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The Work of Tschirnhaus, La Hire and Leibniz on Catacaustics and the Birth of the Envelopes of Lines in the 17th Century

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Abstract

The aim of this paper is to examine the work of Tschirnhaus, La Hire and Leibniz on the theory of caustics, a subject whose history is closely linked to geometrical optics. The curves in question were examined by the most eminent mathematicians of the 17th century such as Huygens, Barrow and Newton and were subsequently studied analytically from the time of Tschirnhaus until the 19th century.

Leibniz was interested in caustics and the subject probably inspired him in his discovery of the concept of envelopes of lines.

Résumé

Le but de cet papier est celui d'analyser l'oeuvre de Tschirnhaus, La Hire et Leibniz sur la théorie de les caustiques, dont l'histoire est reliée a l'Optique géométrique. Cettes courbes furent etudiés par les mathématiciens les plus éminent de le XVIIe siécle comme Huygens, Barrow et Newton jusqu'a le XIXe siécle. Aussi Leibniz, interlocuteur habitué de Tschirnhaus, s'occupe de cette théorie et probablement les caustiques lui inspirent le méthode des enveloppes.

Sommario

In questo lavoro si prendono in esame i contributi di Tschirnhaus, La Hire e Leibniz alla teoria delle caustiche, la cui storia è strettamente legata all'Ottica geometrica. Considerate dai maggiori matematici del secolo XVI e XVII, come Huygens, Barrow e Newton, le caustiche furono trattate analiticamente da Tschirnhaus in poi, fino alla fine del secolo XIX. Leibniz venne probabilmente influenzato dallo studio delle caustiche a introdurre il metodo dell'inviluppo di linee.

1. Introduction

The theory of caustic curves was studied by many outstanding mathematicians in the 17th century, such as Christiaan Huygens, Isaac Barrow and Isaac Newton, and has been continuously studied through the 19th century[1]. The history of these curves is closely linked to geometrical optics. One of the most important questions in optics at that time concerned focusing the sun's rays in an attempt to control its effects: lighthouses, burning mirrors, reverberatory furnaces, and so on[2]. These curves introduced many important examples of envelopes into the history of mathematics, and the caustic of a circle (catacaustic) also became one of the first cases of rectificable curves, as Leibniz had tried to demonstrate before presenting his *Nova methodus* in *Acta eruditorum* in 1684.[3]

In fact, when caustic curves were discovered, they were determined as the envelope of the secondary light rays, either reflected (catacaustic) by a polished surface, or refracted (diacaustic) by a given surface. The first geometrical applications of calculus were in the rectification or the quadrature of each caustic. Furthermore, inverse problems were studied, such as reconstructing the shape (anticaustic) of a mirror by assigning both the position of the light source and the caustic itself, and so on. All this was done towards the end of the 17th century. As a consequence, the subject of caustics (deriving from the Greek $\kappa\alpha\iota\zeta\omega$: to burn) became a matter of interest in all its aspects: experimental, geometrical and algebraic, and the most eminent European mathematicians dedicated themselves to its study.

Ehrenfried Walter Tschirnhaus (1651–1708) in 1682 published a paper on the catacaustic curve. Huygens (1629–1695) showed interest in the same subject in his *Traité de la lumiére*, written in 1678, read to the Royal Academy of Sciences in the same year, and published in 1690. Using his wave theory of light he was able to describe caustic curves by means of the theory of involutes and evolutes, a theory which he had first presented in 1673 in his *Horologium oscillatorium*. He discovered that rays are always normal to the wave front and that they are tangent to the caustic. In this way, he demonstrated that the caustic is the evolute of the wave front. [4] In the reflection of a plane wave in a spherical mirror, he found that the caustic was an epicycloid, but he did not determine the curve for the equivalent case of refraction. He recalled that Barrow, in his *Optical Lectures* of 1669, which were devoted to the mathematical theory of image location, had determined the points on this curve. As Jakob Bernoulli later pointed out, Barrow's investigations into image location, where oblique pencils of rays were concerned, became a fundamental principle in optical theory. Indeed, his image point was nothing other than the tangent of the reflected or refracted ray to the caustic.

Even though a brief account of Tschirnhaus's discovery was given by Kracht and Kreyszig, it seems that the whole subject needs to be examined in more detail, in regard to its mathematical aspects[5]. Tschirnhaus's initial interest in such a question may have arisen (as Montucla seems to have thought) from his experiments with large burning mirrors, produced in one of his factories[6]. If this is the case, we are dealing with a

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[1] Matthes 1837. See also Shapiro 1990.

[2] Tschirnhaus 1691, 519, where he states: "Quoniam ergo vitri hujus ope ... ignis excitatur qui efficacia sua omnes alios nobis notos antecellit. [Due to this glass ... a fire burns with an effectiveness that was previously unheard of]."

[3] Gerhardt 1971, 493–96.

[4] Shapiro 1990, 156–159.

[5] Kracht and Kreyszig 1990, 20–22.

[6] Montucla 1802, 387–390.

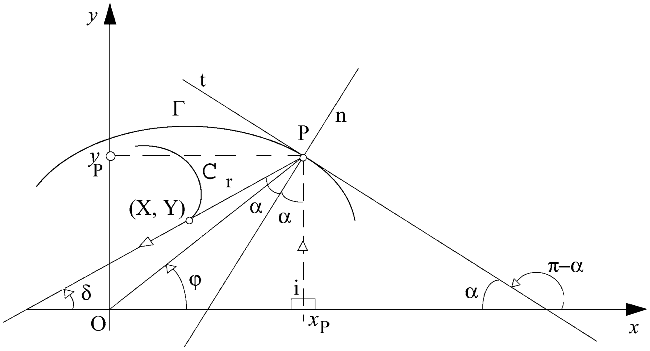

Fig. 1. The generation of catacaustic C by a mirror reflecting light rays parallel to Oy

mathematical discovery arising from the observation of a physical phenomenon (an optical one in this case).

It is our aim, therefore, to examine the way in which Tschirnhaus introduced the concept of catacaustic, his initially erroneous construction of the curve, the subsequent amendments he himself made, and those made by Johann Bernoulli and Philippe de La Hire. The latter was one of the members of the Royal Academy of Sciences who, with Jean Dominique Cassini and Edme Mariotte, was charged with examining Tschirnhaus's results. We will also examine Leibniz's demonstration of the rectificability of the catacaustic, which he attempted without the help of the newly discovered differential form of calculus. Indeed, it was only in 1692 and 1694 that Leibniz presented in *Acta eruditorum* his general method for calculating the envelopes of a family of curves[7].

2. The equation of the catacaustic

It will be useful if we first determine the equation of the catacaustic in respect to a given curve Γ in modern form. For the sake of brevity we will treat a pencil of parallel light rays from an infinitely distant source whose direction will be assumed to be the y-axis.

Let P be the point (Fig. 1) of the curve Γ, $y = y(x)$, from which the incident ray i is reflected into the ray r, and let t and n be the tangent and perpendicular to Γ at P.

The equation of the reflected ray r is:

$$Y - y_P = \tan\delta \cdot (X - x_P).$$

X and Y are the coordinates of any point of r whose slope is

$$\tan\delta = \frac{1 - \tan^2\alpha}{2\tan\alpha} = \frac{y_P'^2 - 1}{2y_P'},$$

[7] Leibniz 1990, 967–969.

where $y'_P = dy_P/dx_P = \tan\alpha$. The equation of r thus becomes

$$2y'_P(Y - y_P) + (X - x_P)(1 - y'^2_P) = 0. \tag{1}$$

Since the catacaustic is the envelope of the reflected rays, its parametric equations can be obtained by eliminating x_P and y_P between (1) and its derivative (2) with respect to x_P:

$$2y''_P(Y - y_P) - 2y''_P y'_P(X - x_P) = 1 + y'^2_P. \tag{2}$$

In this way one obtains

$$X = x_P - \frac{y'_P}{y''_P} \tag{3}$$

$$Y = y_P + \frac{1 - y'^2_P}{2y''_P}, \tag{4}$$

namely, the parametric equations of the envelope, where x_P is the abscissa of the incident ray, and y_P, y'_P, y''_P are known functions of x_P.

From (3) and (4) one can see that, if the mirror, namely the curve of equation $y_P = f(x_P)$ is given, one will immediately obtain the parametric equations of the catacaustic. On the contrary, if the equations of the catacaustic are given, we can calculate the shape of the mirror from (3) and (4), by integrating a nonlinear second order differential equation for $y_P(x_P)$ (the inverse problem of the caustic).

As for the direct problem, if Γ is a circle of radius R, centered in the origin of reference, one has $x_P = R\cos\varphi$, $y_P = R\sin\varphi$, and so:

$$y'_P = -\frac{1}{\tan\varphi} \text{ and } y''_P = \frac{1}{R\sin^3\varphi}, \tag{5}$$

whence (3) and (4) become

$$X = \frac{R}{4}(\cos 3\varphi + 3\cos\varphi) \tag{6}$$

$$Y = \frac{R}{4}(\sin 3\varphi + 3\sin\varphi). \tag{7}$$

To understand what (6) and (7) really mean, it is necessary to remember that if a circle C (*rolling curve*) of center Ω and radius r, rolls without slipping on a second fixed circle (*base curve*) C' of center A and radius $(A + r)$, and if Q is any point of the first circle, the locus of Q will have the parametric equations

$$X = (A + r)\cos\varphi + r\cos\left(\frac{A + r}{r}\varphi\right) \tag{8}$$

$$Y = (A + r)\sin\varphi + r\sin\left(\frac{A + r}{r}\varphi\right), \tag{9}$$

namely, an epicycloid. In the special case $A = R/2$ and $r = R/4$, (6) and (7) will coincide respectively with (8) and (9), and then the catacaustic of a circular mirror of radius R is a bicuspidate epicycloid (*nephroid*), produced by a peripheral point of the circle of radius $R/4$ rolling without slipping on a fixed base-circle having radius $R/2$. Its Cartesian equation can be obtained by squaring and adding (8) and (9):

$$X^2 + Y^2 = \frac{R^2}{8}(5 + 3\cos\varphi\cos 3\varphi + 3\sin\varphi\sin 3\varphi)$$

and finally

$$[4 \cdot (X^2 + Y^2) - R^2]^3 = 27R^4X^2,$$

which is a sixth order curve. However, the curve calculated by Tschirnhaus was wrong, as he had obtained the quartic:

$$Y = \frac{1}{2}\left[\sqrt{1 - X^2} + \sqrt{X - X^2}\right].$$

If the reflecting mirror is the parabola $y_P^2 = 2px_P$, with $2p$ the *latus rectum*, we obtain

$$y'_P = p/y_P,\ y''_P = -p^2/y_P^3\ ,$$

and then (3) and (4) will give

$$X = 3x_P \tag{10}$$

$$Y = \frac{y_P\left(3p^2 - y_P^2\right)}{2p^2}. \tag{11}$$

By substituting $y_P^2 = 2px_P$ in the right hand side of (11) and by using (10), the catacaustic curve with respect to a parabolic reflecting mirror is easily found to be

$$54pY^2 = X \cdot (9p - 2X)^2,$$

or equivalently

$$Y = \pm\,(9p - 2X)\sqrt{\frac{X}{54p}}.$$

This is a double-branched curve (see Fig. 2) which was studied by Johann Bernoulli, by the Marquis de l'Hospital and, after them, by Eugène-Charles Catalan in 1832[8].

It is called either a Catalan trisettrix or Tschirnhaus cubic.[9]

[8] Of course, if the rays were parallel to *Ox*, no caustic would be the locus of the reflected rays because these would degenerate in the focus of the given parabola.

[9] Bernoulli 1742, 109–112, 471–472. L'Hospital 1696, 109–112. Teixeira 1971, 353–355.

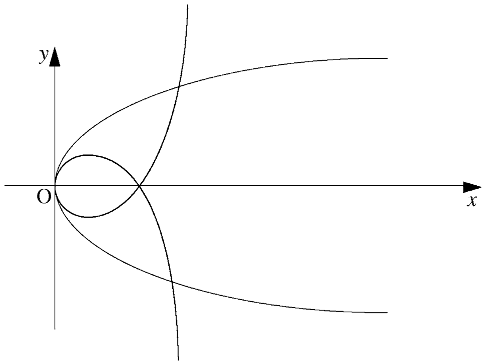

Fig. 2. Parabolic mirror reflecting rays parallel to *Oy*, and the relevant caustic

3. Tschirnhaus and catacaustic curves

The catacaustic of the spherical mirror appeared for the first time in Huygens's (1629–1695) *Traité de la lumière*.[10] Tschirnhaus's first paper describing the same curve dates back to 1682, but the *Traité* had been finished in 1678, when the author read it at the Royal Academy of Sciences. In his preface[11] Huygens tells us:

> I wrote this Treatise during my sojourn in France twelve years ago, and I communicated it in the year 1678 to the learned persons who then composed the Royal Academy of Science, to the membership of which the King had done me the honour of calling me. Several of that body who are still alive will remember having been present when I read it, and above the rest those amongst them who applied themselves particularly to the study of Mathematics; of whom I cannot cite more than the celebrated gentlemen Cassini, Römer, and de la Hire. And although I have since corrected and changed some parts, the copies which I had made of it at that time may serve for proof that I have yet added nothing to it save some conjectures touching the formation of Iceland Crystal, and a novel observation on the refraction of Rock Crystal.

The drawing of the catacaustic (Fig. 3) can be seen on the last page of his *Traité*, and its determination is explained by his wave fronts:

> The curve AFE can be seen in smoke, or in flying dust, when a concave mirror is held opposite the sun. And it should be known that it is none other than that curve which is described by the point E on the circumference of the circle EB, when that circle is made to roll within another whose semi-diameter is ED and whose center is D. So that it is a kind of Cycloid, of which, however, the points can be found geometrically. Its length is exactly equal to $\frac{3}{4}$ of the diameter of the sphere, as can be found and demonstrated by means of these waves, nearly in the same way as the mensuration of the preceding curve;

[10] Shapiro 1973, 263–265.
[11] Huygens 1962, page V.

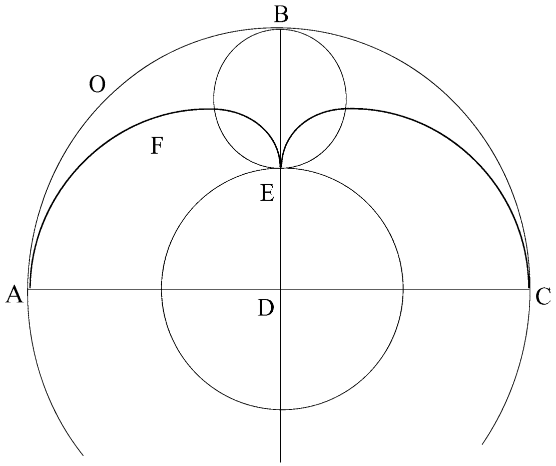

Fig. 3. Following Huygens, *Treatise*, p. 127

though it may also be demonstrated in other ways, which I omit as outside the subject. The area AOBEFA, comprised between the arc of the quarter-circle, the straight line BE, and the curve EFA, is equal to the fourth part of the quadrant DAB. [12]

When Huygens read his *Traité* publicly at the Academy on May 13, 1678, Tschirnhaus was travelling in Italy, where he received – we do not know how or when – news about the *Traité*.[13] In a letter[14] to Leibniz, on April 7, 1681, in which Tschirnhaus wrote about his discovery of the catacaustic of a circle, he asked Leibniz if someone had already discovered a solution to the problem, perhaps Huygens, whose *Dioptrica*[15] he believed had appeared not long before: "I wish to know if such things have been discovered by some mathematician, in particular by Mr. Huygens, whose Dioptrica now happens to be brought to light." Whatever the case, it would appear that the catacaustic of a circle was discovered by Tschirnhaus during his experiments with burning mirrors in his factory. Moreover, Descartes also referred to it in his *Géométrie* when dealing with ovals.[16]

As already mentioned, the first published reference to catacaustics was in a note outlining Tschirnhaus's curriculum in *Acta eruditorum* of 1682, but Tschirnhaus had dealt with the subject earlier, in a private letter on April 7, 1681 to Leibniz.[17]

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[12] Huygens 1962, 127–128.

[13] An account of the fate of Huygens's *Traité* is given by Shapiro 1973. The *Traité* attracted considerable attention from his contemporaries, and it was published 11 years after his initial presentation in 1679. Philippe de La Hire, who had heard the original presentation wrote to Huygens (on January 17, 1691) expressing his opinion that "... this part of your treatise seems to be a chef d'oeuvre both in mathematics and physics...".

[14] Gerhardt 1971, 484.

[15] During his life time Huygens's *Dioptrica* remained in an unpublished manuscript. It was bequeathed to the Leiden Library and published in 1703. Shapiro 1984, 408.

[16] Adam and Tannery 1897–1913.

[17] Tschirnhaus 1682, 364.

Kracht and Kreyszig gave a short account of Tschirnhaus's discovery,[18] but a better interpretation of the subject might be to consider the whole affair as an academic dogfight between the supporters of Leibniz's analytical techniques, such as Johann and Jakob Bernoulli, l'Hospital, Varignon, *chargé de la défense des infiniment petits*[19] and their opponents.

In *Acta eruditorum* in 1682, Tschirnhaus reported his discovery of the catacaustic to the Royal Academy of Sciences in Paris during his admittance speech to the Academy.[20] He had, however, previously disclosed it in a letter to Leibniz, written in that strange language, common in scientific correspondence, which was a mixture of German and Latin:

> I have discovered a new way of manufacturing large mirrors with little effort and more impressive results than those at the Paris Library. I found that from a spherical surface, carved in wood, it is possible to obtain glass surfaces of a round or plane shape, and of such a size that they function as burning glasses. Thanks to my experiments on the subject, I have succeeded in solving the following problem. If *AEC* (Fig. 4) is an arc of a circle and the lines *DE* signify the sun's rays and *EF* the reflected ones, the problem is to determine the curve *FRC* resulting from the crossing of the reflected rays.

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[18] Kracht and Kreyszig 1990. We support Kracht and Kreyszig in rejecting the charge of plagiarism against Tschirnhaus raised by Uylenbroek in 1830, (quoted by the Authors). Uylenbroek implied that Tschirnhaus obtained his results on catacaustics from Huygens, while in Paris, 1678. This is patently impossible, since Tschirnhaus was travelling in Italy from 1677 to 1679.

[19] In a letter to Johann Bernoulli, 6 August 1697, Varignon wrote: "M. le Marquis de l'Hospital est encore à la campagne de sorte que je me trouve seul ici chargé de la défense des infiniment petits, dont je sui vray martyr tant j'ay desja soutenat d'assauts pour eux contre certains mathématiciens du vieux stile ... [The Marquis de l'Hospital stays in the country yet, and I have been left alone in supporting the new calculus against all the attacks from the old fashioned mathematicians ...]". Among the old fashioned mathematicians ("anciens") there was Michel Rolle, still remembered for his theorem on derivable functions named after him (presented in his booklet *Méthode pour résoudre les égalités*, 1691). He judged infinitesimal calculus to be "a stack of ingenious errors".

[20] Tschirnhaus 1682. Tschirnhaus was born at Kieslingswalde, near Görlitz in 1651, and studied mathematics, philosophy and medicine at Leiden University. After having taken part in some military campaigns with the Dutch army, he started to travel around Europe in 1675, visiting England, where he met, among others, Wallis, Collins, Papin, Boyle and Oldenburg, the secretary of the Royal Society. Having been provided with some letters of introduction to Huygens and Leibniz by Oldenburg [Kracht and Kreyszig 1990, 16–35], in the summer of that year he met both in Paris and co-operated scientifically for more than one year with Leibniz, on algebraic problems. Between 1677 and 1679 he travelled in Italy, visiting Rome and Milan. Returning to Paris in 1682, he became a member of the Academy of Sciences and announced his discovery of catacaustics. In 1687 he published *Medicina mentis et corporis* (2nd ed. 1695), a book similar to Nicolas de Malebranche's *Recherche de la Vérité* [Montucla 1802, 387–393]. Later in life he retired to his estate, where he continued his studies and managed his glass factory, which produced very large burning mirrors and porcelain, whose manufacturing process was kept secret till his death in Dresden in 1708. See Montucla 1802, 387–393.

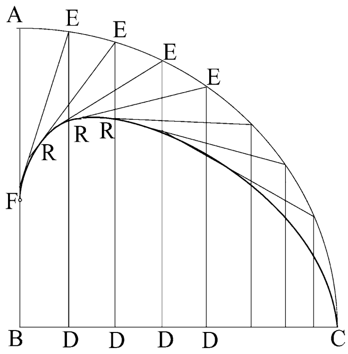

Fig. 4. Following Gerhardt 1971, Fig. 108

In using the word "crossing", Tschirnhaus did probably not realize that the caustic was in reality an envelope: he conceived the caustic as a generalization of the focus, something similar to a locus obtained through the crossing of rays. Later he answered the question that he himself had raised: "I discovered that this curve is a geometrical one, as described by Descartes. And so, given the curve of the mirror, one will arrive at the geometric curve *FRC* too".

In the last part of the letter he mentioned a more important and general fact:

> Furthermore, I have discovered another approach, much easier than Sluse's, and it is impossible to think of another which is easier than mine, consisting as it does of simply drawing tangents to geometrical and mechanical curves.

This new method, together with the quadrature of algebraic curves, probably weakened the friendship between Tschirnhaus and Leibniz. According to Kracht and Kreyszig,[21] Leibniz, fearing that he would be overtaken in the field of tangent methods, decided to publish his *Nova Methodus* in *Acta eruditorum* in 1684. This coolness can also be inferred from Christian Wolff's recollections. As a philosopher and mathematician, Wolff was a professor at Halle and Marburg universities, and he had interviewed Tschirnhaus in 1705, three years before Tschirnhaus's death.[22] We read:

> For the Easter fair, 1705, I made a journey to Leipzig to speak to Mr.Tschirnhaus. We discussed some passages from his "Medicina mentis" which I had had some trouble with, adding how I had succeeded in explaining them. He was very happy about this, but when I asked him about the nature and establishments of the axioms, he acknowledged that this was indeed the heart of the matter. As I had eagerly attempted to learn calculus (that time little known), I then asked him what I could do to improve myself. He did not give any definite advice, but highlighted that he was basing his advice on some points from Barrow's "Geometrical lectures", even though this latter only gave a summary of the problem rather than explain a suitable way of dealing with the method. He himself was planning to explain the correct approach in a successive tome of his treatise, where the rules explained

[21] Kracht and Kreyszig 1990, 16–17.
[22] Gerhardt 1971, 422–23.

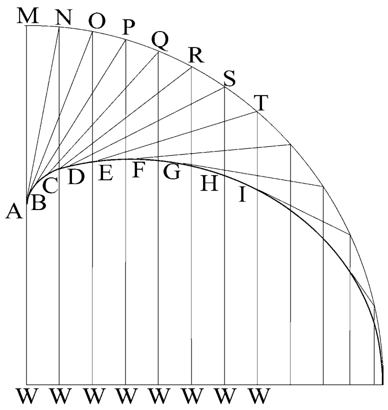

Fig. 5. Following *Acta eruditorum* 1682, Table XIX, Fig. 1

> in the first would be applied to mathematics. In a third volume the same rules would be applied to physical aspects, a feat which would astonish everybody.
>
> Before I left him, Tschirnhaus suggested some books for me to read to improve my knowledge: the aforementioned Barrow, "Analysis" by Nieuwentijt and "Algebra" by Ozanam.

Thus, Tschirnhaus, twenty years after the publication of *Nova methodus* by his friend Leibniz and after all the discoveries made in the calculus, advised Wolff to study his treatise and some old-fashioned authors such as Nieuwentijt and Ozanam, both opponents of Leibniz. It should also be noted, however, that Tschirnhaus had also been an opponent of the new Leibnizian calculus: "The introduction of this language of monstrous signs was entirely needless and the methods in use before that time would have allowed, after some improvement, all the problems of higher mathematics to be solved."[23]

4. The report of 1682 and Johann Bernoulli's amendment

Tschirnhaus's report in *Acta* of 1682, was taken from a second letter to Leibniz on May 27, 1682.[24] Leibniz's answer will be examined later. The main idea of the paper is that the curve given by the crossing of the rays reflected from a concave surface and struck by a pencil of parallel rays, is tangent to the reflected rays. The curve can easily be drawn and rectified.

> Let NB, OC, PD, ... be the rays reflected from a concave surface struck by the parallel rays MW, NW, ..., and A, B, C, D, E, F, ... the vertices of the polygon (Fig. 5) having infinitesimal sides like AB, BC, CD, DE, ...

[23] Ibid., 422.
[24] Ibid., 487–490.

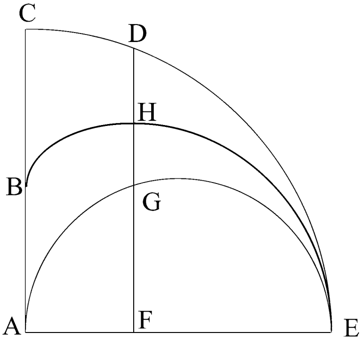

Fig. 6. Following *Acta eruditorum* 1682, Table XIX, Fig. II

> If such distances MN, NO, OP, PQ , and so on, are deemed as infinitely small, the polygon A , B, C, D, E, F, etc., will become a curved line, whose tangents will be the reflected rays; and the point A will be the focus where the reflected rays will all burn together inside the cavity.[25]

Subsequently Tschirnhaus describes the catacaustic more precisely as a line which touches all the reflected rays, while in the quoted letter of the previous year he incorrectly considered it as the crossing of the reflected ones. The caustic curve is one of the first examples of an envelope of a family of curves, even if the general method for calculating it would only be given in some papers by Leibniz that appeared in *Acta eruditorum* in 1692 and 1694.[26]

After having given this qualitative description of the curve, Tschirnhaus provided a geometrical construction in the case of a spherical reflecting mirror (Fig. 6). Given a quarter of a circle $ACDE$, he drew the half-circle AG with diameter AE, and then an arbitrary straight-line FD, parallel to AC .

He asserted that one point of the caustic would have to be the middle point of DG and that the burning point would fall a fortiori on the middle of AC, point B. As we can see, this way of drawing caustics gave rise to the objections of many contemporary mathematicians, but Tschirnhaus accepted them quite late.[27]

The problem attracted the attention of Johann Bernoulli, who corrected Tschirnhaus's mistake and calculated the principal caustics of mirrors. We do not know whether this was done under Leibniz's insistence or not, but, whatever the case, Johann Bernoulli included[28] all his relevant research in his *Lectiones mathematicae de methodo integralium* of 1691–92 written for G. de l'Hospital. This is a peculiar treatise for many reasons. First, only a very small part is devoted to integration: indeed in 1692 only rational algebraic functions were integrable. Second, the work is full of practical examples, and 11

[25] Tschirnhaus 1682, 364.
[26] Leibniz 1971, **II**: 166; **III**: 967–969.
[27] Montucla 1802, 389.
[28] Bernoulli 1692, 30.

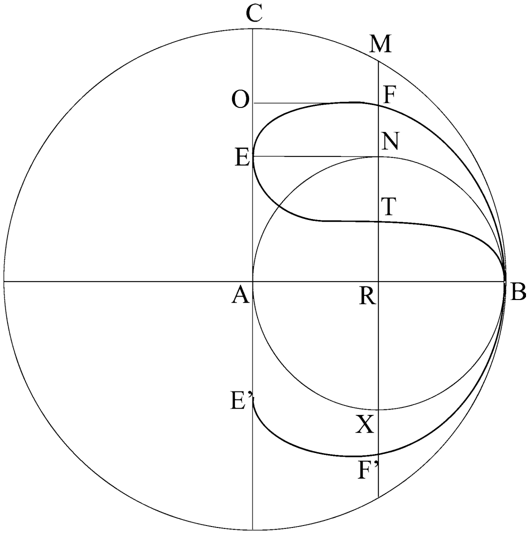

Fig. 7. Following Bernoulli, *Lectio* XXVII, Fig. 105

of the 57 lessons are devoted to caustics, a subject with no integration at all. The wealth of practical applications would have made the book invaluable, had it been published in 1692. However, it remained in manuscript for 50 years, and when it was finally published, it was out of date. Johann's work was, of course, immediately echoed by his brother Jakob, who sent the *Acta eruditorum* three papers on the same subject. [29]

Another eminent scholar in this field was the Marquis de l'Hospital, who, in June 1696, anonymously published his *Analyse des infiniment petits pour l'intelligence des lignes courbes*. The success of this book, which was to become seminal in the diffusion of the new calculus, roused Johann Bernoulli's jealousy. According to Johann Bernoulli,[30] Tschirnhaus extended the distinctive character of the caustic's focus to points which fell exactly in the middle of AC. Bernoulli argued that Tschirnhaus probably did not realize that according to his geometrical construction (Fig. 7), every other point of the caustic could be drawn by taking the semicircle AXB into account. Thus the middle of MX, point T would be a point of the caustic too. But then, the lower caustic would be ETB, which is clearly wrong: the right one is $BF'E'$, symmetrical to BFE in respect to AB. Furthermore, Bernoulli demonstrated that the correct caustic could not be drawn by ruler and compass, because of the presence of irrationals with cubic roots: "and then these two curves are not only different, but of an entirely different nature."[31]

At the end of his paper Tschirnhaus took an arbitrary curve CFE as an example (Fig. 8), "either Geometrical, according to Mr. Descartes, or Mechanical, as the Cycloid, the Quadratrix, and so on, or even a hand-sketched curve", and he established that the curve

[29] Bernoulli, Jakob 1692a, 1692b, 1693.
[30] Bernoulli 1742, 469.
[31] Ibid., 469.

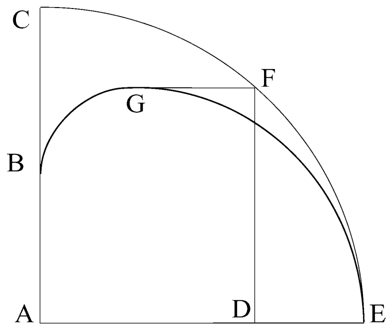

Fig. 8. Following *Acta eruditorum* 1682, Table XIX, Fig. III

BGE is formed both by rays emanating from the source and by their reflections, as can be seen in Fig. 8. Finally, he concluded that the portion GE of the curve would be equal to the sum of both the arc-lengths GF and FD. Therefore, the entire curve BGE would equal the sum of the arc-lengths CA and AB. This, however, he did not demonstrate.

5. The papers of 1690

For the next eight years, while engaged in his *Medicina mentis*, Tschirnhaus did not work on caustics, though he probably did receive several queries about the 1682 paper. After publishing his *Medicina mentis* in 1687, however, he returned to the old subject once more in 1690, publishing another, more compelling paper in the *Acta eruditorum*[32]: "Since many people have asked me many questions about the method briefly exposed in the Acta of November, 1682, I have decided, despite being very busy, to give a short account of it."

The main amendments were as follows:

1. Returning to the subject of 1682, Tschirnhaus describes, rather fleetingly, a general method for calculating the caustic of any given plane curve. To achieve this aim, given (Fig. 9) a curve ALM, lines BL, CM, ..., and an incident pencil of parallel light rays, he calculated the curve AIK as the locus of crossing points of all the reflected rays such as LF and MG.

Furthermore he evaluated the length of the reflected ray $LI = p$, considering the two reflected ones, LB and MC, as being infinitely close to each other; if $BL = y,\ BC = o$, $NM = e$, $BD = b$, $CE = d$, he argued that "using well known analytical procedures we can neglect the upper powers of the infinitely small o and e quantities". In this way the relationship for p will be

$$p = \frac{oyy + 2dey - ddo \text{ in } yy + bb}{dy - by - be \text{ in } -2yy - 2ey - 2bd},$$

[32] Tschirnhaus 1690a, 68–73.

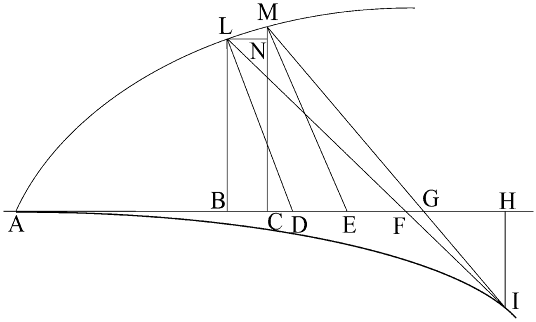

Fig. 9. Following *Methodus*, p. 69, Fig. I

or, in modern notation

$$p = \frac{\left(oy^2 + 2dey - d^2o\right) \cdot (y^2 + b^2)}{(dy - by - be) \cdot (-2y^2 - 2ey - 2bd)}.$$

It should be noted that Tschirnhaus calculated the length p of a segment of the reflected ray. In this way he was able to construct the caustic point by point. This last formula is obtained using his own method[33] of tangents, stemming in some way from Barrow's infinitesimals. He did not explain it in any detail, but he did give an example[34] where the ratio o/e for a curve is obtained – rather than p as might be expected - which today we would say is "implicitly" given by the equation:

$$y^3 + axy - a^3 - x^2y - x^3 - ax^2 = 0.$$

Here, however, he made a mistake by obtaining:

$$\frac{o}{e} = \frac{3y^2 + ax - x^2}{3x^2 + 2ax - ay}$$

where "$+2xy$" is missing from the denominator.

2. The author applied his formula when the (meridian) curve of the mirror is a parabola, a hyperbola and (as in his previous paper) a circle.

When the equation of the parabola is $2rx = y^2$, where $2r$ is the *latus rectum*, the subnormal will be a constant equal to half of the *latus rectum*, then $d = b = r$. Thus

$$p = \frac{oy^4 + 2rey^3 + 2r^3ey - r^4o}{2rey^2 + 2r^3e}.$$

Dividing by e, and $o/e = r/y$, one gets

$$\frac{y}{r} = \frac{2rpy^2 + 2pr^3 - 2ry^3 - 2r^3y}{y^4 - r^4}.$$

[33] See the letter to Leibniz on April 7, 1681, footnote 14.
[34] Tschirnhaus 1690a, 69–70.

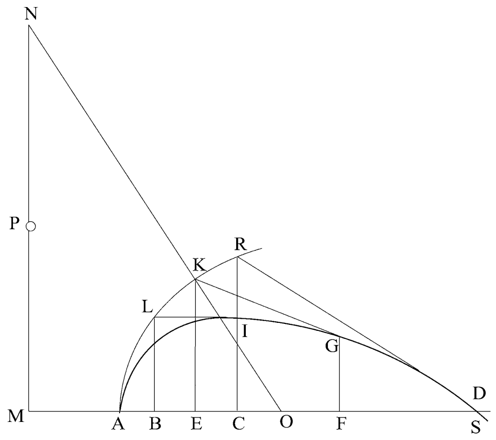

Fig. 10. Following Tschirnhaus *Acta eruditorum*, 1690, Fig. 2

Whence, by cross multiplying one obtains

$$p = \frac{y^3 + r^2 y^2}{2r^2},$$

which gives the length of the reflected ray as p, if y is the length of the light ray falling on the parabola.

Tschirnhaus next described how the caustic of a parabolic mirror can be drawn (Fig. 10):

> Given the parabolic arc *ALKR*, let the straight line *KO* which is normal to the parabola be extended until it meets *N* on *MN*, which is parallel to *KE*; if *AE* =*AM*, half of *MP* is always equal to *KG* , which itself touches the curve *AIGDS* at *G*, and, with *KO*, makes the angle *GKO* which is the same as *OKE* . The following consequences hold: 1) The curve *AIG* will always equal the sum of the lengths of *EK* and *KG*, i. e., of the crossing and reflected rays. 2) If *B* is the focus of the parabola, and *BC* is equal to one half of the latus rectum, *CI* will be the maximum height of the curve, and in the same way will equal one half of the latus rectum. 3) The curve *AIGDS*, as far as point *D,* where it is cut by the axis, [has a length which] is triple *CR*, passing through the vertex of the curve as explained in item 1.

When dealing with the equation of a hyperbola, $2rx + x^2 = y^2$, in a similar way, where the latus rectum is equal to the transverse axis, he found for p the value:

$$p = \frac{2y^3 + rry}{2rr}.$$

3. For the caustic of a circular mirror, Tschirnhaus presented a drawing (Fig. 11) which was different from that of 1682, and now correct. Based upon the general formula of p, it showed that the reflected ray segment is equal to one half of the incident ray segment enclosed in a quarter of the circle, namely, $FE = \frac{1}{2}EC$, or $p = y/2$. We can

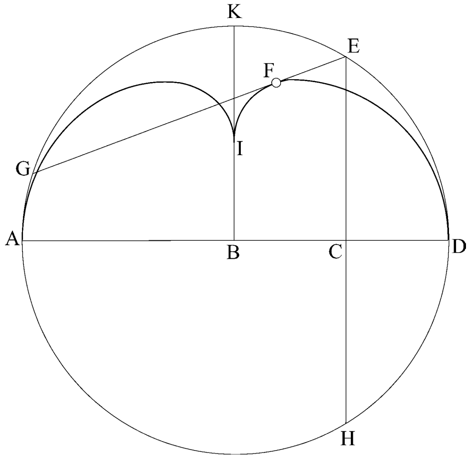

Fig. 11. Following *Methodus curvas* p. 71, Fig. 3

neither exclude nor affirm that the author knew de la Hire's work of 1730, which was certainly written some time after 1687, and in which Tschirnhaus's erroneous explanation of 1682 is corrected. Tschirnhaus wrote[35]:

> As far as the circle is concerned, Mr. Bernoulli, a very proficient man in such studies and well renowned for his researches, has recently highlighted that the curve, formed by the reflected rays, seems to be of the sixth degree, while I deduced last time it should be of the fourth one. [...] and here I am amending the mistake, offering a completely new analytical approach not only for finding the curve, but also for its geometrical drawing, which – on the contrary – is very difficult to obtain from the sixth degree equation with all its implicitness.

Bernoulli's *observatio*, however, would not appear in *Acta eruditorum* until two years later (January 1692). Consequently, Tschirnhaus probably made the amendment using his own methods.

Whatever the case, at the end of his paper, Tschirnhaus took the general Cartesian equation of the conic sections:

$$y^2 = rx + rx^2 + \frac{rx^2}{q},$$

and obtained for p:

$$p = \frac{4qy^3 \pm 4ry^3 + qr^2y}{2qr^2}.$$

For the special case of the circle $2rx - x^2 = y^2$ he obtained

$$p = \frac{4y^3 - 4y^3 + r^2y}{2r^2},$$

[35] Ibid., 71.

and then, since $q + r = 0$, $p = y/2$. Except for the parabolic mirror (and even here with some obscure and ambiguous notation), Tschirnhaus did not obtain the caustic equation, but, following both a synthetic and a Cartesian approach, he disregarded Leibnizian calculus. In his paper one can find neither reference axes, nor coordinates. Furthermore, it seems that he never mastered analytic geometry. His third paper,[36] devoted to the geometric properties of the circle's catacaustic (an epicycloidal curve nowadays called a nephroid) came out in the issue of *Acta eruditorum* of April 1690, almost immediately after the second one, with the following motivation:

> Nowadays several scholars carefully study the geometric curve called a "cycloid": nevertheless, the first to show that such a curve describes [37] itself through its own evolution, was the famous Christiaan Huygens, in his Horologium Oscillatorium. But this characteristic is so peculiar that the supreme mathematician in the same book asks himself if – beyond this curve – any other line could have the same property. As a consequence, I deemed that scholars would not disregard my discovery of the same property in a curve belonging to the family of the cycloid, but different from it, whose points can be determined geometrically. This curve is generated by the sun's rays which are reflected by a spherical mirror. The drawing, which I explained in the last issue of Acta eruditorum in February, established the theory of the curves formed by such rays. Even if its demonstration is a bit longer than for other lines, the treatment I am going to give doesn't require any special knowledge of algebra from the reader, and I shall be thankful to all friends who compelled me to issue it.

First of all, he showed (Fig. 12) that, if LN is a chord parallel to the rays, and if point G belongs to the caustic EGB, then LG will be equal to one half of LP. As a first step he showed that the arc-length HI is triple that of KL; as a second step those of GL and IG are in the same ratio as KL to HI; then, if OP approaches zero,[38] LG and GH will be in the same ratio, and their sum will be LN. LG will then be one-fourth of LN, and, as a consequence, one-half of LP. After this Tschirnhaus showed that, given the caustic INF of the circle $ALFD$ (Fig. 13), its evolute will be the caustic FPM of the circle MGH, the new cuspidal axis being rotated by 90° to the old one.

After having recalled Newton's investigations into the properties of the cycloid in *Principia*,[39] he showed that his caustic could be considered an epicycloid generated by a circle rolling on another one, and with radii in the ratio of 2 : 1. Finally he stated 16

[36] Tschirnhaus 1690b, 169–172.

[37] Tschirnhaus refers to the fact that Huygens in the treatise of 1673 had shown that the evolute of a cycloid is a cycloid too, which would be proven analytically by Euler in 1764.

[38] The original text of this critical passage is: *si concipiantur radii MK & NL indefinite parvo intervallo distare* [if the rays MK and NL are supposed to be distant from each other by an indefinitely small interval].

[39] In Book I, Sect. X, Newton quotes Huygens, and in connection with the acceleration caused by gravity on the movement of the bodies, he analyzes the main properties of many cycloidal curves, and evaluates the epicycloidal arclength. See also Book II, Sect. VI.

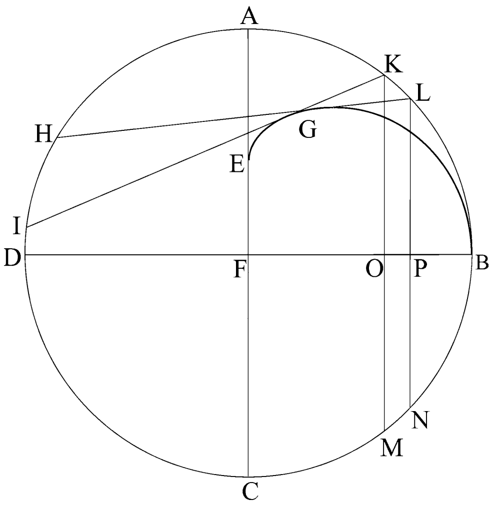

Fig. 12. Following *Curva Geometrica ...*, *Acta eruditorum*, 1690, Fig. 1, p. 169

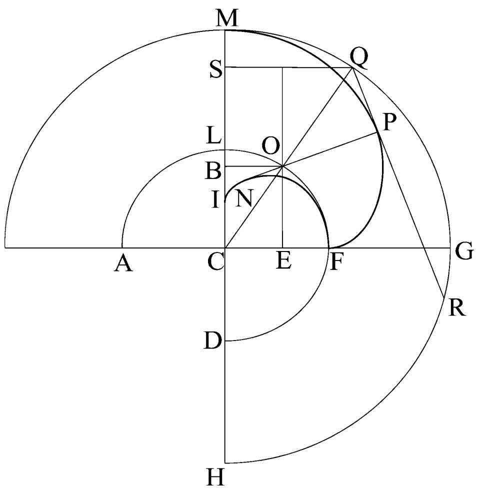

Fig. 13. Following *Curva geometrica ...*, *Acta eruditorum*, 1690, Fig. 2, p. 170

theorems: "so showing to the cultured reader the path for constructing very easy proofs, all concerning epicycloid rectification and the value of several relevant areas." [40] In November 1691 this paper was followed by a further report[41] describing some effects of burning mirrors, but due to limitations of space we will not discuss it in this article.

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[40] Tschirnhaus 1690, 171–172.
[41] Tschirnhaus 1691, 517–520.

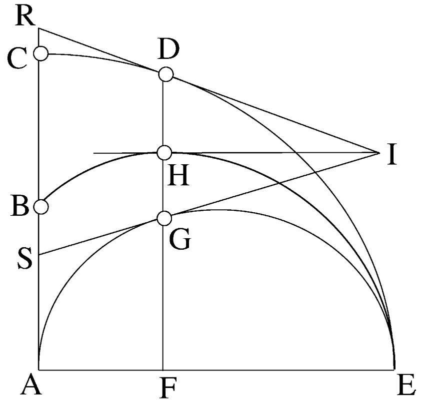

Fig. 14. Following La Hire, *Examen* Fig. 2, p. 450

6. La Hire's counter-arguments

Some time after 1687, La Hire[42] reviewed Tschirnhaus's paper of 1682, but his report was only to be printed posthumously in 1730. He referred to Tschirnhaus's first synthetic geometrical study, not only amending the procedure for drawing the curve, but also adding some further results which he himself had calculated.

As far as the caustic of the circle was concerned, La Hire demonstrated six new theorems. The first one explained that if one draws the tangents (Fig. 14) at points D and G, and if these tangents meet in I, the bisector of their angle DIG will touch Tschirnhaus's curve at the point H.[43]

The second theorem showed, after a long and convoluted explanation, that (Fig. 14) IH is the trajectory of a ray reflected by a quarter of a circle, and that, even if IH touches Tschirnhaus's curve, point H cannot belong to the envelope of the reflected rays (i. e. , to the true reflecting curve) because the absurd situation would follow in which a certain angle would be simultaneously smaller and greater than the double of another one. [44] As a consequence, Tschirnhaus's way of drawing the catacaustic is wrong.

It is useful to compare the locus of midpoints H of the segments DG, when F is moving on AE, with the curve obtained by Tschirnhaus. That curve is a quartic while the true catacaustic is a sextic; but the curves, Fig. 15, both embedded between the semicircles in the figure, are practically undistinguishable, and Tschirnhaus's quartic dominates, even if only slightly, the true catacaustic for $x < 0,2$ only.

42 La Hire's reputation in geometry is due to his treatises on conic sections issued in 1673, 1679, 1685, where, following his teacher Desargues, he studied the relationship between poles and polars.

43 La Hire 1730, 451–453.

44 La Hire 1730, 456–457.

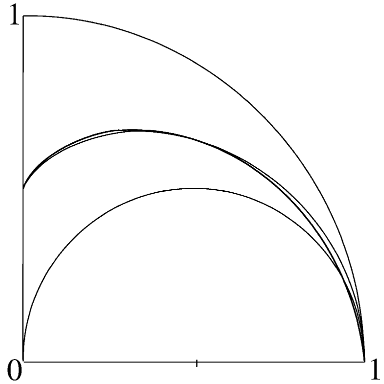

Fig. 15. The true catacaustic (thick line) and Tschirnhaus's quartic

La Hire asserted that the catacaustic met the segment DF below$H$. Moreover, referring to a passage in *Medicina mentis*, he claimed that Tschirnhaus did not prove that the catacaustic could be rectified: [45]

> There is nobody who can be in doubt that the curves generated by the crossing of the reflected sun rays, when they are incident onto a curve, shall not be equal to straight-lines no more than any kind of curve and the circle itself; but the difficulty is in proving that this straight-line has a length equal to the curve in ratio with some other given straight line, as well as we succeed in knowing the circumference of the circle in ratio with its diameter.[46]

In the third theorem, he demonstrated how the points of the curve formed by the reflected rays[47] could be found: let ML be any ray (Fig. 16) incident on the quadrant AEC, and MF reflected ray. Project the midpoint N of AM on MF: the point H will be a point of the caustic.

Since $MH = MI$, the caustic of the circle can easily be drawn point by point, which shows Tschirnhaus's mistake.

La Hire's fourth theorem proved that the catacaustic is an epicycloid generated by a point of a circle such as $NIMH$ of a given radius, which rolls without slipping on another doubled-radius circle. Subsequently (Fig. 17) he put a quarter of the concentric circle ABD, whose radius AD is equal to one half of the radius AE of the mirror, and the circle $NIMH$, with diameter AD, in the quarter circle AEC. When the latter circle

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[45] This enables us to conclude that La Hire certainly wrote his tract after 1687.

[46] La Hire 1730, 459: "Il n'y a personne qui puisse douter que les courbes formées par les intersections des raïons du Soleil reflechis lorsqu'ils tombent au dedans d'une courbe, ne soient égales à des lignes droites, non plus que toute autre sorte de courbe & le cercle même; mais la difficulté est de démontrer quelle est la grandeur de cette ligne droite égale à la courbe par rapporte à quelque ligne droite connuë & donnée, comme de connoître la circonference du cercle par rapport à son diametre."

[47] Ibid., 461–468.

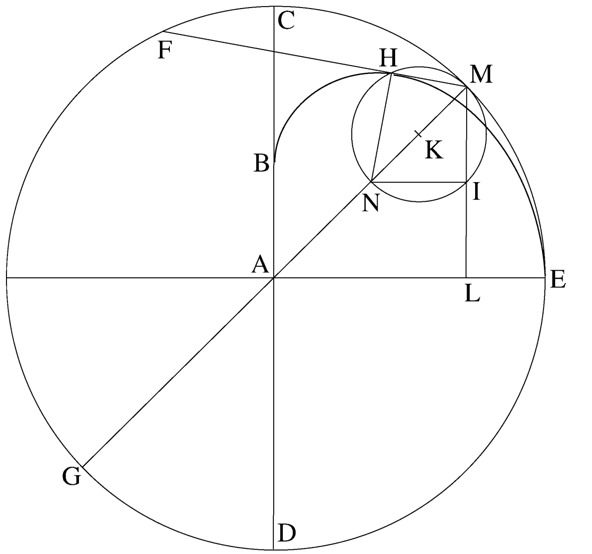

Fig. 16. Following La Hire, *Examen*, p. 462

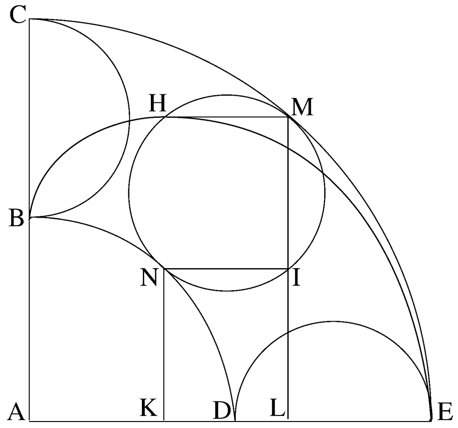

Fig. 17. Following La Hire, *Examen*, p. 467

rolls on the circle ABD, point H which was initially E, will describe the epicycloid EHB.

In the fifth theorem La Hire listed the main properties of this epicycloidal catacaustic, showing that its arclength, HE, equalled the sum of the segments HM and ML, emphasizing how Tschirnhaus had not shown this aspect. The rectification of the catacaustic of the circle is possible because in the special case studied by La Hire the point of the rolling curve which generates the epicycloid is exactly at the same distance from

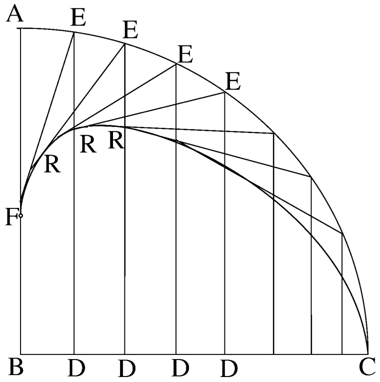

Fig. 18. Following Gerhardt 1971, Fig. 108

the circular base as the radius of the rolling circle. It is well known, however, that in the general case this requires (as for the cycloid) elliptic integrals.

Lastly, in the sixth theorem, La Hire showed that the points of the curve drawn by Tschirnhaus differed significantly from those of the real catacaustic, because each reflected ray always meets the real catacaustic before Tschirnhaus's curve.

7. Leibniz's contribution

As explained earlier, Tschirnhaus referred to the catacaustic in a letter to Leibniz on April 7, 1681:

> Let *AEC* (Fig. 18) be an arc of a round curve; let the sun's parallel rays *DE* be reflected through *EF*; we need to know the nature and form of the curve *FRC* as formed by the reflected rays which cross each other. I detected that this curve could be a geometric one-in the sense of Descartes- and even, given a mirror meridian geometric curve, its relevant caustic is geometric too. I want to know if, till now, this has been obtained by some other mathematician, especially by Mr. Huygens whose Dioptrics has been published recently. I have discovered another approach which is easier than Sluse's (and I don't believe that anyone could find another approach which is easier than mine) by which the tangents can be drawn not only to the geometrical curves, but also to the mechanical ones.

Leibniz, in his answer to Tschirnhaus on May 13, did not display any clear idea about how the caustic curve could be conceived physically.[48]

On May 27 Tschirnhaus replied to Leibniz, who had asked him about the contents of the works presented to the Royal Academy of Sciences in Paris. First of all, he referred to his discovery of the caustic of the circle, with the (wrong) curve drawn and rectified. On the basis of this letter Tschirnhaus prepared a short paper which would be published in the same year in *Acta eruditorum*. Subsequently, Leibniz realized that when a pencil

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[48] Gerhardt 1971, 485.

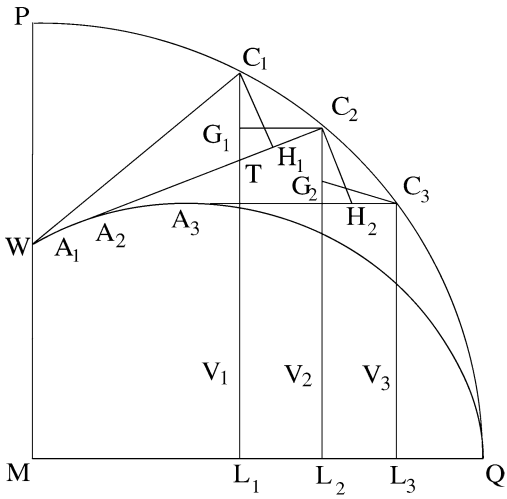

Fig. 19. Following Gerhardt 1971 IV, Fig. 115

of parallel rays hits a circle, the locus of the reflected rays is a curve,[49] and he provided Tschirnhaus with a (supposed) demonstration about how the caustic could be rectified.

We write "supposed", because Leibniz's demonstration seems to be spoiled by the initial assumption that on each line segment (Fig. 19) C_1L_1, C_2L_2, ... , it is possible to consider the points V_1, V_2,... , such that

$$A_1C_1 + C_1V_1 = A_1Q$$

$$A_2C_2 + C_2V_2 = A_2Q$$

....

After this, he shows that points V_1, V_2, ..., must necessarily coincide with L_1, L_2, ... , because otherwise the simultaneous equations would be true:

$$C_1G_1 = C_1V_1 - C_2V_2C_1G_1 = C_1L_1 - C_2L_2$$

$$C_2G_2 = C_2V_2 - C_3V_3C_2G_2 = C_2L_2 - C_3L_3$$

....

This could happen if and only if points V_1, V_2, ... , coincide with L_1, L_2, ...

It is possible that Tschirnhaus and/or Leibniz quickly recognized the mistake, because this demonstration was not referred to again. It is surprising that Leibniz, a much better mathematician than Tschirnhaus, did not find the real locus of the reflected rays. However, Leibniz studied the false quartic (whose rectificability he believed to have shown), and not the real one. As far as *envelopes* were concerned (even though

49 Ibid., 493.

this word was to be used for the first time by G. Monge in his *Application de l'analyse à la géométrie*, 1795), Leibniz had expressed[50] his ideas in 1689[51] and 1692, though only qualitatively. In 1694, however, his most important paper regarding the treatment of envelopes using differentials appeared in *Acta eruditorum*.[52]

While being extremely cautious about the parameters, he gave the following example: to determine a line touching an infinite number of others at a given position. He took into account the bi-parametric family of circles having center $Z(b, 0)$ (Fig. 20) and radius c, whose equation is:

$$x^2 + y^2 + b^2 = 2bx + c^2. \tag{12}$$

Later he defined a functional relationship (*aequatio secundaria*) between the radius c and the abscissa b of Z: $f(b, c) = 0$; e.g., he assumed that f was a parabola with a being the constant, $c^2 = ab$.

At this point, the family of circles becomes single-parametric, so that knowing the center, the radius $c = \sqrt{ab}$ can be found immediately. If b increases, the radius will increase too (Fig. 21), and by replacing the value of c^2 in equation (12), the (single-parametric) family of circles is obtained

$$x^2 + y^2 + b^2 = 2bx + ab,$$

whose equation, derived with respect to b, gives:

$$b = x + \frac{a}{2},$$

which, when inserted in (12), gives:

[50] At the end of his paper, Leibniz displayed his deep knowledge of the current state of the art: "Ego inde longius progressus sum... lineas opticas inveniendas, quarum ope radii redderentur ad datum punctum convergentes vel divergentes aut etiam inter se paralleli, quod alia etiam ratione prestitere Newtonus in Principiis, Hugenius in libro De Lumine....Coeterum ab Hugenio in tractato De Lumine et Tschirnhausio in Actis notatum est causticam illam a speculo concavo sphaerico radios solares reflectente formatam simul esse cycloidalem, provolutione circuli super circulo generatam. [From this point on, I made several improvements in the detection of optical lines causing the light rays to converge in, or diverge from, a fixed point, or to keep parallel paths. All this was investigated by Newton in Principia and by Huygens in his treatise De lumine... Moreover, it was indicated by Huygens in De Lumine and by Tschirnhaus in Acta eruditorum that a spherical concave mirror will produce a cycloidal caustic, namely, the same curve that results when a circle rolls on a fixed one]."

[51] Leibniz wrote: "Lineas illas opticas Cartesius habuit sed celavit. Eas postea ab Hugenio, (sed qui nondum edidit) et nunc a Newtono inventas intelligo. Etiam mihi, sed per diversam viam innotuere. Et habebam quidem methodos generales dudum, sed proprias perelegantes eruendi occasionem dedit egregium inventum D.ni Tschirnhausii nostri in Actis publicatum qui integras lineas tanquam focos considerat. [Those optical lines were well known by Descartes, but he concealed them. After him, they have been found by Huygens and Newton, as I see. They were known to me too, even if through a different path. I had very good methods for drawing them, and the remarkable discovery of Tschirnhaus, who in Acta deals with such lines as foci, gave me the opportunity of knowing his very smart researchs]."

[52] Leibniz 1694, 311.

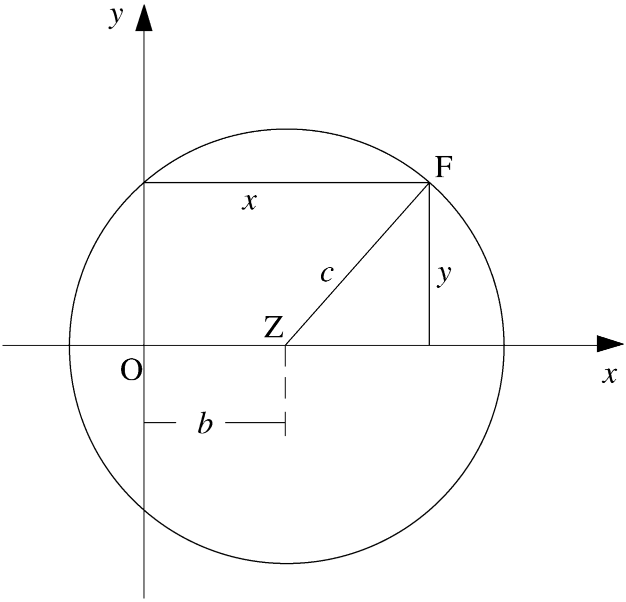

Fig. 20. A circle of the bi-parametric family

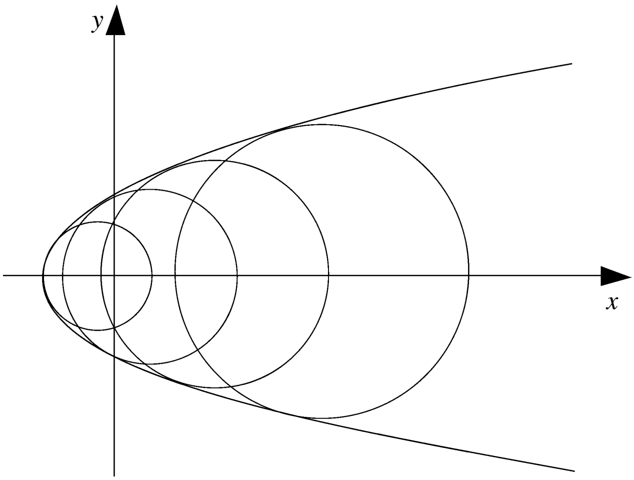

Fig. 21. Leibniz's example: the parabolic envelope of a family of circles

$$y^2 = ax + \frac{a^2}{4}$$

which is the equation of the sought curve, namely a parabola as in Fig. 21.

Leibniz did not give any explanation of the preceding, though he was conscious of its importance:"But this method is especially suitable for dealing with problems of pure or applied geometry, and questions of mechanics or physics."

He ended his paper with the claim[53] that:

> Thus, in my theoretical treatment of the optical lines, I have discovered an approach that showed that lines are capable of making the rays either convergent, divergent or parallel if they are given by a law, or if they come from a mirror of a certain shape. In such a way, if the rays are required to converge, a special line is formed by the aggregation of the ellipses; the same rule applies if they are parallel or divergent.

Leibniz left some points obscure, but no doubt he understood the relationship between catacaustics and envelopes.

8. Conclusions

The subject of caustics appeared on the mathematical scene with Huygens, but it was Tschirnhaus, an "amateur" in the world of academics, who highlighted its importance. His ideas were based on his own experiments and on vague theories gleaned from writings by Descartes and Huygens, while Leibniz, who worked alongside him, seemed more interested in the implications that the subject might have for tangents and never really analyzed the matter deeply enough.

La Hire was the last of the old-school mathematicians. He was an opponent of Varignon and de l'Hospital, but it is difficult to understand his acrimony towards Tschirnhaus. Whatever the reason, Tschirnhaus's work was marred by his rush to publish his insights without giving detailed supporting evidence. La Hire attacked his mistakes in geometry, while in the field of analysis his opponent was Johann Bernoulli.[54] However, it should also be said that Tschirnhaus honestly acknowledged his mistakes and tried to amend them. Even though he was an opponent of the new form of calculus, he consciously addressed the field of mathematical physics in regard to modeling. He devoted himself to catacaustics only, but the subject would be broadened to include anticaustics and diacaustics later by Jakob Bernoulli. In this light, therefore, Tschirnhaus the optics scholar, can be seen as a *trait d'union* between the new and old analytical schools. Johann Ber-

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[53] "Hac ratione jam olim in schediasmate de lineis opticis inveni modum lineas exhibendi, quae radios ordinatim positione datos, seu a datae figurae speculo venientes, reddant convergentes aut divergentes, aut parallelos. Formatur enim talis linea ellipsium concursu, si radii debeant fieri convergentes; eadem methodus valet, si sint parallelae aut divergentes."

[54] Tschirnhaus's "assertiones" also attracted criticism from Johann Bernoulli. For example, in the fifth part of his *Lectiones* , entitled *Inventio curvarum, quae unicum habeant spatium quadrabile*, Bernoulli began by observing: "*Ad confirmationem ejus quod diximus, dari spatium curvilineum geometricum, quod unico modo sit quadrabile, & non infinitis, ut Dn. Tschirnhaus asserit* (Acta, sept. 1687, p. 526) &c. [All this confirms what I have already explained, namely, the value of the area under a given curve can be squared in only one way, and not in infinitely many, as Tschirnhaus asserts]." Also in the next lesson (*Continuatio ejusdem argumenti*), Bernoulli writes, even more expressly: " *tamen generaliter omnes curvas, quae unum spatium quadrabile habent, infinita alia habere, D.nus Tschirnhaus male asseruerit*. [nevertheless, Tschirnhaus wrongly claimed that each curve which can be squared, could be squared in an infinity of other ways]."

noulli, a less innovative but much more keen-minded and deeper mathematician, has only been fleetingly mentioned. It was he who, from 1692 onwards, improved the researchs on caustics by his great analytical skills. In a future paper we will investigate the work on caustics carried out by Jakob and Johann Bernoulli. By the end of the seventeenth century these initial applications of geometrical optics were to become the founding principles of differential geometry, and the basis for a new field of mathematical-physics.

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