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On the Structure of Graphs with Bounded Asteroidal Number

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Abstract. A set $A \subseteq V$ of the vertices of a graph $G = (V, E)$ is an *asteroidal set* if for each vertex $a \in A$, the set $A \backslash \{a\}$ is contained in one component of $G - N[a]$. The maximum cardinality of an asteroidal set of G, denoted by $\mathsf{an}(G)$, is said to be the *asteroidal number* of G. We investigate structural properties of graphs of bounded asteroidal number. For every $k \geq 1$, $\mathsf{an}(G) \leq k$ if and only if $\mathsf{an}(H) \leq k$ for every minimal triangulation H of G. A dominating target is a set D of vertices such that $D \cup S$ is a dominating set of G for every set S such that $G[D \cup S]$ is connected. We show that every graph G has a dominating target with at most $\mathsf{an}(G)$ vertices. Finally, a connected graph G has a spanning tree T such that $d_T(x, y) - d_G(x, y) \leq 3 \cdot |D| - 1$ for every pair x, y of vertices and every dominating target D of G.

1. Introduction

Asteroidal triples (short AT) where introduced in [11] as triples of vertices such that between any two of the vertices there is a path avoiding the neighbourhood of the third. It was shown in [11] that chordal graphs without AT are exactly the interval graphs.

Graphs without an asteroidal triple are called asteroidal triple-free (short AT-free) and attained much attention recently. Möhring has shown that every minimal triangulation of an AT-free graph is AT-free and thus an interval graph. This implies that for every AT-free graph the treewidth and the pathwidth of the graph are equal [14]. This has been extended to the following characterization of AT-free graphs: A graph G is AT-free if and only if every minimal triangulation H of G is an interval graph [5, 15]. Furthermore a collection of interesting structural and algorithmic properties of AT-free graphs has been obtained by Corneil, Olariu and Stewart, among them an existence theorem for so-called dominating pairs in con-

Dedicated to Professor Gerd Wechsung on the occasion of his 60th birthday.

nected AT-free graphs, called 'Dominating Pair Theorem', and a linear time algorithm to compute a dominating pair for connected AT-free graphs (see [5, 6]).

The class of graphs of bounded asteroidal number extends the class of AT-free graphs, based on a natural way of generalizing the concept of asteroidal triples to so-called asteroidal sets introduced by Walter in [18]: A set of vertices A of a graph G is called an asteroidal set, if for every vertex $a \in A$ all vertices of $A \setminus \{a\}$ are contained in the same component of $G - N[a]$. Walter, Prisner and Lin et al. used asteroidal sets to characterize certain subclasses of the class of chordal graphs [12, 16, 18]. The asteroidal number of a graph G, denoted by $\mathsf{an}(G)$, is the maximum cardinality of an asteroidal set in G. Notice that AT-free graphs are exactly those graphs with asteroidal number at most 2.

We consider the question whether important structural properties of AT-free graphs have natural analogues for graphs of bounded asteroidal number. This is in fact the case for two fundamental structural properties of AT-free graphs.

In Section 4 we show that every graph $G = (V, E)$ has a *dominating target* of cardinality at most $\mathsf{an}(G)$, i.e., a set of vertices $D \subseteq V$ with $|D| \leq \mathsf{an}(G)$ such that, if $S \subseteq V$ and $G[D \cup S]$ connected then $D \cup S$ is a dominating set of G. Notice that this theorem (called 'Dominating Target Theorem') contains the 'Dominating Pair Theorem' as a special case, since a dominating target of cardinality at most 2 is a dominating pair. Therefore our proof can be considered as a new and simple proof for the 'Dominating Pair Theorem'. In Section 5 we show that, for all $k \geq 1$, the graph G has asteroidal number at most k if and only if every minimal triangulation of G has asteroidal number at most k.

In Section 6 we consider additive tree spanners. The Dominating Target Theorem of Section 4 enables us to construct for every connected graph G a $(3 \cdot \mathsf{dt}(G) - 1)$-additive tree spanner, where $\mathsf{dt}(G)$ denotes the minimum cardinality of a dominating target of G.

2. Preliminaries

Throughout the paper, let G denote a graph with vertex set V and edge set E. We denote the number of vertices of G by n, the number of edges of G by m and the size of a maximum independent set in G by $\alpha(G)$. For a proper subset $W \subset V$, $G - W$ denotes the subgraph of $G = (V, E)$ induced by the vertex set $V \setminus W$. For a vertex $x \in V$, we write $G - x$ instead of $G - \{x\}$. For $\phi \neq W \subseteq V$, $G[W]$ denotes the subgraph of G induced by the vertices of W. For a vertex x of a graph $G = (V, E)$, $N_G(x)$ is the neighborhood of x in G and $N_G[x] = \{x\} \cup N_G(x)$ is the closed neighborhood of x in G. Furthermore $N_G[A] = \bigcup_{v \in A} N_G[v]$ for $A \subseteq V$. We do not write the index indicating the graph, if there is no ambiguity.

Definition. An independent set $A \subseteq V$ is called an *asteroidal set* of G if for each $a \in A$ the vertices of $A \setminus \{a\}$ are contained in one component of $G - N[a]$. The maximum cardinality of an asteroidal set of G is denoted by $\mathsf{an}(G)$, and is called the *asteroidal number* of G.

By definition, any asteroidal set is an independent set, thus $\mathsf{an}(G) \leq \alpha(G)$. Various algorithms for NP-complete graph problems on AT-free graphs can be extended to graphs of bounded asteroidal number. The graph problems INDEPENDENT SET and INDEPENDENT DOMINATING SET can be solved by polynomial time algorithms when restricted to graphs of bounded asteroidal number [2]. Furthermore, the graph problems TREEWIDTH, MINIMUM FILL-IN and VERTEX RANKING can be solved in polynomial time on graph classes having a polynomial number of minimal separators and a bounded asteroidal number [3].

Although the decision problem: 'Given a graph $G = (V, E)$ and a positive integer k, decide whether $\mathsf{an}(G) \geq k$', is NP-complete and it even remains NP-complete when restricted to triangle-free 3-connected 3-regular planar graphs [9], there are polynomial time algorithms to compute the asteroidal number for graphs in some special classes like HHD-free graphs (including all chordal graphs), claw-free graphs, circular-arc graphs and circular permutation graphs [9].

The modules of a graph will play a crucial role in Sections 3 and 4. Therefore we review some definitions and basic properties. For more information we refer to [13].

For each set $M \subseteq V$, M is a *module* of the graph $G = (V, E)$ if $N(a)\backslash M = N(b)\backslash M$ for all $a, b \in M$, i.e., all vertices of M have the same neighbourhood outside of M. A module M is *trivial* if $|M| \leq 1$ or $M = V$. The graph G is *prime* if G has only trivial modules. Otherwise G is (modular) *decomposable*. For a nontrivial module $M \subset V$ we denote by $G\langle M\rangle$ the graph obtained from G by shrinking M to a single vertex $m \in M$, i.e. $G\langle M\rangle$ is isomorphic to $G - (M\backslash\{m\})$ for every vertex $m \in M$.

3. Extremities and Separators

In this section we introduce extremities and we establish Dirac-type lemmas concerning the existence of extremities in prime graphs. These results are of interest in their own. We emphasize the strong relation to Dirac's theorem on the existence of simplicial vertices in chordal graphs [7] and the recent work of Berry and Bordat on moplexes in graphs [1]. Furthermore some of our results in this section are crucial for the proof of our main theorem.

Definition. Let $G = (V, E)$ be a graph. Then a vertex $v \in V$ is an *extremity* of G if $G - N[v]$ is connected.

The set L of the extremities of a graph G is of interest since any independent set of the graph $G[L]$ is an asteroidal set of G.

We start with some preliminaries on vertex separators in graphs. $S \subseteq V$ is a *separator* of the graph $G = (V, E)$ if $G - S$ is disconnected. We mention that we usually consider components of a graph as vertex sets. $S \subseteq V$ is an *a, b-separator* of $G = (V, E)$ if a and b are in different components of $G - S$. An a, b-separator S of G is a *minimal a, b-separator* if no proper subset of S is an a, b-separator. Then S is said to be a *minimal separator* of G if it is a minimal a, b-separator of G for nonadjacent vertices a and b. Finally, S is an *inclusion minimal separator* of G if S is a separator of G and no proper subset of S is a separator of G. Notice that every inclusion minimal separator is a minimal separator but not every minimal separator is an inclusion minimal separator.

There is a useful characterization of minimal separators and inclusion minimal separators in terms of so-called full components. Let S be a separator of G. Then a component C of $G - S$ is said to be a *full component* if every vertex of S has a neighbour in C. Now S is a minimal separator of G iff $G - S$ has at least two full components. Furthermore S is an inclusion minimal separator of G iff all components of $G - S$ are full.

Finally a minimal a, b-separator S of G is *close to* a, if S contains only neighbors of a. For nonadjacent vertices a and b, there is a unique minimal a, b-separator S close to a [8].

Lemma 1. *Let S be a minimal separator of a prime graph G and let C be a component of $G - S$. Then either C contains an extremity of G or $N(c) \cap S$ is a separator of $G - C$ for every $c \in C$.*

Proof. Let G be a prime graph and let S be a minimal separator of G. Let C be a component of $G - S$ and let a be a vertex of a full component of $G - S$ different from C. For every $c \in C$, we denote by $S_{a,c}$ the unique minimal a, c-separator of G with $S_{a,c} \subseteq N(c)$. Note that $N(c) \cap S \subseteq S_{a,c}$. Furthermore for every $c \in C$ we define $B(c)$ to be the set of all vertices $b \in C \backslash S_{a,c}$ which do not belong to the component of $G - S_{a,c}$ containing a. Thus for every $c \in C$, $c \in B(c)$ and $N(b) \backslash B(c) \subseteq S_{a,c}$ for all $b \in B(c)$. Consequently $b \in B(c)$ implies $B(b) \subseteq B(c)$.

We consider all sets $B(c)$ with $c \in C$ and we choose a vertex $c' \in C$ such that $B(c')$ is minimal with respect to set inclusion. Let $B = B(c')$. Notice that $B(b) = B$ for all $b \in B$ by the minimality of B. This implies $N(b) \backslash B \subseteq N(b')$ for all $b, b' \in B$ and therefore $N(b) \backslash B = N(c')$ for all $b \in B$. Consequently B is a module of G which implies $|B| = 1$ since G is prime. In other words, $B = \{c'\}$, $N(c') = S_{a,c'}$ and no component of $G - S_{a,c'}$ except those containing a or c' contains a vertex of C. Hence either c' is an extremity of G or $N(c') \cap S$ is a separator of $G - C$.

Suppose C contains no extremity of G. Consequently $N(c) \cap S$ is a separator of $G - C$ for all vertices $c \in C$ for which $B(c)$ is minimal with respect to set inclusion. Consider any vertex $d \in C$ for which there is a $c \in C$ with $B(c) \subset B(d)$. As we have shown above $c \in B(d)$ implies $N(c) \backslash B(d) \subseteq S_{a,d}$. Since a is a vertex of a full component of $G - S$ we get $S \cap N(d) \subseteq S_{a,d}$. Consequently $S \cap N(c) \subseteq S \cap N(d)$ which implies that $N(d) \cap S$ is a separator of $G - C$. This completes the proof. □

Corollary 2. *Let G be a prime graph and S a minimal separator of G. Let C be a component of $G - S$ without an extremity. Then $C \subseteq N(S)$.*

Lemma 3. *Let G be a prime graph and S a minimal separator of G. Then there is a component of $G - S$ that contains an extremity of G. Furthermore, if S is an inclusion minimal separator of G, then at least two components of $G - S$ contain an extremity of G.*

Proof. Let G be a prime graph. Suppose there is a minimal separator S of G such that no component of $G - S$ contains an extremity. We choose a component C of $G - S$ such that $N(C)$ is minimal. Then by Lemma 1, for every vertex $c \in C$, $N(c) \cap S$ is a separator of $G - C$, implying $S \subseteq N(c)$ for every vertex $c \in C$. Hence C is a module of G and thus $|C| = 1$, since G is prime.

Since $N(C)$ is a separator of $G - C$, there is a component C' of $G - S$ such that $C \neq C'$ and $N(C') \subseteq N(C)$. Thus the choice of C implies $N(C) = N(C')$. Hence the above arguments applied to C' show that $C' = \{c'\}$. Finally this implies that $\{c, c'\}$ is a nontrivial module of G, contradicting the primality of G.

If S is an inclusion minimal separator of G then $N(C) = S$ for every component C of G. Thus every component without an extremity of G consists of one vertex. As we have seen above, a prime graph can have at most one component of this type. Consequently all but one component of $G - S$ contain an extremity of G. Finally if $G - S$ has exactly two components then both of them contain an extremity of G. □

Corollary 4. *Every non complete prime graph has at least two extremities.*

4. Repulsive Asteroidal Sets and Dominating Targets

In this section we extend the concept of a dominating pair, introduced by D.G. Corneil, S. Olariu and L. Stewart for AT-free graphs in [5], to graphs with bounded asteroidal number and we obtain a generalization of the Dominating Pair Theorem.

Definition. A set $D \subseteq V$ in a graph $G = (V, E)$ is said to be a *dominating target*, if $D \cup S$ is a dominating set in G for every set $S \subseteq V$ for which $G[D \cup S]$ is connected. We denote the minimum cardinality of a dominating target of G by $\mathrm{dt}(G)$.

Obviously every dominating set of a graph is also a dominating target.

Lemma 5. *Let D be a dominating target of $G = (V, E)$. Then $D \subseteq D' \subseteq V$ implies that D' is also a dominating target of G.*

Proof. Consider any $S' \subseteq V$ with $G[D' \cup S']$ connected. Choosing $S = S' \cup (D' \backslash D)$ we get $D' \cup S' = D \cup S$. Hence $G[D \cup S]$ is connected. Thus the fact that D is a dominating target of G implies that $D \cup S = D' \cup S'$ is also a dominating set of G. □

Notice that the notion of a dominating target generalizes the notion dominating pair since every dominating target D with $|D| \leq 2$ forms a dominating pair of G.

Definition. An asteroidal set A of a graph $G = (V, E)$ is *repulsive* if for every vertex $v \in V \backslash N[A]$ not all vertices of A are contained in one component of $G - N[v]$.

Clearly every repulsive asteroidal set (short RAS) is a maximal asteroidal set.

The following theorem presents a structural property that is of great importance for graphs with bounded asteroidal number.

Theorem 6 (RAS Theorem). *Every graph $G = (V, E)$ has a repulsive asteroidal set.*

Proof. We prove the theorem by induction on the number of vertices of the graph. Clearly all graphs on at most two vertices have a RAS. For graphs G on at least three vertices we distinguish between three cases.

Case 1. G is prime.

By Corollary 4, the set L of all extremities of G is non empty. Now let I be any maximal independent set of $G[L]$.

We claim that I is a RAS of G. Clearly I is an asteroidal set of G since $G - N[v]$ is connected for all $v \in I$. Suppose I is not a RAS of G, thus there is a vertex $u \in V \backslash N[I]$ such that $I \subseteq C$ for a component C of $G - N[u]$. Since I is a maximal independent set of $G[L]$, we have $L \subseteq N[I]$. Hence $N[u]$ is a separator of G. Let S be an inclusion minimal separator of G with $S \subseteq N[u]$. By Lemma 3 there is an extremity w of G which belongs to a component of $G - S$ that does not contain any vertex of C. Therefore $I \cup \{w\}$ is an independent set of $G[L]$, contradicting the choice of I. Consequently I is a RAS of G.

Case 2. M is a nontrivial module of G and $G\langle M\rangle$ is complete.

Let A be a RAS of $G[M]$. We consider a vertex $v \in V \backslash N[A]$. Since $V \backslash M \subset N[m]$ for every vertex $m \in M$, we have $v \in M \backslash N[A]$, and every component of $G[M] - N[v]$ is a component of $G - N[v]$. Consequently, A is a RAS of G.

Case 3. M is a nontrivial module of G and $G\langle M\rangle$ is not complete.

Let $m \in M$ be the vertex representing M in $G\langle M\rangle$, and let A be a RAS of $G\langle M\rangle$. We claim that A is a RAS of G if $m \notin A$. Otherwise, $(A \backslash \{m\}) \cup \{m'\}$ is a RAS of G for every vertex $m' \in M$.

First we consider a vertex $v \in V \backslash (N[A] \cup M)$. If m and v are adjacent in $G\langle M\rangle$, then $M \subset N[v]$ and the components of $G\langle M\rangle - N[v]$ are exactly the components of $G - N[v]$. If m is an isolated vertex of $G\langle M\rangle - N[v]$, then the components of $G - N[v]$ are exactly the components of $(G\langle M\rangle - N[v]) - m$ and the components of $G[M]$. If m and w are two different vertices in one component C of $G\langle M\rangle - N[v]$, then the components of $G - N[v]$ are exactly the components of $G\langle M\rangle - N[v]$ except C and a component D containing M and the remaining vertices of C.

Next we consider a vertex v in $M \backslash N[A]$. Then every component of $G\langle M\rangle - N[m]$ is a component of $G - N[v]$.

In all subcases the graph $G - N[v]$ has two different components containing vertices in A since $G\langle M\rangle - N[v]$ has two different components containing vertices in A. □

Remark 1. There are graphs G that do not have a RAS A with $|A| = \mathsf{an}(G)$. For example, the graph G in Fig. 1 has asteroidal number three, but no maximum asteroidal set of G is a RAS.

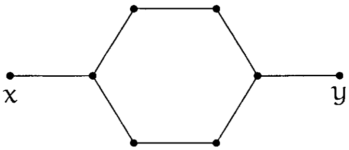

Fig. 1. The RAS $\{x, y\}$ is not a maximum asteroidal set

Now we are ready to present our main theorem which is an immediate consequence of the RAS Theorem.

Theorem 7 (Dominating Target Theorem). *Every graph $G = (V, E)$ has a dominating target D with $|D| \leq \mathsf{an}(G)$. Furthermore, every repulsive asteroidal set of a graph G is a dominating target of G.*

Proof. We show that every repulsive asteroidal set of a graph $G = (V, E)$ is a dominating target of G.

Let D be a RAS of G. Trivially any RAS D of a disconnected graph contains vertices of different components and is thus a dominating target since there is no set $S \subseteq V$ with $G[D \cup S]$ connected.

Thus we may assume that G is connected. Consider any set $S \subseteq V$ such that $G[D \cup S]$ is connected. We claim that $D \cup S$ is a dominating set of G. Let $v \in V \backslash N[D]$. Since D is a RAS the vertices of D are not all contained in one component of $G - N[v]$. Thus there are two vertices a and b of D in different components of $G - N[v]$, implying that any a, b-path contains an internal vertex of $N[v]$. Consequently $D \cup S$ is a dominating set of G.

Therefore any RAS D is a dominating target of G. Furthermore $|D| \leq \mathsf{an}(G)$ since D is an asteroidal set. □

Note that this theorem immediately implies the Dominating Pair Theorem for asteroidal triple-free graphs since $\mathsf{an}(G) \leq 2$ for all AT-free graphs. Therefore we also obtain a simple proof of the Dominating Pair Theorem.

Remark 2. There are graphs G that do not have a RAS A with $|A| = \mathsf{dt}(G)$. For example, the graph G in Fig. 2 has a dominating target of size two, but every RAS of G has at least three vertices.

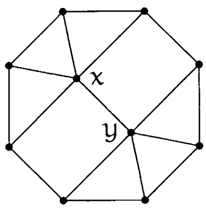

Fig. 2. The dominating target $\{x, y\}$ is not a RAS

5. Minimal Triangulations

We start with some preliminaries on minimal triangulations and minimal separators of graphs.

Definition. A graph H is *chordal* if it does not contain a chordless cycle of length at least four as an induced subgraph.

Definition. A *triangulation* of G is a graph H with the same vertex set as G such that H is chordal and G is a subgraph of H. A triangulation H of G is called *minimal* if there is no proper subgraph H' of H which is also a triangulation of G.

In the following lemma we mention two useful characteristics of minimal triangulations (see e.g. [10]).

Lemma 8. *If H is a minimal triangulation of a graph G then*

1. *If a and b are nonadjacent in H, then every minimal a, b-separator in H is also a minimal a, b-separator in G.*
2. *If S is a minimal separator in H and if C is the vertex set of a component of $H - S$, then C induces also a component in $G - S$.*

As mentioned already in the introduction, it can be shown that a graph G is AT-free if and only if every minimal triangulation of G is AT-free. Now we prove that graphs of bounded asteroidal number have an analogous characterization. We start with the easy part.

Lemma 9. *Let $k \geq 1$. If $\mathsf{an}(H) \leq k$ for every minimal triangulation H of a graph G, then $an(G) \leq k$.*

Proof. Assume $A = \{a_1, \ldots, a_\ell\}$ is a maximum asteroidal set in G. Construct a graph H of G by making a clique of $V \backslash A$. H is a triangulation of G since H is a split graph. Notice that A is an asteroidal set of H, since H is a supergraph of G with $N_H[a_i] = N_G[a_i]$ for all $i \in \{1, 2, \ldots, \ell\}$. Since H is a triangulation of G there exists a spanning subgraph H' of H that is a minimal triangulation of G. Consequently, $N_H[a_i] = N_{H'}[a_i] = N_G[a_i]$ for all $i \in \{1, 2, \ldots, \ell\}$, thus A is an asteroidal set of H', implying $\mathsf{an}(G) = \ell \leq \mathsf{an}(H') \leq k$. □

Lemma 10. *Let $k \geq 1$. If $\mathsf{an}(G) \leq k$ then $\mathsf{an}(H) \leq k$ for every minimal triangulation H of G.*

Proof. Let H be a minimal triangulation of G and let I be an independent set in H. Let $a \in I$ such that $I \backslash \{a\}$ is contained in one component of $H - N_H[a]$. We claim that $I \backslash \{a\}$ is contained in one component of $G - N_G[a]$.

To prove this claim, let $b \in I \backslash \{a\}$ and let S be the unique minimal a, b-separator close to a in H. Then $S \subseteq N_H[a]$ implies that $I \backslash \{a\}$ is contained in the component of $H - S$ containing b. By Lemma 8, S is a minimal a, b-separator in G, and the vertex sets of the components containing b in $H - S$ and $G - S$ are the same. Clearly the component of $G - S$, that contains $I \backslash \{a\}$, does not contain any neighbor of a. Hence its vertices are contained in a component of $G - N_G[a]$. This proves the claim.

Now let A be an asteroidal set of maximum cardinality in any minimal triangulation H of G. By the claim, A is also an asteroidal set of G. Thus $\mathsf{an}(H) \leq \mathsf{an}(G) \leq k$. □

From Lemmas 9 and 10 we conclude:

Theorem 11. *Let $G = (V, E)$ be a graph and $k \geq 1$. Then $\mathsf{an}(G) \leq k$ if and only if $\mathsf{an}(H) \leq k$ for every minimal triangulation H of G.*

6. Tree Spanners

This section shows the power of the insight in the structure of graphs of bounded asteroidal number that we established in previous sections.

We consider additive tree spanners of connected graphs.

Definition. Let d_G denote the distance metric on the connected graph $G = (V, E)$. A spanning tree T of G is called *c-additive tree spanner* of G whenever $d_T(x, y) \leq d_G(x, y) + c$ for every pair $x, y \in V$.

For more information on tree spanners see e.g. [4, 17]. In this section we will prove the following theorem.

Theorem 12. *Let D be a dominating target of a connected graph G. Then G has a $(3|D| - 1)$-additive tree spanner.*

In what follows we consider a connected graph $G = (V, E)$ and a dominating target $D \subseteq V$ of G. We choose a set S of minimum cardinality such that $G[D \cup S]$ is connected.

Let T be an arbitrary spanning tree of $G[D \cup S]$. For a leaf $s \in S$ of T the graph $G[D \cup S \backslash \{s\}]$ is still connected; contradicting the choice of S. Hence every leaf of T is a vertex in D. By B we denote the set of *branch points* in T, these are vertices of degree at least three.

Lemma 13. *Every path in T contains at most $|D|$ vertices in $D \cup B$.*

Proof. Let $b \in B$ be a branchpoint of a path P in T. Then there is a component of $T - b$ containing no vertex of P. This component contains a leaf in D. Consequently, the number of vertices of $D \cup B$ on P is at most $|D|$. □

Let $(v_1, v_2, \ldots, v_l)$ be a shortest path in G. If $v_1 \in D \cup S$ then $u = v_1$, otherwise we fix a vertex $u \in N(v_1) \cap (D \cup S)$. If $v_l \in D \cup S$ then $w = v_l$, otherwise we fix a vertex $w \in N(v_l) \cap (D \cup S)$. Let $(x_1, x_2, \ldots, x_q)$ be the (u, w)-path in T, $u = x_1$ and $w = x_q$. We define a linear ordering on the vertex set of the (u, w)-path in T by $a \prec b$ if $a = x_i$, $b = x_j$ and $i < j$. We write $a \preccurlyeq b$ if $a \prec b$ or $a = b$.

For every vertex $y \in D \cup S$ let y' denote the first common vertex of two paths in T, namely the (y, w)-path and the (u, w)-path. For $i = 1, \ldots, l$ we fix vertices u_i and w_i in $N[v_i] \cap (D \cup S)$ such that $u_i' \preccurlyeq w_i'$ and the (u_i', w_i')-path in T has maximum length. Especially let $u = u_1 = u_1'$ and $w = w_l = w_l'$.

Definition. A subpath of the (u_i, w_j)-path in T is called (i, j)-*strip* if the endvertices are different and belong to $\{u_i, w_j\} \cup D \cup B$, all inner vertices belong to $S \backslash B$, and $i \leq j$.

Hence the vertices of $D \cup B$ split the (u_i, w_j)-path in T into (i, j)-strips such that every edge on the (u_i, w_j)-path belongs to exactly one (i, j)-strip.

Lemma 14. *If the (a, b)-path in T is an (i, j)-strip then $d_T(a, b) \leq 2 + d_G(v_i, v_j)$.*

Proof. Let W be the set of inner vertices on the (a, b)-path in T. Let $S' = \{v_i, v_{i+1}, \ldots, v_j\} \cup (S \backslash W)$. Then $G[D \cup S']$ is connected, and, by the choice of S, $|S| \leq |S'|$ implies $d_T(a, b) = 1 + |W| \leq 2 + j - i = 2 + d_G(v_i, v_j)$. □

Let M be the set of $(1, l)$-strips. To every strip $s \in M$ we assign indices i and j such that s is an (i, j)-strip and $j - i$ is minimum.

Lemma 15. *Let $i \leq j$ and $p \leq q$ be pairs of indices assigned to different strips s and t in M with $i \leq p$. Then the paths $(v_i, v_{i+1}, \ldots, v_{j-1})$ and $(v_p, v_{p+1}, \ldots, v_{q-1})$ have no edge in common.*

Proof. Assume the contrary. Then $i + 2 \leq j$, $p + 2 \leq q$, and $3 \leq l$. Let a and b be the endvertices of s and let c and d be the endvertices of t, $a \prec b$ and $c \prec d$.

First we assume $i = p$. If $u_{i+1}' \preccurlyeq a$ then s is an $(i + 1, j)$-strip. If $b \preccurlyeq u_{i+1}'$ then $b \preccurlyeq w_{i+1}'$ and s is an $(i, i + 1)$-strip. Both contradicts the choice of i and j. Hence $a \prec u_{i+1}' \prec b$ and $u_{i+1}' = u_{i+1}$ since no inner vertex of s is in B. Similarly, u_{i+1} is an inner vertex of t. This contradicts the choice of s and t that have no inner vertex in common. Hence $i \neq p$. Analogously we prove $j \neq q$.

Next we assume $i < p < q < j$. If $b \preccurlyeq c$ then s is an (i, q)-strip, since $u_i' \preccurlyeq a \prec b \preccurlyeq c \prec d \preccurlyeq w_q'$. If $d \preccurlyeq a$ then s is an (p, j)-strip, since $u_p' \preccurlyeq c \prec d \preccurlyeq a \prec b \preccurlyeq w_j'$. Both contradicts the choice of i and j since $j - i > q - i$ and $j - i > j - p$.

Finally let $i < p < j < q$. We consider the neighbours of v_{j-1} in $D \cup S$. Note that $p < j - 1 < j$ since we assume that the paths $(v_i, v_{i+1}, \ldots, v_{j-1})$ and $(v_p, v_{p+1}, \ldots, v_{q-1})$ have an edge in common. Since s is not a $(i, j - 1)$-strip we have $w_{j-1}' \prec b$. Since t is not a $(j - 1, q)$-strip we have $c \prec u_{j-1}'$. Together this implies $c \prec u_{j-1}' \preccurlyeq w_{j-1}' \prec b$. Hence $d \preccurlyeq a$. Both s and t are (p, j)-strips, since $u_p \preccurlyeq c \prec d \preccurlyeq a \prec b \preccurlyeq w_j$. This contradicts our rule to assign indices to strips in M since $i < p < j < q$. □

Now we prove Theorem 12.

Proof. Let $D \subseteq V$ be a dominating target of the connected graph $G = (V, E)$. We choose a set S of minimum cardinality such that $D \cup S$ is connected. Let $T = (D \cup S, F)$ be an arbitrary spanning tree of $G[D \cup S]$. Since D is a dominating target there is a spanning tree $T' = (V, F')$ of G with $F \subseteq F'$, such that every vertex in $V \backslash (D \cup S)$ is a leaf of T'. We will show $d_{T'}(v, v') \leq d_G(v, v') + 3|D| - 1$ for all $v, v' \in V$.

For fixed $v, v' \in V$ let $(v_1, \ldots, v_l)$ be a shortest (v, v')-path in G. Now it is sufficient to show $d_T(u_1, w_l) \leq d_G(v_1, v_l) + 3(|D| - 1)$, for vertices u_1 and w_l in $D \cup S$ adjacent to v_1 and v_l in T', respectively. Clearly we may assume $u_1 \neq w_l$. For $1 \leq i \leq j \leq l$ we choose vertices u_i and w_j as done above, and define (i, j)-strips.

Let a and b be vertices on the (u_1, w_l)-path in T such that the (a, b)-path in T is a strip in $s \in M$ to which we assigned indices i and j. We apply Lemma 14. If $i = j$ then we obtain $d_T(a, b) \leq 2$. If $i < j$ we obtain $d_T(a, b) \leq 3 + d_G(v_i, v_{j-1})$. Summing over all $s \in M$, Lemma 15 implies $d_T(u_1, w_l) \leq d_G(v_1, v_l) + 3(r + 1)$ where r denotes the number of inner vertices on the (u_1, w_l)-path in T that belong to $D \cup B$. We extend the (u_1, w_l)-path in T to a path connecting two leaves of T. By Lemma 13 this new path contains at most $|D|$ vertices in $D \cup B$. Hence we have $d_T(u_1, w_l) \leq d_G(v_1, v_l) + 3(|D| - 1)$. □

Corollary 16. *Every graph G has a* $(3 \cdot \mathrm{dt}(G) - 1)$*-additive tree spanner.*

Remark 3. Corollary 16 is almost the best possible, since, for every $k \geq 1$, $\mathrm{dt}(C_{3k}) = k$, and every tree spanner of C_{3k} is $(3k - 2)$-additive.

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