

Domination Graphs with Nontrivial Components

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Abstract. A tournament is an oriented complete graph. Vertices x and y dominate a tournament T if for all vertices $z \neq x, y$, either (x, z) or (y, z) are arcs in T (possibly both). The domination graph of a tournament T is the graph on the vertex set of T containing edge $\{x, y\}$ if and only if x and y dominate T . In this paper we determine which graphs containing no isolated vertices are domination graphs of tournaments.

A *tournament* is an oriented complete graph. Let $V(D)$ and $A(D)$ denote the vertex and arc sets of a digraph D respectively. An arc from vertex x to y is denoted by (x, y) . If D is a digraph and $(x, y) \in A(D)$, we say x *beats* y . Let $V(G)$ and $E(G)$ denote the vertex and arc sets of a graph G respectively. An edge between vertices x and y is denoted by $\{x, y\}$. A *trivial component* (or graph) is a single vertex.

Vertices x and y *dominate* a tournament T if for all vertices $z \neq x, y$, either x beats z or y beats z (possibly both). The *domination graph* of a tournament T , denoted $\text{dom}(T)$, is the graph on vertices $V(T)$ with $\{x, y\} \in E(\text{dom}(T))$ if and only if x and y dominate T (see Figure 1).

Domination graphs were introduced by Fisher et al. [1] in conjunction with competition graphs. The competition graph of a digraph D is the graph on the same vertices as D with an edge between two vertices if they beat a common vertex in D . The domination graph of a tournament is the complement of the competition graph of its reversal (see [1]). See Lundgren [6] or Roberts [9] for more about competition graphs and Moon [7] or Reid [8] for more on tournaments.

The *domination digraph* $\mathcal{D}(T)$ of a tournament T is the digraph with the same vertices as T where vertex x beats vertex y in $\mathcal{D}(T)$ if x and y dominate T and x beats y in T . Thus, $\mathcal{D}(T)$ is the orientation of $\text{dom}(T)$ induced by T (see Figure 1).

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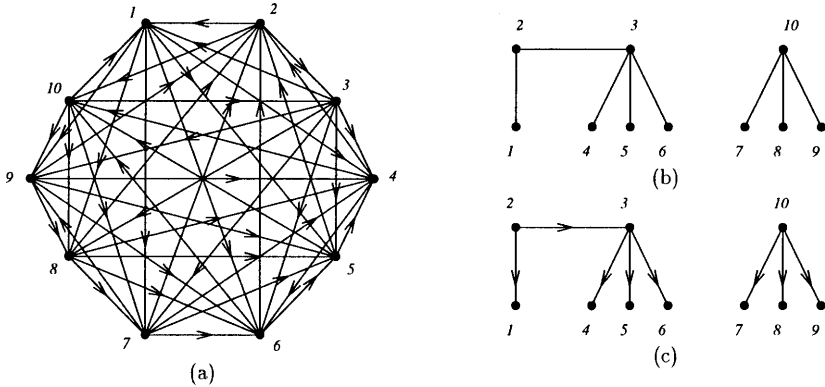


Fig. 1. A tournament (a) with its domination graph (b) and domination digraph (c)

1. Preliminaries

In this paper we extend previous work [1, 2, 3, 4] on domination graphs. A vertex of a graph is *pendant* if it is adjacent to exactly one vertex. A *caterpillar* is a connected graph whose nonpendant vertices form a (possibly trivial) path. A trivial caterpillar is a single vertex. If a caterpillar has nonpendant vertices, the path they form is called the *spine*. The only caterpillar with all vertices pendant is K_2 . The spine of K_2 is either single vertex. A *star*, denoted $K_{1,n}$ is a caterpillar with exactly one vertex adjacent to the other n vertices. A *spiked cycle* is a connected graph whose nonpendant vertices form a cycle. Fisher et al. [1] determined necessary conditions for a graph to be the domination graph of a tournament.

Proposition 1.1. [1] *The domination graph of a tournament is either a spiked odd cycle perhaps with some isolated vertices, or a graph whose components are all caterpillars.*

Proposition 1.2. [2] *In the domination digraph of a tournament, a vertex loses to at most one vertex and beats at most one vertex that beats other vertices.*

A typical caterpillar may be pictured as shown in Figure 2, namely as a path (the *spine*) with pendant vertices (*clusters*) attached to the vertices of the path. A vertex on the spine of a caterpillar is an *end vertex* if it is adjacent to at most one other vertex on the spine. Observe that the end vertices of the spine are not pendant. For example, a path with at least three vertices has two clusters, each containing a single vertex. We say that a caterpillar has a *triple end* if at least one of the end vertices of the spine has degree at least four. Observe that for $n \geq 4$, $K_{1,n}$ has a triple end, but $K_{1,3}$ does not. We say a caterpillar is *triple end-free* if the degree of each end vertex of the spine is at most 3. Fisher et al. [3] have characterized connected domination graphs.

Proposition 1.3. [3] *A connected graph is the domination graph of a tournament if and only if it is a spiked odd cycle, a star, or a caterpillar with a triple end.*

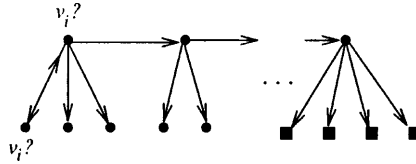


Fig. 2. Orientation of a caterpillar

In this paper we determine which graphs on $n \geq 4$ components, without isolated vertices, are domination graphs of tournaments. Thus all components of these graphs are nontrivial.

2. Domination Graphs on $n = 4$ or $n \geq 6$ Nontrivial Components

In this section we consider those graphs G that are domination graphs of tournaments with at least two nontrivial components. Let G be such a graph and T a tournament such that $\text{dom}(T) = G$. By Proposition 1.1, G is the union of nontrivial caterpillars G_1, \dots, G_n . It will be useful to *properly color* such graphs with two colors, say red and blue, where a proper coloring is an assignment of one color to each vertex such that if two vertices are adjacent, then they are assigned different colors. Each G_i can be so colored in one way (up to interchanging the two colors). The next result follows from Proposition 1.2 and the proof of Theorem 5 of Fisher et al. [3]. Let T_i denote the subtournament of T induced by $V(G_i)$.

Proposition 2.1. *If G , the union of n nontrivial caterpillars G_1, \dots, G_n , is the domination graph of a tournament T and is properly colored with two colors, then the following conditions hold in T :*

1. *The subdigraph of $\mathcal{D}(T)$ on $V(G_i)$ is as shown in Figure 2 (one edge of G_i may be directed either way as indicated by the two arrows).*
2. *Remaining arcs of T_i that are not between vertices in the same cluster of G_i are directed to the right between vertices of different colors and to the left between vertices of the same color.*
3. *Arcs between vertices in the same nonend cluster of each G_i may be directed in any way.*

The reader is warned that Proposition 2.1 does not say that arcs within every cluster may be oriented arbitrarily. Let v_i denote the only vertex in the subdigraph of the domination digraph on $V(G_i)$ with indegree 0 (there are two possibilities, marked “ $v_i?$ ” in Figure 2). If v_i is a pendant vertex, then every vertex in the same cluster with v_i must beat v_i .

Observe that if the spine has an even number of vertices and v_i is pendant, then the vertices of the rightmost cluster (squares in the figure) have a different color than v_i . If v_i is not pendant (and the spine has an even number of vertices), then the vertices of the rightmost cluster have the same color as v_i . On the other hand,

if the spine has an odd number of vertices and v_i is pendant (respectively, not pendant), then the vertices of the rightmost cluster and v_i have the same color (respectively, different colors).

It is also useful to observe that if G_i is not $K_{1,3}$ and is triple end-free, then there is a blue vertex of G_i that beats every other blue vertex of G_i in T_i as well as a red vertex of G_i that beats every other red vertex of G_i in T_i . Furthermore, the orientation of all arcs in T between T_i and T_j is completely determined by the orientation of the arcs between v_i and v_j , as shown in the next lemma.

Lemma 2.2. *Let G be the domination graph of a tournament T where G_1, \dots, G_n denote the properly colored components of G such that v_i and v_j have the same color for all i, j . If an arc is directed from v_i to v_j , and if u and w are vertices of G_i and G_j , respectively, then there is an arc from u to w if and only if u and w are the same color.*

Proof. Assume G is properly colored with red and blue so v_i is red for all i . Suppose v_i beats v_j and u and w are vertices of G_i and G_j , respectively. Then there are directed paths $v_i, u_1, u_2, \dots, u_{m-1}, u$ from v_i to u and $v_j, w_1, w_2, \dots, w_{l-1}, w$ from v_j to w in the subdigraphs of $\mathcal{D}(T)$ induced by $V(G_i)$ and $V(G_j)$ respectively. The proof of this lemma is by induction on l , the length of the path from v_j to w .

Suppose $l = 1$. Then w is blue. Since v_i beats v_j and v_j and w dominate T , w beats v_i , a red vertex. Then since v_i and u_1 dominate T , u_1 , a blue vertex, beats w . Then since v_j and w dominate T , v_j beats u_1 . And since u_1 and u_2 dominate T , u_2 beats v_j . Then w must beat u_2 , a red vertex. Continuing in this way we show that for all $x \in \{v_i, u_1, u_2, \dots, u\}$, x beats w if and only if x and w have the same color. In particular, this holds if $x = u$.

So assume the statement is true for $l \leq k$ and suppose $l = k + 1$. By the induction hypothesis, u beats w_{l-1} if and only if u and w_{l-1} have the same color. We consider two cases.

Suppose w_{l-1} beats u . Then w and u have the same color. Since u_{m-1} and u dominate T , u_{m-1} beats w_l . Since w and w_{l-1} dominate T , w beats u_{m-1} . Then u must beat w .

On the other hand, suppose u beats w_{l-1} . Then w and u are different colors. Since w_{l-1} and w dominate T , w beats u .

Thus, u beats w if and only if u and w are the same color. \square

By Proposition 1.3, any triple end-free caterpillar alone, except $K_{1,3}$, is not the domination graph of a tournament. As we will see, whether or not a collection of n triple end-free caterpillars is the domination graph of a tournament depends on n . Furthermore, whether or not $K_{1,3}$ can be a component of such a domination graph depends on n and the nature of the other components. Before continuing, we need some definitions.

Two vertices u and v are *dominated* in a digraph if there is a vertex w such that (w, u) and (w, v) are arcs. Two vertices u and v are *paired* in a digraph if there is a vertex w such that (u, w) and (v, w) are arcs, or (w, u) and (w, v) are arcs. Two vertices are *distinguished* if there is a vertex w such that (u, w) and (w, v) are arcs,

or (w, u) and (v, w) are arcs. A digraph is *well-covered* if every two distinct vertices u and v are paired and distinguished.

Lemma 2.3. *If there is a well-covered tournament on $n \geq 2$ vertices, then the union of any n nontrivial caterpillars is the domination graph of a tournament.*

Proof. Let G be such a union of caterpillars; call them $G_i, i = 1, \dots, n$. To construct a tournament T having domination graph G , properly color G with red and blue. Proposition 2.1(1) establishes at least two possibilities for the domination digraph of T . Choose the one in which all v_i are nonpendant. Without loss of generality, assume that v_i is red for all i . Add arcs to complete the tournament as follows. Within each cluster, add arcs in any way. Proposition 2.1 then determines the remaining arcs within each subtournament on $V(G_i)$.

Connect the vertices $\{v_i | i = 1, \dots, n\}$ so that the resulting n -vertex tournament is well-covered. Then Lemma 2.2 determines any remaining arcs in T . Now we must verify that all pairs of vertices, except those that are adjacent in G , are dominated.

1. It is easy to check that no vertex beats pairs of vertices that are adjacent in G .
2. Consider two vertices in the same caterpillar, of different colors but not adjacent in G , call them v and w with w to the right of v . There must be a vertex $u \neq v$, adjacent to w in G and with an arc directed from u to w . Thus u is the same color as v and to the right of v , so u beats both v and w .
3. Consider two vertices, v and w , in the same caterpillar, say G_i , of the same color. Pick any other caterpillar, say G_j . Depending on the direction of the arc between v_i and v_j in T , and on the color of v and w , either all red vertices of G_j beat v and w in T , or all blue vertices do. Since G_j must have at least one red and at least one blue vertex, v and w are dominated.
4. Consider two vertices, v and w , in different caterpillars, say G_i and G_j , and of the same color. Pick k so that v_k pairs v_i and v_j . Depending on the direction of the arc between v_k and v_i in T , and on the color of v and w , either all red vertices of G_k beat v and w in T , or all blue vertices do.
5. Consider two vertices, v and w , in different caterpillars, say G_i and G_j , and of different colors. Pick k so that v_k distinguishes v_i and v_j . Depending on the direction of the arc between v_k and v_i in T , and on the colors of v and w , either all red vertices of G_k beat v and w in T , or all blue vertices do.

Thus all pairs, except those adjacent in G , are dominated. □

We use this lemma to characterize a large class of domination graphs with nontrivial components.

Theorem 2.4. *If G is the union of n nontrivial caterpillars where $n = 4$ or $n \geq 6$, then G is the domination graph of a tournament.*

Proof. It is well known that for $n = 3$ and $n \geq 5$, there is a tournament on n vertices such that every arc is in a 3-cycle. For example, regular tournaments on an odd number of vertices have this property. To obtain such a tournament on an

even number of vertices, first construct the following tournament T , on $2m + 1$ vertices, $m \geq 3$. Let the vertices be labeled with Z_{2m+1} , the integers modulo $2m + 1$. For each $i \in Z_{2m+1}$, let i beat $i + j$ for all $j \in \{1, 2, \dots, m - 1, m + 1\}$. This tournament is regular, so every arc is in a 3-cycle. Moreover, every arc not involving vertex 0 is in a 3-cycle that does not involve 0. So the $(2m)$ -subtournament of T on vertices labeled with $i \in Z_{2m+1} - \{0\}$ has the desired property.

Take any tournament such that every arc is contained in a 3-cycle, add a vertex, and direct all arcs from the new vertex. It is easy to check that the resulting tournament is well-covered. Hence, for $n = 4$ and $n \geq 6$, G is the domination graph of a tournament. □

The next result will help establish that if G is the domination graph of a tournament with n nontrivial triple end-free components, none of which is $K_{1,3}$, then in fact n must be 4 or at least 6.

Lemma 2.5. *If G is the union of $n \geq 2$ nontrivial triple end-free caterpillars, G_1, \dots, G_n , none of which is $K_{1,3}$, and G is the domination graph of a tournament T , then there is a well-covered tournament on n vertices.*

Proof. Let G be such a graph and T a tournament such that $\text{dom}(T) = G$. Properly color G with red and blue so that v_i is red for all i . Proposition 2.1 determines many arcs in T . We must assume that arcs within any cluster could be oriented in any way, except the end clusters, which are guaranteed to contain no more than two vertices. Without loss of generality, assume that the rightmost vertex in each end cluster beats any other vertex in that cluster. Let T_i denote the subtournament of T induced by $V(G_i)$. Depending on the subtournament on $\{v_i | i = 1, \dots, n\}$, Lemma 2.2 determines arcs among the T_i .

We claim that the tournament induced on $\{v_i | i = 1, \dots, n\}$ is well covered. Suppose not. Then there are vertices, without loss of generality v_1 and v_2 , that are either not paired or not distinguished. Without loss of generality, assume v_2 beats v_1 . To arrive at a contradiction, and finish the proof, we will show that some pair of vertices, other than the endpoints of an edge of G , is not dominated. There are two cases: either v_1 and v_2 are not paired or not distinguished.

Suppose v_1 and v_2 are not paired. Let v be the red vertex in T_2 that beats every other red vertex in T_2 . We show that no vertex beats v_1 and v .

Consider w in T_1 such that w beats v_1 in T . Suppose that w is blue. Then w is to the left of v_1 , a contradiction since v_1 is red and has indegree 0 in $\mathcal{D}(T)$. Therefore w is red. Hence v beats w in T and so no vertex in T_1 beats v_1 and v .

Consider w in T_2 such that w beats v in T . Since v beats every red vertex in T_2 , we conclude that w is blue. Therefore v_1 beats w so that no vertex in T_2 beats v_1 and v .

Finally, consider w in T_i where $i > 2$ such that w beats v in T . If v_i beats v_2 , then w must be red. Since v_1 and v_2 are not paired, v_1 beats v_i . Consequently, v_1 beats w and so w does not dominate v_1 and v . If v_2 beats v_i , then w must be blue. Since v_1 and v_2 are not paired, v_i beats v_1 . Consequently, v_1 beats w and so w does not dominate v_1 and v . So no vertex in T_i where $i > 2$ beats v_1 and v .

Thus, v_1 and v are not dominated, so v and v_1 are adjacent in $\text{dom}(T) = G$, a contradiction, completing the first case.

Suppose v_1 and v_2 are not distinguished. Let v be the blue vertex in T_1 that beats every other blue vertex in T_1 . We show that no vertex beats v_2 and v .

Consider w in T_1 such that w beats v in T . Since v beats every blue vertex in T_1 , we conclude that w is red. Therefore v_2 beats w so no vertex in T_1 beats v_2 and v .

Consider w in T_2 such that w beats v_2 . Suppose w is blue. Then w is to the left of v_2 . This is a contradiction since v_2 has indegree 0 in $\mathcal{D}(T)$. Therefore w is red. Then since v is blue and v_2 beats v_1 we conclude that v beats w and so no vertex in T_2 beats v_2 and v .

Consider w in T_i where $i > 2$ such that w beats v in T . If v_i beats v_1 , then since v_1 and v_2 are not distinguished, v_i beats v_2 . Consequently w must be blue, so that v_2 beats w and therefore w does not beat v_2 and v . On the other hand, if v_1 beats v_i , then since v_1 and v_2 are not distinguished, v_2 beats v_i . Consequently w must be red, so v_2 beats w so w does not beat v_2 and v . Thus, no vertex of T_i where $i > 2$ beats v_2 and v in T . □

Theorem 2.6. *If G is the union of $n \geq 2$ nontrivial triple end-free caterpillars, none of which is $K_{1,3}$, then G is the domination graph of a tournament T if and only if n is not 2, 3 or 5.*

Proof. It is easy to see that no tournament on 2 or 3 vertices is well-covered. It is also the case that no 5-vertex tournament is well covered. This has been checked by hand and can be verified using the list of all 5-vertex tournaments given in Moon [7]. Thus, if G is the domination graph of a tournament T , then n is not 2, 3 or 5.

On the other hand, if $n = 4$ or $n \geq 6$, then G is the domination graph of a tournament by Theorem 2.4. □

3. Domination Graphs on 5 Nontrivial Components

We now consider domination graphs (of tournaments) with 5 components, all of which are nontrivial.

Theorem 3.1. *If G is the union of five nontrivial caterpillars, at least one of which has a triple end, then G is the domination graph of a tournament.*

Proof. There is a tournament on 5 vertices in which every two vertices are distinguished (for example, see Figure 3), and every pair of vertices, except one, is paired. Let us call such a tournament *almost well-covered*.

Let G be a graph as in the statement of the theorem and denote by G_i , $i = 1, \dots, 5$ the five caterpillars where G_2 is a caterpillar with a triple end. Create tournament T as follows. For $i \neq 2$, color G_i with red and blue so the pendant vertices are blue. These create subtournaments on each $V(G_i)$ consistent with Proposition 2.1 so that each v_i is red. Now, properly color G_2 and orient the edges

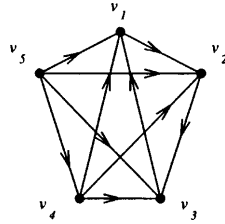


Fig. 3

so that a triple end is the rightmost cluster and so that v_2 and the vertices in the rightmost cluster have the same color, red. Add arcs among the vertices of the rightmost cluster so that every vertex in this set has an arc from at least one of the others (this is possible because G_2 has a triple end). Complete the subtournament of T on $V(G_2)$ so that it is consistent with Proposition 2.1.

Add arcs to complete the tournament as follows. Connect the vertices $\{v_i | i = 1, \dots, 5\}$ so that the resulting 5-vertex tournament is almost well covered. We may assume that v_1 and v_2 are not paired and that (v_2, v_1) is an arc. Then Lemma 2.2 determines the remaining arcs of T .

Now we must verify that all pairs of vertices, except those that are adjacent in G , are dominated. This done as in Theorem 2.3, except in Case 4, for two vertices in G_1 and G_2 of the same color.

If the two vertices, $v \in V(G_1)$ and $u \in V(G_2)$, are blue, then the two are dominated by v_1 . On the other hand, if the two vertices are red, then there is a red vertex in the rightmost cluster of G_2 that beats every red vertex in G_1 as well as u .

Thus, all pairs, except those adjacent in G are dominated. □

Theorem 3.2. *If G is the union of five nontrivial caterpillars, at least one of which is a $K_{1,3}$, then G is the domination graph of a tournament.*

Proof. Let G be such a graph. Denote by G_i , $i = 1, \dots, 5$, the five caterpillars where G_4 is a $K_{1,3}$. For $i \neq 4$, create a subtournament of T on $V(G_i)$ consistent with Proposition 2.1. Define the subtournament of T on $V(G_4)$ as follows: let one vertex (v_4) beat the other three and the subtournament on the other three be a directed cycle. Color the vertices of G_i with red and blue, where v_i is red for all i .

Add arcs to complete the tournament as follows. Connect the vertices $\{v_i | i = 1, \dots, 5\}$ as shown in Figure 3. Then Lemma 2.2 determines the remaining arcs in T .

Now we need to verify that all pairs of vertices, except those that are adjacent in G are dominated. Cases 1, 2 and 3 of the proof for Lemma 2.3 hold for this tournament. Notice that the tournament on $\{v_i | i = 1, \dots, 4\}$ is well-covered. Furthermore, every pair of vertices in the subtournament on $\{v_i | i = 1, \dots, 5\}$ is paired and v_4 and v_5 are the only two vertices that are not distinguished. Thus, Case 4 also holds and Case 5 holds for all pairs except $u \in V(G_4)$ and $v \in V(G_5)$. All that remains is to verify pairs of vertices $u \in V(G_4)$ and $v \in V(G_5)$ of different colors.

1. If v is blue and u is red (so $u = v_4$), then they are dominated by v_5 .
2. If v is red and u is blue, then there is another blue vertex in G_4 that beats u and v .

Thus, all pairs, except those adjacent in G , are dominated. \square

Corollary 3.3. *If G is the union of five nontrivial components, then G is the domination graph of a tournament if and only if each of the components is a caterpillar and at least one of the components has a triple end or is $K_{1,3}$.*

4. The Main Theorem

We have completed all the work necessary to characterize domination graphs, G , (of tournaments) having n components, all of which are nontrivial. If $n = 1$, then G is connected and by Proposition 1.3, G is either a spiked odd cycle, a star, or a caterpillar with a triple end. If $n \geq 2$, then each component is a caterpillar. The cases $n = 4$ and $n \geq 6$ are treated in Theorem 2.4. The case $n = 5$ is treated in Corollary 3.3. The cases $n = 2$ and 3 are treated in [4], and although lengthy, are very similar to the case $n = 5$.

Theorem 4.1. *If G is the union of n nontrivial components, then G is the domination graph of a tournament if and only if*

$n = 1$ and G is a spiked odd cycle, a star or a caterpillar with a triple end or $n \geq 2$, each component is a caterpillar and one of the following occurs:

1. $n = 2$ and either both caterpillars have a triple end or one has a triple end and the other is $K_{1,3}$;
2. $n = 3$ and either all three caterpillars have a triple end, or one caterpillar is $K_{1,3}$ and the other two have a triple end, or two caterpillars are $K_{1,3}$ and the other one has a triple end;
3. $n = 5$ and at least one of the caterpillars has a triple end or is a $K_{1,3}$;
4. $n = 4$ or $n \geq 6$.

Domination graphs (of tournaments) with trivial components are considered separately in [5].

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