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Edge-Colorings with No Large Polychromatic Stars

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Abstract. Given a graph G and a positive integer r, let $f_r(G)$ denote the largest number of colors that can be used in a coloring of $E(G)$ such that each vertex is incident to at most r colors. For all positive integers n and r, we determine $f_r(K_{n,n})$ exactly and $f_r(K_n)$ within 1. In doing so, we disprove a conjecture by Manoussakis, Spyratos, Tuza and Voigt in [4].

Key words. Edge-coloring, Rainbow subgraph

1. Introduction

Let F and G be two graphs, and c be a coloring of $E(G)$ (the edge set of G). F is said to be a *rainbow subgraph* (or *polychromatic subgraph*) of G, if G contains a subgraph isomorphic to F, all of whose edges are assigned distinct colors. Several recent papers study conditions on c that ensure the existence of a rainbow subgraph F in G (see [2]–[4]). There has been particular interest in determining the least number of colors in a coloring of $E(K_n)$ that forces a certain rainbow subgraph to occur. Such problems are sometimes called *Anti-Ramsey* problems.

We study a problem of this type. A *star* with l edges is an $(l+1)$-vertex graph with l edges, in which one vertex is adjacent to all the other l vertices. Given a graph G and a positive integer r, let $f_r(G)$ denote the maximum number of colors that can be used in a coloring of $E(G)$ without forcing a rainbow star with $r+1$ edges. In other words, $f_r(G)$ denotes the maximum number of colors that can be used in a coloring of $E(G)$ such that each vertex is incident to at most r colors.

When $G = K_n$, determining $f_r(G)$ becomes an Anti-Ramsey problem, and is the main focus of this paper. Trivially, for $r \geq n-1$, $f_r(K_n) = \frac{1}{2}n(n-1)$. Hence we will assume that $r \leq n-2$. For $r \leq n-2$, previous bounds on $f_r(K_n)$ were obtain by Manoussakis et al. [4]. They showed that $\lfloor \frac{1}{2}(n(r-1)+2) \rfloor \leq f_r(K_n) \leq \lfloor \frac{1}{2}(n(r-1)+r+1) \rfloor$. They further conjectured the lower bound to be best possible. In section 3, we disprove their conjecture by proving

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Theorem 1. *Given positive integers n and r, where* $r \leq n-2$,

$$f_r(K_n) = \left\lfloor \frac{1}{2} n(r-1) \right\rfloor + \left\lfloor \frac{n}{n-r+1} \right\rfloor + \epsilon,$$

where $\epsilon = 0$ *or* 1 *if n is odd, r is even, and* $\lfloor \frac{2n}{n-r+1} \rfloor$ *is odd*; $\epsilon = 0$ *otherwise.* □

In section 2, we consider the case where $G = K_{n,n}$. Trivially, if $r \geq n$ then $f_r(K_{n,n}) = n^2$. Hence we will assume that $r \leq n-1$. We prove the following theorem.

Theorem 2. *Given positive integers n and r, where* $r \leq n-1$,

$$f_r(K_{n,n}) = n(r-1) + \left\lfloor \frac{n}{n-r+1} \right\rfloor. \qquad \square$$

We are going to prove Theorem 2 first to illustrate some of the key ideas used in the proof of Theorem 1.

We consider only simple graphs. We denote the vertex set and edge set of G by $V(G)$ and $E(G)$, respectively. The *neighborhood* of a vertex $v \in V(G)$, written $N_G(v)$, is the set of all vertices adjacent to v in G. The degree of a vertex $v \in V(G)$ in G is $d_G(v) = |N_G(v)|$. Subscripts will be omitted whenever appropriate. The subgraph of G induced by a subset $S \subseteq V(G)$ is denoted by $G[S]$. A graph is *k-regular* if every vertex in it has degree k. A subgraph $H \subseteq G$ is a *spanning subgraph* of G if $V(H) = V(G)$. A k-regular spanning subgraph of G is called a *k-factor* of G. Given a positive integer m, $[m]$ denotes the set $\{1, \ldots, m\}$.

2. The Bipartite Case

Let $G = K_{n,n}$. Let $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ be the two bipartite sets of G. In our proof, we need the following lemma which can be found in any introductory graph theory book.

Lemma 1. $K_{p,p}$ *has a t-factor for all* $t \in [p]$.

Lower bound. $f_r(G) \geq n(r-1) + \lfloor \frac{n}{n-r+1} \rfloor$, *where* $r \leq n-1$.

Proof. To prove the lower bound, we present a coloring of $E(G)$ using $n(r-1) + \lfloor \frac{n}{n-r+1} \rfloor$ colors, such that at each vertex at most r colors are used on its incident edges.

Let $m = \lfloor \frac{n}{n-r+1} \rfloor$. Partition X into m subsets $X_1, \ldots, X_m$, each of size at least $n-r+1$. Partition Y into $Y_1, \ldots, Y_m$, such that $|Y_i| = |X_i|$ for each $i \in [m]$. By Lemma 1 each $G[X_i \cup Y_i]$ contains an $(n-r+1)$-factor, call it F_i. Let $F = F_1 \cup \cdots \cup F_m$. Then F is an $(n-r+1)$-factor of G, and thus $G - E(F)$ is an $(r-1)$-factor. We now color the edges of G by assigning color i to the edges in F_i

for each $i \in [m]$, and assigning distinct new colors to the remaining edges in G. It is easy to verify that at each vertex exactly r colors are used on the incident edges, and that altogether $n(r-1)+m = n(r-1) + \lfloor \frac{n}{n-r+1} \rfloor$ colors are used. □

Upper bound. $f_r(G) \leq n(r-1) + \lfloor \frac{n}{n-r+1} \rfloor$, *where* $r \leq n-1$.

Proof. Let c be a coloring of $E(G)$, such that at each vertex at most r colors are used on the incident edges. We use $|c|$ to denote the number of colors used in c. For each $i \in [n]$, let A_i and B_i denote the set of colors used on the edges incident to x_i and to y_i, respectively; clearly, $|A_i|, |B_i| \leq r$. Let m be the largest integer such that there are m pairwise disjoint A_i's. Without loss of generality, suppose that $A_1, \ldots, A_m$ are pairwise disjoint. For $j > m$, we have $A_j \cap (\cup_{i=1}^m A_i) \neq \emptyset$ and therefore $|A_j - \cup_{i=1}^m A_i| \leq r-1$. It follows that

$$|c| = |\cup_{i=1}^n A_i| = |\cup_{i=1}^m A_i| + |\cup_{j=m+1}^n A_j - \cup_{i=1}^m A_i| \leq mr + (n-m)(r-1). \quad (2.1)$$

On the other hand, for each $j \in [n]$, since $A_1, \ldots, A_m$ are pairwise disjoint, $c(x_1 y_j), \ldots, c(x_m y_j)$ are all distinct, and they belong to $\cup_{i=1}^m A_i$. Since at most r distinct colors are allowed on the edges incident to y_j, we have $|B_j - \cup_{i=1}^m A_i| \leq (r-m)$. Hence

$$|c| = |\cup_{j=1}^n B_j| = |\cup_{i=1}^m A_i| + |\cup_{j=1}^n B_j - \cup_{i=1}^m A_i| \leq mr + n(r-m). \quad (2.2)$$

We can rewrite (2.1) and (2.2) as $|c| \leq n(r-1)+m$ and $|c| \leq n(r-1)+n+m(r-n)$, respectively. Combining the two, we have

$$|c| \leq n(r-1) + \min\{m, n+m(r-n)\}. \quad (2.3)$$

Notice that m increases as m increases, and $n+m(r-n)$ decreases as m increases, and the two quantities are equal when $m = \frac{n}{n-r+1}$. Hence, $\min\{m, n+m(r-n)\} \leq \frac{n}{n-r+1}$. Since $|c|$ is an integer, we have $|c| \leq n(r-1) + \lfloor \frac{n}{n-r+1} \rfloor$. □

3. The Complete Graph Case

Let $G = K_n$. We need the following two elementary lemmas in our proof.

Lemma 2. *K_{2p} has an l-factor for all $l \in [2p-1]$,* □

Lemma 3. *Every $2p$-regular graph contains a $2t$-factor for all $t \in [p]$.* □

Lower bound. $f_r(G) \geq \lfloor \frac{1}{2} n(r-1) \rfloor + \lfloor \frac{n}{n-r+1} \rfloor$, *where* $r \leq n-2$.

Proof. To prove the lower bound, we present a coloring of $E(G)$ using $\lfloor \frac{1}{2} n(r-1) \rfloor + \lfloor \frac{n}{n-r+1} \rfloor$ colors, such that at each vertex at most r colors are used on the incident edges. Let $m = \lfloor \frac{n}{n-r+1} \rfloor$. Partition $V(G)$ into m subsets $A_1, \ldots, A_m$, where $|A_i| = n-r+1$ for $i \in [m-1]$, and $|A_m| = n-(m-1)(n-r+1) \geq n-r+1$. For $i \in [m-1]$, $G[A_i]$ is $(n-r)$-regular, let $F_i = G[A_i]$.

Case 1. n and r have the same parity.

In this case $n-r$ is even. $G[A_m]$ is a complete graph on at least $n-r+1$ vertices. By Lemma 2 and Lemma 3, it contains an $(n-r)$-factor, call it F_m. Let $F=\cup_{i=1}^m F_i$. Then F is an $(n-r)$-factor of G, and $G-E(F)$ is an $(r-1)$-factor of G. Now color the edges of G by assigning color i to the edges in F_i, and assigning distinct new colors to the remaining edges in G. The coloring defined this way uses $\frac{1}{2}n(r-1)+m=\lfloor\frac{1}{2}n(r-1)\rfloor+\lfloor\frac{n}{n-r+1}\rfloor$ colors, and at each vertex exactly r colors are used on the incident edges.

Case 2. n is even and r is odd.

In this case $|A_m|=n-(m-1)(n-r+1)$ is even, and is at least $n-r+1$. By Lemma 2, $G[A_m]$ has an $(n-r)$-factor, call it F_m. We can then color the edges of G as in the previous case.

Case 3. n is odd and r is even.

In this case $|A_m|=n-(m-1)(n-r+1)$ is odd, and is at least $n-r+2$. Let $t=|A_m|$. Suppose that $A_m=\{v_1,\dots,v_t\}$. Let C denote the spanning cycle $v_1v_2\dots v_tv_1$ of $G[A_m]$. Then $G[A_m]-E(C)$ is $(t-3)$-regular, where $t-3$ is even. Since $n-r-1$ is even, and $n-r-1\le t-3$, by Lemma 3 $G[A_m]-E(C)$ contains an $(n-r-1)$-factor, call it F'. Let F'' be the spanning subgraph of $G[A_m]$ consisting of edges $v_tv_1, v_1v_2, v_3v_4,\dots,v_{t-2}v_{t-1}$. F'' is a spanning subgraph of $G[A_m]$ in which v_1 has degree 2, and all the other vertices have degree 1. Let $F_m=F'\cup F''$. F_m is a spanning subgraph of $G[A_m]$ in which v_1 has degree $n-r+1$, and all the other vertices have degree $n-r$. We now color the edges of G by assigning color i to the edges in F_i for each $i\in[m]$, and assigning distinct new colors to the remaining edges in G. Notice that in $G-E(\cup_{i=1}^m F_i)$, every vertex except v_1 has degree $r-1$, and v_1 has degree $r-2$. Hence exactly $\frac{1}{2}(n(r-1)-1)=\lfloor\frac{1}{2}n(r-1)\rfloor$ new colors are used in addition to colors $1,\dots,m$. Altogether, $\lfloor\frac{1}{2}n(r-1)\rfloor+m=\lfloor\frac{1}{2}n(r-1)\rfloor+\lfloor\frac{n}{n-r+1}\rfloor$ colors are used. Furthermore, at each vertex at most r colors are used on the incident edges. □

Upper bound. $f_r(G))\le\lfloor\frac{1}{2}n(r-1)+\frac{1}{2}\lfloor\frac{2n}{n-r+1}\rfloor\rfloor$, *where* $r\le n-2$.

Proof. Let c be a coloring of $E(G)$, such that at each vertex at most r colors are used on the incident edges. We use $|c|$ to denote the number of colors used in c. Let H be a spanning subgraph of G containing exactly one edge from each color class (H may contain isolated vertices). Write $V=V(G)=V(H)$. By the definition of H, $d_H(v)\le r$ for all $v\in V$ and $e(H)=|c|$. Let $S=\{v\in V: d_H(v)=r\}$, and let $m=|S|$. If S is empty, then $\Delta(H)\le r-1$, and therefore $e(H)\le\lfloor\frac{1}{2}n(r-1)\rfloor$. Hence we may assume that S is nonempty. For each vertex $u\in V$, let $A_G(u)$ and $A_H(u)$ denote the set of colors used on the edges incident to u in G and in H, respectively. Clearly, $A_H(u)\subseteq A_G(u)$, and if $u\in S$ then $A_G(u)=A_H(u)$.

Claim 1. *For every* $u,v\in V$, $A_H(u)\cap A_H(v)=\emptyset$, *if* $uv\notin E(H)$, *and* $A_H(u)\cap A_H(v)=\{c(uv)\}$ *if* $uv\in E(H)$. *Furthermore,* S *induces a clique in* H.

Proof. Since c assigns distinct colors to $E(H)$, it follows immediately that $A_H(u) \cap A_H(v) = \emptyset$ if $uv \notin E(H)$ and that $A_H(u) \cap A_H(v) = \{c(uv)\}$ if $uv \in E(H)$. For $u, v \in S$, $c(uv) \in A_G(u) \cap A_G(v) = A_H(u) \cap A_H(v)$, hence $uv \in E(H)$. Therefore, S induces a clique in H. □

Claim 2. *Let u, v be two vertices in S and w be a vertex in $V - S$. If $c(uw) = c(vw)$, then $c(uw) = c(vw) = c(uv)$.*

Proof. Suppose $c(uw) = c(vw) = \alpha$, then $\alpha \in A_G(u) \cap A_G(v) = A_H(u) \cap A_H(v) = \{c(uv)\}$. Hence, $\alpha = c(uv)$.

Claim 3. *Suppose $u \in S, v \notin S$, and $uv \notin E(H)$, then $c(uv) \notin A_H(v)$.*

Proof. since $uv \notin E(H)$, by Claim 1 $A_H(u) \cap A_H(v) = \emptyset$. On the other hand, $c(uv) \in A_G(u) = A_H(u)$. Hence $c(uv) \in A_H(v)$. □

Claim 4. *For all $v \notin S$, $d_H(v) \leq r - \frac{|S - N_H(v)|}{2}$.*

Proof. Let $k = |S - N_H(v)|$. Suppose that $S - N_H(v) = \{u_1, \ldots, u_k\}$. By Claim 3, $c(u_i v) \notin A_H(v)$ for $i \in [k]$. Furthermore, by Claim 2, if $c(u_i v) = c(u_j v)$, then $c(u_i v) = c(u_j v) = c(u_i u_j)$. This implies that no three of the edges $u_1 v, \ldots, u_k v$ can have the same color, since otherwise S would contain a monochromatic triangle. Hence at least $\frac{k}{2}$ distinct colors are used on the edges $u_1 v, \ldots, u_k v$, and those colors are not used in $A_H(v)$. Altogether, at least $|A_H(v)| + \frac{k}{2} = d_H(v) + \frac{k}{2}$ distinct colors are used on the edges incident to v in G. Since at most r colors are allowed on those edges, we have $d_H(v) + \frac{k}{2} \leq r$, which yields $d_H(v) \leq r - \frac{k}{2} = r - \frac{|S - N_H(v)|}{2}$. □

Now, $\sum_{v \in V} d_H(v) = \sum_{v \in S} d_H(v) + \sum_{v \notin S} d_H(v) \leq mr + \sum_{v \notin S} \left(r - \frac{|S - N_H(v)|}{2} \right) = nr - \sum_{v \notin S} \frac{|S - N_H(v)|}{2}$. Notice that $\sum_{v \notin S} |S - N_H(v)|$ counts exactly the number of non-edges in H between S and $V - S$. Since each vertex in S has exactly $n - 1 - r$ non-neighbors in H outside S, we have $\sum_{v \notin S} |S - N_H(v)| = m(n - 1 - r)$. Therefore,

$$\sum_{v \in V} d_H(v) \leq nr - \frac{m(n-1-r)}{2} = n(r-1) + n - \frac{m(n-1-r)}{2}. \tag{3.1}$$

On the other hand, for each $v \notin S, d_H(v) \leq r - 1$ by the definition of S, hence

$$\sum_{v \in V} d_H(v) \leq mr + (n - m)(r - 1) = n(r - 1) + m. \tag{3.2}$$

Combining (3.1) and (3.2), we have

$$\sum_{v \in V} d_H(v) \leq n(r-1) + \min\left\{ m, n - \frac{m(n-1-r)}{2} \right\}. \tag{3.3}$$

Notice that m increases as m increases, while $n - \frac{m(n-1-r)}{2}$ decreases as m increases, and the two quantities are equal when $m = \frac{2n}{n-r+1}$. Hence, $\min\{m, n - \frac{m(n-1-r)}{2}\} \leq \frac{2n}{n-r+1}$. Since $\sum_{v \in V} d_H(v)$ is an integer, we have $\sum_{v \in V} dH(v) \leq n(r-1) + \lfloor \frac{2n}{n-r+1} \rfloor$. Hence, $|c| = e(H) \leq \lfloor \frac{1}{2} n(r-1) + \frac{1}{2} \lfloor \frac{2n}{n-r+1} \rfloor \rfloor$. □

Upper bound vs. lower bound.

It is easy to verify that the upper bound $\lfloor \frac{1}{2} n(r-1) + \frac{1}{2} \lfloor \frac{2n}{n-r+1} \rfloor \rfloor$ and the lower bound $\lfloor \frac{1}{2} n(r-1) \rfloor + \lfloor \frac{n}{n-r+1} \rfloor$ differ by at most 1. Indeed, they are equal unless n is odd, r is even, and $\lfloor \frac{2n}{n-r+1} \rfloor$ is odd.

4. Further Discussions

In section 3, we have completely determined $f_r(K_n)$ except for the case when n is odd, r is even, and $\lfloor \frac{2n}{n-r+1} \rfloor$ is odd, in which case we have determined $f_r(K_n)$ within 1. It is likely that by using more sophisticated counting arguments and case analysis, one may be able to improve the upper bound by 1, and hence completely settle the issue. For small values of n and r, ad hoc analysis suggest that the lower bound $\lfloor \frac{1}{2} n(r-1) \rfloor + \lfloor \frac{n}{n-r+1} \rfloor$ is best possible.

There are several possible directions for further investigation. In our constructions in both Theorem 1 and Theorem 2, at each vertex, one of the color classes contains a lot of edges while each of the other color classes contains only one edge. One could therefore ask to maximize the number of colors used in a coloring of $E(K_n)$, such that at each vertex, at most r colors are used on the incident edges, and that the color classes on the incident edges differ by at most 1 in size. Another generalization is as follows: Given integers n, r, d, where $dr \geq n-1$, determine the maximum number of colors that can be used in a coloring of $E(K_n)$ such that at each vertex, at most r colors are used on the incident edges, and that each of the colors is used at most d times. One could ask similar questions for $K_{n,n}$. More generally, one can study $f_r(G)$ for other interesting classes of graphs G.

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