics, *La meccanica de' corpi naturalmente estesi trattata col calcolo delle variazioni*, dates back to 1833. The title is ambiguous because at Piola's time '*estesi*' (extended) meant both rigid and deformable, while Piola in this paper studied only rigid bodies, qualifying them as solid, a term used by Euler and Lagrange as synonymous with rigid. Piola maintained this ambiguity throughout the paper, since he used notations which can be extended to deformable bodies. The reason stems from his declared intention, which he did not fulfill, to study deformable bodies in a sequel.[33]

Piola started the paper by characterising rigid motions globally and locally. The physical points of the body under consideration are labelled by two sets of Cartesian co-ordinates. The first (following Lagrange in the *Mécanique*

[32] This is the complete correspondence among the formul here and in (Piola 1833).

Our eq. no.	Piola's eq. no.	Our eq. no.	Piola's eq. no.
(1)	[3] p. 208	(10)	[20] p. 216, [22] p. 217
(2)	[4], [5] p. 209	(11)	[23]–[27] pp. 218–219
(3)	[6]–[8] p. 209	(12)	[28] p. 219
(4)	[9] p. 210	(13)	[29]–[30] pp. 219–220
(5)	[10]–[12] pp. 210–211	2nd addend of l.h.s. of (14)	[31] p. 221
(6)	[13], [14] pp. 211–212	(15)	[33] p. 222, [35] p. 223
(7)	Unnumbered eq. p. 214	(16-1)	[36] p. 223, [38], [40] p. 224
(8)	[16] p. 215	(16-2)	[45] p. 226
2nd addend of l.h.s. of (9)	[18] p. 215	(17)	[46] p. 226

[33] The title of (Piola 1833) contains the statement '*Memoria prima*', i.e., first paper (of a series). Piola spoke, for instance on p. 227, about a '*successiva memoria*', i.e. a further paper which was intended to complete the study; such a paper never appeared in the journal.

Analytique[34]) refers to axes called a, b, c that are rigidly attached to the body, so their position with respect to points of the body does not vary with time, and the second to axes called x, y, z, fixed in the ambient space and to which the motion of the body is referred. Here these quantities will be denoted by the lists a_i and x_i, $i = 1, 2, 3$, respectively, or, in short, $\boldsymbol{a}$ and $\boldsymbol{x}$, respectively. The link between $\boldsymbol{x}$ and $\boldsymbol{a}$ represents the global rigidity condition and is given, provided the subscript 0 refers to a chosen point of the body, by

$$\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{Q}(\boldsymbol{a} - \boldsymbol{a}_0), \tag{2}$$

where $\boldsymbol{Q}$ is a matrix made up of the cosines of the angles between the x_i- and the a_i-axes, satisfying the orthogonality conditions

$$\boldsymbol{Q}^\top \boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{Q}^\top = \boldsymbol{I}, \quad \boldsymbol{Q}^\top = \boldsymbol{Q}^{-1}. \tag{3}$$

Piola proved[35] that only six of the scalar equations implied by Eqs. (3) represent independent conditions on the components of $\boldsymbol{Q}$. He also remarked that in the rigid motion provided by Eq. (2) the partial derivatives of the present coordinates with respect to the invariable ones coincide with the director cosines which are the components of $\boldsymbol{Q}$, viz.

$$\boldsymbol{F} = \boldsymbol{Q}, \quad \boldsymbol{F} = \mathrm{Grad}\, \boldsymbol{x} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}} \tag{4}$$

The differential operators introduced here will be indicated with upper- or lower-case initials depending on the derivatives being made with respect to $\boldsymbol{a}$ or $\boldsymbol{x}$, respectively. Consequently, the fields operated upon will be considered as functions of $\boldsymbol{a}$ or $\boldsymbol{x}$. Similar considerations govern the the fields appearing in integrals evaluated in the reference or in the present configuration, respectively.

From this point onwards Piola's expressions remain somewhat ambiguous, because he did not make a clear distinction between the components of $\boldsymbol{F}$, which can be defined in a generic motion, and the components of $\boldsymbol{Q}$, characterising a rigid motion. On the basis of Eqs. (3) and (4) Piola was able to eliminate $\boldsymbol{x}_0$ and $\boldsymbol{Q}$, the global characteristic parameters of a rigid motion, and obtain '*equazioni di condizione*' [condition equations, i.e., constraint equations; such a terminology is still present, for instance, in (Todhunter and Pearson 1893)] which express a local condition of rigidity. These relate $\boldsymbol{x}$ and $\boldsymbol{a}$ in differential form and are equivalent to

$$\boldsymbol{C} = \boldsymbol{F}^\top \boldsymbol{F} = \boldsymbol{I}, \quad \boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^\top = \boldsymbol{I}, \quad \boldsymbol{F}^{-1} = \boldsymbol{F}^\top. \tag{5}$$

[34] (Lagrange 1815), Sect. XI, art. 4.

[35] (Piola 1833), '*Nota* I^A' in the Appendix, pp. 228–230.

Due to the symmetry of $\boldsymbol{B}$ and $\boldsymbol{C}$, Eqs. (5–1) and (5–2) provide two sets of six independent scalar differential constraint equations.[36] Since Eqs. (5–1) and (5–2) provide local conditions of rigid motion, it is appropriate for their first-order variation to be inserted into the '*equazione generalissima*' to obtain the required local balance equations:

$$\delta \boldsymbol{C} = (\delta \boldsymbol{F}^{\top})\boldsymbol{F} + \boldsymbol{F}^{\top}(\delta \boldsymbol{F}) = \boldsymbol{0}, \quad \delta \boldsymbol{B} = (\delta \boldsymbol{F})\boldsymbol{F}^{\top} + \boldsymbol{F}(\delta \boldsymbol{F}^{\top}) = \boldsymbol{0}. \tag{6}$$

Piola tried to understand if there can be less than six independent scalar equations expressing the local rigidity constraint. He advanced the hypothesis that the minimum number of scalar equations of inner constraint is three, and that the other three make the problem undetermined. From a modern point of view, we accept that a rigid three-dimensional body is internally statically undetermined; it is highly probable, however, that such a fact disturbed Piola, who advanced some obscure statements to infer that there are only three independent inner constraints.[37]

Piola then began to use the methods of analytical mechanics and wrote the '*momenti delle forze acceleratrici*', i.e., the virtual work of the mass density of distance interactions $\boldsymbol{f}$ (including both bulk and inertial actions) as an integral over the mass of body $\mathcal{B}$

$$\int_{\mathcal{B}} [\boldsymbol{f} \cdot (\delta \boldsymbol{x})] \mathrm{d}m, \tag{7}$$

where dm is the body mass element. Piola stated that Eq. (7) can be reduced to a volume integral ('*integrale triplicato*') defined over the domain κ of the unvariable co-ordinates a_i

$$\int_{\kappa} \varrho J [\boldsymbol{f} \cdot (\delta \boldsymbol{x})] \mathrm{d}V. \tag{8}$$

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[36] In the first edition of the *Méchanique Analitique* (1788) Lagrange studied statics and dynamics of elastic and incompressible fluids. The incompressible case is developed in an unnecessary wide general notation, which allowed Lagrange to extend the results he obtained to compressible fluids. Piola followed this approach when he started talking about rigid bodies, hoping to extend the results to deformable bodies, ignoring the relation $\boldsymbol{F} = \boldsymbol{Q}$. Lagrange introduced the local rigidity in hydrostatics (Lagrange 1811, Sect. VII, art. 13). The local incompressibility constraint is provided by $dV = const.$, where dV is an infinitesimal of volume in the XVIIIth century sense (actual infinity). Lagrange obtained the condition equation for the virtual displacements from this rigidity condition. With his symbols (Lagrange 1811, Sect. VII, art. 11):

$$\delta \mathrm{d}V = \mathrm{d}V \left(\frac{\mathrm{d}\delta x}{\mathrm{d}x} + \frac{\mathrm{d}\delta y}{\mathrm{d}y} + \frac{\mathrm{d}\delta z}{\mathrm{d}z} \right).$$

The result was obtained first by Euler vith the use of velocity instead of variation (Dugas 1950, p. 290; Truesdell 1991, p. 101). Though Piola could not accept Lagrange's reasoning, based on actual infinity, he could accept the conclusion: the local condition equation must be in differential form.

[37] This is put into evidence also in (Todhunter and Pearson 1893), art. 762, p. 420.

where $\mathrm{d}V = \mathrm{d}a_1\mathrm{d}a_2\mathrm{d}a_3$ is the volume element in κ, which can be interpreted as a reference configuration, ϱ is the mass density in the present configuration χ, and $J = \det \boldsymbol{F}$.[38]

The variation $\delta\boldsymbol{x}$ in Eq. (8) cannot be arbitrary but must be subjected to the rigidity constraint. The usual way to take these constraints into account is with the introduction of Lagrange multipliers and constraint equations. Here the constraint equations are the global rigidity conditions expressed by Eq. (2): their use leads to six scalar global balance equations. According to Piola, using the local rigidity conditions expressed by Eqs. (5) and (6), local balance equations can be obtained. Piola started using Eqs. (5–2) and (6–2), which he considered more suitable for reasons which are not explained but which will become clear at the end of the paper. By adding the first-order variation of the constraints given by Eq. (6–2) to Eq. (8), the following unconstrained variational problem results:

$$\int_{\kappa} \varrho J[\boldsymbol{f} \cdot (\delta\boldsymbol{x})]\mathrm{d}V + \int_{\kappa} [\boldsymbol{T} \cdot (\delta\boldsymbol{B})]\mathrm{d}V = 0, \tag{9}$$

where $\boldsymbol{T}$ is a symmetrical matrix of Lagrange multipliers, each associated to a scalar component of Eq. (6–2). The components of $\boldsymbol{T}$ are denoted by Piola in such a way as to reproduce Cauchy's notation[39] for the stress components (see Table 1). To obtain expressions in which the variation $\delta\boldsymbol{x}$ is not affected by derivatives, Piola applied integration by parts, claiming to follow the rules of

[38] The use of a reference frame other than the present one was introduced in hydrodynamics by Lagrange. Since the differential problem for dynamics can be more difficult than that for statics, Lagrange tried to simplify it by pulling the equations back to the reference configuration, in which the co-ordinates of a fluid point are a, b, c (like Piola used): all quantities in the balance equations shall then be functions of a, b, c. In particular, Lagrange pulled back the volume element from the present to the reference configuration (Lagrange 1815, Sect. XI, artt. 4–7):

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \Theta\,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c,$$

where Θ (corresponding with the Jacobian J in Eq. (8), called '*sestinomio*' by Piola) is the coefficient which makes it possible to invert the expressions

$$\mathrm{d}\xi = \frac{\partial\xi}{\partial a}\mathrm{d}a + \frac{\partial\xi}{\partial b}\mathrm{d}b + \frac{\partial\xi}{\partial c}\mathrm{d}c, \qquad \xi = x, y, z.$$

Lagrange recognised that for incompressible fluids $\Theta = 1$; in spite of this, he never simplified that factor, and so did Piola for his '*sestinomio*'. Lagrange commented:

> It must be remarked that this value of $\mathrm{D}x\,\mathrm{D}y\,\mathrm{D}z$ is that which we must employ in the triple integrals with respect to x, y, z, when we want to substitute, in place of the variables x, y, z, assigned functions of other variables a, b, c (Il est bon de remarquer que cette valeur de $\mathrm{D}x\,\mathrm{D}y\,\mathrm{D}z$ est celle qu'on doit employer dans les intégrals triples relatives à x, y, z, lorsqu'on veut y substituer, à la place des variables x, y, z, des fonctions données d'autres variables a, b, c) (Lagrange 1813, pp. 284–285).

[39] (Cauchy 1827), p. 108.

the calculus of variations.[40] The variational problem (9) leads to two systems of definite integrals, one on κ and the other on its surface. Piola studied the first one and claimed he would deal with the second in a further paper. After some work, Piola obtained

$$\mathrm{Div}\,(\boldsymbol{T}\boldsymbol{F}) + \varrho J \boldsymbol{f} = \mathbf{0} \tag{10}$$

but, without giving any interpretation of this result, moved on to prove that Eq. (10) may be reduced to the form of the local balance equations provided by Cauchy and Poisson.[41] For this purpose, Piola obtained a theorem[42] which lets us transform differential operators with respect to $\boldsymbol{a}$ into differential operators with respect to $\boldsymbol{x}$, and can be read as

$$\mathrm{Div}[\boldsymbol{L}_\kappa (J\boldsymbol{F}^{-\top})] = J\,\mathrm{div}\,\boldsymbol{L}_t, \tag{11}$$

where $\boldsymbol{L}$ is a matrix function and the subscripts κ and t denote its descriptions in terms of $\boldsymbol{a}$ or $\boldsymbol{x}$, respectively. Note that Eq. (11) is general; on the other hand, in art. 14 Piola restricted himself assuming

$$\boldsymbol{F}^\top = J\boldsymbol{F}^{-1} \tag{12}$$

which is true only when $\boldsymbol{F} = \boldsymbol{Q}$. Then, inserting Eqs. (11) and (12) into Eq. (10) yields

$$\mathrm{div}\,\boldsymbol{T} + \varrho \boldsymbol{f} = \mathbf{0} \tag{13}$$

and Piola commented upon this result in art. 15:

> Observe the perfect agreement of this result with those obtained by the two celebrated geometers cited at the start of the introduction [Cauchy and Poisson] by absolutely different means and treating the two cases of equilibrium and motion separately. I recommend to note that in my analysis A, B, C, D, E, F [the components of $\boldsymbol{T}$] are not pressures exerted on different planes, but are coefficients to which in the following I also will attach a representation of forces, when it will seem more natural to me: they are functions of x, y, z, t of a still unknown form, but of which we know that nothing is changed by passing from one part of the body to the other. One can object that these equations [...] were found by the methods of the A. M. only in the case of systems of rigid bodies, while those of the two famous Frenchmen refer to elastic and deformable systems. I reply that in the next memoir I will show how they can be generalised to include all

40 (Lagrange 1813), Sect. IV, artt. 14 and 15.

41 The papers to which Piola refers are (Cauchy 1827), (Cauchy 1828), (Poisson 1829), (Poisson 1831a,b). It is remarkable that the model of the body is continuous in Cauchy's papers, while it is discrete in those of Poisson.

42 (Piola 1833), '*Nota III*^{A}' in the Appendix, pp. 234–236.

the cases considered by the cited Authors without leaving the analytical methods by Lagrange.[43]

Piola was probably dissatisfied with the results that he obtained by implicitly taking Eq. (12) into account, and therefore he tried a different approach. He now maintained that in order to extend his results to the general case of deformable bodies it is useful to examine the variational problem obtained using Eq. (6–1)

$$\int_{\kappa} \varrho J[\boldsymbol{f} \cdot (\delta \boldsymbol{x})]\mathrm{d}V + \int_{\kappa} [\boldsymbol{P}_2 \cdot (\delta \boldsymbol{C})]\mathrm{d}V = 0, \tag{14}$$

where $\boldsymbol{P}_2$ is another symmetrical matrix of Lagrange multipliers, each associated to a scalar component of Eq. (6–1), and different from the Lagrange multipliers listed in $\boldsymbol{T}$. Piola did not say why he thought that this procedure is more general, as he did not explain why he took Eqs. (5–2) and (6–2) in the first place. By applying integration by parts on Eq. (14) and ignoring the contribution of the surface integrals, Piola obtained

$$\mathrm{Div}\,(\boldsymbol{F}\boldsymbol{P}_2) + \varrho J \boldsymbol{f} = \boldsymbol{0}. \tag{15}$$

He did not comment upon this equation, but said that he wanted to adjust the first addend so as to apply the transformation rule in Eq. (11). For this purpose, he introduced quantities which are the components of a new symmetric matrix $\boldsymbol{S}$ which is such that

$$\boldsymbol{S}J\boldsymbol{F}^{-\top} = \boldsymbol{F}\boldsymbol{P}_2 \quad \left(J\boldsymbol{S} = \boldsymbol{F}\boldsymbol{P}_2\boldsymbol{F}^{\top}, \quad \boldsymbol{P}_2 = J\boldsymbol{F}^{-1}\boldsymbol{S}\boldsymbol{F}^{-\top}\right) \tag{16}$$

Piola wrote exhausting and lengthy passages to prove the equations corresponding to Eq. (16–2). Then, by using Eqs. (11) and (16), Eq. (15) reads

$$\mathrm{div}\,\boldsymbol{S} + \varrho \boldsymbol{f} = \boldsymbol{0} \tag{17}$$

which is formally equivalent to Eq. (13), as Piola himself remarked.

Table 1 compares some of our equations with the corresponding writing in the *Meccanica de' corpi naturalmente estesi*, so as to be able to capture the spirit of the full expressions, which we have shortened.

[43] Osservisi la perfetta coincidenza di questo risultato con quello ottenuto dai due celebri geometri citati dal principio dell'introduzione dietro ragionamenti affatto diversi e nei due casi dell'equilibrio e del moto trattati separatamente. Raccomando di notare che nella mia analisi le A, B, C, D, E, F non sono pressioni che si esercitino sopra diversi piani, ma sono coefficienti, cui nel seguito attaccherò io pure una rappresentazione di forze secondo mi sembrerà più naturale: sono funzioni delle x, y, z, t di forma ancora incognita, ma di cui sappiamo che non cambia passando dall'una all'altra parte del corpo. Mi si può obbjettare che queste equazioni [...] sono state trovate coi metodi della M. A. nel solo caso dei sistemi solidi rigidi, laddove quelle dei due chiarissimi francesi si riferiscono anche a' solidi elastici e variabili. Rispondo che nella seguente memoria farò vedere come esse si generalizzano ad abbracciare tutti i casi contemplati dai citati Autori senza dipartirsi dagli andamenti analitici insegnati da Lagrange (Piola 1833, p. 220).

Table 1 Some comparison among the expressions here and in (Piola 1833)

Our eq. no.	Piola's eq. no.	Expression in full
(2)	[4] p. 209	$x = f + \alpha_1 a + \beta_1 b + \gamma_1 c$ $y = g + \alpha_2 a + \beta_2 b + \gamma_2 c$ $z = h + \alpha_3 a + \beta_3 b + \gamma_3 c$
(5–1)	[10] p. 210	$(\frac{dx}{da})^2 + (\frac{dy}{da})^2 + (\frac{dz}{da})^2 = 1$ $(\frac{dx}{db})^2 + (\frac{dy}{db})^2 + (\frac{dz}{db})^2 = 1$ $(\frac{dx}{dc})^2 + (\frac{dy}{dc})^2 + (\frac{dz}{dc})^2 = 1$ $(\frac{dx}{da})(\frac{dx}{db}) + (\frac{dy}{da})(\frac{dy}{db}) + (\frac{dz}{da})(\frac{dz}{db}) = 0$ $(\frac{dx}{da})(\frac{dx}{dc}) + (\frac{dy}{da})(\frac{dy}{dc}) + (\frac{dz}{da})(\frac{dz}{dc}) = 0$ $(\frac{dx}{db})(\frac{dx}{dc}) + (\frac{dy}{db})(\frac{dy}{dc}) + (\frac{dz}{db})(\frac{dz}{dc}) = 0$
(8)	[16] p. 215	$SdaSdbSdc \cdot \Gamma H \left\{ \left[\left(\frac{d^2x}{dt^2} \right) - X \right] \delta x + \left[\left(\frac{d^2y}{dt^2} \right) - Y \right] \delta y + \left[\left(\frac{d^2z}{dt^2} \right) - Z \right] \delta z \right\}$
Matrix ***T***	in [18] p. 215	$\begin{pmatrix} A & F & E \\ F & B & D \\ E & D & C \end{pmatrix}$
(13)	[30] p. 220	$\Gamma \left[X - \left(\frac{d^2x}{dt^2} \right) \right] + \left(\frac{dA}{dx} \right) + \left(\frac{dF}{dy} \right) + \left(\frac{dE}{dz} \right) = 0$ $\Gamma \left[Y - \left(\frac{d^2y}{dt^2} \right) \right] + \left(\frac{dF}{dx} \right) + \left(\frac{dB}{dy} \right) + \left(\frac{dD}{dz} \right) = 0$ $\Gamma \left[Z - \left(\frac{d^2z}{dt^2} \right) \right] + \left(\frac{dE}{dx} \right) + \left(\frac{dD}{dy} \right) + \left(\frac{dC}{dz} \right) = 0$

3.2 Piola's *Nuova analisi*

The article *Nuova analisi per tutte le questioni della meccanica molecolare* was submitted in 1835 and published in 1836. It could be seen as a turning point in Piola's mechanical conceptions, the passage from an "ancient" and continuous to a "modern" and discrete model of matter. The discrete model was well attested in the French scientific community. Poisson paid particular attention to it because he wanted to develop a system of physical mechanics based on a model of matter made up of particles interacting with each other, juxtaposed with the analytical mechanics of continua:[44]

> [...] Lagrange came as far as one can conceive when he replaced the physical connections of bodies with equations between the coordinates of their different points: that is what constitutes *Analytical mechanics*; but alongside this admirable conception one can now raise *Physical mechanics*,

[44] Note that Piola nearly neglected Cauchy's formulations by attributing the molecular conception to Poisson only. This can in part be explained by the fact that Piola personally knew and had a high regard for Cauchy, so probably he did not want to criticise him. In (Dahan 1993), p. 291, a letter by Cauchy to Piola is quoted where the French scientist contests Piola's continuum approach to mechanics and the extended use of variational calculus.

> whose sole principle will be to reduce everything to molecular actions which transmit the action of given forces from a point to another and which mediate their equilibrium.[45]
> The use which Lagrange has made of this calculus [the calculus of variations] in the *Mécanique analytique* is only really valid for continuous masses, and the calculations with which the results so found are extended to natural bodies, must be rejected as inadequate.[46]

In fact the *Nuova analisi* represents only a temporary digression, which was more suffered than accepted. Piola somehow reluctantly accepted Poisson's claim that continuum mechanics starts from a consideration of molecular actions, but did not want to abandon Lagrange's techniques completely:

> […] Mr. Poisson […] would reduce everything to molecular actions only. I accept this point of view by admitting only a reciprocal action of attraction and repulsion apart from external forces […]. I do not believe that in so doing I have abandoned the other way used by Lagrange; indeed, I am convinced that with it we can fruitfully treat many modern problems, and I have already published an essay which can partially prove this claim of mine.[47]
> Some new theorems have been obtained, but a large part of the advantages and beauties of an analysis elaborated by our teachers by means of long studies has been lost.[48]

So, even though Piola gave up the continuum model, he retained virtual work as the basic principle from which to derive the equation of motion of the corpuscular model of matter:

> […] to show how in this way a great part of the analyses of d'Alembert, Euler and Lagrange is still valid, by supposing, with moderns, the matter as discontinuous: to maintain the treasure of science transmitted from our

[45] […] Lagrange est allé aussi loin qu'on puisse le concevoir, lorsqu'il a remplacé les liens physiques des corps par des équations entre les coordonnées de leurs différents points: c'est là ce qui constitute la *Mécanique analytique*; mais à coté de cette admirable conception, on pourrait maintenant élever la *Mécanique physique*, dont le principe unique serait de ramener tout aux actions moléculaires, qui transmettent d'un point à un autre l'action des forces données, et sont l'intermédiaire de leur équilibre (Poisson 1829, p. 361).

[46] L'usage que Lagrange a fait de ce calcul dans la *Mécanique Analytique* ne convient réellement qu'à des masses continues; et l'analyse d'après laquelle on étend les résultats trouvés de cette manière aux corps de la nature, doit être rejetée comme insuffisante (Poisson 1829, p. 400).

[47] […il] Sig. Poisson […] vorrebbe ridurre tutto alle sole azioni molecolari. Io mi conformo a questo voto non ammettendo appunto oltre le forze esterne, che un'azione reciproca di attrazione e repulsione […]. Non è già che io creda da abbandonarsi l'altra maniera usata da Lagrange, ché anzi io sono d'avviso che eziandio con essa si possano vantaggiosamente trattare molte moderne questioni, ed ho già pubblicato un saggio di un mio lavoro che può in parte provare questa mia asserzione (Piola 1836, p. 171).

[48] Si guadagnarono alcuni nuovi teoremi, ma si perdette gran parte dei vantaggi e delle bellezze di un'analisi elaborata con lungo studio dai nostri maestri (Piola 1836, p. 155).

predecessors and meanwhile to progress along with the enlightenment of our century.[49]

Almost all of the mathematical procedures contained in this paper stem from the attempt to lead the results obtained by the physical and mathematical discrete model of matter back to the mathematical continuum model. Indeed, Piola was not satisfied by the passage used also by Lagrange in which, by interpreting molecules as infinitesimal volume elements, infinite sums are turned directly into definite integrals.[50]

In order to be as thorough as possible, Piola introduced an original concept which he was to use again in the work of 1848. The reference configuration of the molecules of a body, labelled by the invariable co-ordinates $\boldsymbol{a}$ as in the paper of 1833, is an imaginary regular reticular disposition, to which Piola gave the intuitive meaning of an

> [...] ideal configuration prior to the true status in which the matter of the body itself was contained in a parallelopipedon [...] and all the *a*s differ by no more than an amount equal to α, the *b*s by no more than β, and the *c*s by no more than γ [...].[51]

The use of this regular lattice allowed Piola to write summations where the spatial difference of coordinates is constant so that from

> [...] the irregularity due to the discontinuity of the matter [...] I obtain a regularity [...] necessary for the calculus used by Lagrange in the Analytical mechanics.[52]

For these kind of summations the application of theorems which link a summation ('*integrale finito definito*') to an integral ('*integrale continuo definito*') is easiest. All the theorems presented in the first two sections of the *Nuova analisi* aim to provide expressions for internal forces among molecules in terms of a suitable series expansion of a non-linear function of the mutual distance between pairs of molecules. Later, again using the principle of virtual work, but with no equations of condition for the present position $\boldsymbol{x}$, Piola obtained the local balance equation in the following form:

$$\mathrm{Div}\,\boldsymbol{P}_1 + \boldsymbol{f} = \boldsymbol{0}. \tag{18}$$

[49] [...] mostrare come si sostenga ancora in gran parte l'analisi di D'Alembert, di Eulero e di Lagrange supponendo coi moderni la materia discontinua: conservare il tesoro di scienza trasmessoci dai nostri predecessori, e nondimeno progredire coi lumi del nostro secolo (Piola 1836, pp. 155–156).

[50] Todhunter and Pearson provide detailed comments upon the mathematical aspects of the first two sections of Piola's paper, with reference to theorems of the calculus of finite differences. See (Todhunter and Pearson 1893), artt. 769–772, pp. 422–425.

[51] [...] disposizione ideale antecedente allo stato vero nella quale la materia del corpo stesso era contenuta in un parallelepipedo [...] e tutte le *a* non diversificano fra loro che di aumenti eguali ad α, le *b* di aumenti eguali a β, le *c* di aumenti eguali a γ [...] (Piola 1836, p. 167).

[52] [...fatta salva] l'irregolarità voluta dalla discontinuità della materia, [...] ottengo una regolarità [...] necessaria pel meccanismo del calcolo quale è adoperato da Lagrange nella Meccanica analitica (Piola 1836, p. 167).

Equation (18) has the same form of Eq. (15) but for the missing product ϱJ, the mass density in the ideal state, now supposed to be uniform and equal to unity. Each of the components of $\boldsymbol{P}_1$ is a non-linear function of the successive derivatives of $\boldsymbol{x}$ with respect to $\boldsymbol{a}$. Piola then pushed the local balance equations forward to the present configuration by the theorem in Eq. (11), presented in the *Meccanica de' corpi naturalmente estesi*

$$\operatorname{div} \boldsymbol{T} + \varrho \boldsymbol{f} = \boldsymbol{0}, \tag{19}$$

where ϱ is the mass density of the body at $\boldsymbol{x}$; $\boldsymbol{T} = \varrho \boldsymbol{P}_1 \boldsymbol{F}^\top$ with $\boldsymbol{T}$ symmetric; $\boldsymbol{F}$ is defined by Eq. (4–2), so that Eq. (19) is similar to that obtained by Poisson.[53]

Piola went on to deal with the simplification of the local balance equations obtained in the previous sections on the basis of constitutive assumptions about molecular interaction. Following Poisson,[54] Piola insisted that the molecular interaction is negligible for sensible distances (the localisation assumption), so that some higher-order terms in his former expressions can be discarded:

> […] the expression of molecular action may have a sensible value for points very close […], molecular action is insensible at sensible distances […].[55]
> […] the radius of the activity sphere of molecular action, though extended to a very large number of molecules, must still be considered as an insensible quantity […].[56]

Under this assumption Piola could then introduces six quantities which are the components of a symmetric matrix $\boldsymbol{P}_2$ such that

$$\boldsymbol{P}_1 = \boldsymbol{F}\boldsymbol{P}_2, \quad \boldsymbol{P}_2 = J\boldsymbol{F}^{-1}\boldsymbol{T}\boldsymbol{F}^{-\top} \tag{20}$$

Unlike the *Meccanica de' corpi naturalmente estesi*, where he did not comment upon Eq. (15) and the coefficients appearing in it, Piola gave a physical meaning to the components of $\boldsymbol{P}_1$ under the localisation assumption. They are functions of $\boldsymbol{a}$ and represent the components of stress on planes through a point $\boldsymbol{x}$ corresponding to planes through the corresponding place $\boldsymbol{a}$ in the ideal state.

It is interesting that, in order to obtain Eq. (19), Piola introduced no particular constitutive assumption except for a generic dependence of the molecular action on the distance among pairs of corpuscles. This approach is different from that taken by Poisson and Cauchy,[57] who introduced linearisation and

53 (Poisson 1829), p. 387; (Poisson 1831b), pp. 578–579.

54 (Poisson 1829) p. 369.

55 […] l'espressione dell'azione molecolare può avere un valore sensibile pei punti estremamente vicini […], l'azione molecolare è insensibile per distanze sensibili […] (Piola 1836, p. 248).

56 […] il raggio della sfera di attività dell'azione molecolare, quantunque si estenda ad un numero grandissimo di molecole, deve ancora considerarsi una quantità insensibile […] (Piola 1836, p. 253).

57 (Cauchy 1827), (Cauchy 1828), (Poisson 1829), (Poisson 1831a,b).

localisation together with some isotropy assumptions. This fact is very important from a theoretical point of view because it makes local balance equations independent of the constitutive law, which seems desirable.

3.3 Piola's *Intorno alle equazioni fondamentali*

Piola's paper *Intorno alle equazioni fondamentali del movimento di corpi qualsivogliono, considerati secondo la naturale loro forma e costituzione* was submitted in 1845 and published in 1848. It contains a mature and complete revision of the article of 1833, Piola having eliminated some mistakes and complications which he himself recognised. As a matter of fact, some twelve years had passed since the publication of the *Meccanica de' corpi naturalmente estesi* and mathematics and mechanics had moved forward somewhat. In mathematics, Cauchy had contributed some important results to the theory of integration and this made the passage from discrete to continuum less problematic for Piola. In mechanics, Cauchy, Green and Saint-Venant had introduced the idea of strain in a clear form. The main differences from the article of 1833 are: the proof of the local balance equation even in the case of deformable bodies; the study of the terms appearing as surface integrals in the application of the calculus of variations; and the particularisation of the general expressions for two- and one-dimensional continua.

Piola began this paper by re-affirming the superiority of Lagrange's approach compared with others, almost apologising for having partially abandoned it in the *Nuova analisi* and for his previous ingenuities:

> I have often written that I do not think it necessary to create a new Mechanics, abandoning the luminous methods of Lagrange's Analytical mechanics […]. I was and I am faced also now by very respectable authorities, in front of which I should surrender […]. But […] I thought it would be convenient […] to collect my opinions on the subject in this Memoir […]. Indeed, I do not now conceal that in my preceding writings some of my ideas were expounded with insufficient maturity; there are some much too fearful and some much too daring. Some parts of these writings could be omitted, […] mainly those which […] I would not repeat anymore.[58]

Piola then replied to Poisson, who had claimed that Lagrange's methods were too abstract:

> I hope to clarify in the following Memoir that the only reason why Analytical mechanics seems inferior in treating some problems is that Lagrange,

[58] Scrissi più volte non parermi necessario il creare una nuova Meccanica, dipartendoci dai luminosi metodi della Meccanica analitica di Lagrange […]. Però mi stettero e mi stanno anche attualmente contro autorità ben rispettabili, davanti alle quali io dovrei darmi per vinto […]. Ma […] credetti convenisse […] [riunire] in questa Memoria i miei pensieri sull'argomento […]. Perocchè non dissimulo accorgermi ora che ne' precedenti miei scritti alcune idee non furono esposte con sufficiente maturità: ve ne ha qualcuna troppo spinta, ve ne ha qualch'altra troppo timorosa: certe parti di quelle scritture potevano essere ommesse, […] a più forte ragione quelle altre che […] non mi sentirei più di ripetere […] (Piola 1848, pp. 1–2).

> while writing about the equilibrium and motion of a solid body, did not give equations that apply to each point of it. If he had done so, and he could without leaving the methods taught in his book, he would have easily reached the same equations as the famous Geometers of our time have arrived at with much effort, and that now are the basis for new theories. But what he did not accomplish […] can be made by others […].[59]

In the first section of *Intorno alle equazioni fondamentali*, Piola gave some preliminary notions, among which he recalled that of ideal disposition with uniform unit mass density. The present position $\boldsymbol{x}$ depends on that in the ideal state $\boldsymbol{a}$, $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{a})$. Piola moved on to bodies with non uniform density and described the density in the present configuration via the Jacobian J of the transformation $\boldsymbol{x}(\boldsymbol{a})$, so that he could express the equation of continuity. As Piola himself remarked, there are passages which denote the use of a mature and up-to-date theory of limits and integration, approaching the "modern" Lacroix and Bordoni[60] rather than the "ancient" Lagrange, when dealing with the passage from the discrete to continuum for the one-dimensional case:

> We have got a theorem which gives us the ability to pass from a finite definite integral [a summation] to a well defined continuous integral [an ordinary definite integral] […].[61]

Two- and three-dimensional cases are reduced to the one-dimensional one.

The second section is devoted to extending the equation of virtual work from the discrete to the continuous, in the case of three-, two-, and one-dimensional bodies. For three-dimensional bodies, Piola eventually obtained the equation of virtual work in the ideal state

$$\int_{\kappa} [\boldsymbol{f} \cdot (\delta \boldsymbol{x})] \mathrm{d}V + \int_{\kappa} [\boldsymbol{P}_2 \cdot (\delta \boldsymbol{L})] \mathrm{d}V + \Omega = 0, \tag{21}$$

where the density is not made explicit since it is supposed to be unitary and uniform in the ideal state; $\boldsymbol{P}_2$ is a symmetric list of Lagrange multipliers; $\boldsymbol{L}$ is a

[59] Spero mettere in chiaro nella seguente Memoria che l'unico motivo pel quale la Meccanica Analitica parve restar addietro nella trattazione di alcuni problemi, fu che Lagrange nello scrivere dell'equilibrio e del moto di un corpo solido, non è disceso fino ad assegnare le equazioni spettanti a un solo punto qualunque di esso. Se questo avesse fatto, e lo potea benissimo senza uscire dai metodi insegnati nel suo libro, sarebbe giunto prontamente alle stesse equazioni cui arrivarono con molta fatica i Geometri francesi del nostro tempo, e che ora servono di base alle nuove teoriche. Però quello ch'egli non fece […] può esser fatto da altri […] (Piola 1848, p. 4).

[60] (Lacroxi 1811), vol. II, p. 97 and (Bordoni 1831), vol. II, p. 489.

[61] Abbiamo un teorema di analisi che ci somministra il mezzo di passare da un integrale finito definito a un integrale continuo parimenti definito […] (Piola 1848, p. 42).
The theorem referred to by Piola is found in (Lacroxi 1811), for example, in the form:

$$\int u \mathrm{d}x = h \sum u + \alpha h^2 \sum \frac{\mathrm{d}u}{\mathrm{d}x} + \beta h^3 \sum \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \dots,$$

where $\alpha, \beta, \dots$ are numerical coefficients and h is a small quantity (the step of discretisation).

symmetric list of scalar equations of condition; and Ω represents the contribution of surface actions, which are not represented by a volume integral.

In the third section, Piola obtained the local balance equations for a rigid body. He used equations of condition equivalent to Eqs. (6–1), which he had obtained in the *Meccanica de' corpi naturalmente estesi*, but which are now more general, since the identity $\boldsymbol{F} = \boldsymbol{Q}$ is no longer assumed. By following the same steps he had used in the paper of 1833, but with some improvements to their form, he got local balance equations in the ideal state, equivalent to Eq. (15). The comments Piola added to this result are particularly revealing; indeed, he declared that equations such as (15) have no physical meaning for him, because the ideal state does not represent an actual physical state of the body:

> [...] it would be useful to transform these equations [...] into others which have no trace of a, b, c and which only contain quantities proper of the real state of the body [...].[62]

By means of his push-forward theorem, which transforms quantities in the reference to the present configuration, expressed by our Eq. (11), Piola then obtained the local balance equations for the present configuration, having the same form of Eq. (17).

In the fourth section, Piola derived the local balance equations for a deformable body. He started by affirming that it is impossible to provide a specific form of local condition equations as he had done in the case of the rigid body, and that some other procedure is needed. This is identified in the introduction of an intermediate configuration χ_p labelled by $\boldsymbol{p}$, so that the path from $\boldsymbol{a}$ to $\boldsymbol{x}$ turns into a path from $\boldsymbol{a}$ to $\boldsymbol{p}$ and from $\boldsymbol{p}$ to $\boldsymbol{x}$; this second step is assumed to be rigid. With this assumption, the mass density ϱ in χ_p is the same as in χ, and Piola could write Eq. (25) in χ_p, by using the transformation $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{a})$

$$\int_{\chi_p} \varrho[\boldsymbol{f}^* \cdot (\delta \boldsymbol{x})] \mathrm{d}V_p + \int_{\chi_p} [\boldsymbol{T}^* \cdot (\delta \boldsymbol{L}^*)] \mathrm{d}V_p \; + \; \Omega^* = 0, \tag{22}$$

where $\boldsymbol{f}^*$ is the volume density of distance interactions seen in χ_p, $\boldsymbol{T}^*$ is a symmetric list of Lagrange multipliers, different from $\boldsymbol{P}_2$; the list of the equations of condition $\boldsymbol{L}^*$ refers to the transformation $\boldsymbol{x}(\boldsymbol{p})$; and Ω^* represents the contribution of surface actions seen in χ_p. Piola's trick reduces the equation of virtual work with unknown condition equations to an equation where the condition equations are known. In fact, the motion from $\boldsymbol{p}$ to χ_p is rigid, hence the equations of condition are the same as in Eqs. (5–1), with $\boldsymbol{F} = \boldsymbol{Q}$, $\boldsymbol{Q}$ being the rotation matrix from the p_i- to the x_i-axes, and the derivatives being formed with respect to $\boldsymbol{p}$. Thus, Piola obtained the local balance equations in χ_p,

$$\operatorname{div} \boldsymbol{T}^* + \varrho \boldsymbol{f}^* = \boldsymbol{0} \tag{23}$$

[62] [...] si vorrebbero tramutare queste equazioni [...] in altre che non contenessero traccia delle a, b, c e non constassero che di quantità spettanti allo stato reale del corpo [...] (Piola 1848, p. 63).

and pushed them forward to χ by means of the theorem in Eq. (11), obtaining

$$\operatorname{div} \boldsymbol{T} + \varrho \boldsymbol{f} = \boldsymbol{0}, \quad \boldsymbol{T} = \boldsymbol{Q}\boldsymbol{T}^{*}\boldsymbol{Q}^{\top} \tag{24}$$

Next, Piola gave an interpretation of the surface integrals arising from the variational procedure in terms of the pressures acting on the boundary of the body. Also, using the calculus of variations he proved a relation which, he said, reproduces Cauchy's theorems on pressures

$$\boldsymbol{T}(\boldsymbol{x})\boldsymbol{n} = \boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n}) \tag{25}$$

with $\boldsymbol{t}$ being the stress vector at a point $\boldsymbol{x}$ and on a surface with unit normal $\boldsymbol{n}$ in the present configuration χ. After some more work, Piola showed that what he had proven for $\boldsymbol{T}$ holds also for $\boldsymbol{T}^{*}$; thus, the components of either $\boldsymbol{T}$ or $\boldsymbol{T}^{*}$ are the analytical expressions containing the effect of all internal actions on a point $\boldsymbol{p}$ or the corresponding $\boldsymbol{x}$

> [...] the six named quantities are in both cases analytical expressions containing the whole effect of all internal actions on the generic point (p, q, r) or (x, y, z) [...].[63]

Note that, though clever, Piola's reasoning is not entirely conclusive. His analysis is based on the possibility of passing from the unknown condition equations of the balance equation (21) to the known condition equations of Eq. (22). This reasoning is inconsistent, however, because it is not possible to define condition equations for a deformable body; put simply, the points of a deformable body have no constraint, they are free.

In art. 60 of the paper Piola gave a summary of his procedure, starting with some considerations about the virtual displacements $\delta\boldsymbol{x}$. Piola's reluctance to introduce virtual displacements as infinitesimals has already been underlined. Piola considered virtual displacements simply as first-order variations of the position of body-points satisfying equations of condition. In this article Piola characterized the virtual displacements explicitly, affirming that they coincide with the variation of the co-ordinates of the same body-point when referred to two co-ordinate systems shifted, one from the other, by a very small amount, and that this consideration may vary from body-point to body-point

> [...] such a principle lies in the simultaneous reference of any system to two triads of orthogonal axes. It can work in two ways and for both of them it is very effective. It is used in the first way [...] to prove the principle of virtual velocity and also those others [principles] regarding conservation of the motion of centre of gravity and of areas. Instead of conceiving the $\delta x, \delta y, \delta z$ of the various points of the system as virtual

[63] [...] le mentovate sei quantità in ambi i casi sono le espressioni analitiche contenenti l'effetto complessivo di tutte le azioni interne sopra il punto generico (p, q, r) ovvero (x, y, z) [...] (Piola 1848, p. 101).

> velocities or infinitesimal spaces described by that fictitious motion (which after [Lazare] Carnot was named geometrical motion[64]), it is more natural and unmysterious to consider them as an increase of the coordinates of such points when the system is referred to another set of three orthogonal axes, displaced very little from the first [...] so it is clear how the increase in the coordinates occurs without changing the reciprocal actions among the parts of the system [...].[65]

In this way, it is apparent that the virtual displacements considered take place without modifying inner actions. In a similar way, the equations of condition in the rigid motion from $\boldsymbol{p}$ to $\boldsymbol{x}$ may be thought of as transformations of equations of condition when the co-ordinate system varies; this, claimed Piola, was certainly an original approach of his

> The simultaneous reference to two triads of orthogonal axes then plays out efficiently in another way [...]. Here we intend to speak about the method which leaves $\delta x, \delta y, \delta z$ completely general and considers constraint equations by introducing indeterminate multipliers. In such a way the use of the two triads is useful for writing down the named constraint equations, which could not otherwise be given in general [...]. Such a point of view seems to me to have escaped Lagrange and the other geometers: all things in the present Memoir which are worthy of attention derive from it [...].[66]

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64 Lazare Carnot introduced the idea of "*mouvement géométrique*" in the last part of the eighteenth century

> [...] if a system of bodies starts to move from a given position, with an arbitrary velocity, but such that it would be also possible for the system to follow another velocity exactly of the same magnitude and opposite, each one of these velocities will be named geometrical velocity [...]. ([...] si un systéme de corps part d'une position donnée, avec un mouvement arbitraire, mai tel qu'il eût été possible aussi de lui en faire prendre un autre tout-à-fait ègal et directement opposè; chacun de ces mouvements sera nommè mouvement géométrique [...] (Carnot 1783, pp. 23–24)).

65 [...] tal principio sta nel riferimento simultaneo di un qualunque sistema a due terne di assi ortogonali: esso può adoperarsi in due maniere e in entrambe produce grandiosi effetti. Si adopera in una prima maniera [...] a fine di dimostrare il principio delle velocità virtuali, ed anche gli altri della conservazione del moto del centro di gravità, e delle aree. Invece di concepire in tal caso le $\delta x, \delta y, \delta z$ dei diversi punti del sistema come velocità virtuali o spazietti infinitesimi descritti in virtù di quel moto fittizio (il quale fu poi altresì detto dopo Carnot un moto geometrico), è assai più naturale e non ha nulla di misterioso il ravvisarle quali aumenti che prendono le coordinate degli anzidetti punti quando il sistema si riferisce ad altri tre assi ortogonali vicinissimi ai primi, come se questi si fossero di pochissimo spostati. [...] allora si capisce chiaro come gli aumenti delle coordinate abbiano luogo senza alterazioni nelle azioni reciproche delle parti del sistema le une sulle altre [...] (Piola 1848, p. 110).

66 Il riferimento simultaneo del sistema a due terne di assi ortogonali giuoca poi efficacemente in un'altra maniera [...]. Qui s'intende parlare di quel metodo che lascia alle $\delta x, \delta y, \delta z$ tutta la loro generalità e tratta le equazioni di condizione, introducendo moltiplicatori indeterminati. In tal caso la contemplazione delle due terne di assi giova per l'impianto delle dette equazioni di condizione, che altrimenti non si saprebbero assegnare in generale [...]. Un tal punto di vista parmi sfuggito a Lagrange e ad altri Geometri: ad esso si riferisce quanto nella presente Memoria può essere più meritevole di attenzione (Piola 1848, p. 111).

The last sections of the *Intorno alle equazioni fondamentali* are dedicated to the motion of fluids, not directly linked with the subject of this paper, and to a reformulation of the treatment of molecular forces in the paper of 1836. Eventually Piola extends the results of previous sections to two- and one-dimensional continua.

Piola died in 1850, and in 1856 Francesco Brioschi, a former pupil of Piola and at that time professor of Rational Mechanics in Pavia, edited the posthumous publication of Piola's last work on continuum mechanics, entitled *Di un principio controverso della Meccanica Analitica di Lagrange e delle sue molteplici applicazioni*. This paper is stated to be directly linked with the *Intorno alle equazioni fondamentali* and to be its natural completion and refinement. In his work of 1848 Piola had said that he felt that Lagrange's technique of indeterminate multipliers of the first-order variation of the equations of condition contained something unclear and unproven. Therefore, he said, in this article he felt compelled to show how to overcome this difficulty. In the first chapter Piola showed that the first-order variation of the equations of condition in the rigid body motion from χ_p to χ can be obtained simply by moving the reference frame of the present configuration. Thus, Piola surpassed the difficulty due to the intermediate configuration χ_p, which is unknown and in principle may not exist, so that it might be meaningless to take derivatives with respect to $\boldsymbol{p}$. Starting from this proof, which turns out to be a terrific tool for Piola, who desired to leave nothing to intuition, in the rest of the paper Piola re-derived the local balance equation, extended the results to two- and one-dimensional continua, re-interpreted the Lagrange multipliers as expressions of inner forces and provided a molecular representation for the latter; moreover, he gave a clear and completely modern interpretation of the components of $\boldsymbol{C}$ as measures of strain, recalled the property of the ellipsoid of finite strain, recovered Cauchy's theorem on stress and wrote down localised and linearised elastic constitutive relations for three-, two-, and one-dimensional continua. In many aspects, this work may be seen as the natural completion of Piola's path in the field of continuum mechanics, yet, probably because it was published posthumously, it is somewhat ignored.

3.4 Solidification principle and generalised forces

Certainly Piola's most relevant contribution to continuum mechanics is the way he introduces internal stresses: they are presented as Lagrange multipliers of constraint equations. Piola applied this approach in all his papers, but in the articles of 1848 and 1856 the concept is made extremely clear.

When dealing with the equilibrium of deformable bodies Piola said he could simply follow Lagrange, who had treated some deformable bodies as they were rigid by using what Poinsot had called '*principle de solidification*'.[67] According to this principle, the active forces present in a deformable body are equivalent to the passive forces obtained assuming '*the same functions that remain constant for rigid bodies*'[68] as condition equations. This is what Lagrange said on the subject:

> One adds to this integral $SF\delta ds$ the integral $SX\delta x + Y\delta y + Z\delta z$ which expresses the sum of the moments [works] of all external forces acting on the thread [...] and by equating all them to zero, we obtain the general equilibrium equation of the elastic thread. Now it is clear that this equation has the same form as that [...] for the case of inextensible thread, and [it is clear too] that by changing F in λ, the two equations will become identical. One therefore has in the present case the same particular equations that we found in the case of art. 31, simply by putting F in place of λ.[69]

In other words, for example in the case of a thread, Lagrange stated that the elastic forces can be treated as constraint reactions by introducing the first-order variation of the inextensibility constraint. Piola was not convinced by this argument:

> [Lagrange] in his A. M. [...] adopted a general principle (§. 9. of Sect. II and 6. of IV[70]) according to which the analytical expression for the

67 (Poinsot 1848). The principle of solidification was used also by Stevin in his study of the equilibrium of fluids *De beghinselen des waterwichts*, 1586, and by Euler to treat hydrostatics in the *Scientia Navalis*, 1738 (Capecchi 2001, pp. 17–18). Cauchy used it in (Cauchy 1823) to introduce the idea of stress. Later on, it was used to study systems of constrained bodies. Lagrange used it to prove the equation of virtual work (Lagrange 1811, Sect. II, art. 1.). Nowadays it is more often derived from the equation of virtual work, as in (Duhem 1891):

> It is not unpleasant to deduce from the Principle of virtual velocity and from the thermodynamic generalization of this principle the following consequence: If a system is in equilibrium when it is subjected to certain constraints, it will persist in equilibrium when it will be subjected not only to these constraints but also to some more [constraints] which are consistent with them [...] (Il n'est pas malaisé de déduire du Principe des vitesses virtuelles et de la généralisation thermodynamique de ce principe la conséquence suivant: Si un système est en équilibre lorsqu'il est assujetti à de certains liaisons, il demeura en équilibre lorsqu'on l'assujettira non seulement à ces liaisons mais encore à des nouvelles liaisons compatibles avec les premières [...] (Duhem 1891, pp. 36–37).).

68 [...] quelle stesse funzioni che rimangono costanti per corpi rigidi (Piola 1848, p. 76).

69 On ajoutera donc cette intégrale $SF\delta ds$ à l'intégrale $SX\delta x + Y\delta y + Z\delta z$, qui exprime la somme des momens de toutes les forces extérieures qui agissent sur le fil [...], & égalent le tout à zéro, on aura l'équation générale de l'équilibre du fil à ressort. Or il est visible que cette équation sera de la même forme que celle [...] pour le cas d'un fil inextensible, & qu'en y changeant F en λ, les deux équations deviendront même identiques. On aura donc dans le cas présent les mêmes équations particulieres pour l'équilibre du fil qu'on a trouvées dans le cas de l'art. 31, en mettant seulement dans celle-ci F à la place de λ [...] (Lagrange 1788, p. 100)

70 Piola's quotation is not entirely correct, since in these sections Lagrange deals with constraints in general.

> effect of the internal active forces is similar to that for passive ones when we have constraints: this is obtained by assuming indeterminate coefficients and by multiplying them by the variation of those functions which remain constant for rigid, inextensible, or liquid bodies. If we adopted such a method, we could immediately generalize the results obtained in the previous chapter: I, however, prefer to abstain, because my appreciation for the great Geometer does not prevent me from recognizing that that principle proposes something obscure and not yet proved.[71]

There are reasons for Piola's rejection of Lagrange's use of the principle of solidification: the first is that this approach stems from intuition, being based on a non formalised procedure; as an analyst, Piola preferred to obtain his result as consequences of a chain of formulae where nothing is left to intuition. Secondly, it requires the ideas of deformation and inner action, which Piola did not provide and did not want to use, at least in the *Intorno alle equazioni fondamentali*. Unconvinced by Lagrange's procedure, Piola looked for a different one, thereby showing his skills and talent.

Piola might have suspected some weakness in his reasoning because he returned to the argument in the posthumous paper of 1856, when he took a different approach, avoiding the use of an intermediate configuration χ_p. Here, he practically adopted Lagrange's use of the solidification principle. At the heart of this reconciliation is Piola's explicit understanding that the coefficients of the matrix $\boldsymbol{C}$ in Eq. (5–1) represent deformations. More precisely, Piola studied metrics in the present configuration with respect to the ideal state. His dislike of infinitesimals seems to have diminished somewhat, probably because of the spread of Cauchy's rigorous concept of differential, which could replace the XVIIIth century concept of infinitesimal; as noted above, Cauchy was someone whose ideas Piola appreciated even though he did not always agree with them. However, Piola did not adopt the differential, though he came close to it. In the metric considerations for the present configuration, where he could comfortably use the infinitesimal element of length ds he preferred to consider the quantity $s' = \sqrt{x'^2 + y'^2 + z'^2}$, which he called '*elemento di arco*', where the prime means derivative with respect to a parameter varying in the ideal configuration.

[71] [Lagrange] nella sua M. A. [...] adottò un principio generale (§. 9. della Sez. II^{a}, e 6. della IV^{a}), mediante il quale l'espressione analitica dell'effetto di forze interne attive riesce affatto analoga a quella che risulta per le passive quando si hanno equazioni di condizione: il che si ottiene assumendo dei coefficienti indeterminati e moltiplicando con essi le variate di quelle stesse funzioni che rimangono costanti per corpi rigidi, o inestensibili, o liquidi. Se ci conformassimo ad un tal metodo, potremmo a dirittura generalizzare i risultamenti ai quali siamo giunti nel capitolo precedente: io però preferisco astenermene, giacché la mia ammirazione pel grande Geometra non m'impedisce di riconoscere come in quel principio rimanga tuttavia alcun che di oscuro e di non dimostrato (Piola 1848, p. 76).

For the three-dimensional case, art. 29 and art. 33, Piola developed geometrical relations of a local character which partially reflect Cauchy's approach,[72] yet have a certain originality. For the element of arcs which in the ideal configuration have, at a given point P, a tangent characterized by director cosines $\alpha_1, \alpha_2, \alpha_3$, the expression of the square of the arc element s' in the present configuration is represented by

$$(s')^2 = \sum_{i,j} C_{ij}\alpha_i\alpha_j \tag{26}$$

where the C_{ij} are the coefficients of the matrix $\boldsymbol{C}$ in Eq. (5–1) evaluated at P. The expression (26) with equal indices coincides with that of the coefficient ε which Cauchy called '*la dilatation linéaire*'.[73] Similar expressions are obtained for the cosines of angles between two curves.

In any case Piola remained critical of Lagrange's approach to deformable systems. He now had explicit reasons for this criticism, claiming that Lagrange had not given the criterion for establishing what and how many components of deformation must be used:

> Indeed there are many possible contemporary expressions for quantities that the internal forces of a system tend to vary. Which of them shall we consider, which shall we neglect? Who will assure us that by using many such functions that are subject of variation because of the action of internal forces, we do not make useless repetitions and by means of some express an effect already written with some some others? And could it not happen that we neglect those [expressions] which are necessary in order that the whole effect of internal forces could be expressed completely?[74]

However, Piola believed he had resolved the question and found how many condition equations are needed and what they are:

> Regarding the problem: *which* of the functions that are varied by internal forces we must use, preferring them to others, I proved that they are those trinomials of derivatives [...]. For the other question: *how many* must such

[72] For instance, they can be found in (Cauchy 1841).

[73] (Cauchy 1841), p. 304.

[74] Infatti molte possono essere contemporaneamente le espressioni di quantità che le forze interne di un sistema tendono a far variare; quali di esse prenderemo, quali ommetteremo? Chi ci assicura che adoperando parecchie di tali funzioni soggette a mutamenti per l'azione delle forze interne, non facciamo ripetizioni inutili, esprimendo per mezzo di alcune un effetto già scritto con altre? E non potrebbe invece accadere che ommettessimo di quelle necessarie ad introdursi affinché l'effetto complessivo delle forze interne venga espresso totalmente? (Piola 1856, p. 391).

functions be […] I answered so many as they are necessary to get the variation of those trinomials equated to zero […].[75]

Once Piola had introduced deformations he could legitimately write Eq. (21) for deformable systems; now $\delta \boldsymbol{L}$ does not represent the variation of condition equations but the variation of deformation and is given by $\delta \boldsymbol{C}$.

The introduction of strain threw new light on Lagrange multipliers. The latter were seen as forces producing displacements associated with the variation of condition equations. Piola extended Lagrange's[76] concepts and conceived of very general inner forces, anticipating modern approaches to internal forces in structured continua, for example Cosserat's.[77] Indeed, it is apparent that when dealing with one-dimensional continua Piola introduced the twist of the line as a strain measure, defining the dual inner action as the corresponding Lagrange multiplier. In Piola's words,

The concept that Lagrange wanted us to have about forces, and which we expounded in the prologue, is more general than what is usually accepted. Everybody easily intends that a force is a cause which, by means of its action, modifies the magnitude of certain quantities. In the most evident case, by making a body or a material point approach another, it modifies distances, and so makes the length of rectilinear lines vary; but it can also modify an angle, a density, etc. In these latter cases the way the forces act remains obscure to us, while it is clear in the former. But, perhaps, the reason of this is independent of the nature of the forces. Actually, even in the former case it is not understood how a force can communicate its action to the body so as to decrease or increase the distance of the body from another; meanwhile, we see the fact continuously; everyday observation decreases the will to look further into this. But, if on scrutiny we find that here also the action of forces is mysterious, no wonder that it is mysterious in the other cases too. To want to reduce the action of forces always to that decreasing a distance, is to reduce a vaster concept, and to recognize but a particular class of forces. Generally speaking, how far can our ideas about causes which are object of measurements be driven? perhaps to comprehend their intimate nature and the true way in which they act? […] When we have given all unknown concepts a unit with which we measure things of the same kind, we say we know the truth when we can assign ratios with such unit, originally assumed arbitrary. Now, when, following Lagrange, we conceive of forces in a more general way, namely as causes which may vary quantities other than lines, we obtain the data necessary to affirm that we can measure them. We have all we can reasonably pretend: if we lack the imagine with which to invest the concept, it is

[75] Circa la questione: *quali* sono le funzioni fatte variare dalle forze interne che si debbono adoperare a preferenza di altre, ho dimostrato che sono que' trinomj alle derivate […]. Relativamente all'altra questione: *quante* poi debbano essere tali funzioni […] ho risposto quante ce ne vogliono per risalire alle variate di que' trinomj poste uguali a zero […] (Piola 1856, p. 421).

[76] (Lagrange 1811), Sect. V.

[77] (Cosserat 1896, 1907).

because we want to describe it in the same way as when forces act along lines. An unknown part always remains both in these more general cases and in the very ordinary one.[78]

This conception of forces led Piola to reconsider the condition equations by investigating what happens to Lagrange multipliers (the forces) when these equations are transformed into others, with some mathematics. Piola examined one-, two-, and three-dimensional cases; in the latter he focussed solely on fluids.

4 Piola's stress tensors and theorem

According to the ideas of the history of science prevailing today, the modern interpretation of a non-contemporary text is something to be avoided at all costs. The aim of the historian is to reconstruct the "actual" thinking of the scientists, and this can be done only by using the categories of the time in which the scientists wrote. From this point of view it is necessary to understand all aspects which could influence the way of thinking, not only in science: the political environment, dominant metaphysics, and so on.

We agree with this approach and in the previous sections we have tried to follow it as much as possible. Nonetheless, the modern interpretation of relatively recent papers such as Piola's are of remarkable interest also to the historian: it may be seen as a complement to the reconstruction of the history of a part of continuum mechanics from Piola's time up to now. In fact, it should not be forgotten that reference to Piola's papers is made in some monographs[79] at the beginning

78 Il concetto che Lagrange voleva ci formassimo delle forze, e che esponemmo nel prologo, è più generale di quello universalmente ammesso. S'intende facilmente da tutti essere la forza una causa che mediante la sua azione altera la grandezza di certe quantità. Nel caso più ovvio, avvicinando un corpo o un punto materiale ad un altro, cambia distanze, ossia fa variare lunghezze di linee rette: ma può invece far variare un angolo, una densità, ecc. In questi altri casi il modo di agire delle forze ci riesce oscuro, mentre ci par chiaro nel primo: ma forse la ragione di ciò è estrinseca alla natura delle forze. Per verità anche in quel primo caso non si capisce come faccia la forza a infondere la sua azione nel corpo sì da diminuirne od accrescerne la distanza da un altro corpo: nondimeno noi vediamo continuamente il fatto: l'osservazione giornaliera sopisce in noi la voglia di cercarne più in là. Se però sottilmente esaminando si trova che qui pure il modo di agire delle forze è misterioso, nessuna meraviglia ch'esso ci appaja oscuro negli altri casi. Voler ridurre in ogni caso, l'azione delle forze a quella che diminuisce una distanza, è impiccolire un concetto più vasto, è un non voler riconoscere che una classe particolare di forze. Generalmente parlando, a qual punto possono essere spinte le nostre cognizioni intorno alle cause che sottoponiamo a misura? forse a comprenderne l'intima natura, e il vero modo con cui agiscono? [...] Radunato tutto quanto vi è d'incognito nella unità di misura della stessa specie, noi diciamo di conoscere la quantità, lorché possiamo assegnarne i rapporti colla detta unità assunta originaliamente arbitraria. Ora eziandio quando si concepiscono le forze alla maniera più generale di Lagrange, cioè siccome cause che fanno variare quantità talvolta diverse dalle linee, concorrono i dati necessari a poter dire che sappiamo misurarle: si ha tutto ciò che ragionevolmente ci è lecito di pretendere: se pare che ci manchi l'immagine con che rivestirne il concetto, è perché vogliamo colorirla come nel caso particolare delle forze che agiscono lungo le rette: un fondo incognito rimane sempre tanto in questi casi più generali, come in quello sì comune (Piola 1856, pp. 456–457).

79 (Müller and Timpe 1906), p. 23; (Hellinger 1914), p. 620.

of the twentieth century, where the formulation of mechanics essentially coincides with today's. Moreover, the reconstruction of the history of continuum mechanics allows us to highlight the aspects that Piola "failed to understand" and try to explain why. These can be internal, that is, they can depend on logical or methodological grounds, or external, such as the lack of time or of attention devoted to a particular subject. For instance, it is interesting to understand why Piola did not attribute the meaning to $\boldsymbol{P}_2$ which is attributed to it today. Understanding facts like this helps us to comprehend how Piola "actually" thought.

In modern continuum mechanics it is customary to attribute the following to Piola: a) two stress tensors, which provide respectively stress in the present configuration and stress in the present configuration pulled back to the reference configuration when applied to the unit normal in the reference configuration;[80] b) a theorem on the derivation of balance equations from the principle of virtual power.[81] In this section we shall try to explain the reasons behind these attributions.

4.1 A modern interpretation of Piola's contributions

Piola's papers contain a lot of interesting hints from the point of view of modern continuum mechanics. Firstly, the a_i co-ordinates are fundamental to all his works. In *Meccanica de' corpi naturalmente estesi* Piola declares a_i to be independent of time. One could thus interpret the same co-ordinates to be those in a reference configuration, but Piola did not affirm that explicitly. However, it is clear from this choice that the description of motion used by Piola is a referential one, and to us the $\boldsymbol{a}$ list will describe a reference configuration κ. In the *Nuova analisi* Piola took a daring step forward, because the introduction of the concept of ideal state and its identification with the $\boldsymbol{a}$ list is exactly the construction of a reference configuration in the sense of Truesdell.[82] This is a shape useful for calculations, but which in principle may not coincide with a shape that the body may have assumed or will assume. Otherwise a purist and an analyst, Piola did not realize the power of such an abstract formulation, which nowadays is the basis of many treatments of finite continuum mechanics. Indeed, in this he followed the norms of his time, and, while in principle he should have realised that his "lagrangean" description of motion would lead him to a different description of balance, he felt urged to focus only on the present configuration χ, as all his contemporaries had done. We remarked upon this in Sect. 3.3, where we quoted a passage in which Piola felt compelled to

[80] (Truesdell and Toupin 1960), pp. 553–554; (Truesdell and Noll 1965), pp. 124–125; (Malvern 1969), pp. 220–224; (Gurtin 1981), pp. 178–180; (Truesdell 1991), pp. 185.

[81] (Müller and Timpe 1906), p. 23; (Hellinger 1914), p. 620; (Truesdell and Toupin 1960), pp. 595–600; (Truesdell and Noll 1965), pp. 124–125; (Malvern 1969), pp. 246–248; (Truesdell 1991), p. 185.

[82] (Truesdell 1991), p. 96.

push the referential balance equations forward to χ, the only "real" state of the body.

Another interesting point is the implicit definition, when Piola dealt with the local rigidity equations of condition, of the right ($\boldsymbol{C}$) and left ($\boldsymbol{B}$) Cauchy–Green strain tensors. As it is well known, $\boldsymbol{C}$ is a measure of the metric in the present configuration with respect to the reference one, while $\boldsymbol{B}^{-1}$ is a measure of the metric in the reference configuration with respect to the present one. Imposing the metric as the identity, as Piola does, amounts to supposing that the metric does not change during the motion. Modern measures of strain are the right and left Green–Saint-Venant tensors,

$$\boldsymbol{M} := \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}), \quad \boldsymbol{N} := \frac{1}{2}(\boldsymbol{B}^{-1} - \boldsymbol{I}) \tag{27}$$

and Piola's local rigidity equations of condition, equivalent to Eq. (5–1), coincide with the vanishing of $\boldsymbol{M}$ and $\boldsymbol{N}$ defined by Eq. (27). This is of course the case of a rigid motion. However, Piola was not interested in defining a deformation measure, and so what could have been a formidable intuition is not presented, even if one may suspect that Piola had something in mind; it was made clear only in his posthumous *Di un principio controverso*.

Moreover, the ambiguity that Piola kept between the expression of a generic motion gradient $\boldsymbol{F}$ and the gradient of a rigid motion $\boldsymbol{Q}$ makes some of his equations generally invalid. Indeed, in a rigid motion it is apparent from Eq. (5) that $\boldsymbol{C} = \boldsymbol{B}(= \boldsymbol{I})$, while in general $\boldsymbol{C} \neq \boldsymbol{B}$ since the product of two tensors is not commutative. In his work of 1833 Piola made no distinction between the equations of condition in terms of either $\boldsymbol{B}$ or $\boldsymbol{C}$, and the Lagrange multipliers he introduced have the meaning of stresses in χ; however, since $\boldsymbol{B}^{-1}$, not $\boldsymbol{B}$, expresses a metric, the local balance equations derived from the use of $\boldsymbol{B}$ have no physical significance. Piola corrected himself in the last part of *Meccanica de' corpi naturalmente estesi*, and in his other papers he seems never to have the same doubts and used only $\boldsymbol{C}$. It seems reasonable, however, to think that Piola offered no metric consideration other than in *Di un principio controverso*, so it is likely that he "felt" he was right in using $\boldsymbol{C}$, rather than having rationally proven this.

In the Lagrange multipliers listed in $\boldsymbol{P}_2$ a modern reader sees the components of the so-called second Piola's stress tensor, and in $\boldsymbol{P}_1 = \boldsymbol{F}\boldsymbol{P}_2$ the so-called first Piola's stress tensor; the list $\boldsymbol{S}$ in Eq. (16) shall be equal to the list $\boldsymbol{T}$ in Eq. (19) and coincide with Cauchy's stress tensor. In his paper of 1833, Piola provided an interpretation neither of the mechanical meaning of any of these quantities, nor of the local balance equation (15). Moreover, he did not introduce the list of components of $\boldsymbol{S}$ as an array of Lagrange multipliers, as they would be supposed to be given the mechanical meaning of the stresses. Indeed, he introduced the components of $\boldsymbol{S}$ merely as a list of auxiliary quantities to which he applied his already described push-forward theorem, Eq. (11). Moreover, Piola gave no mechanical interpretation of the surface integrals which follow from the applications of the techniques of the calculus of variations. Thus, many capital results he found remain somehow hidden. In his later works, some changes are made with regard to this: from the treatment of molecular interactions it immediately becomes clear that the components of lists like $\boldsymbol{P}_1$ have the meaning of internal actions and that the equations obtained are actually local balance equations. Indeed, Piola remarked that

> The general equations of motion of any point (x, y, z) of the body are Eqs. (56) [coinciding with Eq. (18)] where L_1, L_2, etc. [the components of $\mathbf{P}_1$] […] are reduced to depending […] only on the unknown $\psi(S)$ corresponding to the molecular action. It is quite true that […] the equations found are related to those for the composition of the x, y, z in terms of a, b, c which is unknown and not assignable. But let us pass to see how, keeping fixed the advantage of equations rigorously obtained, we can pass over the cited difficulty with respect to the effects […].[83]

That is, Piola introduced a virtual ideal state and felt compelled to push his equations forward to the present configuration, which he considered to be the only true one. It is remarkable how in the *Nuova analisi* Piola introduced an intermediate configuration, which is the one assumed by the body at the initial time. But, rather than generalising his results to this reference configuration, he focussed his attention on the present one and derived local balance equations in Cauchy's form in terms of quantities (the components of $\boldsymbol{P}_2$) which are expressions of molecular forces. The expressions he provided, which are summarised in Eq. (20), are those which are commonly accepted in modern continuum mechanics.[84] Moreover, the interpretation of the components of $\boldsymbol{P}_1$ in terms of contact actions is the one which is currently accepted. It was clear in Piola's mind that these results match those of Cauchy on surface and internal actions.

It is interesting to remark that the local rigidity equations of condition cannot in principle be used to derive balance equations for deformable bodies, for which constraint equations do not exist, unless one invokes the princile of solidification. Piola did not use condition equations directly, though, but used their first-order variation. These would be nowadays interpreted as describing a virtual strain velocity, and expressions such as Eq. (21) would be read as follows: the total mechanical power spent on a virtual rigid velocity field vanishes. While in his work of 1833 Piola wrote with some imprecision, as he himself admitted later, in other articles, especially in those of 1848 and 1856, his treatment was

[83] Le equazioni generali del moto di un punto qualunque (x, y, z) del corpo sono le (56) ove le L_1, L_2, ec. […] si riducono a dipendere […] [dal]la sola incognita $\psi(S)$ relativa all'azione molecolare. Ben è vero, che […] le trovate equazioni si rapportano a quella composizione delle x, y, z in a, b, c che è ignota anzi inassegnabile; ma passiamo ora a vedere in qual modo, fermato il vantaggio di formole ottenute rigorosamente, si sormonta in quanto agli effetti l'accennata difficoltà […] (Piola 1836, p. 202).

[84] (Truesdell and Toupin 1960), pp. 553–554; (Truesdell and Noll 1965), pp. 124–125; (Malvern 1969), pp. 224–225; (Gurtin 1981), pp. 178–180.

acute and he was sure that this approach was original. And, indeed, this is a rather modern view, and at the beginning of the twentieth century credit is given to Piola for this formulation,[85] which is called Piola's theorem in well known monographs on continuum mechanics.[86] In short, credit is given to Piola to have proven balance equations via only the description of the present configuration, the principle of solidification and the well accepted principle of the vanishing of virtual power spent on a rigid body motion. In particular, Hellinger highlighted the fact that Piola's approach requires only the knowledge of external forces, without compromising on the nature of inner actions, which are simply Lagrange multipliers:

> […there is] another formulation of the Principle of virtual displacements that from the start considers only the proper *forces*, mass forces X, Y, Z and surface forces $\bar{X}, \bar{Y}, \bar{Z}$ as given; it is the following simple position of the formulation by *G. Piola*: *for equilibrium it is necessary that the virtual work of the acting forces*
>
> $$\underset{(V)}{\int\int\int}(X\delta x + Y\delta y + Z\delta z)\mathrm{d}V + \underset{(S)}{\int\int}(\bar{X}\delta x + \bar{Y}\delta y + \bar{Z}\delta z)\mathrm{d}S$$
>
> *vanishes for every* [*rigid motion*] *of the entire domain V* […*so that*] *the components of the tension dyad appear as Lagrange's factors of certain rigidity conditions.*[87]

The treatment in the paper of 1856, moreover, is basically more modern, since Piola started to introduce the idea of generalized deformations of continua, regardless of the dimension of their geometrical support. From the study of this posthumous paper, it seems that Piola could have anticipated some later developments of continuum mechanics; yet his provincialism made his work almost unknown outside Italy. Indeed, in the references to Piola's work in the twentieth century, no mention was made to *Di un principio controverso*, and many Italian scientists did not quote Piola at all when dealing with continuum mechanics.

85 (Müller and Timpe 1906), pp. 23–24; (Hellinger 1914), p. 620.

86 (Truesdell and Toupin 1960), pp. 596–597; (Malvern 1969), pp. 246–248; (Truesdell 1991), p. 185.

87 […es gibt] eine andere Auffassung des Prinzipes der virtuellen Verrückungen, die von vornherein nur die eigentlichen *Kräfte*, die Massenkräfte X, Y, Z und die Flachenkräfte $\bar{X}, \bar{Y}, \bar{Z}$ als gegeben betrachtet; es ist die folgende leichte Fortbildung der Formulierung von *G. Piola*: *Für das Gleichgewicht ist notwendig, dass die virtuelle Arbeit der angeführten Kräfte*

$$\underset{(V)}{\int\int\int}(X\delta x + Y\delta y + Z\delta z)\mathrm{d}V + \underset{(S)}{\int\int}(\bar{X}\delta x + \bar{Y}\delta y + \bar{Z}\delta z)\mathrm{d}S$$

verschwindet für alle [starren Bewegungen] *des ganzen Bereiches V* […so dass] die *Komponenten der Spannungsdyade als Lagrangesche Faktoren gewisser Starrheitsbedingungen* erweisen (Hellinger 1914, p. 620).

4.2 The Piola–Kirchhoff stress tensors

The introduction of Piola's name to qualify the stress tensors pulled back to the reference configuration is in (Truesdell and Toupin 1960): from the preceding sections it is clear why, and Truesdell and Toupin often refer to the works we have examined in this paper. Frequently, Kirchhoff is mentioned in the same breath as Piola, and this attribution is also due to Truesdell and Toupin; we shall highlight why. Even though the present paper is focused on Piola's contributions, it also summarises Kirchhoff's contribution for a more complete study of the subject. In fact, unlike Piola, Kirchhoff was conscious of introducing a new idea, the stress pulled back in the reference state to study finite deformations. Unfortunately, Kirchhoff's mathematical treatment is not as good as Piola's: so the complementarity of understanding and misunderstanding of mathematical and physical concepts by the two scientists justifies Truesdell's lexical juxtaposition.

In 1852, Kirchhoff published a paper[88] in which he studied the problem of elastic equilibrium in presence of finite displacements. Kirchhoff maintained that he was inspired by Saint-Venant,[89] who had formulated a clear definition of a finite measure of strain (which is now indeed called Green–Saint-Venant strain tensor) and had given some hints about how to obtain balance equations for non-infinitesimal displacements, claiming that

> [...] when pressures are taken over planes slightly inclined to those into which the three material planes are transformed that initially were rectangular and parallel to the coordinates, we have, for the six components, the same expressions, as functions of dilatations and distortions [the components of the Green–Saint-Venant strain tensor], as we have when the displacements are very small [...].[90]

The conclusion drawn by Saint-Venant in this passage does not seem so clear to a modern reader, and is probably the cause of Kirchhoff's uncertainties in the considered paper. Quite surprisingly, in fact, Kirchhoff's article is somewhat obscure and presents incorrect expressions according to modern standards. It is not clear from the text whether Kirchhoff intended to follow an approximate reasoning, or if he made genuine errors. According to Todhunter and Pearson,[91] Kirchhoff himself later realised the weakness of this paper and did not want to re-publish it in his *Gesammelte Abhandlungen*.

These are Kirchhoff's words on how he claimed to derive local balance equations in the case of finite displacements:

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88 (Kirchhoff 1852).

89 (Saint-Venant 1847).

90 [...] lorsque les pressions sont prises sur les planes légèrement obliques dans lesquels se sont changés les trois plans matériels primitivement rectangulaires et parallèles aux coordonnées, on a, pour les six composantes, les mêmes expressions, en fonction des dilatations et des glissements, que lorsque les déplacements sont très petits [...] (Saint-Venant 1847, p. 261).

91 (Todhunter and Pearson 1893), art. 1244, p. 50.

I will call ξ, η, ζ the coordinates of a point after deformation, x, y, z the coordinate of the same point before the same [deformation]. In the natural state of the body I think of three planes, parallel to the coordinate planes, through the point (x, y, z); the parts of these planes, which lie infinitely close to the named point, are transformed by the deformation into planes that are skew to the coordinate planes and meet them at finite angles, but meet each other at angles that differ infinitely little from 90^o. The pressure that these planes endure after the deformation I think of as broken up into components on the coordinate axes, and I call these components: $\boldsymbol{X}_x, \boldsymbol{Y}_x, \boldsymbol{Z}_x, \boldsymbol{X}_y, \boldsymbol{Y}_y, \boldsymbol{Z}_y, \boldsymbol{X}_z, \boldsymbol{Y}_z, \boldsymbol{Z}_z$, in such a way that e.g. $\boldsymbol{Y}_x$ is the component along y of the pressure that the plane which was orthogonal to the x axis before the deformation endures. These nine pressures are in general skew to the planes on which they work, and there are not three equal to other three, as in the case with infinitely small displacement. If one writes down the conditions that one part of the body that before the deformation is an infinitely small parallelepipodon with edges parallel to the coordinate axes and of length dx, dy, dz is in equilibrium, then one gets the equations:

$$\left.\begin{aligned} \rho\boldsymbol{X} &= \frac{\partial \boldsymbol{X}_x}{\partial x} + \frac{\partial \boldsymbol{X}_y}{\partial y} + \frac{\partial \boldsymbol{X}_z}{\partial z} \\ \rho\boldsymbol{Y} &= \frac{\partial \boldsymbol{Y}_x}{\partial x} + \frac{\partial \boldsymbol{Y}_y}{\partial y} + \frac{\partial \boldsymbol{Y}_z}{\partial z} \\ \rho\boldsymbol{Z} &= \frac{\partial \boldsymbol{Z}_x}{\partial x} + \frac{\partial \boldsymbol{Z}_y}{\partial y} + \frac{\partial \boldsymbol{Z}_z}{\partial z} \end{aligned}\right\} \ldots \qquad 1)$$

when one denotes by ρ the density of the body and by $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ the components of the accelerating force that acts on the body at the point (ξ, η, ζ). One comes to these equations by considering that the edges and the angles of the parallelepipodon have altered infinitely little, however on the other hand one uses the same considerations by which these equations are proved in the case of infinitely small displacements.[92]

[92] Ich werde die Coordinaten eines Punktes nach der Formänderung ξ, η, ζ nennen, die Coordinaten desselben Punktes vor derselben, x, y, z. Im natürlichen Zustande des Körpers denke ich mich durch den Punkt (x, y, z) drei Ebenen gelegt, parallel den Coordinaten-Ebenen; die Theile dieser Ebenen, welche unendlich nahe an den genannten Punkte liegen, gehen bei der Formänderung in Ebenen über, die mit den Coordinaten-Ebenen schiefe, endliche Winkel bilden, mit einander aber Winkel, die unendlich wenig von 90^o verschieden sind. Die Drucke, die diese Ebenen nach der Formänderung auszuhalten haben, denke ich mich in Componenten nach den Coordinaten-Axen zerlegt, und nenne diese Componenten: $\boldsymbol{X}_x, \boldsymbol{Y}_x, \boldsymbol{Z}_x, \boldsymbol{X}_y, \boldsymbol{Y}_y, \boldsymbol{Z}_y, \boldsymbol{X}_z, \boldsymbol{Y}_z, \boldsymbol{Z}_z$, in der Art, dass z. B. $\boldsymbol{Y}_x$ die y Componente des Druckes ist, den die Ebene auszuhalten hat, die von der Formänderung senkrecht zur x Axe war. Diese neun Drucke sind im Allgemeinem schief gegen die Ebenen gerichtet, gegen die sie wirken, und es sind nicht drei von ihnen drei anderen gleich, wie es bei unendlich kleinen Verschiebung der Fall ist. Stellt man die Bedingungen dafür auf, dass ein Theil des Körpers sich im Gleichgewichte befindet, der vor der Formänderung ein unendlich kleines Parallelepipedum ist, dessen Kanten parallel den Coordinaten-Axen sind, und die Längen dx, dy, dz haben,so

Thus, Kirchhoff focused on three infinitesimal faces which are parallel to fixed co-ordinate planes and pass through a generic point which undergoes a finite displacement. He then projected the stresses arising after the deformation on those faces onto the fixed co-ordinate axes and wrote the local balance equations with respect to the same axes. Kirchhoff's Eqs. 1) above seem inconsistent when what has been said in the previous section is considered. Indeed, although they have a similar form to that of our Eqs. (15) and (18), they do not coincide with them for two reasons: (i) it is not clear how the components $\boldsymbol{X}_x, \boldsymbol{Y}_x, \ldots$ may coincide with those of the first Piola stress tensor. Indeed, no information is provided either on how the area affected by the stress changes during deformation, or on the change of metric between the present and the reference configuration; and (ii) it is not clear where ρ is measured. If ρ is the mass per unit volume in the present configuration, as seems to be implied by Kirchhoff's words, this is again inconsistent with our Eq. (15), since the mass density is required to be measured in the reference configuration.

It is strange that a sharp expert in physics and a well-educated mathematician like Kirchhoff wrote such inconsistencies. This may perhaps be explained by the fact that Kirchhoff was studying a problem of finite displacements with infinitesimal strain, as explicitly stated on p. 762 and quoted above, and as conjectured by Saint-Venant:

> [...] mutual distances of points very close together vary only in a small ratio [...].[93]

One may then suppose that Kirchhoff considered the body as almost undistorted so that areas and volumes do not vary. In this case, it is still possible to derive local balance equations for the stress components in the present configuration, projected on the fixed co-ordinated axes, by means of standard procedures. This should be represented by Kirchhoff's equations (1), if ρ is taken as the density in the reference configuration.

(Footnote 92 continued)
kommt man zu den Gleichungen:

$$\left.\begin{aligned} \rho\boldsymbol{X} &= \frac{\partial \boldsymbol{X}_x}{\partial x} + \frac{\partial \boldsymbol{X}_y}{\partial y} + \frac{\partial \boldsymbol{X}_z}{\partial z} \\ \rho\boldsymbol{Y} &= \frac{\partial \boldsymbol{Y}_x}{\partial x} + \frac{\partial \boldsymbol{Y}_y}{\partial y} + \frac{\partial \boldsymbol{Y}_z}{\partial z} \\ \rho\boldsymbol{Z} &= \frac{\partial \boldsymbol{Z}_x}{\partial x} + \frac{\partial \boldsymbol{Z}_y}{\partial y} + \frac{\partial \boldsymbol{Z}_z}{\partial z} \end{aligned}\right\} \ldots \qquad 1)$$

wenn man mit ρ die Dichtigkeit des Körpers, mit $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ die Componenten der beschleunigenden Kraft bezeichnet, die auf den Körper im Punkte (ξ, η, ζ) wirkt. Man kommt zu diesen Gleichungen, indem man benützt, dass die Winkel und die Kanten des Parallelepipedums sich nur unendlich wenig geändert haben, übrigens aber dieselben Betrachtungen anstellt, durch die man bei unendlich kleinen Verschiebungen diese Gleichungen beweist (Kirchhoff 1852, pp. 762–763; some evident typographical errors have been amended).

93 [...] les distances mutuelles de points très-rapprochées ne varient que dans une petite proportion [...] (Saint-Venant 1847, p. 261).

It is remarkable how the developments by Piola and Kirchhoff are somehow specular images. In the second derivation of the local balance equations which Piola presented in the *Meccanica de' corpi naturalmente estesi*, he first introduced what we now call Piola's second stress tensor: its components are the Lagrange multipliers of his variational problem. Then, he introduces what we now call Piola's first stress tensor simply as a mathematical stratagem with which to write the local balance equations in the present configuration; no mechanical meaning is given to its components. On the other hand, Kirchhoff began by considering from a physical point of view the quantities that we now call the components of Piola's first stress tensor. Later, he introduced the components of what we now call Piola's second stress tensor only to obtain a constitutive relation for the components of the first.

5 Conclusions

Even in Italy Piola's work seems to have fallen into oblivion soon after his death, as he is not referred to in the papers and textbooks of the most famous Italian mechanicians. Piola's name is not found in (Cesàro 1894), (Marcolongo 1905), (Signorini 1930a,b), (Grioli 1960); the local balance equations pulled back in the reference configuration are labelled as Kirchhoff's equations in (Signorini 1952), (Signorini 1960). The same equations are attributed to Boussinesq in (Signorini 1930b), (Brillouin 1960) and also in (Truesdell and Toupin 1960). The paper in question is (Boussinesq 1872), in which the equations of motion are obtained for continua in terms of the reference configuration in order to study the periodic waves in a liquid infinite domain.[94]

Nor is credit given to Piola is given in the well-known monograph by Love.[95] As we have already said, perhaps the first to have attributed Piola's name to the description of stress in the reference configuration and the "lagrangean" way to derive the local balance equations was Truesdell in (Truesdell and Toupin 1960). Most likely, Truesdell came in contact with Piola's works via Walter Noll and the German school of mechanics, which produced both (Müller and Timpe 1906) and (Hellinger 1914). The Germans knew Piola probably due to the well established links between the German and the Italian school of mechanics and applied mathematics. Indeed, the fact that in (Müller and Timpe 1906) among the basic references works by Castigliano,[96] Cesàro and Marcolongo are found is certainly of importance. However, it seems that the posthumous paper of 1856 was not well known to the scientific community, if indeed it was known at all; no reference is made to it in either (Müller and Timpe 1906) or (Hellinger 1914) or (Truesdell and Toupin 1960), in spite of the tremendous amount of literature cited and consulted especially in the last mentioned one.

What we can be certain of is Piola's strong abilities as a mechanician and skills as a mathematician, and the fact that he had some powerful intuitions, some of

94 (Boussinesq 1872), eqs. (3) and (3*bis*) in §I, pp. 513–517.

95 (Love 1944).

96 (Castigliano 1879).

which he did not develop, others among which he developed in a very modern manner. In many ways he belonged to the Italian isolated cultural environment, and we can almost certainly attribute much of his uncompleted work to the lack of confrontation with the international scientific community. However, a bright intuition and a powerful tool to study mechanics are found throughout his writings, which thoroughly deserve the attention of mechanicians and historians.

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