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Einstein Equations and Hilbert Action: What is missing on page 8 of the proofs for Hilbert's First Communication on the Foundations of Physics?

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Abstract

The history of the publication of the gravitational field equations of general relativity in November 1915 by Einstein and Hilbert is briefly reviewed. An analysis of the internal structure and logic of Hilbert's theory as expounded in extant proofs and in the published version of his relevant paper is given with respect to the specific question what information would have been found on a missing piece of Hilbert's proofs. The existing texts suggest that the missing piece contained the explicit form of the Riemann curvature scalar in terms of the Ricci tensor as a specification of the axiomatically underdetermined Lagrangian in Hilbert's action integral. An alternative reading that the missing piece of the proofs already may have contained the Einstein tensor, i.e. an explicit calculation of the gravitational part of Hilbert's Lagrangian is argued to be highly implausible.

1. Introduction

In contrast to Einstein's discovery of special relativity in 1905, his path towards the theory of general relativity is documented by a rich historical record. Not only did Einstein publish quite a few papers on earlier versions of a generalized theory of relativity, we also have a number of research manuscripts from crucial periods of his search, and we have an extensive correspondence from the relevant years. Hilbert's involvement in the discovery of general relativity is less abundantly documented but also here we have a few key documents that shed light on his work. Compared to other episodes in the history of science, the history of general relativity is very well written, and specifically the competition between Einstein and Hilbert in the final weeks before the publication of generally covariant field equations of gravitation in late 1915 has been commented on extensively (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] and further references cited therein). Nevertheless, much of the historical literature on the Einstein-Hilbert competition took sides in what was perceived as a priority debate and it still seems worthwhile to come to a succinct and balanced assessment of the respective contributions of both authors in the final establishment of the general theory of relativity. In this respect, a set of proofs of Hilbert's relevant paper are of some significance and with those proofs the fact that a piece of them is missing. Although the fact that a piece of those proofs is missing is well-known and was briefly commented on by several authors, the question naturally arises as to whether that missing part could have contained information that would compel us to reassess the historical account?

2. The context

Before focussing on some minor yet significant details of the historical record, let me briefly review the broader historical context. In 1907, Einstein first formulated his equivalence hypothesis according to which no physical experiment can distinguish between the existence of a homogeneous, static gravitational field in a Newtonian inertial frame of reference and a uniformly and rectilinearly accelerated frame of reference that is free of any gravitational field. The hypothesis linked the problem of generalizing the special theory of relativity to accelerated motion with the problem of a relativistic theory of gravitation. In 1912, Einstein realized that such a relativistic theory of gravitation could not be achieved using a scalar gravitational potential but required the introduction of the metric tensor as the crucial mathematical object for a generalized theory of relativity. Together with his mathematician friend Marcel Grossmann, Einstein published an "Outline of a Generalized Theory of Relativity and a Theory of Gravitation" in 1913 [13]. The theory of this "Outline" has already many features of the final theory of general relativity except for one "dark spot." Einstein and Grossmann did not succeed in finding gravitational field equations for the components of the metric tensor that were both generally covariant and acceptable from the point of view of Einstein's understanding of the requirements for a satisfactory theory of gravitation.

The final episode of Einstein's path towards General Relativity began in the fall of 1915 when Einstein lost faith in the validity of the field equations of his "Outline" and reverts to a reassessment of the mathematics of general covariance as developed in the work of Riemann, Christoffel, Ricci and Levi-Civita. The final steps were taken in four successive communications to the Prussian Academy of Sciences, all of them presented for publication in the month of November 1915 [14, 15, 16, 17]. On November 4, Einstein advanced field equations that are based on the Ricci tensor but that are not yet generally covariant [14]. Instead, by stipulation of a restrictive condition on the admissible coordinates, he split off a part of the Ricci tensor and equated the remaining part to an unspecified energy-momentum tensor as the source of the gravitational field. In an addendum to this paper, presented a week later on November 11 [15], Einstein temporarily entertains the speculation that all matter might be of electromagnetic origin. This assumption allowed him to advance a generally covariant field equation of gravitation where the Ricci tensor is directly set proportional to the energy-momentum tensor. Another week later, Einstein presented a paper to the Berlin Academy in which he successfully computed the anomalous advance of the perihelion of Mercury on the basis of his new equations [16]. And yet another week later, Einstein realized that he can add a trace term to the right hand side of his field equations which turns them into what we now refer to as the Einstein equations [17].

David Hilbert's path towards general relativity is a rather different one. Half a generation older than Einstein, Hilbert in 1900 formulated his famous 23 problems of mathematical research of the coming century to the International Congress of Mathematicians in Paris. The sixth of these problems asked for an axiomatization of physics. After working on the theory of integral equations in the first decade of the century, Hilbert himself then turned to an intense study of all fields of theoretical physics. In the course of his study of contemporary physics literature he soon became interested in an attempt by the German physicist Gustav Mie to generalize Maxwellian electrodynamics so as to turn it into a theory of matter. Mie's idea was to take Maxwellian electrodynamics in its variational formulation but to search for a generalized Lagrangian entering the action, keeping the requirement of Lorentz covariance but allowing for the Lagrangian to depend explicitly on the electromagnetic vector potential. Mie's hope was to find a modified Lagrangian that would produce modified Maxwell equations which, on microscopic scales, would allow for particle-like solutions. Around that time, Hilbert also became interested in Einstein's recent work on a relativistic theory of gravitation and invited Einstein to give a series of lectures on his new theory to the Göttingen mathematicians and physicists. After Einstein presented his theory in Göttingen in July 1915, Hilbert left Göttingen for his summer vacations and began pondering on Einstein's "Outline" theory. Shortly after coming back to Göttingen at the beginning of the winter term, Hilbert himself then presented a paper to the Göttingen Academy of Sciences. In this communication, Hilbert presented a theory of the "Foundations of Physics" which combined Mie's idea of a generalized electrodynamics with Einstein's idea of a generally covariant theory of gravitation.

The dateline on Hilbert's *First Communication* on the *Foundations of Physics* [18] says that it was presented to the Göttingen Academy of Sciences on 20 November 1915. The dateline on Einstein's note on *The Field Equations of Gravitation* [17] says that it was presented to the Berlin Academy of Sciences on 25 November 1915. From a comparison of the two publications, it appears that Hilbert preceeded Einstein with the publication of the final gravitational field equations of general relativity by five days, notwithstanding the fact that both authors arrived at these equations along very different routes.

The question as to where the correct field equations of gravitation are first found in print is in need of some qualification. The gravitational field equations of general relativity may be written in two very different yet essentially equivalent ways. Einstein published his final field equations of 25 November [17, p. 845],

$$G_{im} = -\kappa \left(T_{im} - \frac{1}{2} g_{im} T \right), \tag{1}$$

as an explicit set of differential equations for the components of the metric tensor g_{im}. Using the Ricci tensor G_{im} as the differential operator acting on the metric and the energy-momentum tensor T_{im} in the source term on the right hand side made sure that his equations retained its form under arbitrary coordinate transformations, i.e. made them generally covariant. Adding a trace term $-(1/2)g_{im}T$ where $T = \sum g^{\rho\sigma} T_{\rho\sigma}$ to the right hand side of his equations in his last November paper did not violate this feature. Hilbert published the gravitational field equations in implicit form in terms of a variational principle. He axiomatically postulated an action integral [18, p. 396]

$$\int H \sqrt{g} d\omega, \tag{2}$$

where $g = |g_{\mu\nu}|$, $d\omega = dw_1 dw_2 dw_3 dw_4$ for space-time coordinates w_i and required that the Lagrangian H that enters into the action of his variational formulation be invariant under arbitrary coordinate transformations. He also assumed that the Lagrangian splits into the sum of two parts, a gravitational part given by the Riemann curvature scalar and a matter part which he left unspecified except for the postulation that it depend only on the components of the metric and the components of the electromagnetic vector potential and its first derivatives. This specification technically renders Einstein's equations equivalent to Hilbert's action, except for some ambiguity in the assumptions on how the source term is to be specified, i.e. on the fundamental constitution of matter. Both Hilbert and Einstein had left the matter term undetermined to some extent. Einstein had not specified his source term at all. Hilbert had axiomatically required that the source term depend only on the electromagnetic variables and hence that all matter is of electromagnetic origin.

But several years ago it was pointed out [5] that a set of proofs for Hilbert's *First Communication* is extant in the Hilbert archives in Göttingen. It bears a printer's stamp of December 6, 1915, and differs in some significant respects from the published version.[1] The main difference pertains to a different treatment of the energy concept that motivated an axiomatic restriction of the general covariance of Hilbert's theory and that was substantially rewritten for the published version. In the published paper, the discussion of the energy concept no longer results in the postulation of a restriction of the general covariance. It was also pointed out that the proofs did not contain the explicit version of the gravitational field equations in terms of the Einstein tensor as does Hilbert's published paper. What we now call the Einstein tensor is obtained by adding a trace term to the Ricci tensor, its covariant divergence vanishes identically, and it is obtained from the explicit variation of the gravitational part of Hilbert's action integral. To be precise, in Einstein's paper of 25 November the trace term was added on the right hand side of the field equation to the source term and not to the Ricci tensor on the left hand side and strictly speaking his paper does not contain the Einstein tensor explicitly but this difference is a minor detail since both variants are trivially equivalent. In view of the differences between the proofs and the published paper general agreement seems to have been reached [6, 7, 8, 9, 10, 11] about the conclusion that the proofs unequivocally rule out the possibility that Einstein may have taken the clue of adding a trace term to his field equations of 11 November [15] from Hilbert's paper [18]. No agreement, however, was reached on the question as to the path along which Hilbert arrived at his finally published theory: by taking the main clues from Einstein's paper, as suggested in [5], or along an independent logic of discovery, as first advocated in explicit response to this claim in [6]. It also remains an open question to what extent Einstein in those weeks of October and November 1915 had heard directly or indirectly about Hilbert's work on his theory and to what extent he may have been influenced by what he heard, e.g. in entertaining temporarily the speculation that all matter is of electromagnetic origin.

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[1] Hilbert's paper was eventually issued only on March 31, 1916, but off-prints of the final version were available to Hilbert already by mid-February [6, note 74]. Einstein's November papers were each published a week after their presentation to the Prussian Academy.

To add to the complexity of the issue, it so happens that a portion of one sheet of the extant proofs for Hilbert's *First Communication* is missing [6, note 75; 7, note 40]. In view of this fact, it seems worthwhile to discuss the question what part of the argument of the proofs is missing and whether an answer to this question may possibly affect our assessment of the Einstein-Hilbert competition in late 1915. In the following, I will argue that an analysis of the internal structure of the text and argument of the proofs and the published version of Hilbert's paper shows that the missing piece in all probability did not contain an explicit version of the Einstein tensor and its trace term. The analysis rather suggests that it contained an explicit form of the Riemann curvature scalar and the Ricci tensor as a specification of the Lagrangian in Hilbert's variational principle.

3. What is missing in the proofs

Axiom I of Hilbert's *First Communication*, as presented on p. 2 of his proofs,[2] introduces an action integral[3]

$$\int H\sqrt{g}d\tau \qquad (3)$$

where $g = |g_{\mu\nu}|$, $d\tau = dw_1dw_2dw_3dw_4$, and H is a Lagrangian density that depends on the components of the metric $g_{\mu\nu}$, its first and second derivatives with respect to the coordinates w_l of the space-time manifold, $g_{\mu\nu l} = \frac{\partial g_{\mu\nu}}{\partial w_l}$ and $g_{\mu\nu lk} = \frac{\partial^2 g_{\mu\nu}}{\partial w_l \partial w_k}$, respectively, and also depends on the components of the electromagnetic vector potential q_s and its first derivatives $q_{sl} = \frac{\partial q_s}{\partial w_l}$. Specifically, the axiom demands that the laws of physics be given by the vanishing of the variation of the action integral with respect to the fourteen potentials $g_{\mu\nu}$ and q_s for some as yet unspecified function H.

Axiom II, immediately following, then demands that H must be an invariant under all coordinate transformations. Other than that, the Lagrangian H is left undetermined by the axioms.

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[2] Niedersächsische Staats- und Universitätsbibliothek (NSUB), Handschriftenabteilung, Cod. Ms. Hilbert 634, f.23-29. Facsimile versions of both Hilbert's proofs and of the published version were made available online by the Max Planck Institute for the History of Science, Berlin, on http://echo.mpiwg-berlin.mpg.de/content/relativityrevolution/hilbert. A facsimile of the published version is also available online from the website of the *Göttinger Digitalisierungszentrum* of the NSUB, see http://gdz.sub.uni-goettingen.de/gdz.

[3] The argument being partly one of textual exegesis, I am keeping strictly to Hilbert's notation. He uses an imaginary time-coordinate and, following standard usage of the time, refers to the Lagrangian density as a Hamiltonian function. Contrary to later and current usage, Hilbert and Einstein at the time also consistently wrote contravariant indices of coordinate differentials as subscript indices. Hilbert also uses subscript indices to denote partial coordinate derivatives without, however, indicating this meaning by separating the index with a comma.

On page 3, Hilbert writes down the "ten Lagrangian differential equations"[4]

$$\frac{\partial\sqrt{g}H}{\partial g^{\mu\nu}} = \sum_k \frac{\partial}{\partial w_k}\frac{\partial\sqrt{g}H}{\partial g_k^{\mu\nu}} - \sum_{k,l}\frac{\partial^2}{\partial w_k \partial w_l}\frac{\partial\sqrt{g}H}{\partial g_{kl}^{\mu\nu}}, \qquad (\mu,\nu = 1,2,3,4) \qquad \text{(4-pr)}$$

which he calls the "fundamental equations of gravitation," and the four Lagrangian differential equations

$$\frac{\partial\sqrt{g}H}{\partial q_h} = \sum_k \frac{\partial}{\partial w_k}\frac{\partial\sqrt{g}H}{\partial q_{hk}}, \qquad (h = 1,2,3,4) \qquad \text{(5-pr)}$$

which he calls the "fundamental equations of electrodynamics or the generalized Maxwell equations." Hilbert then proceeds to discuss the concept of energy in the theory by looking at what we would now call Lie variations of the action, i.e. variations of the metric that arise from pure coordinate transformations. In the course of this discussion he introduces the notational "abbreviation"

$$[\sqrt{g}H]_{\mu\nu} = \frac{\partial\sqrt{g}H}{\partial g^{\mu\nu}} - \sum_k \frac{\partial}{\partial w_k}\frac{\partial\sqrt{g}H}{\partial g_k^{\mu\nu}} + \sum_{k,l}\frac{\partial^2}{\partial w_k \partial w_l}\frac{\partial\sqrt{g}H}{\partial g_{kl}^{\mu\nu}} \qquad (4)$$

which he calls "the Lagrangian variational derivative of $\sqrt{g}H$ with respect to $g^{\mu\nu}$." He observes that the fundamental equations of gravitation (4-pr) may now compactly be written as

$$[\sqrt{g}H]_{\mu\nu} = 0. \qquad \text{(8-pr)}$$

Hilbert's discussion of the energy concept in the proofs does not provide any further specifications of the Lagrangian H, although it does lead to a third axiom that restricts the covariance of the generally covariant equations (4-pr), (5-pr), by demanding that the physically admissible coordinates for the theory obey a set of equations that are not generally covariant.[5]

It is towards the end of the discussion of the problem of the energy concept and the significance of his third axiom, which runs until the bottom of page 7, that we find two passages missing in the proofs, since the top portion of the sheet that contains pages 7

[4] Hilbert tended to use equation numbers only for those equations that he actually referred to in his text. I will use his own equation numbers whenever an equation was given one and indicate this fact by adding "-pr" resp. "-pu" to the number, depending on whether it is the equation number used in the proofs or the published version, respectively.

[5] Contrary to the discussion in [12], this condition is conceptually very different from what we now call a coordinate condition since it pertains to *any* possible application of the field equations. In [19], such restricting equations are called "coordinate restrictions" as opposed to "coordinate conditions." Nonetheless, there is a significant difference between Einstein's use of "coordinate restrictions" prior to his final version of the general theory of relativity and Hilbert's third axiom in the proofs. Einstein used "coordinate restrictions" to derive field equations that are covariant only under a correspondingly restricted group of coordinate transformations. Hilbert kept the generally covariant field equations as fundamental field equations and only postulated a limitation of the physically admissible coordinate systems.

and 8 was cut off.[6] Without any further discussion of Hilbert's treatment of the energy concept (see [6] and [7]), I will assume that the missing portion on the top of p. 7, i.e. on the verso of the top of p. 8, is not in any way relevant to the question under investigation in this note. But what is missing on page 8?

On page 8 of the proofs, immediately following the excised portion, Hilbert asserts: "Since K depends only on $g^{\mu\nu}$, $g_k^{\mu\nu}$, $g_{lk}^{\mu\nu}$, the ansatz (17-pr) allows us to express the energy E [...] solely as a function of the gravitational potentials $g^{\mu\nu}$ and their derivatives, if only we assume L not to depend on $g_s^{\mu\nu}$, but only on $g^{\mu\nu}$, q_s, q_{sk}." In the next sentence, Hilbert states that he would make that latter assumption in the following.

We observe that the quantities K and L had not been used earlier in the proofs,[7] and we may conclude that K must have been introduced just before as a function of the components of the metric and its derivatives only, and that L must have been introduced just before as a function of the electromagnetic potential, its derivatives as well as of the components of the metric and its first derivatives, although the dependence of L on the derivatives of the metric is immediately assumed away for the rest of the text. We also observe that the previous page has an equation that is numbered (16-pr) and that the next line gives an equation that is numbered (18-pr). The equation with number (17-pr) is referred to a few pages later, on p. 11, where Hilbert writes that "because of (17-pr)" the fundamental equations of gravitation (8-pr) take the form

$$[\sqrt{g}K]_{\mu\nu} + \frac{\partial\sqrt{g}L}{\partial g^{\mu\nu}} = 0, \tag{26-pr}$$

and the fundamental equations of electrodynamics take the form

$$[\sqrt{g}L]_h = 0. \tag{27-pr}$$

Spelling out $[\sqrt{g}K]_{\mu\nu}$ in terms of the definition (4), Eq. (26-pr) reads

$$\frac{\partial\sqrt{g}K}{\partial g^{\mu\nu}} + \frac{\partial\sqrt{g}L}{\partial g^{\mu\nu}} - \sum_k \frac{\partial}{\partial w_k}\frac{\partial\sqrt{g}K}{\partial g_k^{\mu\nu}} + \sum_{k,l}\frac{\partial^2}{\partial w_k \partial w_l}\frac{\partial\sqrt{g}K}{\partial g_{kl}^{\mu\nu}} = 0. \tag{5}$$

Assuming that the missing piece introduced the quantities K and L by specifying H as some function of these quantities, $H = H(K, L)$, and taking into account that $L = L(g^{\mu\nu}, q_s, q_{sk})$ was assumed not to depend on $g_k^{\mu\nu}$ and $g_{kl}^{\mu\nu}$, we conclude that, in all probability, Eq. (17-pr) must have been of the form:

$$H = \zeta(K + L) \tag{6}$$

with some constant ζ that may well have been set equal to 1. Clearly, Eq. (27-pr) is consistent with this conclusion. We also note that later in the text the quantities K and L are referred to as "invariants" (L on p. 9 and on p. 10, K on p. 11).

Taking together these bits of information from the text of the proofs, we can draw the following preliminary conclusions about the content of the missing piece:

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[6] For a description of the physical appearance of the proofs, see [6, note 75].

[7] The choice of characters seems to have been motivated by alphabetical order. After denoting the generic "Hamiltonian function" as H, some invariant expression is denoted on p. 4 as $J^{(h)}$. Later, on p. 10, the electromagnetic field tensor is denoted by $M_{ks} = q_{sk} - q_{ks}$.

1. It must have contained an equation of the form (6) that was given the number (17-pr).
2. The missing piece introduced a quantity K in such a way that the definition or characterization of K, whatever it was, implied that $K = K(g^{\mu\nu}, g_l^{\mu\nu}, g_{kl}^{\mu\nu})$ is an invariant and only depends on the components of the metric and its first and second derivatives.
3. The missing piece introduced a quantity L in such a way that the definition or characterization of L, whatever it was, implied that $L = L(q_s, q_{sl}, g^{\mu\nu}, g_k^{\mu\nu})$ is an invariant and depends on the components of the electromagnetic vector potential and its first derivatives as well as on the metric components and its first derivatives.

It should be noted that these conclusions emerge from looking at the existing text of the proofs alone, without taking recourse to the published version or any other historical source.

4. What is contained in the published version

Let us now take further account of Hilbert's published version of his *First Communication* [18]. As was indicated above, the published version differs significantly from the proofs in several respects, the main difference being a completely revised discussion of the energy theorem. Specifically, with respect to the gravitational and electrodynamical field equations, however, the differences are not significant, as we will see, apart from the fact that the explicit evaluation of the variational derivative of the gravitational part of the Lagrangian K is found only in the published version and not in the existing part of the proofs. Whether it may have been on the missing part of the proofs will be discussed below.

The formulation of the first two axioms is the same, and in the published version, Hilbert again wrote down the fundamental equations (4-pr), and (5-pr), albeit in a slightly different form as

$$\frac{\partial\sqrt{g}H}{\partial g^{\mu\nu}} - \sum_k \frac{\partial}{\partial w_k}\frac{\partial\sqrt{g}H}{\partial g_k^{\mu\nu}} + \sum_{k,l}\frac{\partial^2}{\partial w_k \partial w_l}\frac{\partial\sqrt{g}H}{\partial g_{kl}^{\mu\nu}} = 0, \qquad \text{(4-pu)}$$

and

$$\frac{\partial\sqrt{g}H}{\partial q_h} - \sum_k \frac{\partial}{\partial w_k}\frac{\partial\sqrt{g}H}{\partial q_{hk}} = 0. \qquad \text{(5-pu)}$$

The equivalence of eqs. (4-pr) and (5-pr), with (4-pu) and (5-pu) is, of course, completely trivial but the form (4-pu), (5-pu) allowed Hilbert to introduce the abbreviated notation $[\sqrt{g}H]_{\mu\nu}$ and $[\sqrt{g}H]_h$ already at this point as the left hand sides of the "fundamental equations" (4-pu) and (5-pu).

The specification of the Lagrangian H in terms of a gravitational part K and an electromagnetic part L appears twice in the published version. The first time the relevant equation appears it is in a context that would fit quite naturally into the missing piece of page 8 of the proofs. The relevant passage reads [18, p. 402]:

> As far as the world function H is concerned, further axioms are needed to determine its choice in a unique way. If the gravitational field equations are to contain only second derivatives of the potentials $g^{\mu\nu}$, then H must have the form

$$H = K + L \tag{7}$$

where K is the invariant that derives from the Riemannian tensor (curvature of the four-dimensional manifold)

$$K = \sum_{\mu\nu} g^{\mu\nu} K_{\mu\nu} \tag{8}$$

$$K_{\mu\nu} = \sum_{\kappa} \left(\frac{\partial}{\partial w_\nu} \begin{Bmatrix} \mu\kappa \\ \kappa \end{Bmatrix} - \frac{\partial}{\partial w_\kappa} \begin{Bmatrix} \mu\nu \\ \kappa \end{Bmatrix} \right) + \sum_{k,\lambda} \left(\begin{Bmatrix} \mu\kappa \\ \lambda \end{Bmatrix} \begin{Bmatrix} \lambda\nu \\ \kappa \end{Bmatrix} - \begin{Bmatrix} \mu\nu \\ \lambda \end{Bmatrix} \begin{Bmatrix} \lambda\kappa \\ \kappa \end{Bmatrix} \right) \tag{9}$$

and where L only depends on $g^{\mu\nu}$, $g_l^{\mu\nu}$, q_s, q_{sk}.

Hilbert then adds the following sentence: "Finally, we will, in the following, make the simplifying assumption that L does not depend on $g_l^{\mu\nu}$." The physical size of the missing piece allows for some ten lines of text or the equivalent of some smaller number of lines of text plus a number of displayed equations, taking into account that a displayed equation would take up more than a single line of text.[8] In view of this restriction, the passage in the published version is clearly too long to be inserted into the missing piece of the proofs. However, we can easily cut down the passage to fit into the size of the missing piece as, e.g., with the following German sentence:

Wir machen im folgenden den Ansatz

$$H = K + L \tag{10}$$

wo K die aus dem Riemannanschen Tensor entspringende Invariante

$$K = \sum_{\mu\nu} g^{\mu\nu} K_{\mu\nu} \tag{11}$$

$$K_{\mu\nu} = \sum_{\kappa} \left(\frac{\partial}{\partial w_\nu} \begin{Bmatrix} \mu\kappa \\ \kappa \end{Bmatrix} - \frac{\partial}{\partial w_\kappa} \begin{Bmatrix} \mu\nu \\ \kappa \end{Bmatrix} \right) + \sum_{k,\lambda} \left(\begin{Bmatrix} \mu\kappa \\ \lambda \end{Bmatrix} \begin{Bmatrix} \lambda\nu \\ \kappa \end{Bmatrix} - \begin{Bmatrix} \mu\nu \\ \lambda \end{Bmatrix} \begin{Bmatrix} \lambda\kappa \\ \kappa \end{Bmatrix} \right) \tag{12}$$

bedeutet und L nur von $g^{\mu\nu}$, $g_l^{\mu\nu}$, q_s, q_{sk} abhängt.[9]

It seems perfectly natural to assume that this passage or some very similar variant of it was the missing piece on page 8 of the proofs. And, as already conjectured in [6, note 82], Hilbert himself may have cut out this piece from his proofs, perhaps to paste it into some other unknown manuscript of his, e.g. into the manuscript for his revised version.

As indicated above, the equation $H = K + L$ appears at one other place in the published version of Hilbert's *First Communication.* This passage reads [18, pp. 404f.]:

It remains to show directly how with the assumption

$$H = K + L \tag{20-pu}$$

[8] See [6, note 75], the length of the type area seems to vary slightly over the different pages of the proofs.

[9] In English: "We now make the *ansatz* [...] where K is the invariant that derives from the Riemannian tensor [...] and where L only depends on $g^{\mu\nu}$, $g_l^{\mu\nu}$, q_s, q_{sk}."

the generalized Maxwell equations (5-pu) put forth above are entailed by the gravitational equations (4-pu).

Using the notation introduced earlier for the variational derivatives with respect to the $g^{\mu\nu}$, the gravitational equations, because of (20-pu), take the form

$$[\sqrt{g}K]_{\mu\nu} + \frac{\partial\sqrt{g}L}{\partial g^{\mu\nu}} = 0. \tag{21-pu}$$

The first term on the left hand side becomes

$$[\sqrt{g}K]_{\mu\nu} = \sqrt{g}\left(K_{\mu\nu} - \frac{1}{2}Kg_{\mu\nu}\right), \tag{13}$$

as follows easily without calculation from the fact that $K_{\mu\nu}$, apart from $g_{\mu\nu}$, is the only tensor of second rank ("Ordnung") and K the only invariant, that can be formed using only the $g^{\mu\nu}$ and their first and second differential quotients $g_k^{\mu\nu}$, $g_{kl}^{\mu\nu}$.

And after this assertion, Hilbert adds the following comment as to the apparent equivalence of his equations to those published by Einstein:

The resulting differential equations of gravitation are, it seems to me, in agreement with the broad ("großzügigen") theory of general relativity established by Einstein in his later papers.

The reference to Einstein's "later papers" is specified in a footnote by citing all four of Einstein's November memoirs [14, 15, 16, 17], including the last one that was presented to the Berlin Academy only on 25 November [17]. The question arises whether the missing piece of the proofs could have contained equation (13), i.e. the explicit form of the variational derivative for some gravitational Lagrangian K. Specifically under the assumption that K was defined or characterized as the Riemannian curvature scalar, it would then have displayed what we now call the Einstein tensor with its trace term $-\frac{1}{2}Kg_{\mu\nu}$. This reading would allow revival of a speculation that a version of the theory as laid out in the proofs may then possibly have inspired Einstein to make the transition of his field equations of his second November memoir of 11 November 1915 [15] to those of his final November paper of 25 November 1915 [17] by adding a similar trace term to the matter term of his previous equation.

However, from the internal logic and structure of both the argument in the proofs and in the published version, this conjecture seems highly unlikely for the following reasons. In addition to equation (13) or some similar equation displaying the explicit form of the variational derivative of the gravitational part of the Lagrangian, the missing piece must still have contained an equation of the form (6), as in (20-pu), and some kind of characterization of the quantities K and L as discussed above on the basis of the proofs alone. In addition, it must also have contained some kind of characterization of the term $K_{\mu\nu}$ which appears in equation (13) but which had not appeared in the proofs before. In view of the physical size of the missing piece, the explicit form of the Ricci tensor $K_{\mu\nu}$, as in (12), could hardly have fitted on it in addition to equation (20-pu), as well as equation (13). Therefore, the quantity K must then have been defined or characterized without using its explicit form, maybe only with words ("die aus dem Riemannschen Krümmungstensor $K_{\mu\nu}$ entspringende Invariante K").

However, there are at least two arguments against the assumption that the missing piece contained equation (13) in addition to equation (20-pu) and some minimal information needed to introduce K and L.

1. Nowhere in the extant parts of the proofs does Hilbert calculate explicitly the result of the variational derivative or argues on this level. Indeed, in and of itself such an explicit calculation would be at odds with the general thrust of his communication which is to draw quite general conclusions from combining variational calculus and invariant theory. And in the published version, the explicit form of the variational derivative of the gravitational part of the Lagrangian is clearly directly motivated by Hilbert's comment on the presumed equivalence of his own equations with those of Einstein's November memoirs, specifically as it seems with the final ones of 25 November 1915.
2. The mathematical assertion captured by equation (13), i.e. the assertion that the Einstein tensor $K_{\mu\nu} - \frac{1}{2}K g_{\mu\nu}$ is obtained by a variation of the Riemann curvature scalar K with respect to the metric $g^{\mu\nu}$, must have been given with even less comments on how this result is obtained and on what assumptions are needed for its validity, as were given in the published version.

To elaborate on the second point, let me finally comment on the derivation of the Einstein tensor from a variation of the Riemann curvature. As pointed out in [5], the fact that Hilbert's assertion quoted above about the uniqueness of the Einstein tensor, if taken literally, is wrong, since there are many invariants that are of second rank and "can be formed using only the $g^{\mu\nu}$ and their first and second differential quotients." However, earlier on, Hilbert had also mentioned the condition that second derivatives are to be contained in the gravitational equations only linearly. This additional condition fixes the tensor to the form $K_{\mu\nu} - \alpha K g_{\mu\nu}$ with some undetermined factor α. This factor α is determined to be equal to $1/2$ if it is further assumed that the covariant divergence of the expression vanishes, an assumption that is never mentioned explicitly in the published version, although it is implied by the contracted Bianchi identities that follow from Hilbert's proto-version of Noether's second theorem in his published communication [6, notes 104 and p.564; 12]. The authors of [5] also point out that, while Hilbert asserts that the result follows "without calculation," he does give a more explicit derivation of the Einstein tensor in his 1924 republication of his *Communications* on the *Foundations of Physics* [20].[10] Nevertheless, we have contemporary evidence that may give a meaning to Hilbert's assertion. It is found in a letter by the mathematician Hermann Vermeil to Felix Klein, dated 2 February 1918.[11] In it Vermeil explicitly addressed the question how the result can be obtained "without calculation." The answer that he found goes like this:

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[10] I disagree with the claim [5] that the 1924 republication was primarily motivated by Hilbert's wish to correct some errors of his 1915 publication. As argued elsewhere [21], it was on the contrary Hilbert's intention to reaffirm his own priority of the field equations after Einstein in his 1923 papers on Eddington's unified field theory had arrived at equations that were essentially equivalent to the gravitational field equations of 1915 in variational form in the context of the unified field theory program.

[11] NSUB Cod. Ms. Klein 22B, f. 28. This letter was discussed extensively at a history of mathematics conference at Oberwolfach in May 2000 in which the Einstein-Hilbert competition was

Assuming that

$$[\sqrt{g}K]_{\mu\nu} \propto \sqrt{g}\left(K_{\mu\nu} - \alpha K g_{\mu\nu}\right) \tag{14}$$

which, as discussed, follows from Hilbert's assumptions if one also demands that second derivatives occur only linearly, Vermeil evaluated $[\sqrt{g}K]_{\mu\nu}$ (see (4) for the scalar $K = \sum_{\rho\sigma} g^{\rho\sigma} K_{\rho\sigma}$, see (8), and obtained

$$\begin{aligned}[\sqrt{g}K]_{\mu\nu} = K\frac{\partial\sqrt{g}}{\partial g^{\mu\nu}} + \sqrt{g}K_{\mu\nu} + \sqrt{g}g^{\rho\sigma}\frac{\partial K_{\rho\sigma}}{\partial g^{\mu\nu}} \\ -\sum_k \frac{\partial}{\partial w_k}\frac{\partial\sqrt{g}K}{\partial g_k^{\mu\nu}} + \sum_{k,l}\frac{\partial^2}{\partial w_k \partial w_l}\frac{\partial\sqrt{g}K}{\partial g_{kl}^{\mu\nu}}.\end{aligned} \tag{15}$$

Using $dg = -g g_{\mu\nu} dg^{\mu\nu}$, this turns into

$$\begin{aligned}[\sqrt{g}K]_{\mu\nu} = \sqrt{g}\left(K_{\mu\nu} - \frac{1}{2}K g_{\mu\nu}\right) \\ +\sqrt{g}g^{\rho\sigma}\frac{\partial K_{\rho\sigma}}{\partial g^{\mu\nu}} - \sum_k \frac{\partial}{\partial w_k}\frac{\partial\sqrt{g}K}{\partial g_k^{\mu\nu}} + \sum_{k,l}\frac{\partial^2}{\partial w_k \partial w_l}\frac{\partial\sqrt{g}K}{\partial g_{kl}^{\mu\nu}}.\end{aligned} \tag{16}$$

where all terms on the second line do not produce terms of the form (14).

While this derivation shows that Hilbert's claim in the published version about the derivation of the Einstein tensor is correct (granting that the postulate that second derivatives occur only linearly was implied) and credible, the question still remains as to why Hilbert should have done this derivation and included its result into the proofs without elaborating at all about the necessary steps and assumptions. Assuming that Hilbert added the explicit evaluation of $[\sqrt{g}K]_{\mu\nu}$ into the published version after seeing the explicit field equations of Einstein's final November paper, on the other hand, makes good sense. Let us not forget after all, that Hilbert in this context does cite Einstein's paper of 25 November.

5. Concluding remarks

What was on the excised piece? Merely requiring continuity with the remaining text constrains the possibilities quite considerably. It is highly unlikely that the missing part contained the explicit result of a variational derivative of the action with respect to the metric and specifically some version of the Einstein tensor. Consistency with the remaining text rather leads virtually uniquely to the conclusion that on the missing piece Hilbert had specified the Lagrangian of his variational principle as a sum of a gravitational part and a matter part, that he had further specified the gravitational part as the Riemann curvature scalar, and that he did so by giving the Ricci tensor in its explicit form.

It still remains true that the proofs of Hilbert's *First Communication* on the *Foundations of Physics* already contain the correct gravitational field equations of general

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a central topic of discussion. The argument is also presented, apparently without knowledge of Vermeil's letter, in [12, p. 611]. For Vermeil's role, see also the discussion in [10, p.417f.].

relativity in implicit form, i.e. in terms of a variational principle and the Hilbert action. The variational formulation is fully equivalent to the explicit Einstein equations published by Einstein a few days later, although the theory of Hilbert's proofs was not yet a fully generally covariant theory. It remains an interesting task to spell out in detail a scenario by which Hilbert would have overcome the restriction implied by the third axiom of his proofs following his own heuristics and logic of discovery.

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