Arch. Hist. Exact Sci. 60 (2006) 353–377
Digital Object Identifier (DOI) 10.1007/s00407-006-0110-z

Taking Latitude with Ptolemy: Jamshīd al-Kāshī's Novel Geometric Model of the Motions of the Inferior Planets

Glen Van Brummelen

Communicated by L. Berggren

The *Almagest* model of the motions of the planets is one of the monumental achievements in the history of science. Its ability to predict the locations of some planets to within a couple of degrees of their true positions centuries into the future would give even modern scientists pause; it reigned as the planetary paradigm for well over a millennium. Nevertheless a number of Muslim astronomers revised or even replaced it with various alternatives, designed not to improve its predictive success, but rather to correspond better to the requirement of uniform circular motion. Although Jamshīd al-Kāshī came after many of these reformers, he did not follow them. Rather than overturning Ptolemy, he revised the planetary model and applied his prodigious mathematical skills to help it conform more closely to his conception of Ptolemy's wishes.

The most characteristic feature of a planet's motion is its regular retrograde arcs, illustrated in Fig. 1 (the motion of Mars from May to December 2003). Ptolemy's epicyclic model of Fig. 2 does a remarkable job of predicting the planet's ***longitude***, its position projected onto the ecliptic.[1] However, as Fig. 1 makes clear, planets exhibit considerable motions above and below the ecliptic. A complete planetary theory must also incorporate these movements; thus, it must handle the planet's ***latitude***. Ptolemy attempted this in Book XIII, the last book of the *Almagest*, by tilting the deferent and epicycle out of the plane of the ecliptic, and setting them in back and forth motions in ways that he considered best suited to reproduce the phenomena (see Fig. 3). For the inferior planets Venus and Mercury he defined three effects. (i) The deferent circle is set in a bobbing motion, starting flat on the ecliptic when the epicycle is at points A and C, and reaching its maximum ***inclination*** when the epicycle is at B and D. The epicycle is itself tilted with respect to the plane of the deferent as follows: (ii) the ***first diameter***,[2] in the line of sight as seen from Earth, has zero tilt when the center of the epicycle is

[1] This model is described in any number of places; see for instance [Pedersen 1974] or [Evans 1998].

[2] Ptolemy used most of these terms except for "first diameter" and "second diameter", which were used by Muslim astronomers.

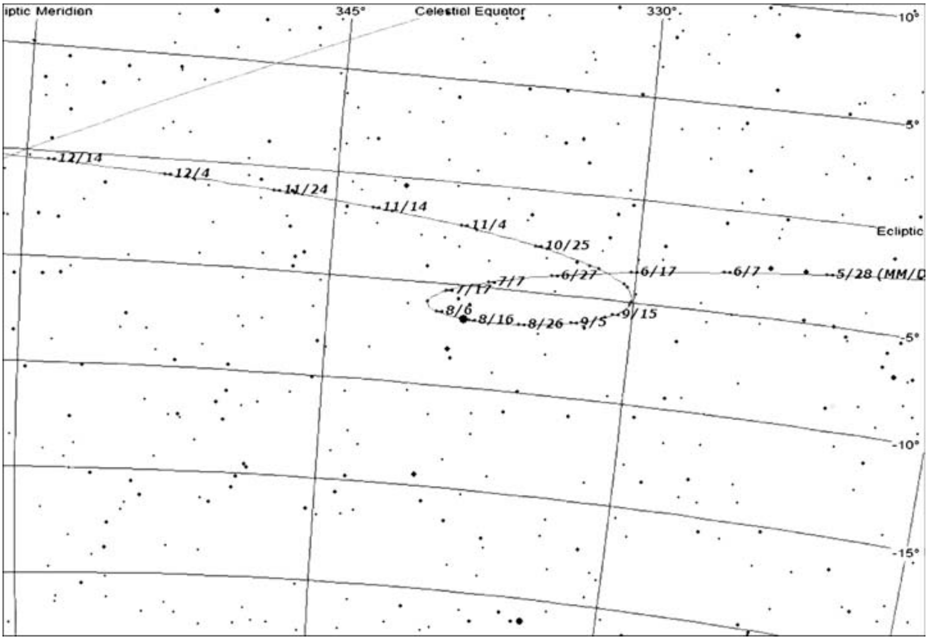

Figure 1

at B and D and reaches its maximum ***deviation*** (*al-mayl*)[3] when at A and C; and (iii) the ***second diameter***, perpendicular to the first, is parallel to the ecliptic plane when the epicycle center is at A and C, and has maximum ***slant*** (*al-inḥirāf*) when at B and D.

Although Ptolemy later published a simpler latitude theory in the *Handy Tables* and the *Planetary Hypotheses*, Muslim astronomers in the Ptolemaic tradition generally followed the *Almagest*. Among those who composed ***zījes*** (astronomical compendia that gave numerical tables for use in predictive astronomy), only a few varied from the geometric model, or the tabular setup, laid out in the *Almagest*.[4] In many cases a *zīj* author would not only adopt Ptolemy's structures, but simply borrow Ptolemy's tables outright in his own work.

Almost the only astronomer who attempted to improve on the *Almagest* with respect to determining latitudes was the early 15th-century Persian mathematician and astronomer Jamshīd al-Kāshī. Eventually a member of Ulugh Beg's scientific court in Samarkand and contributor to the great *Sulṭānī Zīj*, al-Kāshī was exceptionally mathematical in his astronomical work. His reputation as a computational genius is already

---

[3] Al-Kāshī's terms for the deviation and slant are identical to those used by Naṣīr al-Dīn al-Ṭūsī, author of the *Īlkhānī Zīj* which al-Kāshī sets out to improve in the *Khāqānī Zīj* (see [Ragep 1993, vol. 2, 424, 449].) In fact, they were used already in the Isḥāq-Thābit translation of the *Almagest* ([Sabra 1979, 389]).

[4] Variants included the use of a scheme taken from the *Handy Tables* or from Indian methods, as well as several alterations to the tables designed to ease the computations. See the valuable summary of Islamic planetary latitude tables in [Van Dalen 1999].

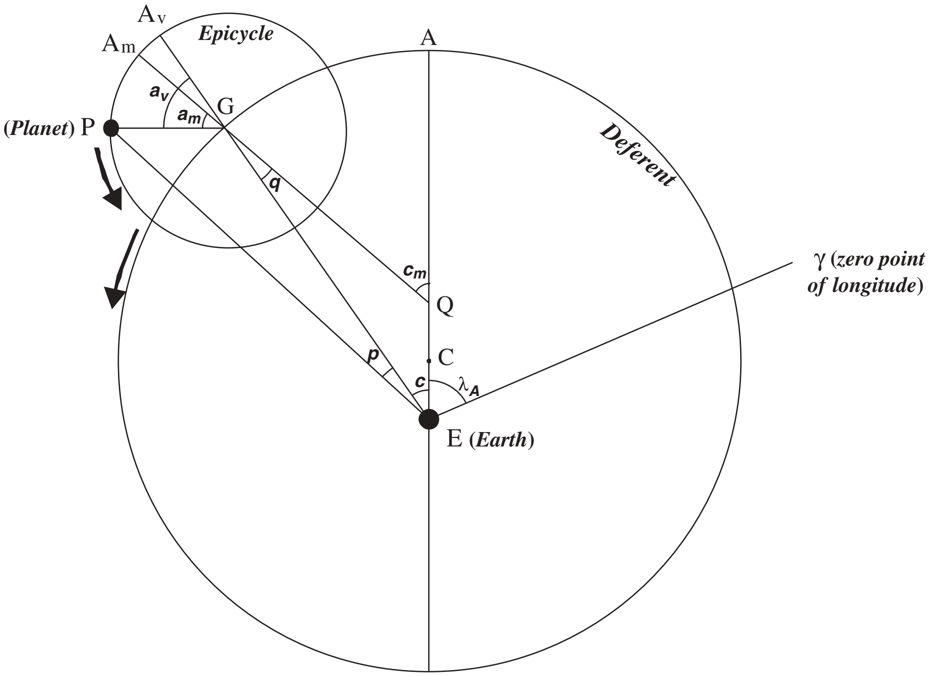

Figure 2

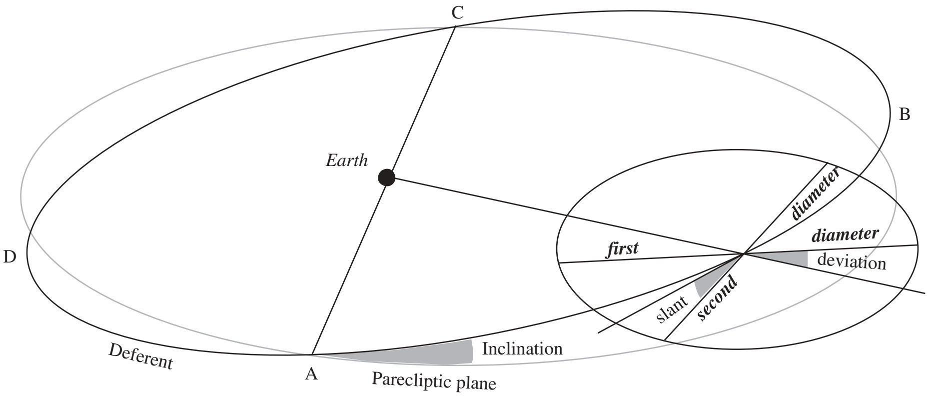

Figure 3

well established: his methods for computing π to the equivalent of 16 decimal places and sin 1° to about the same accuracy (see [Luckey 1953] for the former, [Aaboe 1954] for the latter, and [Van Brummelen 1998b] for a popular account of both) are his most recognized achievements. His contributions to mathematical astronomy go much further. Among them are a number of instruments to compute the positions of the Sun, Moon, and planets, including an ingenious device that could find planetary latitudes (see [Kennedy 1950] and especially [Kennedy 1951]). This "plate of zones" (*ṭabaq* *al-manāṭiq)*, however, relied on Ptolemy's methods and contains none of the innovations found here.

The *Khāqānī Zīj*, written as an improvement to Naṣīr al-Dīn al-Ṭūsī's *Īlkhānī Zīj* before al-Kāshī joined the scientific court of Ulugh Beg, is exceptional in its inclusion of extensive and sophisticated mathematical arguments to support the astronomy. Several studies of parts of the *zīj* have appeared in recent years, most notably [Kennedy 1998], a section-by-section survey of the entire *zīj* and our understanding of its contents thus far. Among the innovations in the material on planetary latitudes was the only major revision ever made to the single-argument tables for Mercury, which we shall publish shortly. The subject of this paper is one of the most complex sections of the *Khāqānī Zīj*, which contains a detailed description of an alternate geometric model for the motions of the inferior planets. It is intended neither to improve upon the fit with the planetary phenomena, nor to replace Ptolemy's model with a superior one. Rather, its purpose is to provide a better mathematical path from the basic description of the planets' motions in the *Almagest* to their predictions.

Al-Kāshī is dissatisfied with Book XIII primarily on mathematical grounds. Ptolemy's mathematical approach to latitudes is uncharacteristically crude and relies on a series of approximations and simplifying assumptions, some justified, some not. In his tabulations for Mercury, al-Kāshī replaces Ptolemy's correction to his earlier assumption that the Earth is at the center of the deferent with a better one.[5] Al-Kāshī's adjusted tabulation is still an approximation, but it improves the fit between geometry and computation by up to $1/4^\circ$ on latitudes whose magnitudes reach a maximum of only just over 4°. However, al-Kāshī is also concerned with the entire Ptolemaic approach, which considers the three latitude effects (inclination, deviation, and slant) to be independent of each other. In fact, the particular value of (say) the deviation at a given moment has a small effect on the value of the slant. Al-Kāshī proposes an entirely different line of attack which requires virtually no approximations, thereby repairing what is among the least satisfactory mathematical topics in the *Almagest*.

The key to al-Kāshī's method is to exploit the sphere within which the epicycle may be enclosed. Ptolemy dealt with the various inclined circles using ordinary trigonometry in three dimensions, rendering the interactions between the latitude effects nearly inaccessible. By using the epicyclic sphere, al-Kāshī accounts for two of the three latitude effects simply by setting certain great circles in motion (see Fig. 4). The ***circle of deviation*** wobbles regularly about the epicyclic equator (the intersection of the deferent with the epicyclic sphere); the ***circle of slant*** wobbles similarly about the circle of deviation. The planet is carried on the circle of slant, which in a sense represents the sum of all three latitude effects. In order to locate a planet at a given time, al-Kāshī needs only to apply spherical trigonometry to these great circles and transfer the results to the context of the complete model.

The idea that the circles in planetary models are actually great circles on spheres was not new; Ptolemy himself had mentioned these spheres occasionally in the *Almagest* and in his cosmological work, the *Planetary Hypotheses* (although they did not play a

[5] Ptolemy's correction is described in [Toomer 1984, 630, 635].

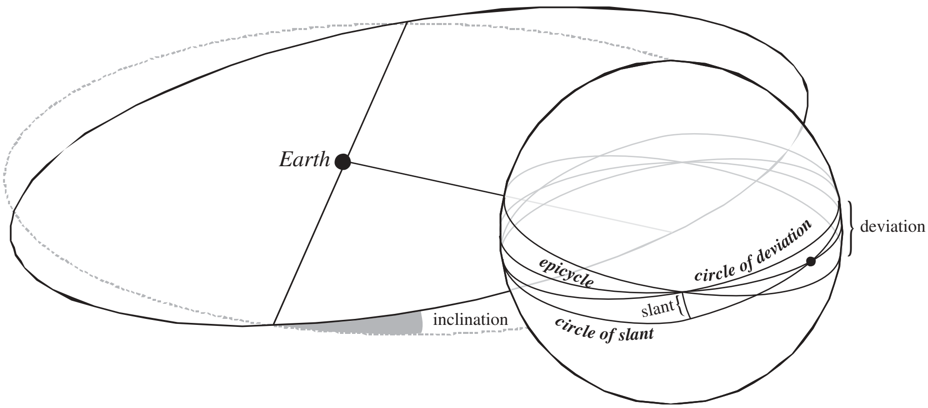

Figure 4

role in his mathematical models of the planets).[6] In medieval Islam the idea was given concrete realization first by Ibn al-Haytham, whose *Maqāla fī ḥarakat al-iltafāf* proposed to produce Ptolemy's latitudinal effects while dealing effectively with philosophical objections to the physical model.[7] Ptolemy had attached the epicycle to two small circles to cause the epicycle to move appropriately;[8] Ibn al-Haytham proposed instead a model that uses a pair of epicyclic spheres, with a design reminiscent of Eudoxus's nested spheres. Naṣīr al-Dīn al-Ṭūsī improved on Ibn al-Haytham's design with a spherical version of his famous "Ṭūsī couple", intended to produce an oscillation of a point along a given spherical arc.[9] Such a couple could be attached to the epicycle, generating the correct motion without the objections to Ptolemy's model.[10]

Al-Kāshī may well have been inspired by this; the *Khāqānī Zīj* repeatedly refers to al-Ṭūsī's work. In fact it seems entirely possible, perhaps almost inevitable, that he received the inspiration for the curvilinear oscillation that drives his model from al-Ṭūsī. However, what al-Kāshī was trying to accomplish was quite different. Both Ibn al-Haytham and al-Ṭūsī wanted to find philosophically acceptable physical models

[6] For discussions of the spherical models in the *Planetary Hypotheses* and their reality in Ptolemy's thinking see, for instance, [Pedersen 1974, 391–397] or [Murschel 1995]. For a translation of the Arabic version of part of the *Planetary Hypotheses*, see [Goldstein 1967].

[7] See [Sabra 1979] for an edition and translation of one of al-Haytham's related works (the original treatise is lost), and [Ragep 2004] for an edition and translation of a passage by al-Ṭūsī on the same topic.

[8] See [Riddell 1978] for a discussion of these circles in the *Almagest*.

[9] See [Ragep 1993, 216–22] for al-Ṭūsī's description of his curvilinear couple. [Ragep 1993, 454–456] is a discussion and analysis that shows that the point actually deviates from the arc by a small amount. For further information on the Ṭūsī couple, see also [Ragep 1987] and [Kennedy and Saliba 1991].

[10] It is also worth mentioning in passing al-Biṭrūjī's 12th-century work *On the Principles of Astronomy* (see the edition in [Goldstein 1971]), in which the entire planetary model exists on the surface of a sphere with the Earth at its center. This was also motivated by philosophic rather than technical reasons, in order to reconcile astronomy with Aristotle's cosmology.

capable of reproducing the motions of the planets. Al-Kāshī, literally, picks up where they left off. There are no nested Eudoxan spheres or Ṭūsī couples. The wobbles that give rise to the deviation and the slant are assumed to be sinusoidal, and no physical model is postulated. Rather, al-Kāshī takes the motions as axiomatic, presumably leaving to others the task of finding the right combination of spheres that produces these motions, and he builds a geometry that leads to a method for determining the planet's position.

This attitude is appropriate to *zīj*es, which generally emphasized computational results, although some dealt also with theoretical matters. *Zīj*es often attempted to improve upon the Ptolemaic schema for calculating planetary positions without altering the underlying model; the extent of their relation to the cosmological literature is an intriguing, insufficiently explored issue. Although the *Khāqānī Zīj* is exceptionally theoretical and the subject treated in this paper is extraordinary for a *zīj*, al-Kāshī nevertheless uses his invention here for an entirely computational purpose: to determine the position of Venus at a given time. Even so, one wonders whether this work might have been a source for the discussions of improvements to Ptolemy's models that George Saliba has recently demonstrated took place later at Ulugh Beg's court.[11]

In terms of complexity, planetary latitudes are the culmination of Ptolemy's mathematical astronomy. Al-Kāshī's remarkable system removes its mathematical flaws, and demonstrates that Muslim astronomers not only mastered this apex of Ptolemaic astronomy, but also perfected its mathematics.

The remainder of this paper is devoted first to a brief description of the mathematics of Ptolemy's latitude model, and then to a technical account of the part of the *Khāqānī Zīj* devoted to al-Kāshī's spherical approach. Al-Kāshī's text falls roughly into three sections: a geometrical description of the spherical model, a mathematical discussion of how one might generate planetary positions from it, and a sample calculation for Venus. A translation by Sergei Tourkin of the passage in which al-Kāshī describes the geometric structure of his model may be found in an appendix.

Ptolemy's model for the latitudes of the inferior planets

The latitudes of the inferior planets are more complicated than those of the superior planets; we shall ignore the latter, since the *Khāqānī Zīj* is not as original with respect to them. We shall follow the notation established in [Pedersen 1974, 261–294 and 355–386]. Since part of the theory of longitudes (which is the same for all planets) is needed, we describe it briefly.

The center of the epicycle revolves around the deferent at a constant rate with respect to the ***equant point*** Q (see Fig. 2). Thus the ***mean centrum*** $c_m = \angle AQG$ changes linearly with time and may be found easily using a mean motion table. The ***true centrum*** $c = \angle AEG$ is the position of the center of the epicycle seen from the Earth; it may be found by either adding or subtracting the ***equation of center***, $q = \angle QGE$, depending

[11] In addition to Ulugh Beg himself, participants in this discussion included al-Qushjī, who devised a reform of the Ptolemaic model for Mercury [Saliba 1993], and al-Shirwānī in a commentary to al-Ṭūsī's *Tadhkira* [Saliba 2004]. Saliba's latter paper contains accounts of several fascinating episodes in the daily academic life of the court.

on whether the center of the epicycle is to the right or left of line AQE. The planet's position on the epicycle is measured by the ***mean anomaly*** $a_m = \angle A_m GP$ (taken from the epicycle's apogee seen from the equant point), which changes linearly with time; it is similarly corrected to the ***true anomaly*** $a_v = \angle A_v GP$ (where the apogee is now seen from the Earth) by adding or subtracting the equation of center. The angular displacement of the planet from the center of the epicycle, as seen from the Earth, is the ***equation of anomaly*** $p = \angle GEP$, a function of both a_v and c_m. Finally, the planet's longitude λ may be found by adding the appropriate quantities to the longitude of the apogee of the deferent λ_A:

$$\lambda = \lambda_A + c \pm p. \tag{1}$$

The model for Mercury varies from this; the interested reader is referred to [Pedersen 1974, pp. 309–328]. See also [Van Brummelen 1998a] and [Duke 2004] for computer animations of each model.

For the inferior planets' latitudes, Ptolemy assumes that the three motions (inclination, deviation, and slant) are independent of each other. We begin with the ***inclination***, which is a tilt of the deferent, and which reaches its maximum when the center of the epicycle is at the deferent's apogee B and perigee D, and zero at the ***nodes*** A and C (Fig. 3). For Venus the deferent's tilt is northward to the right when the epicycle is on the right of Fig. 3, and northward to the left when the epicycle is on the left, changing sinusoidally throughout.[12] The position of the epicycle is measured by the true centrum c, taken counterclockwise from node A. Note that the Earth, not the center of the deferent, is fixed in place; thus the deferent is not cut perfectly in half by the plane of the ecliptic, which passes through the center of the Earth. The maximum inclinations for both planets are small: $i_{\max} = 0;10°$ for Venus, and $i_{\max} = 0;45°$ for Mercury.[13]

Ptolemy begins his computation of the ***first latitude*** β_1 (the latitude caused by the effect of the inclination) by finding the altitude of the apogee of the deferent when the center of the epicycle is at position c:

$$i = i_{\max} \cos c.^{14} \tag{2}$$

The correct formula is

$$\sin i = \sin i_{\max} \cos c, \tag{3}$$

but since i and $i_{\max}$ are very small, the approximation is reasonable. Then the first latitude is found by moving from the apogee to the planet's current position:

$$\beta_1 = i \cos c = i_{\max} \cos^2 c. \tag{4}$$

[12] The motions of Mercury are in the reverse direction to those of Venus; to avoid confusion we will stick with Venus.

[13] We use the standard notation for representing numbers in sexagesimal format: for instance, $1,23;45,6,7 = 1 \cdot 60 + 23 + \frac{45}{60} + \frac{6}{60^2} + \frac{7}{60^3}$. Thus, for Venus, $i_{\max} = 1/6°$.

[14] To be precise, Ptolemy uses the chord function rather than sines and cosines, and uses a circle of radius 60 for his trigonometry rather than the unit circle.

Again, the correct formula should be

$$\sin \beta_1 = \sin i \cos c, \tag{5}$$

but only an astronomer with an obsession for precision would complain about this slight difference.

The determination of the ***second latitude*** β_2, caused by the deviation (the tilt of the epicycle's first diameter), begins similarly. The first diameter is attached to a small circle placed at right angles to the deferent circle at the perigee of the first diameter; as the small circle rotates it causes the first diameter to wobble so that it reaches its maximum and minimum displacement from the plane of the deferent when the center of the epicycle is at the nodes A and C, and returns to zero at the deferent's apogee B and perigee D. Thus the ***deviation*** j is

$$\sin j = \sin j_{\max} \sin c, \tag{6}$$

which Ptolemy simplifies to the sinusoidal motion

$$j = j_{\max} \sin c, \tag{7}$$

where $j_{\max} = 2; 30^\circ$ for Venus (al-Kāshī uses $j_{\max} = 3; 30^\circ$, the same as the value in Ptolemy's *Planetary Hypotheses*) and $j_{\max} = 6; 15^\circ$ for Mercury (al-Kāshī uses 7°). The computation of β_2 from j is a complicated function of both c and a_v, which we omit here since it is not necessary to our discussion. The interested reader is referred to [Pedersen 1974, 377–379].

Finally, the ***third latitude*** β_3, caused by the slant (the tilt of the epicycle's second diameter), uses the same structure as that for the second latitude, although the second diameter is parallel to the ecliptic when the center of the epicycle is at the nodes A and C, not at the apogee B and perigee D. When the center of the epicycle reaches the apogee, the second diameter reaches its maximum slant $k_{\max}$ (with the leading edge tilting northward); at the deferent's perigee the slant is in the other direction. Thus the ***slant*** k is

$$\sin k = \sin k_{\max} \cos c, \tag{8}$$

which Ptolemy simplifies to

$$k = k_{\max} \cos c, \tag{9}$$

where $k_{\max} = 3; 30^\circ$ for Venus and 7° for Mercury. Ptolemy estimates β_3 by combining the effect of the slant with an appropriate scaling of the equation of anomaly for $c_m = c_m^0 \approx 90^\circ$:[15]

[15] The value c_m^0 is chosen so that the distance ρ from the Earth to the center of the epicycle is exactly 60; this occurs when c_m is slightly greater than 90°. In practice c_m^0 does not have to be computed; one simply needs to know that $\rho = 60$.

$$\beta_3 \approx k \cos c \cdot p(a_v, c_m^0). \tag{10}$$

At this stage, in the case of Mercury, Ptolemy decides that the varying distance of the epicycle center from the Earth must be taken into account. If the epicycle is on the apogee side of the deferent then the planet is further from the Earth than the calculations assume; he therefore instructs that β_3 is to be decreased by a factor of $\frac{1}{10}$. Similarly, if the epicycle is on the perigee side, then β_3 is to be increased by a factor of $\frac{1}{10}$.[16] The *Khāqānī Zīj* innovation in the tabulation of Mercury's latitudes to which we referred earlier is equivalent to allowing the $\frac{9}{10}/\frac{11}{10}$ factor to change continuously as the epicycle traverses the deferent.

This theory is subject to many criticisms, not least of which is the quality of its fit to the phenomena. However, this was not al-Kāshī's, or in fact other Muslim astronomers', concern. The complexity was not an issue either: although Ptolemy himself simplified the model considerably and composed simpler latitude tables in his *Handy Tables* (see [Stahlman 1960, pp. 143–155, 325–334]), most Muslim astronomers adhered closely to the *Almagest* model. Al-Kāshī's complaint about the *Almagest* is rather that Ptolemy does not stick closely enough to his own geometry when he turns to the mathematics. For instance, in formulas involving the sines of small arcs, Ptolemy often replaces the sine with the arc itself, such as the use of (4) rather than (5). The errors in Ptolemy's approximations for the effect of slant are particularly egregious: the varying distance from the Earth to the center of the epicycle is handled crudely for Mercury, and ignored entirely for Venus (albeit with some computations to justify the claim that its effect is negligible). Finally, Ptolemy assumes that the three latitude effects work independently of each other. He does not attempt to measure the interdependence of the inclination, deviation and slant; indeed, using plane trigonometry in three dimensions, this would be a huge undertaking.

Al-Kāshī's latitude model

The third treatise of the *Khāqānī Zīj* deals with the determinations of the positions of the planets and is divided into two chapters, the first describing the use of the tables located at the end of the treatise, and the second containing geometric proofs of the operations. This account is based on Sect. 8 of the second chapter. The following manuscripts were consulted:

- India Office (London), MS 430 (Ethé 2232), f. 104v–108v
- Dār al-Kutub (Cairo), MS TR 149, pp. 180–187
- Aya Sofia (Istanbul), MS 2692, f. 73r–77r

There are very few differences between the India Office and Cairo manuscripts, but some text is missing from the Aya Sofia manuscript, especially in the geometric description of

[16] One of the few Muslim innovations in tables for planetary latitudes was to tabulate $\frac{9}{10}\beta_3$ and $\frac{11}{10}\beta_3$ rather than tabulating just β_3 itself and asking the reader to perform the multiplication (see [Van Dalen 1999]).

the model. The India Office manuscript, the basis of most modern studies, was composed about AD 1500. All folio numbers given in this paper refer to the India Office manuscript.

After describing the model and mathematics for the superior planets in a conventional fashion, al-Kāshī begins his discussion of the inferior planets by detailing the geometric model. The Appendix contains a translation of this passage; we follow roughly its order of presentation here.

Al-Kāshī's treatment of the inclination works identically to Ptolemy's (see Fig. 3) and he does not discuss it here. The deferent wobbles above and below the ***parecliptic plane*** (the plane containing the ecliptic). For Venus, the deferent has no tilt when the center of the epicycle is at nodes A and C, reaches its maximum northward tilt to the right when it is at B, and reaches its maximum northward tilt to the left when it is at D. Note that this keeps the center of the epicycle on or above the parecliptic plane at all times.

At this stage the epicycle is enclosed in a sphere, as in Figs. 4 and 5. The ***epicyclic equator***[17] is the intersection of the epicyclic sphere with the plane of the deferent. The ***circle of deviation*** wobbles with respect to the epicyclic equator, moving the ***apparent apogee and perigee*** above and below it. The axis of this motion is the diameter parallel to the parecliptic plane, at right angles to the line joining the apogee and perigee. (Al-Kāshī describes this axis as perpendicular to the "plane of latitude of the epicycle center", a vertical plane through the Earth and the epicycle center in Fig. 4.) The motion itself follows Ptolemy's prescription: the circle of deviation coincides with the epicyclic equator when the epicycle's center is at either of the points B and D of Fig. 3, and reaches its maximum deviation from the epicyclic equator when it is at either of the nodes A and C. Figure 5 illustrates a typical situation, the deviation being arc j.

The ***circle of slant***[18] moves with respect to the circle of deviation, around an axis that is the diameter through the apogee and perigee. Again the motion follows Ptolemy's instructions: when the epicycle center is at either of the nodes A and C of Fig. 3, the circle of slant coincides with the circle of deviation. When the epicycle is at either of points B and D, the slant (labeled arc k on Fig. 5) is at its maximum. The planet is carried along the circle of slant.

Next al-Kāshī gives the maximum values of the deviation and the slant; for Venus we have $j_{\max} = k_{\max} = 3;30$, and for Mercury we have $j_{\max} = k_{\max} = 7;0$. To find the deviation j and slant k from the planet's true centrum al-Kāshī uses the formulas

$$\frac{\operatorname{Sin} 90^\circ}{\operatorname{Sin} j_{\max}} = \frac{\operatorname{Sin} c}{\operatorname{Sin} j} \quad \text{and} \quad \frac{\operatorname{Sin} 90^\circ}{\operatorname{Sin} k_{\max}} = \frac{\operatorname{Cos} c}{\operatorname{Sin} k}, \tag{11}$$

equivalent to (6) and (8) above, eschewing Ptolemy's approximations (7) and (9). (The notation "Sin" refers to the medieval sine function, which is R (=60) times the modern sine; similarly for the other trigonometric functions.)

The model itself is now completely specified, but the procedure to determine the planet's position requires the definition of an extra circle. The planet's position is known

---

[17] This term is not found in our passage, and is used here for convenience.

[18] Again, al-Kāshī uses the same term for slant (*al-inḥirāf*) as al-Ṭūsī; see [Ragep 1993, vol. 2, 449].

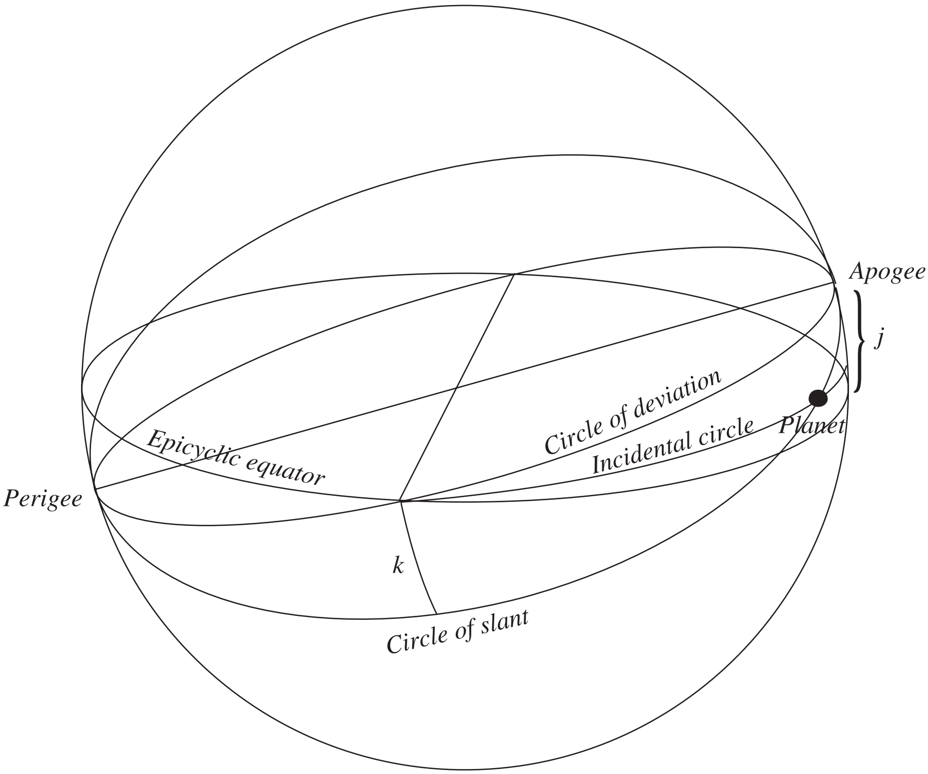

Figure 5

by moving from the apogee along the circle of slant by an arc equal to the true anomaly a_v. The ***incidental circle*** is then drawn to pass through the endpoints of the axis of rotation of the circle of deviation (see Fig. 5) and the planet. As we shall see, al-Kāshī converts the true anomaly, measured along the circle of slant from its apogee, to an ***incidental anomaly*** a, measured along the incidental circle from its apogee.

Mathematical discussion for determining the positions of the planets

This conversion is a problem in spherical trigonometry, a subject with which al-Kāshī was more than familiar. In the fourth of the six treatises in the *Khāqānī Zīj*, al-Kāshī gives trigonometric solutions to the standard problems of spherical astronomy.[19] Of the twenty-six problems solved in that section, twenty-three apply the Rule of Four Quantities (hereafter called R4Q), the "workhorse of his trigonometric stable" [Kennedy 1985, p. 3]. This rule asserts that, for two right spherical triangles sharing an acute angle, the ratios of the sines of the adjacent sides to the sines of the hypotenuses are equal; i.e., in Fig. 6,

[19] See [Kennedy 1985] for an analysis.

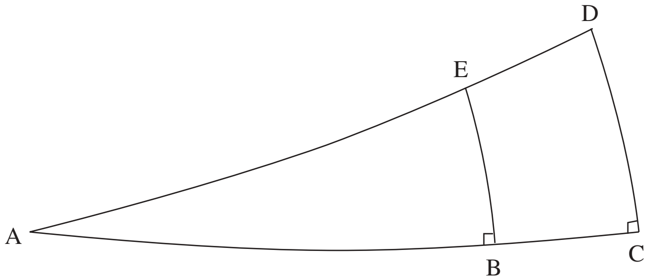

Figure 6

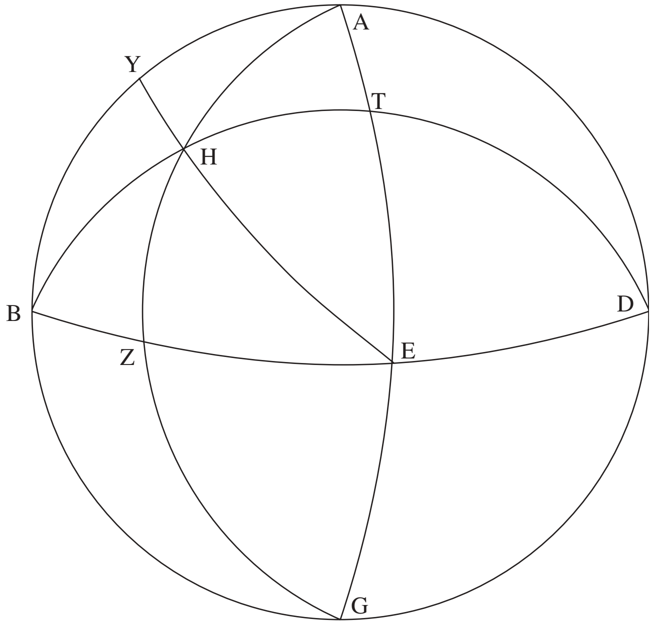

Figure 7

$$\frac{\text{Sin}\,\overset{\frown}{BE}}{\text{Sin}\,\overset{\frown}{AE}} = \frac{\text{Sin}\,\overset{\frown}{CD}}{\text{Sin}\,\overset{\frown}{AD}}. \tag{12}$$

Moreover, in all but one of al-Kāshī's uses of R4Q in the fourth treatise, he applies it to figures for which $\overset{\frown}{AC}$ and $\overset{\frown}{AD}$ are quadrants. The same holds here: R4Q is the principal tool of all his inquiries, and he applies it almost always with this restriction.

In Fig. 7 (f. 106v),[20] $ABGD$ is the circle of deviation, broken into quadrants by the four named points; A and G are the epicyclic apogee and perigee respectively. We shall

[20] Starting with Fig. 7, each diagram in this paper is a copy of one in the manuscripts (other than the lower part of Fig. 8). We use [Kennedy 1991/92] to transliterate the letters in the diagrams.

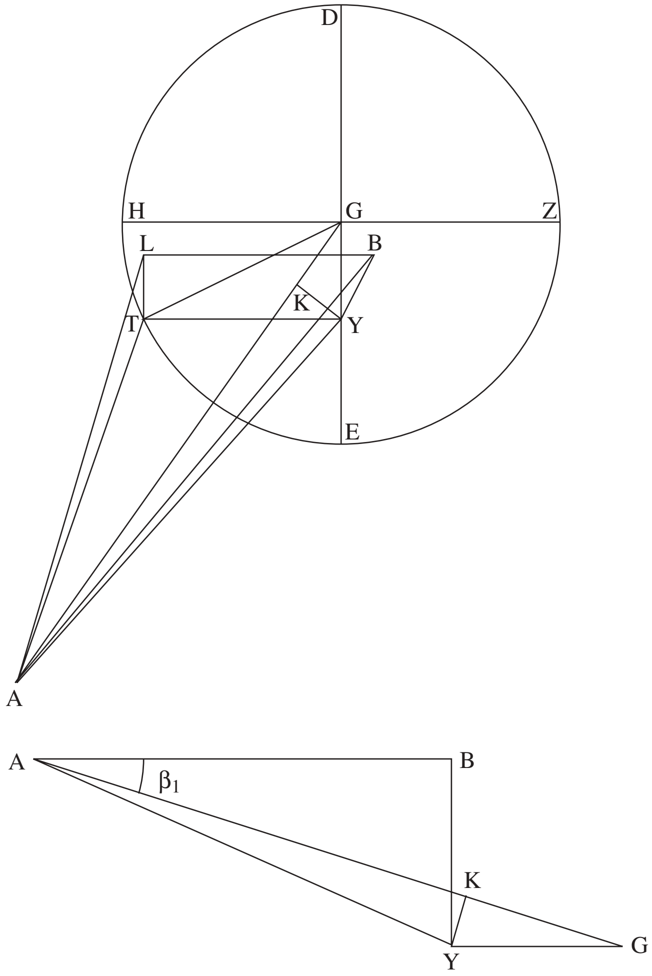

Figure 8

refer to great circle AEG, where E is the pole of the circle of deviation, as the ***epicyclic meridian*** (although al-Kāshī does not name it). Great circle AZG is the circle of slant, with Z elevated above B by arc $\overset{\frown}{BZ} = k$. The planet's current position H is determined by moving an amount $\overset{\frown}{AH} = a_v$ along the circle of slant from A. The ***incidental circle*** is the great circle through B, the planet H, and D; and the ***incidental anomaly*** a is measured from its apogee T (the intersection of the incidental circle with the epicyclic meridian) to the planet H.

Our first step is to determine the inclination of the incidental circle. Join E to H, and extend that arc to point Y on the circle of deviation. Arcs $\overset{\frown}{AH} = a_v$ and $\overset{\frown}{BZ} = k$ are known from the usual planetary longitude calculations and (11). Applying R4Q to figure $AYBZH$ gives

$$\frac{\text{Sin } 90^\circ}{\text{Sin } \widehat{BZ}} = \frac{\text{Sin } \widehat{AH}}{\text{Sin } \widehat{HY}}, \tag{13}$$

which determines $\widehat{HY}$. Next, apply R4Q to $EZBYH$ (since the angles at B and Z are right), giving

$$\frac{\text{Cos } \widehat{HY}}{\text{Cos } \widehat{AH}} = \frac{\text{Sin } 90^\circ}{\text{Cos } \widehat{AY}}. \tag{14}$$

(The cosines arise as the sines of the complements of arcs.) Thus $\widehat{AY}$ is known, and the original arc $a_v = \widehat{AH}$ has been converted to "coordinates" with respect to the circle of deviation. To measure the anomaly $a = \widehat{TH}$ measured along the incidental circle, apply R4Q to $ETAYH$:

$$\frac{\text{Sin } 90^\circ}{\text{Sin } \widehat{AY}} = \frac{\text{Cos } \widehat{HY}}{\text{Sin } \widehat{TH}}; \tag{15}$$

this gives $a = \widehat{TH}$. Finally al-Kāshī calculates $\widehat{AT}$, the inclination of the incidental circle to the circle of deviation, by applying R4Q to $BYATH$:

$$\frac{\text{Cos } \widehat{TH}}{\text{Sin } \widehat{HY}} = \frac{\text{Sin } 90^\circ}{\text{Sin } \widehat{AT}}. \tag{16}$$

We now turn to the diagram of Fig. 8,[21], similar to one in the *Almagest*. The points in this figure are in two layers. In the surface of the paper $DHEZ$ is the epicycle[22] and Y is the projection of the planet, T, onto the first diameter of the epicycle. Points A, B, and L are the same distance above the surface. A, the Earth, should actually appear on the page on the extension of line DGE; al-Kāshī moves A to the left so that the reader can see triangles GKY and AYB. Thus Fig. 8b represents the plane perpendicular to the page through DGE. Segments BY and LT are perpendicular to the plane of the paper; points B and L are projections of Y and T respectively onto the parecliptic plane, in which triangle ALB lies. The planet T is displaced from the apogee by $a = \widehat{DT}$. K is the perpendicular projection of Y onto AG.

Al-Kāshī asserts that we may determine $\angle YAK$ using a plane trigonometric argument borrowed from the corresponding passage on the superior planets. We follow al-Kāshī by not repeating it here; the interested reader will find the details in the upcoming sample calculation for Venus. Since

[21] Fig. 8 is actually a representation of the diagram on f. 108r, a few pages later in the text. It is almost identical to the diagram on f. 105r, to which al-Kāshī refers here. The only differences are that the epicycles are in different quadrants in the two figures, and that the diagram on f. 105r does not connect one line segment and labels one extra point, none of which affect us here.

[22] Both here and in the sample calculation for Venus, when al-Kāshī moves from the spherical epicycle diagram to the planar one, the incidental circle takes on the role of the epicycle.

$$\frac{\text{Sin } 90^\circ}{\text{Sin}(90^\circ - c)} = \frac{\text{Sin } i_{\max}}{\text{Sin } i} = \frac{\text{Sin } i}{\text{Sin } \beta_1},^{23} \tag{17}$$

knowledge of c allows the calculation of $\beta_1 = \angle GAB$. Then $\angle BAY$, the latitude of the projection of the planet onto the first diameter of the epicycle, may be found by adding or subtracting $\angle YAK$ and β_1, depending on the northward or southward direction of these two angles. One final step is needed to convert from $\angle BAY$ to the latitude $\beta = \angle TAL$, but al-Kāshī does not comment on it here, leaving it to the sample calculation for Venus.

The sample calculation for Venus

To demonstrate his new method al-Kāshī works his way through a sample computation of the position of Venus for a given moment, using none of the approximations used in tabular methods. Almost all his calculations are correct to within one or (occasionally) several units in the last sexagesimal place; virtually none are in error by as much as ten units. Al-Kāshī begins by extracting the values of the following three parameters for a certain (unstated) moment, presumably obtained from mean motion tables:

$$\begin{aligned} c_m &= 86;52,41^\circ, \\ a_m &= 129;55,18^\circ, \text{ and} \\ \lambda_A &= 80;15,6^\circ, \end{aligned} \tag{18}$$

where c_m is the mean centrum, a_m the mean anomaly, and λ_A the longitude of the apogee.[24]

The first few steps are typical of a Ptolemaic procedure and have as their goal the determination of the basic quantities associated with longitude, including those needed to enter the epicyclic sphere, namely the true centrum $c = \angle AEB$ and the true anomaly $a_v = \overset{\frown}{YKL}$ in Fig. 9 (f. 107r). ABG is the deferent of Venus with center D, B is the center of the epicycle, YKL is the epicycle, L is Venus, and Z is the equant point; hence $\angle AZB = c_m = 86;52,41^\circ$. Draw BD (the radius of the deferent, assumed to be 60 units long), EBY, and KBZ. Drop perpendiculars ET and DH onto the extension of BZ.

From c_m, we know that $\frac{DH}{DZ} \cdot R = \text{Sin}(86;52,41^\circ) = 59;54,39,26$ and $\frac{HZ}{DZ} \cdot R = \text{Cos}(86;52,41^\circ) = 3;16,3,39$. But DZ is the eccentricity of Venus, which for al-Kāshī is $e = 1;3,4,30$;[25] hence $DH = 1;2,58,53$ and $HZ = 0;3,26,7$. By the Pythagorean Theorem,

[23] This is equivalent to the precise versions (3) and (5) of Ptolemy's (2) and (4) respectively; al-Kāshī derives these equalities in the previous Sect. II.7.

[24] These angles are given first in terms of numbers of signs (units of 30°); we have converted to pure degree measurements throughout.

[25] This number is taken to be half the value of al-Kāshī's solar eccentricity, $e = 2;6,9$.

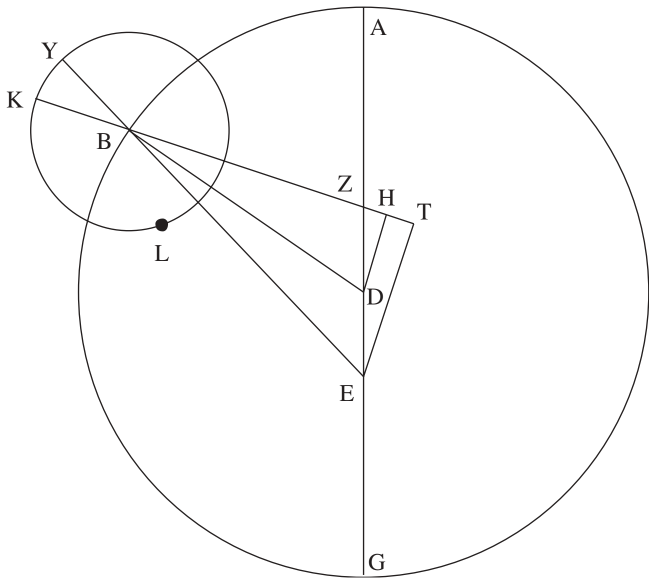

Figure 9

$$BH^2 = BD^2 - DH^2$$
$$= 60^2 - (1;6,6,39,19,49)^2$$
$$= 59,58;53,53,20,40,11,^{26} \qquad (19)$$

and so

$$BH = 59;59,26,56.$$

Now $HT = HZ$ (since $ED = DZ = e$ and DH is parallel to ET), so $BT = BH + HZ = 60;2,53,3.$ [27] Therefore $BT^2 = 1,0,5;46,14,19,6,18,9$, and $ET^2 = (2DH)^2 = 4;24,26,37,16,59,16$, so $BE^2 = BT^2 + ET^2 = 1,0,10;10,40,56,23,17,25$. So the distance from the Earth to the center of the epicycle is

$$\rho = BE = 60;5,5,7. \qquad (20)$$

We now have the information needed to find the equation of center $q = \angle EBT$: $\text{Sin}\, q = \text{Sin}\, \frac{ET}{BE} = \frac{2(1;2,58,53)}{60;5,5,7} = 2;5,47,6$, so $q = \text{arcSin}(2;5,47,6) = 2;0,8,24$.

---

[26] All three manuscripts have a spurious digit near the end of the value for BH^2 : ..., 20, 40, 45, 11.

[27] Here and occasionally elsewhere, al-Kāshī writes sixty as 60 rather than 1,0.

Therefore the true centrum is $c = \angle AEB = c_m - q = 84; 52, 32, 36$, and the true anomaly is $a_v = \overset{\frown}{YKL} = a_m + q = 131; 55, 26, 24$.

With the true centrum in hand, we are in a position to find the first latitude β_1. We have Sin $c = 59; 45, 36, 41$, Cos $c = 5; 21, 32, 18$, and Sin $i_{\max} = \text{Sin } 10' = 0; 10, 28, 19$. Therefore, by (17),

$$\text{Sin } i = \frac{\text{Cos } c \cdot \text{Sin } i_{\max}}{R} = 0; 0, 56, 7, \tag{21}$$

and again by (17),

$$\text{Sin } \beta_1 = \frac{\text{Cos } c \cdot \text{Sin } i}{R} = 0; 0, 5, 1, \tag{22}$$

which gives $\beta_1 = 0; 0, 4, 47$ in the northerly direction.

Before entering the epicyclic sphere we must first determine j and k, the angles of inclination of the circles of deviation and slant respectively. Since (for Venus) Sin $j_{\max} =$ Sin $k_{\max} = \text{Sin } (3; 30) = 3; 39, 46, 30$, by (6),

$$\text{Sin } j = \frac{\text{Sin } c \cdot \text{Sin } j_{\max}}{R} = 3; 38, 53, 48; \tag{23}$$

therefore $j = 3; 29, 9, 33$. Also, by (8),

$$\text{Sin } k = \frac{\text{Sin } k_{\max} \cdot \text{Cos } c}{R} = 0; 19, 37, 46, \tag{24}$$

so $k = 0; 18, 44, 40$.

We now possess the quantities needed to begin calculations on the epicyclic sphere. In Fig. 10 (f. 107v), we have the following:

$ABGD =$ the epicyclic equator;
$ZBHD =$ the circle of deviation, with pole E and inclined to the epicyclic equator by the amount $j = \overset{\frown}{GH}$;
$AEGH =$ the epicyclic meridian, the intersection of the epicyclic sphere with the vertical plane through the Earth and the center of the epicycle;
$BED =$ the great circle through points B, E, and D, at right angles to the circle of deviation;
$ZTH =$ the circle of slant, inclined to the circle of deviation by the amount $k = \overset{\frown}{BT}$ (where T is the intersection of the circle of slant with BED);
$K =$ the planet Venus, its position on the circle of slant determined by moving an amount $a_v = \overset{\frown}{ZTK}$ along the circle of slant from the epicyclic apogee Z;
$BKMD =$ the incidental circle, the great circle through B, K, and D; and
$EKL =$ the arc through E and K, extended to point L on the circle of deviation.

The goal in what follows is to transfer knowledge of the adjusted anomaly a_v on the circle of slant to the incidental anomaly a on the incidental circle. The information is transferred in two stages, first to the circle of deviation, and then to the incidental circle, following the procedure previously outlined in Eqs. (13) through (16).

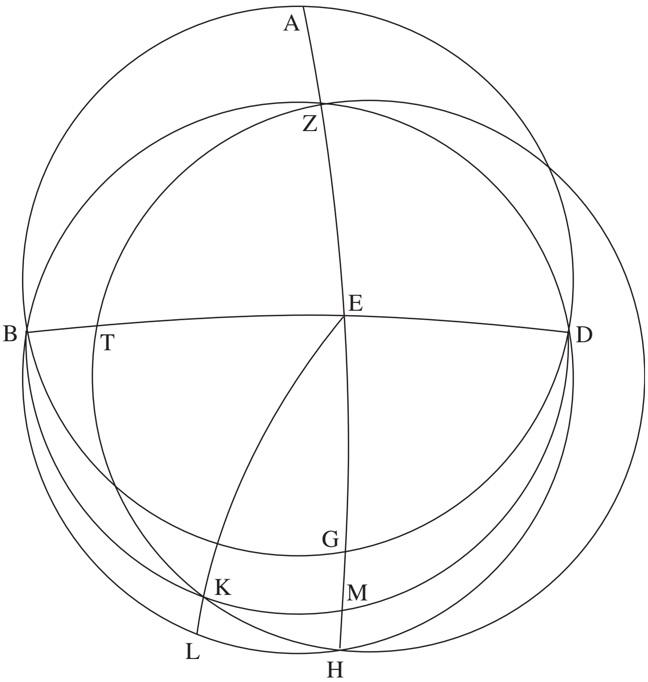

Figure 10

We begin by noting that $\overset{\frown}{HK} = 180° - \overset{\frown}{ZTK} = 180° - a_v = 48; 4, 33, 36$, so $\text{Sin}\ \overset{\frown}{HK} = 44; 38, 30, 52$. From (24) we already have $\text{Sin}\ \overset{\frown}{TB} = \text{Sin}\ k = 0; 19, 37, 46$. Thus, applying R4Q to $HLBTK$, we get

$$\frac{\text{Sin}\ \overset{\frown}{KL}}{\text{Sin}\ \overset{\frown}{KH}} = \frac{\text{Sin}\ \overset{\frown}{TB}}{R}, \tag{25}$$

which gives $\text{Sin}\ \overset{\frown}{KL} = 0; 14, 36, 18$, so $\overset{\frown}{KL} = 0; 13, 56, 49$. Therefore $\overset{\frown}{EK}$, its complement, is $89; 46, 3, 11$. Now $\overset{\frown}{TK} (= a_v - 90°) = 41; 55, 26, 24$. Therefore, applying R4Q to $ETBLK$, we have

$$\frac{\text{Sin}\ \overset{\frown}{BL}}{R} = \frac{\text{Sin}\ \overset{\frown}{TK}}{\text{Sin}\ \overset{\frown}{EK}}, \tag{26}$$

so that

$$\text{Sin}\ \overset{\frown}{BL} = \frac{R \cdot \text{Sin}\ \overset{\frown}{TK}}{\text{Sin}\ \overset{\frown}{EK}} = \frac{R \cdot 40; 5, 15, 9}{59; 59, 58, 14} = 40; 5, 20, 20 \tag{27}$$

and hence $\overset{\frown}{BL} = 41; 55, 27, 55$. Since $\overset{\frown}{BL}$ is $90°$ less than $\overset{\frown}{ZBL}$, the arc on the circle of deviation corresponding to a_v, the first half of the determination of a is complete.

Next we transfer the anomaly from the circle of deviation to the incidental circle. $\widehat{HL}$, the complement of $\widehat{BL}$, is 48;4,32,5. Thus, applying R4Q to the figure $EMHLK$,

$$\frac{\mathrm{Sin}\,\widehat{MK}}{\mathrm{Sin}\,\widehat{EK}} = \frac{\mathrm{Sin}\,\widehat{HL}}{R}, \tag{28}$$

so

$$\begin{aligned}\mathrm{Sin}\,\widehat{MK} &= \frac{\mathrm{Sin}\,\widehat{EK} \cdot \mathrm{Sin}\,\widehat{HL}}{R} \\ &= \frac{(59;\,59,\,58,\,14) \cdot (44;\,38,\,29,\,48)}{R} = 44;\,38,\,28,\,29,\end{aligned} \tag{29}$$

which gives $\widehat{MK} = 48;\,4,\,30,\,12$. But $\widehat{MK}$ is the supplement of the incidental anomaly, so $a = 180° - \widehat{MK}$ is known. Finally, $\widehat{BK} = 90° - \widehat{MK} = 41;\,55,\,29,\,48$.

Before returning to more familiar Ptolemaic ground, al-Kāshī must determine the amount by which the incidental circle is inclined to the epicyclic equator, i.e., $\widehat{GM}$. Apply R4Q to $BLHMK$:

$$\frac{\mathrm{Sin}\,\widehat{HM}}{R} = \frac{\mathrm{Sin}\,\widehat{KL}}{\mathrm{Sin}\,\widehat{BK}}, \tag{30}$$

so

$$\mathrm{Sin}\,\widehat{HM} = \frac{R \cdot \mathrm{Sin}\,\widehat{KL}}{\mathrm{Sin}\,\widehat{BK}} = \frac{R \cdot 0;\,14,\,36,\,18}{40;\,5,\,21,\,48} = 0;\,21,\,51,\,31. \tag{31}$$

Hence $\widehat{HM}$, the amount by which the incidental circle is inclined to the circle of deviation, is 0; 20, 52, 25. From (23) above we know that $\widehat{GH} = j = 3;\,29,\,9,\,33$; therefore $\widehat{GM}$, the southward inclination of the perigee of the incidental circle from the epicyclic equator, is $\widehat{GH} - \widehat{HM} = 3;\,8,\,17,\,8$.

The remaining calculations take place on Fig. 8 (f. 108r). In this figure, as we said above, A is the Earth, $DHEZ$ is the epicycle,[28] T is Venus, Y is the projection of T onto the first diameter of the epicycle, and B and L are projections of Y and T respectively onto the parecliptic plane. GT, the radius of the epicycle, is taken to be Ptolemy's value 43;10.

The immediate goal is to determine $\angle BAY$, the latitude of the projection of Venus onto the first diameter of the epicycle. Note that $a = \angle DGT = 180° - \angle TGY$. So, from ΔGYT,

$$TY = \frac{GT \cdot \mathrm{Sin}\,a}{R} = \frac{(43;\,10) \cdot (44;\,38,\,28,\,29)}{R} = 32;\,7,\,0,\,50 \tag{32}$$

and

[28] i.e., the incidental circle of the epicyclic sphere. See also note 22.

$$GY = \frac{GT \cdot \text{Cos}\, a}{R} = \frac{(43;10) \cdot (40;5,21,48)}{R} = 28;50,31,31. \tag{33}$$

Let K be the projection of Y onto AG . Then $\angle YGK$, the tilt of the epicycle to the south, is 3;8,17,8 (as determined above). Therefore, in ΔYGK,

$$YK = \frac{GY \cdot \text{Sin}\, \angle YGK}{R} = \frac{(28;50,31,31) \cdot (3;17,4,26)}{R} = 1;34,43,1^{29} \tag{34}$$

and

$$GK = \frac{GY \cdot \text{Cos} \angle YGK}{R} = \frac{(28;50,31,31) \cdot (59;54,36,6)}{R} = 28;47,55,50. \tag{35}$$

From (20) we know that AG, the distance from the Earth to the center of the epicycle, is $60;5,5,7$. Therefore $AK = AG - GK = 31;17,8,17;$[30] thus

$$\begin{aligned} AY^2 &= AK^2 + YK^2 \\ &= 16,18;47,27,16,46,36,49 + 2;29,31,17,58,26,1 \\ &= 16,21;16,58,34,45,2,50, \end{aligned} \tag{36}$$

and so $AY = 31;19,31,36$. Hence

$$\angle YAK = \text{arcSin}(YK/AY) = \text{arcSin}(3;1,25,7) = 2;53,18,55 \tag{37}$$

in the southerly direction. Now from (22) $\angle GAB = \beta_1$, the latitude of the epicycle's center, is $0;0,4,47$ in a northerly direction. Therefore the latitude of the projection of Venus onto the first diameter of the epicycle is $\angle BAY (= \angle YAK - \angle GAB) = 2;53,14,8$ to the south.[31]

Next we must determine AL and BL, from which the equation of anomaly may be found. Since ΔBAY is right,

$$BY = \frac{AY \cdot \text{Sin} \angle BAY}{R} = \frac{(31;19,31,36) \cdot (3;1,20,6)}{R} = 1;34,40,24. \tag{38}$$

Therefore

$$\begin{aligned} AB^2 &= AY^2 - BY^2 \\ &= 16,21;16,58,34,45,2,50 - 2;29,23,2,24,9,36 \\ &= 16,18;47,35,32,20,53,14. \end{aligned} \tag{39}$$

---

[29] The correct value is 1;34, *44*,1, but the calculations after this confirm that al-Kāshī uses 1;34, *43*,1. All three manuscripts have a 10 instead of a 1 in the last digit.

[30] The correct value is 31;17, *9*,17, but all three manuscripts and the following calculations support the use of 31;17, *8*,17.

[31] The India Office manuscript gives the last two places of this number as 18,55, presumably copied accidentally from the value for $\angle YAK$ in the preceding line. The other two manuscripts have the correct value.

And since $BL(= TY) = 32; 7, 0, 50$,

$$AL^2 = AB^2 + BL^2$$
$$= 16, 18; 47, 35, 32, 20, 53, 14 + 17, 11; 29, 42, 31, 40, 41, 40$$
$$= 33, 30; 17, 18, 4, 1, 34, 54, \quad (40)$$

so $AL = 44; 50, 10, 28$. Hence

$$\text{Sin } p = \frac{R \cdot BL}{AL} = 42; 58, 44, 10 \quad (41)$$

where $p = \angle BAL$ is the equation of anomaly, and so $p = 45; 45, 4, 2$.[32]

For the final computation of the longitude al-Kāshī revisits the true centrum $c = 84; 52, 32, 36$, deeming it necessary to correct its value very slightly to account for the inclination of the deferent. From

$$\text{Sin } \hat{c} = \frac{R \cdot \text{Sin } c}{\text{Cos } \beta_1} = \frac{R \cdot 59; 45, 36, 41}{59; 59, 59, 59, 56} = 59; 45, 36, 41, \quad (42)$$

he arrives at $\hat{c} = 84; 52, 32, 36, 43$,[33] which he rounds up to 84;52,32,37. Al-Kāshī is certainly pressing a point in accounting for the difference between c and $\hat{c}$, especially considering that his values for Sin $\hat{c}$ and Sin c are identical! Finally, the longitude of Venus is

$$\lambda = \hat{c} + p + \lambda_A = 210; 52, 42, 41, \quad (43)$$

where $\lambda_A = 80; 15, 6$ is the longitude of the apogee.

The final latitude computation proceeds as follows:

$$AT^2 = AL^2 + TL^2 = AL^2 + BY^2$$
$$= 33, 32; 46, 41, 6, 25, 44, \quad (44)$$

so the distance from the Earth to Venus is $AT = 44; 51, 50, 23$. Therefore

$$\text{Sin } \beta = \frac{R \cdot TL}{AT} = 2; 6, 36, 50, \quad (45)$$

and $\beta = 2; 0, 55, 56$ in the southerly direction.

---

[32] Al-Kāshī could have avoided calculating AL by using BL, AB, and an arctangent to calculate p, but chose not to. Perhaps he trusted using an arcsine more than an arctangent for accuracy, or he wished to stay close to *Almagest* procedures (with only a chord table, Ptolemy had not been able to use the equivalent of an arctangent).

[33] The value Sin(84; 52, 32, 36) = 59; 45, 36, 41 is obtained by linear interpolation from al-Kāshī's sine table earlier in the *zīj*. The arc Sine appears to have been generated by the reverse process and some creative rounding.

Al-Kāshī concludes by remarking that he is the first person ever to compute longitudes and latitudes by this method. Although his longitude calculations are hardly distinct from standard Ptolemaic methods, his latitude calculations are entirely unique, and are easily the most precise mathematical approach designed within the Ptolemaic system of planetary astronomy.

Acknowledgments. The author thanks the Deutscher Akademischer Austausch Dienst, which supplied a grant to visit the Institut für Geschichte der Naturwissenschaften in Frankfurt in 2001, when this research was begun. In addition to providing hospitality, Prof. David King, director of the Institut, gave access to manuscripts and helped in dating them. Prof. Fuat Sezgin also made some manuscripts available. Dr. Benno van Dalen was of great help in various conversations on astronomical topics, as was Prof. J. L. Berggren, my doctoral supervisor and mentor, who provided valuable suggestions. Some of the early work was done using a rough translation of the *Khāqānī Zīj* by Prof. E. S. Kennedy. Prof. Jamil Ragep provided highly valued critical suggestions. The translation in the appendix is by Sergei Tourkin. Prof. Norman Derby created the animation that led to Fig. 1, using the software *Starry Night*. All remaining shortcomings and omissions are my responsibility.

Appendix: Translation of the geometric description of the model

by Sergei Tourkin, Institute of Oriental Studies, St. Petersburg, Russian Academy of Sciences

[f. 105v:16] The inferior planets

Several introductory statements need to be explained. We say that [there is] a great circle in the sphere of the epicycle that passes through the apparent apogee and perigee and [through] the two sides of the diameter that is parallel to the parecliptic plane if the epicycle center is not in the parecliptic plane, except when that diameter is in the parecliptic plane. That diameter will be perpendicular to the plane of the latitude circle of the epicycle center. We call this the circle of deviation of the epicycle, and that [circle] is always perpendicular to the plane of the circle of latitude of the center of the epicycle. And when the center of the epicycle is in the middle between the two nodes, then this circle will coincide with a plane that passes through the center of the universe and the epicycle center, and which will be at right angles to the plane of the latitude circle of the epicycle center. Otherwise, two points (*mawdi'*) will deviate from that plane. The axis of this motion will be a diameter that is perpendicular to the plane of the latitude circle of the epicycle center and parallel to the ecliptic plane, because the apparent apogee and perigee are always [f. 106r] in the plane of the latitude circle of the epicycle center. Then if there was no slant movement, the plane of the circle of the epicycle would always be in the plane of this circle, as it is in the case of the superior planets. However, in the case of the inferior planets, a diameter which passes through the two mean distances slants (*munharif mishavad*) from the diameter that is parallel to the parecliptic plane and that is perpendicular to the latitude [circle] of the epicycle center. And we call the [great] circle that passes through the two sides of this diameter and through the apparent apogee and perigee the circle of slant. The plane of the circle of the epicycle is in the plane of this circle and when the epicycle center is in one of the two nodes the plane of this circle will coincide with the plane of the circle of deviation of the epicycle. Otherwise, two points (*mawdi'*) would slant from that plane, and the axis of this motion is a diameter that passes through the apparent apogee and perigee. Otherwise [when the epicycle center is at a node], the two nodes and the two extreme points of the circle of deviation of the epicycle would intersect with the plane that passes through the center of the universe and the epicycle center and that is perpendicular to the [latitude] circle of the epicycle center at the epicycle center. And the common section between the two will be the diameter of the epicycle that is parallel to the ecliptic plane and perpendicular to the above-mentioned plane of the latitude [circle]. And the circle of slant, or rather, the circle of the epicycle, intersects the plane of the circle of deviation of the epicycle at the epicycle center, and the common section is a diameter that passes through the apogee and perigee. And the extreme of both deviation and slant, according to the new/recent observations and the opinion of the recent scholars, has been found to be 3;30 for Venus and 7;0 for Mercury, as we have mentioned before. Then by the sine theorem and according to the motions which have been proved, the ratio of the largest sine to the sine of both the extreme of the deviation and the extreme of the slant is equal to the ratio of both the sine of the adjusted center and its cosine at all times, to both the sine of the deviation and the sine of the slant at that time respectively. Then both the deviation and the slant become known at each time. If we imagine a [great] circle that passes through the planet and through the two sides of the diameter that is perpendicular to the plane of the circle of latitude of the epicycle center, that is, the axis of the motion of the deviation, we call it the incidental circle of the epicycle. This circle intersects the plane that passes through the center of the universe and through the epicycle center, and is perpendicular to the plane of the circle of latitude of the epicycle center.[34] The common section between these two is the same common section between the circle of deviation and the above-mentioned plane, that is, the diameter that is perpendicular to the plane of the circle of latitude of the epicycle center. And this circle is always also perpendicular to the above-mentioned latitudinal plane. Then we can know the deviation of this circle from the above-mentioned plane, that is, the angle of intersection of the diameter of this circle that is the common section between the plane of this circle and the above-mentioned latitudinal plane, and the line that passes through the center of the universe and the epicycle center. And we [f. 106v] call it the deviation of the incidental circle. And likewise, we can know the arc of this circle that is located between the planet and its intersection with the above-mentioned latitude circle on the further side. And we call it the incidental anomaly. [f. 106v:2]

References

Aaboe, Asger. Al-Kāshī's Iteration Method for the Determination of sin 1°, Scripta Mathematica **20** (1954), 24–29.

Duke, Dennis. Computer Animations of Ancient Greek and Arabic Planetary Models, Journal for the History of Astronomy **35** (2) (2004), 225–228.

Evans, James. The History and Practice of Ancient Astronomy, New York/Oxford: Oxford University Press, 1998.

[34] The redundant phrase "and to the epicycle center" appears here.

Goldstein, B. R. The Arabic Version of Ptolemy's *Planetary Hypotheses*, Philadelphia: American Philosophical Society, 1967.

Goldstein, B. R. Al-Biṭrūjī: *On the Principles of Astronomy*, 2 vols., New Haven/London: Yale University Press, 1971.

Kennedy, E. S. A Fifteenth-Century Planetary Computer: al-Kāshī's "Tabaq al-Manateq" I. Motion of the Sun and Moon in Longitude, Isis **41** (1950), 180–183.

Kennedy, E. S. An Islamic Computer for Planetary Latitudes, Journal of the American Oriental Society **71** (1951), 13–21.

Kennedy, E. S. Spherical Astronomy in Kāshī's *Khāqānī Zīj*, Zeitschrift für Geschichte der Arabisch-Islamischen Wissenschaften **2** (1985), 1–46.

Kennedy, E. S. Transcription of Arabic Letters in Geometric Figures, Zeitschrift für Geschichte der Arabisch-Islamischen Wissenschaften **7** (1991/92), 21–22.

Kennedy, E. S. On the Contents and Significance of the *Khāqānī Zīj* by Jamshīd Ghiyāth al-Dīn al-Kāshī, Frankfurt: Institute for the History of Arabic-Islamic Science, 1998.

Kennedy, E. S.; and Saliba, George. The Spherical Case of the Ṭūsī Couple, Arabic Sciences and Philosophy **1** (1991), 285–291.

Kennedy, E. S.; and Ukashah, Walid. Al-Khwārizmī's Planetary Latitude Tables, Centaurus **14** (1) (1969), 86–96.

Luckey, Paul. Der Lehrbrief über den Kreisumfang von Gamshid b. Mas'ud al-Kashi, Berlin: Akademie Verlag, 1953.

Murschel, Andrea. The Structure and Function of Ptolemy's Physical Hypotheses of Planetary Motion, Journal for the History of Astronomy **26** (1995), 33–61.

Pedersen, Olaf. A Survey of the *Almagest*, Odense: Odense University Press, 1974.

Ptolemy, Claudius. Ptolemy's *Almagest*, Gerald Toomer, tr., London: Duckworth / New York: Springer-Verlag, 1984.

Ragep, F. J. The Two Versions of the Ṭūsī Couple, in David A. King and George Saliba, eds., From Deferent to Equant: A Volume of Studies in the History of Science in the Ancient and Medieval Near East in Honor of E. S. Kennedy (New York: New York Academy of Sciences, 1987), 329–356.

Ragep, F. J. Naṣīr al-Dīn al-Ṭūsī's *Memoir on Astronomy (al-Tadhkira fī ͨ ilm al-hay'a)*, 2 vols., New York: Springer-Verlag, 1993.

Ragep, F. J. Ibn al-Haytham and Eudoxus: The Revival of Homocentric Modeling in Islam, in Charles Burnett, Jan P. Hogendijk, Kim Plofker, and Michio Yano, eds., Studies in the History of the Exact Sciences in Honour of David Pingree (Leiden/Boston: Brill, 2004), 786–809.

Riddell, R. C. The Latitudes of Venus and Mercury in the *Almagest*, Archive for History of Exact Sciences **19** (1978), 95–111.

Sabra, A. I. Ibn al-Haytham's Treatise: Solution of Difficulties Concerning the Movement of Iltifāf, Journal for the History of Arabic Science **3** (1979), 388–422.

Saliba, George. Al-Qushjī's Reform of the Ptolemaic Model for Mercury, Arabic Sciences and Philosophy **3** (1993), 161–203.

Saliba, George. Reform of Ptolemaic Astronomy at the Court of Ulugh Beg, in Charles Burnett, Jan P. Hogendijk, Kim Plofker, and Michio Yano, eds., Studies in the History of the Exact Sciences in Honour of David Pingree (Leiden/Boston: Brill, 2004), 810–824.

Stahlman, W. D. The Astronomical Tables of Codex Graecus Vaticanus 1291, doctoral dissertation, Brown University, 1960.

Tichenor, Mark J. Late Medieval Two-Argument Tables for Planetary Longitudes, Journal of Near Eastern Studies **26** (2) (1967), 126–128.

Toomer, Gerald J. Ptolemy's *Almagest*, London: Duckworth / New York: Springer-Verlag, 1984.

Van Brummelen, Glen R. Computer Animations of Ptolemy's Geometric Models of Lunar and Planetary Motion, Journal for the History of Astronomy **29** (1998), 271–274.

Van Brummelen, Glen R. Jamshid al-Kāshī: Calculating Genius, Mathematics in School **27** (1998), 40–44.

Van Dalen, Benno. Tables of Planetary Latitude in the *Huihui Li* (II), in Yung Sik Kim and Francesca Bray, eds., Current Perspectives in the History of Science in East Asia (Seoul: Seoul National Univ. Press, 1999), 318–328.

Glen Van Brummelen
Bennington College
Bennington, VT 05201
USA
gvanbrum@bennington.edu

(Received January 10, 2006)