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On Vertex-Disjoint Complete Bipartite Subgraphs in a Bipartite Graph

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Abstract. We show that, if $G = (X, Y; E)$ is a bipartite graph with $|X| = |Y| = 4s$ and $\delta(G) \geq 4s - 3$ for any integer $s \geq 2$, then G contains four vertex-disjoint copies of $K_{s,s}$. This constitutes a partial answer to a conjecture in [4].

1. Introduction

Hajnal and Szemerédi [3] proved that if G is of order sk with minimum degree at least $(s-1)k$, then G contains k vertex-disjoint complete subgraphs of order s, where $s \geq 3$ and $k \geq 1$ are integers. The case $s = 3$ was first obtained by Corrádi and Hajnal [2]. In [4] and [5], we have considered a similar problem in bipartite graphs and proposed a conjecture as follows:

Conjecture 1. [4] *Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = sk$ where s and k are integers with $s \geq 2$ and $k \geq 1$. If the minimum degree of G is at least $(s-1)k + 1$, then G contains k vertex-disjoint subgraphs isomorphic to $K_{s,s}$.*

We verified this conjecture for the case $k \leq 3$ in [4]. For $s \in \{2, 3\}$, we proved the following:

Theorem 2. [4] *Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = 3k$. If the minimum degree of G is at least $2k + 1$, then G contains k vertex-disjoint hexagons such that each of them has two chords in G.*

Theorem 3. [5] *Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = 2k$, where k is a positive integer. Suppose that the minimum degree of G is at least $k + 1$. Then G contains $k - 1$ vertex-disjoint quadrilaterals and a path of order 4 such that the path is sertex-disjoint from all the $k - 1$ quadrilaterals.*

The condition on the minimum degree of G in Theorem 3 is also sharp. In this paper, we verify the conjecture for the case $k = 4$, proving the following:

Theorem 4. *Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = 4s$ where s is an integer with $s \geq 2$. If the minimum degree of G is at least $4s - 3$, then G contains four vertex-disjoint complete subgraphs isomorphic to $K_{s,s}$.*

We shall use the following terminology and notation. Let G be a graph. For two disjoint subsets A and B of $V(G)$, $e(A, B)$ is the number of edges of G between A and B. For a vertex u of G and a subset (resp. a subgraph) X of $V(G)$ (resp. G), $N(u, X)$ is the set of vertices in X that are adjacent to u in G. Let $d(u, X) = |N(u, X)|$. Thus $d(u, G) = d(u, V(G)) = d_G(u)$ which is the degree of u in G. For convenience, we consider a bipartite graph G as an ordered triple $(V_1, V_2; E)$ with $V_1 \cup V_2$ as a fixed bipartition and E the edge set of G. Thus if G has another bipartition $V_1' \cup V_2'$ but $(V_1, V_2) \neq (V_1', V_2')$ as two ordered pairs, we regard $(V_1, V_2; E) \neq (V_1', V_2'; E)$. Let $G = (V_1, V_2; E)$ be a given bipartite graph. For a subgraph $H = (U_1, U_2; F)$ of G, we write $G \supseteq H$ if $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$. If $\mathscr{H}$ is a set of subgraphs of G, we write $G \supseteq \mathscr{H}$ if $G \supseteq H$ for all $H \in \mathscr{H}$. For a bipartite graph H, we use kH to denote a set of k vertex-disjoint copies of H. If $X \subseteq V_1$ and $Y \subseteq V_2$, we use (X, Y) to denote the subgraph $(X, Y; F)$ of G induced by $X \cup Y$. For positive integers a and b, we use $E_{a,b}$ to denote a bipartite graph $(A, B; \emptyset)$ with no edges such that $|A| = a$ and $|B| = b$. If we write $G = E_{a,b}$, it means that $|V_1| = a$, $|V_2| = b$ and $E = \emptyset$. The bi-complement of G is the bipartite graph $(V_1, V_2; E')$ with $E' = \{xy \notin E | x \in V_1 \text{ and } y \in V_2\}$. Unexplained terminology and notation are adopted from [1].

2. Proof of Theorem 4

Considering the bi-complement of bipartite graphs in Theorem 4, an equivalent statement of Theorem 4 is as follows:

Theorem 4$'$. *Let $G = (X, Y; E)$ be a bipartite graph with $|X| = |Y| = 4s$ where s is an integer with $s \geq 2$. If the maximum degree of G is at most 3, then G contains four vertex-disjoint copies of $E_{s,s}$.*

Suppose, for a contradiction, that the theorem does not hold. Let $G = (X, Y; E)$ be a minimal counter-example to Theorem 4$'$, i.e., $G \not\supseteq 4E_{s,s}$ but $G - xy \supseteq 4E_{s,s}$ for every $xy \in E$. Let $x'y' \in E$ with $x' \in X$ and $y' \in Y$. By the minimality of G, there exist partitions $X = X_1' \cup X_2' \cup X_3' \cup X_4'$ and $Y = Y_1' \cup Y_2' \cup Y_3' \cup Y_4'$ such that $(X_i', Y_i') = E_{s,s}$ for every $i \in \{2, 3, 4\}$ and (X_1', Y_1') contains $x'y'$ but no other edges of G. As $\Delta(G) \leq 3$, $d(x', Y_2' \cup Y_3' \cup Y_4') \leq 2$, and therefore $d(x', Y_i') = 0$ for some $i \in \{2, 3, 4\}$. W.l.o.g., say $d(x', Y_4') = 0$. Let $X_1 = X_1' - \{x'\}$, $X_2 = X_2'$, $X_3 = X_3'$, $X_4 = X_4' \cup \{x'\}$ and $Y_i = Y_i'$ for every $i \in \{1, 2, 3, 4\}$. Then $(X_1, Y_1) = E_{s-1,s}$, $(X_2, Y_2) = E_{s,s} = (X_3, Y_3)$ and $(X_4, Y_4) = E_{s+1,s}$. Set

$$\Theta = \{(X_1, Y_1), (X_2, Y_2), (X_3, Y_3), (X_4, Y_4)\}. \qquad (1)$$

For convenience, we introduce the following notation. If σ is a permutation of Y and $x_1, x_2, \ldots, x_n$ are distinct vertices of X with $x_1 \in X_4$ and $x_n \notin X_1$, then

$\Theta_{\sigma}^{[x_1,x_2,\ldots,x_n,X_1]} = \{(A_i, B_i) \mid 1 \le i \le 4\}$ such that

$$B_i = \sigma^{-1}(Y_i) = \{y \in Y \mid \sigma(y) \in Y_i\},\ i \in \{1,2,3,4\};$$

$$A_j = X_j \cup \{x_r | x_{r+1} \in X_j; 1 \le r \le n-1\} - \{x_{r+1} \in X_j \mid 1 \le r \le n-1\},\ j \in \{2,3,4\};$$

$$A_1 = X_1 \cup \{x_n\} \cup \{x_r | x_{r+1} \in X_1; 1 \le r \le n-2\} - \{x_{r+1} \in X_1 \mid 1 \le r \le n-2\}.$$

Note that if $\sigma = (y_1, y_2, \ldots, y_m)$, then $\sigma(y_i) = y_{i+1}$ for $i \in \{1,2,\ldots,m-1\}$ and $\sigma(y_m) = y_1$. In the following argument, we manage to choose σ and $\{x_1, x_2, \ldots, x_n\}$ carefully such that $\Theta_{\sigma}^{[x_1,x_2,\ldots,x_n,X_1]} = 4E_{s,s}$ to obtain a contradiction.

Clearly, $e(Y_1, X_2 \cup X_3 \cup X_4) \le \sum_{y \in Y_1} d(y) \le 3s < 3s+1 = |X_2| + |X_3| + |X_4|$. This implies that $d(u, Y_1) = 0$ for some $u \in X_2 \cup X_3 \cup X_4$. As $G \not\supseteq 4E_{s,s}$, we see that $u \notin X_4$. W.l.o.g., say $u \in X_2$. If $d(x, Y_2) = 0$ for some $x \in X_4$, then $\Theta^{[x,u,X_1]} = 4E_{s,s}$, a contradiction. Therefore we have

$$d(x, Y_1) \ge 1 \quad \text{and} \quad d(x, Y_2) \ge 1 \quad \text{for all } x \in X_4. \tag{2}$$

We divide our proof into Part I and Part II according to whether there exists $v \in X_3$ such that $d(v, Y_1) = 0$ or $d(v, Y_2) = 0$.

Part I. There exists $v \in X_3$ such that $d(v, Y_1) = 0$ or $d(v, Y_2) = 0$.

We may assume that $d(v, Y_1) = 0$. For if $d(v, Y_2) = 0$, we redefine (X_1, Y_1) and (X_2, Y_2) by moving u to X_1 and then change the subscripts accordingly. As in (2), we have $d(x, Y_3) \ge 1$ for all $x \in X_4$. As $\Delta(G) \le 3$, we obtain

$$d(x, Y_1) = d(x, Y_2) = d(x, Y_3) = 1 \quad \text{for all } x \in X_4. \tag{3}$$

By (3), for each $i \in \{1,2,3\}$, $e(Y_i, X_4) = s+1$ and therefore Y_i contains a vertex adjacent to at least two vertices of X_4. Let b_i be such an arbitrary vertex in Y_i for each $i \in \{1,2,3\}$. We shall show that $N(b_1, X_4) = N(b_2, X_4) = N(b_3, X_4)$ and then complete the proof in this part. To do so, we will prove a number of claims in the following. Case 1 in the proof of (4) contains the basic idea. Most of the other cases following Case 1 are dealt similarly. We first claim

$$N(b_2, X_4) \supseteq N(b_3, X_4) \quad \text{or} \quad N(b_3, X_4) \supseteq N(b_2, X_4). \tag{4}$$

Proof of (4). Suppose (4) false. Let $N(b_2, X_4) \supseteq \{d_1, d_2\}$ and $N(b_3, X_4) \supseteq \{x_1, x_2\}$ with $d_1 \ne d_2$ and $x_1 \ne x_2$ such that $x_1 b_2 \notin E$ and $d_1 b_3 \notin E$. Then we have either $d(b_2, X_3) = 1$ or $d(b_3, X_2) = 1$ for otherwise $\Theta_{(b_2,b_3)}^{[d_1,u,X_1]} = 4E_{s,s}$. We divide the proof into the following two cases according to the values of $d(b_2, X_3) + d(b_3, X_2)$.

Case 1: $d(b_2, X_3) + d(b_3, X_2) = 2$, i.e., $d(b_2, X_3) = d(b_3, X_2) = 1$.

In this case, $d(b_2, X_1) = d(b_3, X_1) = 0$. We also have either $d(b_1, X_2) = 0$ or $d(b_1, X_3) = 0$. W.l.o.g., say $d(b_1, X_3) = 0$. Then we have

$$N(b_1, X_4) = N(b_3, X_4) = \{x_1, x_2\} \tag{4.1}$$

for otherwise $\Theta_{(b_1,b_3)}^{[x',X_1]} = 4E_{s,s}$ for any $x' \in N(b_1, X_4) - N(b_3, X_4)$. Then $d(b_1, X_2) = 1$ for otherwise $\Theta_{(b_1,b_2)}^{[d_1,u,X_1]} = 4E_{s,s}$. We claim

$$d(y, X_i) = 1 \text{ for all } y \in Y_3 - \{b_3\} \text{ and } i \in \{1,2,4\} \text{ and } d(u, Y_3) = 0 \tag{4.2}$$

Proof of (4.2). First, $b_3u \notin E$ for otherwise $\Theta^{[d_1,u,X_1]}_{(b_1,b_3,b_2)} = 4E_{s,s}$. Let y be an arbitrary vertex in $Y_3 - \{b_3\}$. If $d(y, X_1) = 0$ then $\Theta^{[x_1,X_1]}_{(b_1,y)} = 4E_{s,s}$, and if $d(y, X_2) = 0$ then $\Theta^{[x_1,X_1]}_{(b_1,y,b_2)} = 4E_{s,s}$, a contradiction. Hence $d(y, X_1) \geq 1$ and $d(y, X_2) \geq 1$. As $\Delta(G) \leq 3$, there exists $d' \in \{d_1, d_2\}$ with $d'y \notin E$. Then $d(y, X_2 - \{u\}) \geq 1$ for otherwise $\Theta^{[d',u,X_1]}_{(b_1,y,b_2)} = 4E_{s,s}$, a contradiction. As y is arbitrary in $Y_3 - \{b_3\}$ and $\Delta(G) \leq 3$, this shows, together (3), that $3s \geq e(Y_3, X_1 \cup X_2 - \{u\}) + e(Y_3, X_4) \geq (2s-1) + (s+1) = 3s$, from which (4.2) follows.

Let $c_1 \in Y_1$ with $c_1d_1 \in E$. We claim

$$\text{if } d(c_1, X_2) > 0, d(y, X_i) = 1 \text{ for all } y \in Y_1 - \{b_1\} \text{ and } i \in \{2,3,4\} \tag{4.3}$$

and

$$\begin{aligned}&\text{if } d(c_1, X_2) = 0, \text{ there exists } c' \in Y_1 - \{b_1, c_1\} \text{ such that } d(y, X_i) = 1\\ &\text{for all } y \in Y_1 - \{c_1, c', b_1\} \text{ and } i \in \{2,3,4\}, N(c_1, X_4) = \{d_1, d_2\} \text{ and}\\ &1 \leq d(c', X_2) \leq 2.\end{aligned} \tag{4.4}$$

Proof of (4.3) *and* (4.4). Let y be an arbitrary vertex in $Y_1 - \{b_1\}$. Then $d(y, X_3) \geq 1$ for otherwise $\Theta^{[x_1,v,X_1]}_{(y,b_3)} = 4E_{s,s}$. If $d(y, X_2) = 0$ and there exists $d' \in \{d_1, d_2\}$ with $d'y \notin E$, then $\Theta^{[d',u,X_1]}_{(y,b_2)} = 4E_{s,s}$, a contradiction. Hence if $d(y, X_2) = 0$, then $y = c_1$ and $N(y, X_4) = \{d_1, d_2\}$. It follows that $3s \geq e(Y_1, X_2 \cup X_3) + e(Y_1, X_4) \geq 2(s-1) + d(c_1, X_2) + (s+1) = 3s - 1 + d(c_1, X_2)$. As $\Delta(G) \leq 3$ and $N(c_1, X_4) \cap N(b_1, X_4) = \emptyset$ by (3), we see that (4.3) and (4.4) hold.

By (4.2–4.4), we have $e(X_2 - \{u\}, Y_1 \cup Y_3) \geq 2(s-1) + 1$ with equality only if $d(c_1, X_2) = 0$ and $d(c', X_2) = 1$. As $\Delta(G) \leq 3$, this implies that there exists $\{w_1, w_2\} \subseteq X_2 - \{u\}$ with $d(w_1, Y_1 \cup Y_2) = d(w_2, Y_1 \cup Y_2) = 3$ such that if $d(c_1, X_2) = 1$ or $d(c', X_2) = 2$ then $w_1 \neq w_2$. Furthermore, we see that, if $d(c_1, X_2) = 1$ then $N(w_1) \cap N(w_2) = \emptyset$, and if $d(c_1, X_2) = 0$ and $d(c', X_2) = 2$ then $N(w_1) \cap N(w_2) \subseteq \{c'\}$. Note that if $d(c_1, X_2) = 0$ then $N(c_1, X_4) = \{d_1, d_2\}$. Then we can choose a vertex $y \in N(\{w_1, w_2\})$ such that $y \notin N(\{d_1, d_2\})$ and $d(y, X_2) = 1$. Say w.l.o.g. that $y \in N(w_1)$. If $y \in Y_1$ then $\Theta^{[d_1,w_1,d_2,u,X_1]}_{(y,b_2)} = 4E_{s,s}$, and if $y \in Y_3$ then $\Theta^{[d_1,w_1,d_2,u,X_1]}_{(b_1,y,b_2)} = 4E_{s,s}$, a contradiction.

Case 2: $d(b_2, X_3) + d(b_3, X_2) = 1$.

W.l.o.g., say $d(b_2, X_3) = 1$ and $d(b_3, X_2) = 0$. Then $d(b_2, X_1) = 0$. Similar to the proofs of (4.2), (4.3) and (4.4), we can prove the following (4.5), (4.6) and (4.7). We omit their proofs.

$$d(y, X_i) = 1 \text{ for all } y \in Y_1 - \{b_1\} \text{ and } i \in \{2,3,4\} \text{ and } d(b_1, X_3) = 1. \tag{4.5}$$

Let $a_2 \in Y_2$ with $a_2x_1 \in E$. Then we have

$$\begin{aligned}&\text{if } d(a_2, X_3) > 0, d(y, X_i) = 1 \text{ for all } y \in Y_2 - \{b_2\} \text{ and } i \in \{1,3,4\} \text{ and}\\ &d(v, Y_2) = 0\end{aligned} \tag{4.6}$$

and

if $d(a_2, X_3) = 0$, there exists $a' \in Y_2 - \{a_2, b_2\}$ such that $d(y, X_i) = 1$ for all $y \in Y_2 - \{a_2, a', b_2\}$ and $i \in \{1, 3, 4\}$. Moreover, $N(a_2, X_4) = \{x_1, x_2\}, 1 \leq d(a', X_3) \leq 2, d(v, Y_2 - \{a'\}) = 0$, and if $d(a', X_3) = 1$ then $va' \notin E$. (4.7)

By (4.5–4.7), $e(X_3 - \{v\}, Y_1 \cup Y_2) \geq 2(s-1) + 1$ with equality only if $d(a_2, X_3) = 0$ and $d(a', X_3) = 1$. As $\Delta(G) \leq 3$, this implies that there exists $\{w_1, w_2\} \subseteq X_3 - \{v\}$ with $d(w_1, Y_1 \cup Y_2) = d(w_2, Y_1 \cup Y_2) = 3$ such that, if $d(a_2, X_3) = 1$ or $d(a', X_3) = 2$, then $w_1 \neq w_2$. Furthermore, we see that when $w_1 \neq w_2$, either $N(w_1) \cap N(w_2) = \emptyset$ or $N(w_1) \cap N(w_2) \subseteq \{a'\}$. Clearly, we can choose $y \in N(\{w_1, w_2\})$ such that $y \notin N(\{x_1, x_2\})$ and $d(y, X_3) = 1$. Say $y \in N(w_1)$. If $y \in Y_1$ then $\Theta^{[x_1, w_1, x_2, v, X_1]}_{(y, b_3, b_2)} = 4E_{s,s}$, and if $y \in Y_2$, then $\Theta^{[x_1, w_1, x_2, v, X_1]}_{(y, b_3)} = 4E_{s,s}$, a contradiction. This proves (4).

We claim

$$N(b_2, X_4) = N(b_3, X_4). \tag{5}$$

Proof of (5). Suppose (5) false. By (4), we may assume w.l.o.g. that $N(b_2, X_4) \supseteq N(b_3, X_4)$. Say $N(b_2, X_4) = \{x_1, x_2, x_3\}$ and $N(b_3, X_4) = \{x_1, x_2\}$. Similar to the proofs of (4.2), (4.3) and (4.4), we can show the following (5.1), (5.2) and (5.3).

$$d(b_3, X_2 - \{u\}) = 1, d(b_1, X_2) = 1 \text{ and } N(b_1, X_4) = \{x_1, x_2\}; \tag{5.1}$$

$$d(y, X_i) = 1 \text{ for all } y \in Y_1 - \{b_1\} \text{ and } i \in \{2, 3, 4\}; \tag{5.2}$$

$$d(y, X_j) = 1 \text{ for all} y \in Y_3 - \{b_3\} \text{ and } j \in \{1, 2, 4\} \text{ and } d(u, Y_3) = 0. \tag{5.3}$$

By (5.1–5.3), $e(X_2 - \{u\}, Y_1 \cup Y_3) \geq 2(s-1) + 2$. As $\Delta(G) \leq 3$, this implies that there exists $w \in X_2 - \{u\}$ with $d(w, Y_1 \cup Y_3) = 3$. Clearly, there exists $y \in N(w)$ such that $y \notin N(\{x_1, x_2\})$ and $d(y, X_2) = 1$. As before, it is easy to see that $G \supseteq 4E_{s,s}$, a contradiction. This proves (5).

We claim

$$N(b_2, X_4) \supseteq N(b_1, X_4) \tag{6}$$

Proof of (6). Suppose (6) false. Let $N(b_1, X_4) \supseteq \{d_1, d_2\}$ and $N(b_2, X_4) \supseteq \{x_1, x_2\}$ such that $d_1 \notin N(b_2, X_4)$, $d_1 \neq d_2$ and $x_1 \neq x_2$. Similar to the proofs of (4.2), (4.3) and (4.4), we can show the following (6.1), (6.2) and (6.3).

$$d(b_2, X_1) = 1 \text{ and } d(b_3, X_1) = 1; \tag{6.1}$$

$$d(y, X_i) = 1 \text{ for all } y \in Y_2 - \{b_2\} \text{ and } i \in \{1, 3, 4\}; \tag{6.2}$$

$$d(y, X_j) = 1 \text{ for all } y \in Y_3 - \{b_3\} \text{ and } j \in \{1, 2, 4\}. \tag{6.3}$$

By (6.1–6.3), $e(X_1, Y_2 \cup Y_3) = 2(s-1) + 2$. This implies that there exists $\{w_1, w_2\} \subseteq X_1$ such that $d(w_1, Y_2 \cup Y_3) = d(w_2, Y_2 \cup Y_3) = 3$ and $w_1 \neq w_2$. Clearly, we can choose $y \in N(\{w_1, w_2\})$ such that $y \notin N(\{d_1, d_2\})$. W.l.o.g., say $y \in Y_2 \cap N(w_1)$. Then either $\Theta^{[d_1, w_1, d_2, X_1]}_{(b_1, y)} = 4E_{s,s}$ with $d(b_1, X_2) = 0$ or $\Theta^{[d_1, w_1, d_2, X_1]}_{(b_1, b_3, y)} = 4E_{s,s}$ with $d(b_1, X_3) = 0$, a contradiction. So (6) holds.

Then we claim

$$N(b_1, X_4) = N(b_2, X_4) \tag{7}$$

Proof of (7). Suppose (7) false. By (6), we may choose $x \in N(b_2, X_4)$ such that $xb_1 \notin E$. Moreover, we have $d(b_2, X_4) = d(b_3, X_4) = 3$. We have $d(b_1, X_2 \cup X_3) \leq 1$. W.l.o.g., say $d(b_1, X_2) = 0$. Then $\Theta^{[x,u,X_1]}_{(b_1,b_2)} = 4E_{s,s}$, a contradiction. So (7) holds.

We are now in the position to complete the proof of the theorem in Part I. The proof of (5) and (7) shows, together with (3), that for each $i \in \{1,2,3\}$, b_i is the unique vertex in Y_i such that $d(b_i, X_4) \geq 2$. Hence we have

$$d(y, X_4) \leq 1 \quad \text{for all } y \in Y_1 \cup Y_2 \cup Y_3 - \{b_1, b_2, b_3\}. \tag{8}$$

We suppose first that $d(b_1, X_4) = 3$. Let $N(b_1, X_4) = \{x_1, x_2, x_3\}$. By (3) and (8), for each $i \in \{1,2,3\}$, there exists $a_i \in Y_i - \{b_i\}$ and a bijection $\tau_i : X_4 - \{x_1, x_2, x_3\} \to Y_i - \{a_i, b_i\}$ such that $x\tau_i(x) \in E$ for all $x \in X_4 - \{x_1, x_2, x_3\}$ and $d(a_i, X_4) = 0$. Since $d(b_i, X_1 \cup X_2 \cup X_3) = 0$ for all $i \in \{1,2,3\}$, we can readily show, as before, that $d(y, X_j) \geq 1$ for all $y \in Y_i - \{b_i\}$ and $\{i,j\} \subseteq \{1,2,3\}$ with $i \neq j$. Consequently, as $\Delta(G) \leq 3$, $d(y, X_j) = 1$ for all $y \in Y_i - \{a_i, b_i\}$ and $\{i,j\} \subseteq \{1,2,3\}$ with $i \neq j$. Clearly, $e(X_1, Y_2 \cup Y_3 - \{b_2, b_3\}) \geq 2(s-1)$. Let x be arbitrary in X_1. If $d(x, Y_2 \cup Y_3) = 3$, we choose $y \in N(x) - \{a_2, a_3\}$. Then we may assume w.l.o.g. that $y \in Y_2$ and see that $\Theta^{[x_1,x,x_2,X_1]}_{(b_1,y)} = 4E_{s,s}$, a contradiction. Hence $d(x, Y_2 \cup Y_3) \leq 2$. This argument shows that $e(X_1, Y_2 \cup Y_3) \leq 2(s-1)$, and consequently, $d(z, Y_2 \cup Y_3 - \{b_2, b_3\}) = 2$ for all $z \in X_1$ and $d(a_2, X_1) = 1 = d(a_3, X_1)$. If $d(x, Y_4) = 0$, then we see that $G \supseteq 4E_{s,s}$ as we can use any $y \in N(x)$ in the above argument. Hence $d(x, Y_4) = 1$. Assume $d(x, Y_3) = 0$. Let $y \in Y_2 - \{b_2\}$ with $y \neq a_2$ and $xy \in E$. Then either $\Theta^{[x_1,x,v,X_1]}_{(b_1,y)} = 4E_{s,s}$ with $vy \notin E$, or $\Theta^{[x_1,v,X_1]}_{(y,b_3)} = 4E_{s,s}$ with $vy \in E$, a contradiction. Hence $d(x, Y_3) \geq 1$. Similarly, $d(x, Y_2) \geq 1$. Thus $d(x, Y_i) = 1$ for all $i \in \{2,3,4\}$ as $\Delta(G) \leq 3$. Then $\Theta^{[x_1,x,u,X_1]}_{(b_1,y)} = 4E_{s,s}$ where $y \in N(x, Y_2)$.

Next, we suppose $d(b_1, X_4) = 2$. Let $N(b_1, X_4) = \{x_1, x_2\}$. As before, for each $i \in \{1,2,3\}$, there exists a bijection $\tau_i : X_4 \to Y_i - \{b_i\}$ such that $x\tau_i(x) \in E$ for all $x \in X_4 - \{x_1, x_2\}$. We claim

$$d(y, X_j) = 1 \text{ for all } y \in Y_i - \{b_i\} \text{ and } \{i,j\} \subseteq \{1,2,3\} \text{ with } i \neq j. \tag{9}$$

Proof of (9). Suppose (9) false, i.e., there exists $i \in \{1,2,3\}$ and $y_i \in Y_i - \{b_i\}$ such that $d(y_i, X_j) = 0$ for some $j \in \{1,2,3\}$ with $j \neq i$. Let $x_0 \in X_4 - \{x_1, x_2\}$ be such that $\tau_i(x_0) = y_i$. Say $N(x_0, Y_r) = \{y_r\}$ for $r \in \{1,2,3\}$. We divide the proof of (9) into the following Case 3, Case 4 and Case 5.

Case 3: $i = 1$, i.e., $d(y_1, X_j) = 0$ for some $j \in \{2,3\}$.

W.l.o.g., say $d(y_1, X_2) = 0$. As $\Theta^{[x_0,X_1]}_{(y_1,b_2)} \neq 4E_{s,s}$, we see that $d(b_2, X_1) = 1$ and $d(b_2, X_3) = 0$. Then we see that $d(b_3, X_1) = 1$ and $d(b_3, X_2) = 0$ as $\Theta^{[x_0,X_1]}_{(y_1,b_2,b_3)} \neq 4E_{s,s}$. Together with $d(b_1, X_2 \cup X_3) \leq 1$, it is easy to see that $(X_2 \cup X_3, Y_2 \cup Y_3 \cup \{b_1\} - \{y\}) \supseteq 2E_{s,s}$ for any $y \in Y_2 \cup Y_3 - \{b_2, b_3\}$. As $(X_4 - \{x_1\}, Y_4) = E_{s,s}$, we obtain $(X_1 \cup \{x_1\}, Y_1 \cup \{y\} - \{b_1\}) \neq E_{s,s}$ for any $y \in Y_2 \cup Y_3 - \{b_2, b_3\}$. This implies that $d(y, X_1) \geq 1$ for all $y \in Y_2 \cup Y_3 - \{b_2, b_3\}$.

If $d(y, X_3) = 0$ for some $y \in Y_2 - \{b_2\}$, then $\Theta^{[x_1,v,X_1]}_{(y,b_3)} = 4E_{s,s}$, and if $d(z, X_2) = 0$ for some $z \in Y_3 - \{b_3\}$, then $\Theta^{[x_1,u,X_1]}_{(b_2,z)} = 4E_{s,s}$, a contradiction. Hence

$d(y, X_3) \geq 1$ for all $y \in Y_2 - \{b_2\}$ and $d(z, X_2) \geq 1$ for all $z \in Y_3 - \{b_3\}$. As $\Delta(G) \leq 3$, this shows, together with (3), that, for each $i \in \{2, 3\}$ and $j \in \{1, 2, 3, 4\} - \{i\}$, $d(y, X_j) = 1$ for all $y \in Y_i - \{b_i\}$. Hence $e(X_1, Y_2 \cup Y_3) = 2(s-1) + 2$. This implies that there exists $w \in X_1$ such that $d(w, Y_2 \cup Y_3) = 3$. We choose $y \in N(w)$ such that $y \notin \{b_2, b_3\}$. W.l.o.g., say $y \in Y_2$. Then either $\Theta^{[x_1, w, x_2, X_1]}_{(b_1, y)} = 4E_{s,s}$ with $d(b_1, X_2) = 0$, or $\Theta^{[x_1, w, x_2, X_1]}_{(b_1, b_3, y)} = 4E_{s,s}$, a contradiction.

Note that as $\Delta(G) \leq 3$, the argument of Case 3 shows (9) for the case $i = 1$.

Case 4: $i \in \{2, 3\}$ and $j = 1$, i.e., $d(y_2, X_1) = 0$ or $d(y_3, X_1) = 0$.

W.l.o.g., say $d(y_2, X_1) = 0$. As $\Theta^{[x_0, u, X_1]}_{(b_1, y_2)} \neq 4E_{s,s}$, we have $d(b_1, X_2) = 1$ and $d(b_1, X_3) = 0$. Then $d(b_3, X_2 - \{u\}) = 1$ and $d(b_3, X_1) = 0$ as $\Theta^{[x_0, u, X_1]}_{(b_1, b_3, y_2)} \neq 4E_{s,s}$. If $d(z, X_2 - \{u\}) = 0$ for some $z \in Y_3 - \{b_3\}$, then either $\Theta^{[x_1, u, X_1]}_{(b_2, z)} = 4E_{s,s}$ with $d(b_2, X_3) = 0$, or $\Theta^{[x_1, u, X_1]}_{(b_1, z, b_2)} = 4E_{s,s}$ with $d(b_2, X_1) = 0$, a contradiction. If $d(z, X_1) = 0$ for some $z \in Y_3 - \{b_3\}$, then $\Theta^{[x_1, X_1]}_{(b_1, z)} = 4E_{s,s}$, a contradiction. Hence $d(z, X_1) \geq 1$ and $d(z, X_2 - \{u\}) \geq 1$ for all $z \in Y_3 - \{b_3\}$. Together with Case 3 and as $\Delta(G) \leq 3$, this shows that $d(y, X_j) = 1$ for all $y \in Y_i - \{b_i\}$, $i \in \{1, 3\}$ and $j \in \{1, 2, 3, 4\} - \{i\}$. Furthermore $d(u, Y_1 \cup Y_3) = 0$. Thus $e(X_2 - \{u\}, Y_1 \cup Y_3) = 2(s-1) + 2$. This implies that there exists $w \in X_2 - \{u\}$ with $d(w, Y_1 \cup Y_3) = 3$. As before, we readily see that $G \supseteq 4E_{s,s}$, a contradiction.

Note that the argument of Case 4 shows that $d(y, X_1) \geq 1$ for all $y \in Y_2 \cup Y_3 - \{b_2, b_3\}$.

Case 5: $\{i, j\} = \{2, 3\}$, i.e., $d(y_i, X_j) = 0$.

W.l.o.g., say $d(y_2, X_3) = 0$. Then $d(b_3, X_2) = 1$ and $d(b_3, X_1) = 0$ as $\Theta^{[x_1, v, X_1]}_{(y_2, b_3)} \neq 4E_{s,s}$. Then we see that $d(b_1, X_2) = 1$ and $d(b_1, X_3) = 0$ as $\Theta^{[x_1, v, X_1]}_{(b_1, y_2, b_3)} \neq 4E_{s,s}$. If $d(z, X_2 - \{u\}) = 0$ for some $z \in Y_3 - \{b_3\}$, then either $\Theta^{[x_1, u, X_1]}_{(b_2, z)} = 4E_{s,s}$ with $d(b_2, X_3) = 0$ or $\Theta^{[x_1, u, X_1]}_{(b_1, z, b_2)} = 4E_{s,s}$, a contradiction. Hence $d(z, X_2 - \{u\}) \geq 1$ for all $z \in Y_3 - \{b_3\}$. Together with Case 3 and Case 4, this shows that $d(y, X_j) = 1$ for all $y \in Y_i - \{b_i\}$, $i \in \{1, 3\}$ and $j \in \{1, 2, 3, 4\} - \{i\}$. Furthermore $d(u, Y_1 \cup Y_3 - \{b_3\}) = 0$. Thus $e(X_2 - \{u\}, Y_1 \cup Y_3) \geq 2(s-1) + 1$. This implies that there exists $w \in X_2 - \{u\}$ with $d(w, Y_1 \cup Y_3) = 3$. As before, we readily see that $G \supseteq 4E_{s,s}$, a contradiction. This proves (9).

By (9), we have $e(X_1, Y_2 \cup Y_3) = 2(s-1) + e(\{b_2, b_3\}, X_1)$. Let yz be arbitrary in E with $y \in Y_2 \cup Y_3 - \{b_2, b_3\}$ and $z \in X_1$. Say $y \in Y_i - \{b_i\}$. By (9), $d(y, X_1) = 1$. We claim

$$d(z, Y_i) \geq 2 \quad \text{and} \quad d(z, Y_4) \geq 1. \tag{10}$$

Proof of (10). Suppose (10) false. W.l.o.g., say $i = 2$. Then either $d(z, Y_2) = 1$ or $d(z, Y_4) = 0$. Assume first $d(z, Y_4) = 0$. Then $d(b_1, X_2) = 1$ and $d(b_1, X_3) = 0$ for otherwise $\Theta^{[x_1, z, x_2, X_1]}_{(b_1, y)} = 4E_{s,s}$. Let $x \in X_4 - \{x_1, x_2\}$ with $\tau_2(x) = y$. Then $d(b_3, X_2 - \{u\}) = 1$ and $d(b_3, X_1) = 0$ for otherwise $\Theta^{[x_1, z, x, u, X_1]}_{(b_1, b_3, y)} = 4E_{s,s}$. If $d(y', X_2 - \{u\}) = 0$ for some $y' \in Y_3 - \{b_3\}$, then either $\Theta^{[x_1, u, X_1]}_{(b_2, y')} = 4E_{s,s}$ with $d(b_2, X_3) = 0$, or $\Theta^{[x_1, u, X_1]}_{(b_1, y', b_2)} = 4E_{s,s}$ with $d(b_2, X_1) = 0$, a contradiction. Hence $d(y, X_2 - \{u\}) \geq 1$ for all $y \in Y_3$. Together with (9), this shows that $e(X_2 - \{u\},$

$Y_1 \cup Y_3) = 2(s-1)+2$. This implies that there exists $w \in X_2 - \{u\}$ such that $d(w, Y_1 \cup Y_2) = 3$. As before, we can readily see that $G \supseteq 4E_{s,s}$, a contradiction. Hence $d(z, Y_4) \geq 1$. Next, assume $d(z, Y_2) = 1$. Then $d(b_1, X_2) = 1$ and $d(b_1, X_3) = 0$ for otherwise $\Theta^{[x_1,z,u,X_1]}_{(b_1,y)} = 4E_{s,s}$. Then for each $y' \in Y_3 - \{b_3\}$, $d(y', X_2 - \{u\}) \geq 1$ as $\Theta^{[x_1,z,u,X_1]}_{(b_1,y',y)} \not\supseteq 4E_{s,s}$ unless $y'z \in E$; but when $y'z \in E$, we have $d(z, Y_3) = 1$ as $\Delta(G) \leq 3$, and consequently, $\Theta^{[x_1,z,v,X_1]}_{(b_1,y')} = 4E_{s,s}$, a contradiction. Then by (9), $e(X_2 - \{u\}, Y_1 \cup Y_3 - \{b_3\}) = 2(s-1)+1$. This implies that there exists $w \in X_2 - \{u\}$ such that $d(w, Y_1 \cup Y_3) = 3$. As before, we can readily see that $G \subseteq 4E_{s,s}$, a contradiction. So (10) holds.

As $\Delta(G) \leq 3$ and by (9) and (10), we see that $d(z, Y_4) = 1$ and either $d(z, Y_2) = 2$ or $d(z, Y_3) = 2$ for all $z \in X_1$. Moreover, $d(b_2, X_1) = d(b_3, X_1) = 0$. Let $z_0 y_0 \in E$ with $z_0 \in X_1$ and $y_0 \in Y_2 \cup Y_3$. W.l.o.g., say $y_0 \in Y_2$. Then we have $d(z_0, Y_4) = 1$ and $d(z_0, Y_2 - \{b_2\}) = 2$. Let $t \in Y_4$ with $z_0 t \in E$. Then $d(t, X_1) \geq 2$ for otherwise $\Theta^{[x_1,z_0,x_2,X_1]}_{(b_1,t)} = 4E_{s,s}$. Therefore $d(t, X_2) = 0$ or $d(t, X_3) = 0$ as $\Delta(G) \leq 3$. First, assume $d(t, X_3) = 0$. Then $d(b_1, X_2) = 1$ and $d(b_1, X_3) = 0$ for otherwise $\Theta^{[x_1,z_0,x_2,X_1]}_{(b_1,y_0)(b_3,t)} = 4E_{s,s}$. Then $d(b_3, X_2) = 1$ and $d(b_3, X_1) = 0$ for otherwise $\Theta^{[x_1,z_0,x_2,X_1]}_{(b_1,t,b_3,y_0)} = 4E_{s,s}$. Then for each $y \in Y_3 - \{b_3\}$, $d(y, X_2 - \{u\}) \geq 1$ as $\Theta^{[x_1,u,X_1]}_{(b_1,y,b_2)} \neq 4E_{s,s}$. By (9), this shows that $e(X_2 - \{u\}, Y_1 \cup Y_3) \geq 2(s-1)+1$. This implies that there exists $w \in X_2 - \{u\}$ such that $d(w, Y_1 \cup Y_3) = 3$. As before, we can readily see that $G \supseteq 4E_{s,s}$, a contradiction. Hence $d(t, X_3) = 1$ and $d(t, X_2) = 0$. Then $\Theta^{[x_1,z_0,x_2,X_1]}_{(b_1,t,y_0)} = 4E_{s,s}$. This completes Part I.

Part II. For every $x \in X_3$, $d(x, Y_1) \geq 1$ and $d(x, Y_2) \geq 1$.

We need the following structure lemma. To state the lemma, we construct the following graphs first.

For each **odd** integer $s \geq 3$, G_s is a bipartite graph of order $4s$ with a bipartition (A, B) such that $|A| = |B| = 2s$ and $d(x) = 1$ for all $x \in A$ and $d(y) = 0$ or $d(y) = 2$ for all $y \in B$.

For each integer $s \geq 4$ with $s \equiv 1 \pmod 3$, H_s is a bipartite graph of order $4s$ with a bipartition (A, B) and a fixed vertex $y_0 \in B$ such that $|A| = |B| = 2s$, $d(x) = 1$ for all $x \in A$, $d(y_0) = 2$, and $d(y) = 0$ or $d(y) = 3$ for all $y \in B - \{y_0\}$.

For each integer $s \geq 2$ with $s \equiv 2 \pmod 3$, I_s is a bipartite graph of order $4s$ with a bipartition (A, B) and two fixed vertices $x_0 \in A$ and $y_0 \in B$ such that $d(x_0, B - \{y_0\}) = 0$, $d(y_0, A - \{x_0\}) = 0$, $d(x) = 1$ for all $x \in A - \{x_0\}$, and $d(y) = 0$ or $d(y) = 3$ for all $y \in B - \{y_0\}$.

Lemma. *Let $H = (X, Y; E)$ be a bipartite graph of order $4s$ with $|X| = |Y| = 2s$ and $s \geq 2$. Suppose that $d(x) \leq 1$ for all $x \in X$ and $d(y) \leq 3$ for all $y \in Y$. Then $H \supseteq 2E_{s,s}$ unless H is isomorphic to one of G_s, H_s and I_s.*

Proof of the Lemma. Suppose $H \not\supseteq 2E_{s,s}$. We shall prove that H is isomorphic to one of G_s, H_s and I_s. Enumerate $Y = \{u_1, u_2, \ldots, u_{2s}\}$ such that

$$3 \geq d(u_1) \geq d(u_2) \geq \cdots \geq d(u_r) > 0 \text{ and } d(u_{r+1}) = d(u_{r+2}) = \cdots = d(u_{2s}) = 0. \tag{11}$$

If $\sum_{i=1}^{r} d(u_i) \le s$, let $N(\{u_1, u_2, \ldots, u_r\}) \subseteq X_1 \subseteq X$ with $|X_1| = s$ and $Y_1 = \{u_{r+1}, \ldots, u_{r+s}\}$. Then $H \supseteq 2E_{s,s} = \{(X_1, Y_1), (X - X_1, Y - Y_1)\}$, a contradiction. Therefore we may let t be the least integer in $\{1, 2, \ldots, r\}$ such that $\sum_{i=1}^{t} d(u_i) > s$. Similarly, it is easy to see that $H \supseteq 2E_{s,s}$ if $\sum_{i=1}^{t-1} d(u_i) = s$. Therefore we have $\sum_{i=1}^{t-1} d(u_i) \le s - 1$. Hence $d(u_1) \ge d(u_2) \ge \cdots \ge d(u_t) \ge 2$. Moreover, as $d(u_t) \le 3$, we see that $\sum_{i=1}^{t-1} d(u_i) = s - 1$ or $\sum_{i=1}^{t-1} d(u_i) = s - 2$. In the latter case, $d(u_t) = 3$. We divide the proof into the following two cases.

Case a: $\sum_{i=1}^{t-1} d(u_i) = s - 1$.

In this case, we claim

$$d(x) = 1 \text{ for all } x \in X, \text{ and } d(y) = 0 \text{ or } d(y) \ge 2 \text{ for all } y \in Y. \tag{12}$$

Proof of (12). Suppose that $d(u) = 1$ for some $u \in Y - \{u_1, u_2, \ldots, u_t\}$. Let $X_1 = N(\{u_1, u_2, \ldots, u_{t-1}, u\})$ and $X_2 = X - X_1$. Clearly, $|X_1| = |X_2| = s$. As $d(x) \le 1$ for all $x \in X$ and by (11), we see that $N(X_2) \subseteq \{u_t, u_{t+1}, \ldots, u_{t+s-1}\}$ and $d(u_{t+s}) = 0$. Let $Y_1 = \{u_t, u_{t+1}, \ldots, u_{t+s}\} - \{u\}$ and $Y_2 = Y - Y_1$. Then $H \supseteq 2E_{s,s} = \{(X_1, Y_1), (X_2, Y_2)\}$, a contradiction. Similarly, we can show that $H \supseteq 2E_{s,s}$ if $d(x) = 0$ for some $x \in X$ as we can take $X_1 = N(\{u_1, u_2, \ldots, u_{t-1}\}) \cup \{x\}$ in the first place. So (12) holds.

Suppose $d(u_{t-1}) = 3$. Then $d(u_i) = 3$ for all $i \in \{1, 2, \ldots, t-1\}$, $s \equiv 1 \pmod 3$ and $2s \equiv 2 \pmod 3$. Together with (12), this implies that there exists $y_0 \in Y$ such that $d(y_0) = 2$. If there exists another vertex $y_0' \in Y - \{y_0\}$ with $d(y_0') = 2$, let $X_1 = N(\{u_1, u_2, \ldots, u_{t-2}, y_0, y_0'\})$, $X_2 = X - X_1$, $Y_1 = \{u_{t-1}, u_t, \ldots, u_{t+s}\} - \{y_0, y_0'\}$ and $Y_2 = Y - Y_1$. Then by (11), we see that $N(X_2) \subseteq Y_1$ and thus $\{(X_1, Y_1), (X_2, Y_2)\} = 2E_{s,s}$, a contradiction. Hence $d(y) = 0$ or $d(y) = 3$ for all $y \in Y - \{y_0\}$, and consequently, $H \cong H_s$.

Next, assume $d(u_{t-1}) = 2$, By (11) and (12), $d(u_t) = d(u_{t+1}) = \cdots = d(u_r) = 2$. As $|N(\{u_1, u_2, \ldots, u_t\})| = s + 1 < 2s = |X|$ and by (12), we see $r > t$. If $d(u_1) = 3$, let $X_1 = N(\{u_2, u_3, \ldots, u_{t+1}\})$, $X_2 = X - X_1$, $Y_1 = \{u_1, u_{t+2}, \ldots, u_{t+s}\}$ and $Y_2 = Y - Y_1$. Then we see that $\{(X_1, Y_1), (X_2, Y_2)\} = 2E_{s,s}$, a contradiction. Hence $d(u_1) = 2$. This shows $d(y) = 0$ or $d(y) = 2$ for all $y \in Y$ and $s - 1 = \sum_{i=1}^{t-1} d(u_i) \equiv 0 \pmod 2$. Hence $H \cong G_s$.

Case b: $\sum_{i=1}^{t-1} d(u_i) = s - 2$.

In this case, $d(u_t) = 3$. By (11), $d(u_i) = 3$ for all $i \in \{1, 2, \ldots, t\}$ and therefore $s \equiv 2 \pmod 3$ and $2s \equiv 1 \pmod 3$. As before, it is easy to see that if there exists $y_0 \in Y$ with $d(y_0) = 2$, or there exist two distinct vertices $y_0, y_0' \in Y$ with $d(y_0) = d(y_0') = 1$, or there exist two distinct vertices $x_0, x_0' \in X$ with $d(x_0) = d(x_0') = 0$, then $H \supseteq 2E_{s,s}$, a contradiction. Therefore there exist $x_0 \in X$ and $y_0 \in Y$ such that $d(x) = 1$ for all $x \in X - \{x_0\}$, and $d(y) = 0$ or $d(y) = 3$ for all $y \in Y - \{y_0\}$. Furthermore, $d(x_0, Y - \{y_0\}) = 0$ and $d(y_0, X - \{x_0\}) = 0$. Thus $H \cong I_s$. This proves the lemma. □

We now turn back to the proof of the theorem. As $\Delta(G) \le 3$ and by the assumption of Part II and (2), we have

$$d(x, Y_3 \cup Y_4) \le 1 \quad \text{for all } x \in X_3 \cup X_4. \tag{13}$$

Moreover, for each $i \in \{1,2\}$ we have $e(X_3 \cup X_4, Y_i) \geq 2s+1$ and therefore there exists $b_i \in Y_i$ such that $d(b_i, X_3 \cup X_4) = 3$ and $d(b_i, X_1 \cup X_2) = 0$. We divide the proof into the following two cases.

Case 1: There exists $w \in X_3 \cup X_4$ such that $H = (X_3 \cup X_4 - \{w\}, Y_3 \cup Y_4) \not\supseteq 2E_{s,s}$.

By the lemma, $H \cong G_s, H_s$ or I_s. We enumerate $Y_3 \cup Y_4 = \{y_0, y_1, \ldots, y_t, y_{t+1}, \ldots, y_{2s-1}\}$ with $d(y_{t+1}, H) = \cdots = d(y_{2s-1}, H) = 0$ such that

(i) when $s \equiv 1 \pmod 2$ and $H \cong G_s$, $t = s-1$ and $d(y_i, H) = 2$ for all $i \in \{0, 1, \ldots, t\}$;

(ii) when $s \equiv 1 \pmod 3$ and $H \cong H_s$, $t = (2s-2)/3$, $d(y_0, H) = 2$ and $d(y_i, H) = 3$ for all $i \in \{1, 2, \ldots, t\}$;

(iii) when $s \equiv 2 \pmod 3$ and $H \cong I_s$, $t = (2s-1)/3$, $d(y_0, H) \leq 1$ and $d(y_i, H) = 3$ for all $i \in \{1, 2, \ldots, t\}$.

Moreover, when $s \equiv 2 \pmod 3$ and $H \cong I_s$, let $x_0 \in X_3 \cup X_4 - \{w\}$ be such that $d(x_0, H - y_0) = 0$ and $d(y_0, H - x_0) = 0$. We may also assume that when the neighbor of w in $N(w, H)$ (if any) is of degree 2 in H, then w is adjacent to y_0. This might happen when $H \cong G_s$ or H_s. We claim

$$s \equiv 1 \pmod 3, \quad H \cong H_s \quad \text{and} \quad wy_0 \in E. \tag{14}$$

Proof of (14). Suppose (14) false. Then $wy_0 \notin E$ if $s \equiv 1 \pmod 3$ and $H \cong H_s$. Choose an arbitrary vertex z_0 from $X_3 \cup X_4 - \{w\}$ such that if $H \cong G_s$ with $s \equiv 1 \pmod 2$ and $wy_0 \in E$ then $z_0 y_0 \notin E$, and if $H \cong I_s$ with $s \equiv 2 \pmod 3$ then $z_0 \neq x_0$. Then $d(z_0, H) = d(z_0, Y_1) = d(z_0, Y_2) = 1$. Note that we have at least four different choices for z_0 if $H \not\cong I_2$ and three different choices for z_0 if $H \cong I_2$. It is easy to see that $H - z_0 + w$ is isomorphic to none of G_s, H_s and I_s. By the lemma, $H - z_0 + w \supseteq 2E_{s,s} = \{(X_3', Y_3'), (X_4', Y_4')\}$. Therefore $(X_1 \cup X_2 \cup \{z_0\}, Y_1 \cup Y_2) \not\supseteq 2E_{s,s}$.

It is clear that there exists $y' \in Y_3' \cup Y_4'$ such that $d(y', X_3' \cup X_4') \geq 2$.

Note that $d(u, Y_1 \cup Y_2) = 0$. If $z_0 \notin N(b_1)$, then $z_0 \notin N(b_2)$ for otherwise $\{(X_1 \cup \{u\}, Y_1 \cup \{b_2\} - \{b_1\}), (X_2 \cup \{z_0\} - \{u\}, Y_2 \cup \{b_1\} - \{b_2\})\} = 2E_{s,s}$. Similarly, we see that if $z_0 \notin N(b_2)$ then $z_0 \notin N(b_1)$. Hence if we must have $z_0 \in N(\{b_1, b_2\})$ for any appropriate choice of z_0, then $H \cong I_2$ and $N(b_1) = N(b_2) = N(y_1)$. In this situation, let $X_1 = \{c_1\}$, $X_2 = \{u, c_2\}$, $N(y_1) = \{d_1, d_2, d_3\}$, $Y_1 = \{a_1, b_1\}$ and $Y_2 = \{a_2, b_2\}$. We may assume $N(w, Y_3 \cup Y_4) \subseteq \{y_0, y_2\}$. Then $G \supseteq 4E_{2,2} = \{(\{d_1, d_2\}, \{y_0, y_2\}), (\{d_3, u\}, \{a_1, a_2\}), (\{c_1, c_2\}, \{b_1, b_2\}), (\{x_0, w\}, \{y_1, y_3\})\}$, a contradiction.

Therefore $z_0 \notin N(\{b_1, b_2\})$. Let $N(z_0, Y_1 \cup Y_2) = \{z_1, z_2\}$ with $z_1 \in Y_1$ and $z_2 \in Y_2$. If $d(z_1, X_2) = 0$, then $\{(X_1 \cup \{z_0\}, Y_1 \cup \{b_2\} - \{z_1\}), (X_2, Y_2 \cup \{z_1\} - \{b_2\})\} = 2E_{s,s}$, a contradiction. It follows that $d(z_1, X_2) \geq 1$ and $d(z_1, X_3' \cup X_4') \leq 1$. W.l.o.g., say $d(z_1, X_4') = 0$. Let $X_1' = X_1 \cup \{z_0\}$ and $Y_1' = Y_1 - \{z_1\}$. Clearly, $(X_1', Y_1') = E_{s,s-1}$. Let $\Theta' = \{(X_1', Y_1'), (X_2, Y_2), (X_3', Y_3'), (X_4', Y_4' \cup \{z_1\})\}$. As $b_2 z_0 \notin E$, we have $d(b_2, X_1') = d(b_2, X_1) = 0$. Since $d(y', X_3' \cup X_4') \geq 2$, we have either $d(y', X_1') = 0$ or $d(y', X_2) = 0$. As in the proof of (2), we must have $y' \in Y_3'$. Then we apply the proof of Part I to Θ' to obtain $G \supseteq 4E_{s,s}$, a contradiction. This proves (14).

Let $H' = H + w$. By (14), $d(y_0, H') = 3$. Clearly, $t \geq 2$ as $s \geq 4$. We may assume w.l.o.g. that $N(b_1) = \{d_1, d_2, d_3\} \subseteq N(\{y_0, y_1, y_2\})$ such that $d(b_1, N(y_0)) \leq d(b_1, N(y_1)) \leq d(b_1, N(y_2))$. Let i be the smallest integer in $\{0, 1, 2\}$ such that $N(b_1) \cap N(y_i) \neq \emptyset$. W.l.o.g., say $d_1 \in N(b_1) \cap N(y_i)$. Then $d(d_1, Y_1) = d(d_1, Y_2) = 1$. Let $\{w_1, w_2, w_3, w_4, w_5\} \subseteq Y_3 \cup Y_4$ be a set of five isolated vertices of H'. It is easy to see that $R = (N(\{y_0, y_1, y_2\}) - \{d_1\}, \{y_0, y_j, b_1, w_1, w_2, w_3, w_4, w_5\}) \supseteq 2E_{4,4}$ where $y_j = y_2$ if $d_1 \in N(y_2)$ and $y_j = y_1$ if $d_1 \notin N(y_2)$. Clearly, $S = H' - (V(R) \cup \{d_1\})$ is isomorphic to none of G_{s-4}, H_{s-4} and I_{s-4} as $d(y, S) = 0$ or $d(y) = 3$ for all $y \in V(S) \cap (Y_3 \cup Y_4)$ and $d(x, S) = 1$ for all $x \in V(S) \cap (X_3 \cup X_4)$. By the lemma, $S \supseteq 2E_{s-4,s-4}$. Hence $R \cup S \supseteq 2E_{s,s}$. Let $i \in \{0, 1, 2\} - \{0, j\}$. As $d(y_i, X_1 \cup X_2) = 0$, it follows that $\{(X_1 \cup \{d_1\}, Y_1 \cup \{y_i\} - \{b_1\}), (X_2, Y_2)\} = 2E_{s,s}$. Therefore $G \supseteq 4E_{s,s}$.

Case 2: $(X_3 \cup X_4 - \{x\}, Y_3 \cup Y_4) \supseteq 2E_{s,s}$ for all $x \in X_3 \cup X_4$.
Then we have

$$(X_1 \cup X_2 \cup \{x\}, Y_1 \cup Y_2) \not\supseteq 2E_{s,s} \quad \text{for all } x \in X_3 \cup X_4. \tag{15}$$

We claim

$$\text{For each } i \in \{1, 2\} \text{ if } y \in Y_i \text{ and } d(y, X_3 \cup X_4) = 3 \text{ then } d(x, Y_i) = 2 \text{ for every } x \in N(y). \tag{16}$$

Proof of (16). Suppose (16) false. We may assume w.l.o.g. that for some $i \in \{1, 2\}$, there exists $x \in N(b_i)$ such that $d(x, Y_i) = 1$. Let $\{i, j\} = \{1, 2\}$ and $N(x, Y_j) = \{x_1, x_2\}$ (maybe $x_1 = x_2$). W.l.o.g., we may assume $i = 2$ as $d(u, Y_1 \cup Y_2) = 0$. Then $d(y, X_2) \geq 1$ for all $y \in Y_1 - \{x_1, x_2\}$ for otherwise $\{(X_1 \cup \{u\}, Y_1 \cup \{b_2\} - \{y\}), (X_2 \cup \{x\} - \{u\}, Y_2 \cup \{y\} - \{b_2\})\} = 2E_{s,s}$ for any $y \in Y_1 - \{x_1, x_2\}$ with $d(y, X_2) = 0$. It follows that $3s \geq e(Y_1, X_2 \cup \{x\}) + e(Y_1, X_3 \cup X_4 - \{x\}) \geq s + 2s = 3s$. Then we see that $d(y) = 3$ for all $y \in Y_1$, $d(x_1, X_2) = d(x_2, X_2) = 0$, $d(z, Y_1) = 1$ for all $z \in X_3 \cup X_4 - \{x\}$ and $d(y, X_2) = 1$ for all $y \in Y_1 - \{x_1, x_2\}$. Hence $b_1 \in \{x_1, x_2\}$. If there exists $x' \in N(\{x_1, x_2\})$ such that $x'b_2 \notin E$, then $x' \neq x$ and $\{(X_1 \cup \{x'\}, Y_1 \cup \{b_2\} - \{x_r\}), (X_2, Y_2 \cup \{x_r\} - \{b_2\})\} = 2E_{s,s}$ where $x_r \in \{x_1, x_2\}$ with $x_r x' \in E$, a contradiction. Hence $N(b_2) \supseteq N(\{x_1, x_2\})$. It follows that $x_1 = x_2 = b_1$ and $N(b_2) = N(b_1)$. This also says that $d(x, Y_1) = 1$. Then similarly, we can show that $d(y, X_1) = 1$ for all $y \in Y_2 - \{b_2\}$ and $d(z, Y_2) = 1$ and $z \in X_3 \cup X_4$. Let z be an arbitrary vertex in $X_3 \cup X_4 - N(b_2)$. Let $y_1 \in Y_1$ and $y_2 \in Y_2$ be such that $N(z, Y_1 \cup Y_2) = \{y_1, y_2\}$. Let $z_1 \in X_1$ be such that $z_1 y_2 \in E$. If $d(z_1, Y_2) = 1$, then $\{(X_1 \cup \{u, x\} - \{z_1\}, Y_1 \cup \{y_2\} - \{b_1\}), (X_2 \cup \{z_1\} - \{u\}, Y_2 \cup \{b_1\} - \{y_2\})\} = 2E_{s,s}$, a contradiction. Hence $d(z_1, Y_2 - \{b_2\}) \geq 2$ and therefore $d(z_1, Y_3 \cup Y_4) \leq 1$. As $(X_3 \cup X_4 - \{x\}, Y_3 \cup Y_4) \supseteq 2E_{s,s}$, we see that there are partitions $X_3 \cup X_4 \cup \{z_1\} - \{x\} = X_3' \cup X_4'$ and $Y_3 \cup Y_4 = Y_3' \cup Y_4'$ such that $(X_3', Y_3') = E_{s,s}$ and $(X_4', Y_4') = E_{s+1,s}$. Let $X_1' = X_1 \cup \{x\} - \{z_1\}$, $Y_1' = Y_1 \cup \{y_2\} - \{b_1\}$, $X_2' = X_2$ and $Y_2' = Y_2 \cup \{b_1\} - \{y_2\}$. Then $(X_1', Y_1') = E_{s-1,s}$ and $(X_2', Y_2') = E_{s,s}$. Furthermore $d(u, Y_1') = 0$ and $d(x', Y_1') = 0$ where $x' \in N(b_1) - \{x\}$. Then we apply the proof of Part I to $\Theta' = \{(X_i', Y_i') | 1 \leq i \leq 4\}$ to obtain $G \supseteq 4E_{s,s}$, a contradiction. This proves (16).

By (16) and as $\Delta(G) \leq 3$, we have $N(y') \cap N(y'') = \emptyset$ for any $y' \in Y_1$ and $y'' \in Y_2$ with $d(y', X_3 \cup X_4) = d(y'', X_3 \cup X_4) = 3$. It follows that $N(y') \cap N(y'') = \emptyset$ for any two distinct vertices $y', y'' \in Y_2$ with $d(y', X_3 \cup X_4) = d(y'', X_3 \cup X_4) = 3$ for otherwise we apply the proof of (16) to $\Theta_{(b_1, y')}$ to show that $d(z, Y_1 \cup \{y'\} - \{b_1\}) = 2$ and $d(z, Y_2 \cup \{b_1\} - \{y'\}) = 2$ and so $d(z) \geq 4$ for each $z \in N(y') \cap N(y'')$, a contradiction. Let $y_1, \ldots, y_t$ be a list of vertices of Y_2 with $d(y_i, X_3 \cup X_4) = 3$ for all $i \in \{1, 2, \ldots, t\}$. Then $t \geq 1$, $d(z, Y_2) = 2$ for all $z \in N(y_i)$ and $i \in \{1, 2, \ldots, t\}$, and $N(y_i) \cap N(y_j) = \emptyset$ for $1 \leq i < j \leq t$. Hence $e(Y_2, X_3 \cup X_4) = e(Y_2, X_3 \cup X_4 - N(\{y_1, \ldots, y_t\})) + e(Y_2, N(\{y_1, \ldots, y_t\})) \geq (2s + 1 - 3t) + 6t = 2s + 3t + 1$. As $\Delta(G) \leq 3$, this implies that Y_2 contains at least $3t + 1$ vertices, each of which is adjacent to three vertices of $X_3 \cup X_4$. Hence $t \geq 3t + 1$, a contradiction. This completes Part II.

By Part I and Part II, the theorem holds.

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