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Spectra of Bipartite *P*- and *Q*-Polynomial Association Schemes

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Abstract. Let $Y = (X, \{R_i\}_{0 \le i \le D})$ denote a symmetric association scheme with $D \ge 3$, and assume Y is not an ordinary cycle. Suppose Y is bipartite P-polynomial with respect to the given ordering $A_0, A_1, \ldots, A_D$ of the associate matrices, and Q-polynomial with respect to the ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents. Then the eigenvalues and dual eigenvalues satisfy exactly one of (i)–(iv).

(i)

$$\theta_0 > \theta_1 > \theta_2 > \theta_3 > \cdots > \theta_{D-3} > \theta_{D-2} > \theta_{D-1} > \theta_D,$$
$$\theta_i^* = \theta_i \quad (0 \le i \le D).$$

(ii) D is even, and

$$\theta_0 > \theta_{D-1} > \theta_2 > \theta_{D-3} > \cdots > \theta_3 > \theta_{D-2} > \theta_1 > \theta_D,$$
$$\theta_i^* = \theta_i \quad (0 \le i \le D).$$

(iii) $\theta_0^* > \theta_0$, and

$$\theta_0 > \theta_1 > \theta_2 > \theta_3 > \cdots > \theta_{D-3} > \theta_{D-2} > \theta_{D-1} > \theta_D,$$
$$\theta_0^* > \theta_1^* > \theta_2^* > \theta_3^* > \cdots > \theta_{D-3}^* > \theta_{D-2}^* > \theta_{D-1}^* > \theta_D^*.$$

(iv) $\theta_0^* > \theta_0$, D is odd, and

$$\theta_0 > \theta_{D-1} > \theta_2 > \theta_{D-3} > \cdots > \theta_3 > \theta_{D-2} > \theta_1 > \theta_D,$$
$$\theta_0^* > \theta_D^* > \theta_2^* > \theta_{D-2}^* > \cdots > \theta_{D-3}^* > \theta_3^* > \theta_{D-1}^* > \theta_1^*.$$

1. Introduction

Let $Y = (X, \{R_i\}_{0 \le i \le D})$ be a P- and Q-polynomial association scheme, with eigenvalues $\theta_0, \theta_1, \ldots, \theta_D$ and dual eigenvalues $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$. We want to find the

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permutations σ, τ of $0, 1, \ldots, D$ for which

$$\theta_{\sigma 0} > \theta_{\sigma 1} > \theta_{\sigma 2} > \cdots > \theta_{\sigma D},$$
$$\theta^*_{\tau 0} > \theta^*_{\tau 1} > \theta^*_{\tau 2} > \cdots > \theta^*_{\tau D}.$$

In this paper we focus on the case where Y is bipartite, and prove the following theorem.

1.1 Theorem. *Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ denote a symmetric association scheme with $D \geq 3$, and assume Y is not an ordinary cycle. Suppose Y is bipartite P-polynomial with respect to the given ordering $A_0, A_1, \ldots, A_D$ of the associate matrices, and Q-polynomial with respect to the ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents. Then the eigenvalues and dual eigenvalues satisfy exactly one of* (i)–(iv).

(i)

$$\theta_0 > \theta_1 > \theta_2 > \theta_3 > \cdots > \theta_{D-3} > \theta_{D-2} > \theta_{D-1} > \theta_D,$$
$$\theta^*_i = \theta_i \quad (0 \leq i \leq D).$$

(ii) *D is even, and*

$$\theta_0 > \theta_{D-1} > \theta_2 > \theta_{D-3} > \cdots > \theta_3 > \theta_{D-2} > \theta_1 > \theta_D,$$
$$\theta^*_i = \theta_i \quad (0 \leq i \leq D).$$

(iii) $\theta^*_0 > \theta_0$, *and*

$$\theta_0 > \theta_1 > \theta_2 > \theta_3 > \cdots > \theta_{D-3} > \theta_{D-2} > \theta_{D-1} > \theta_D,$$
$$\theta^*_0 > \theta^*_1 > \theta^*_2 > \theta^*_3 > \cdots > \theta^*_{D-3} > \theta^*_{D-2} > \theta^*_{D-1} > \theta^*_D.$$

(iv) $\theta^*_0 > \theta_0$, *D is odd, and*

$$\theta_0 > \theta_{D-1} > \theta_2 > \theta_{D-3} > \cdots > \theta_3 > \theta_{D-2} > \theta_1 > \theta_D,$$
$$\theta^*_0 > \theta^*_D > \theta^*_2 > \theta^*_{D-2} > \cdots > \theta^*_{D-3} > \theta^*_3 > \theta^*_{D-1} > \theta^*_1.$$

2. Preliminaries

A D-class *symmetric association scheme* is a pair $Y = (X, \{R_i\}_{0 \leq i \leq D})$, where X is a non-empty finite set, and where:

(i) $\{R_i\}_{0 \leq i \leq D}$ is a partition of $X \times X$;
(ii) $R_0 = \{xx | x \in X\}$;
(iii) $R^t_i = R_i$ for $0 \leq i \leq D$, where $R^t_i = \{yx | xy \in R_i\}$;
(iv) there exist integers p^h_{ij} such that for all $x, y \in X$ with $xy \in R_h$, the number of $z \in X$ with $xz \in R_i$ and $zy \in R_j$ is p^h_{ij}.

The constants p^h_{ij} are called the *intersection numbers* of Y. By a *scheme*, we mean a symmetric association scheme.

The Bose-Mesner Algebra M

Let $Y = (X, \{R_i\}_{0 \le i \le D})$ denote a scheme, and let $\mathrm{Mat}_X(\mathbb{R})$ denote the algebra of matrices over $\mathbb{R}$ with rows and columns indexed by X. For each integer i $(0 \le i \le D)$, let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{R})$ with x, y entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } xy \in R_i, \\ 0 & \text{otherwise} \end{cases} \quad (x, y \in X). \tag{1}$$

From (1) we obtain:

$$A_0 = I, \tag{2}$$

$$A_i^t = A_i \qquad (0 \le i \le D), \tag{3}$$

$$A_i A_j = \sum_{h=0}^{D} p_{ij}^h A_h \quad (0 \le i, j \le D), \tag{4}$$

$$A_0 + A_1 + \cdots + A_D = J, \tag{5}$$

where I is the identity matrix, and J is the all 1's matrix. We refer to A_i as the i^{th} *associate matrix for* Y $(0 \le i \le D)$.

By (2)–(4), $A_0, \ldots, A_D$ is a basis for a subalgebra M of $\mathrm{Mat}_X(\mathbb{R})$. M is known as the *Bose-Mesner algebra* for Y.

By [2, p. 45], the algebra M has a second basis $E_0, \ldots, E_D$ such that

$$E_0 = |X|^{-1} J, \tag{6}$$

$$E_i^t = E_i \quad (0 \le i \le D), \tag{7}$$

$$E_i E_j = \delta_{ij} E_i \quad (0 \le i, j \le D), \tag{8}$$

$$E_0 + E_1 + \cdots + E_D = I. \tag{9}$$

We refer to E_i as the i^{th} *primitive idempotent for* Y $(0 \le i \le D)$.

By the *Krein parameters* of Y, we mean the real scalars $\{q_{ij}^h | 0 \le h, i, j \le D\}$ satisfying

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q_{ij}^h E_h \quad (0 \le i, j \le D), \tag{10}$$

where $\circ$ denotes the entry-wise matrix product [1, p. 64]. In [2, p. 49], it is shown that

$$q_{ij}^h \ge 0 \quad (0 \le h, i, j \le D). \tag{11}$$

The Dual Bose-Mesner Algebra M^*

Let $Y = (X, \{R_i\}_{0 \le i \le D})$ denote a scheme, and fix any $x \in X$. For each integer i $(0 \le i \le D)$, let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{R})$ with y, y entry

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } xy \in R_i, \\ 0 & \text{otherwise} \end{cases} \quad (y \in X). \tag{12}$$

From (12) we obtain

$$E_i^{*t} = E_i^* \qquad (0 \le i \le D), \tag{13}$$

$$E_i^* E_j^* = \delta_{ij} E_i^* \quad (0 \le i, j \le D), \tag{14}$$

$$E_0^* + E_1^* + \cdots + E_D^* = I. \tag{15}$$

We refer to E_i as the i^{th} *dual idempotent for* Y *with respect to* x $(0 \le i \le D)$.

By (14)–(15), $E_0^*, \ldots, E_D^*$ is a basis for a subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{R})$. M^* is known as the *dual Bose-Mesner algebra for* Y *with respect to* x.

For each integer i $(0 \le i \le D)$, let $A_i^* = A_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{R})$ with y, y entry

$$(A_i^*)_{yy} = |X|(E_i)_{xy} \quad (y \in X). \tag{16}$$

We note that $A_0^*, \ldots, A_D^*$ form a second basis for M^* [8, p. 379].

By (16), we obtain

$$A_0^* = I, \tag{17}$$

$$A_i^{*t} = A_i^* \qquad (0 \le i \le D), \tag{18}$$

$$A_i^* A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^* \quad (0 \le i, j \le D), \tag{19}$$

$$A_0^* + A_1^* + \cdots + A_D^* = |X| E_0^*. \tag{20}$$

We refer to A_i^* as the i^{th} *dual associate matrix of* Y *with respect to* x $(0 \le i \le D)$.

Eigenvalues and Dual Eigenvalues

Let $Y = (X, \{R_i\}_{0 \le i \le D})$ denote a scheme. By [8, pp. 377, 379], there exist real scalars $p_i(j)$, $q_i(j)$ $(0 \le i, j \le D)$ which satisfy

$$A_i = \sum_{j=0}^{D} p_i(j) E_j \qquad (0 \le i \le D), \tag{21}$$

$$E_i = |X|^{-1} \sum_{j=0}^{D} q_i(j) A_j \quad (0 \le i \le D), \tag{22}$$

$$A_i^* = \sum_{j=0}^{D} q_i(j) E_j^* \qquad (0 \le i \le D), \tag{23}$$

$$E_i^* = |X|^{-1} \sum_{j=0}^{D} p_i(j) A_j^* \quad (0 \le i \le D). \tag{24}$$

We refer to $p_i(j)$ (resp. $q_i(j)$) as the j^{th} *eigenvalue* (resp. j^{th} *dual eigenvalue*) associated with A_i (resp. A_i^*). For simplicity, we define

$$k_i := p_i(0) \quad (0 \le i \le D), \tag{25}$$

$$m_i := q_i(0) \quad (0 \le i \le D). \tag{26}$$

In [2, p. 45] it is shown that

$$p_0(j) = 1 \qquad (0 \le j \le D), \tag{27}$$

$$k_i \ge p_i(j) \quad (0 \le i, j \le D), \tag{28}$$

and dually,

$$q_0(j) = 1 \qquad (0 \le j \le D), \tag{29}$$

$$m_i \ge q_i(j) \quad (0 \le i, j \le D). \tag{30}$$

By [2, p. 45] and the construction,

$$m_i = rank(E_i) \quad (0 \le i \le D), \tag{31}$$

$$k_i = rank(E_i^*) \quad (0 \le i \le D). \tag{32}$$

And the following useful identities are from [1, p. 65].

$$q_{ij}^h = \frac{m_i m_j}{|X|} \sum_{r=0}^{D} \frac{1}{k_r^2} p_r(i) p_r(j) p_r(h) \quad (0 \le h, i, j \le D), \tag{33}$$

$$p_{ij}^h = \frac{k_i k_j}{|X|} \sum_{r=0}^{D} \frac{1}{m_r^2} q_r(i) q_r(j) q_r(h) \quad (0 \le h, i, j \le D). \tag{34}$$

The Terwilliger Algebra $T(x)$

Let $Y = (X, \{R_i\}_{0 \le i \le D})$ denote a scheme, fix any $x \in X$, and write $M^* = M^*(x)$. By the *Terwilliger algebra* for Y with respect to x, we mean the sub-algebra $T = T(x)$ of $\mathrm{Mat}_X(\mathbb{R})$ generated by M and M^*. The following result gives some relations in T.

2.1 Lemma [8, p. 379]. *Let $Y = (X, \{R_i\}_{0 \le i \le D})$ denote a scheme. Fix $x \in X$ and write $A_i^* = A_i^*(x)$, $E_i^* = E_i^*(x)$ $(0 \le i \le D)$. Then for all integers h, i, j $(0 \le h, i, j \le D)$,*

$$p_{ij}^h = 0 \quad \textit{if and only if} \quad E_i^* A_j E_h^* = 0, \tag{35}$$

$$q_{ij}^h = 0 \quad \textit{if and only if} \quad E_i A_j^* E_h = 0. \tag{36}$$

We now consider the modules for T. Let V denote the vector space $\mathbb{R}^{|X|}$ (row vectors), where the coordinates are indexed by X. Then T acts on V by right multiplication. We endow V with the inner product

$$\langle u, v \rangle = uv^t \quad (u, v \in V). \tag{37}$$

By a *T-module*, we mean a subspace W of V such that $WT \subseteq W$. A T-module W is said to be *irreducible* whenever $W \ne 0$, and W contains no T-modules other than 0 and W. Let W denote a T-module. Then by [4, Lem. 3.3],

$$W^\perp := \{v \in V | \langle v, w \rangle = 0 \text{ for all } w \in W\} \tag{38}$$

is also a T-module. It follows that V can be decomposed into an orthogonal direct sum of irreducible T-modules.

2.2 Lemma [8, p. 381]. *Let* $Y = (X, \{R_i\}_{0 \le i \le D})$ *denote a scheme. Fix* $x \in X$, *and write* $E_i^* = E_i^*(x)$ $(0 \le i \le D)$, $T = T(x)$.

(i) *There exists a unique irreducible T-module* W_0 *containing* VE_0 *and* VE_0^*.
(ii) $dim(W_0 E_i) = dim(W_0 E_i^*) = 1 \quad (0 \le i \le D)$.

We refer to W_0 as the *trivial T-module*.

Proof. Existence is shown in [8], and uniqueness follows from (6) and (12), since $rank(E_0^*) = rank(E_0) = 1$. □

2.3 Lemma. *Let* $Y = (X, \{R_i\}_{0 \le i \le D})$ *denote a scheme. Fix* $x \in X$, *and write* $E_i^* = E_i^*(x)$ $(0 \le i \le D)$, $T = T(x)$. *Let W be any irreducible T-module distinct from* W_0. *Then* $W \subseteq W_0^{\perp}$.

Proof. Fix any vectors $u \in W$ and $v \in W_0$. We must show that $\langle u, v \rangle = 0$. It follows from Lemma 2.2 that the T-module $W_0 E_0^* T$ is nonzero. Since W_0 is irreducible and contains this T-module, $W_0 E_0^* T = W_0$. So for some $w \in W_0$ and $B \in T$, we may write $v = wE_0^* B$. Observe $W \cap W_0 = 0$, since W, W_0 are irreducible. So $WE_0^* = 0$, otherwise W, W_0 have nonzero intersection by Lemma 2.2(i). It follows that

$$\langle u, v \rangle = \langle u, wE_0^* B \rangle \tag{39}$$

$$= \langle uB^t E_0^*, w \rangle, \tag{40}$$

which is zero, since $uB^t E_0^* \in WTE_0^* = WE_0^* = 0$, and we are done. □

2.4 Lemma. *Let* $Y = (X, \{R_i\}_{0 \le i \le D})$ *denote a scheme. Fix* $x \in X$, *write* $E_i^* = E_i^*(x)$ $(0 \le i \le D)$, *and let* W_0 *be as in Lemma* 2.2. *Then*

(i) $dim(W_0^{\perp} E_i) = m_i - 1 \quad (0 \le i \le D)$.
(ii) $W_0^{\perp} E_0 = 0$.
(iii) $dim(W_0^{\perp} E_i^*) = k_i - 1 \quad (0 \le i \le D)$.
(iv) $W_0^{\perp} E_0^* = 0$.

Proof of (i). Immediate from (31) and Lemma 2.2(ii). □

Proof of (ii). Immediate from part (i) and lines (26), (29). □

Proof of (iii). Immediate from (32) and Lemma 2.2(ii). □

Proof of (iv). Immediate from part (iii) and lines (25), (27). □

3. The *P*-Polynomial Property

Let $Y = (X, \{R_i\}_{0 \le i \le D})$ denote a scheme. We say that Y is *P-polynomial* (with respect to the given ordering $A_0, \ldots, A_D$ of the associate matrices) whenever $D \ge 1$, and for all integers h, $i, j (0 \le h, i, j \le D)$,

$$p_{ij}^h = 0 \text{ if one of } h, i, j \text{ is greater than the sum of the other two,} \tag{41}$$

$$p_{ij}^h \ne 0 \text{ if one of } h, i, j \text{ equals the sum of the other two.} \tag{42}$$

When Y is P-polynomial, we set

$$\theta_i := p_1(i) \quad (0 \le i \le D), \tag{43}$$

$$c_i := p_{1i-1}^i \quad (1 \le i \le D), \tag{44}$$

$$a_i := p_{1i}^i \quad (0 \le i \le D), \tag{45}$$

$$b_i := p_{1i+1}^i \quad (0 \le i \le D-1), \tag{46}$$

and define $c_0 = c_{D+1} = b_D = b_{-1} = 0$. We note $a_0 = 0, c_1 = 1$, and

$$a_i + b_i + c_i = k \quad (0 \le i \le D), \tag{47}$$

where $k := k_1$. Moreover, by (25) and (28),

$$\theta_0 = k \tag{48}$$

$$\ge \theta_i \quad (0 \le i \le D). \tag{49}$$

For the remainder of this section, $Y = (X, \{R_i\}_{0 \le i \le D})$ will denote a scheme which is P-polynomial with respect to the given ordering $A_0, \ldots, A_D$ of the associate matrices.

3.1 Lemma [1, p. 190].

$$\theta_i \ne \theta_j \quad \text{if} \quad i \ne j \quad (0 \le i, j \le D). \tag{50}$$

3.2 Lemma [2, p. 45]. *For any integers i, j $(0 \le i, j \le D)$,*

$$c_{j+1}p_{j+1}(i) + a_j p_j(i) + b_{j-1}p_{j-1}(i) = \theta_i p_j(i), \tag{51}$$

where $p_{-1}(i), p_{D+1}(i)$ are indeterminants.

3.3 Lemma [2, p. 45]. *For any integers i, j $(0 \le i, j \le D)$,*

$$c_j q_i(j-1) + a_j q_i(j) + b_j q_i(j+1) = \theta_i q_i(j), \tag{52}$$

where $q_i(-1), q_i(D+1)$ are indeterminants.

3.4 Lemma. *For any integer i $(0 \le i \le D)$,*

(i)

$$\frac{\theta_i}{k} = \frac{q_i(1)}{q_i(0)}. \tag{53}$$

(ii) *Suppose $D \geq 2$, and that $i \neq 0$. Then $q_i(0) > q_i(1)$. Moreover,*

$$\frac{1+\theta_i}{b_1} = \frac{q_i(1) - q_i(2)}{q_i(0) - q_i(1)}. \tag{54}$$

Proof of (i). Set $j = 0$ in (52) and simplify, noting $a_0 = c_0 = 0$, $b_0 = k$. □

Proof of (ii). The first assertion follows from (49), (50), and (53). To get (54), set $j = 1$ in (52) to obtain

$$c_1 q_i(0) + a_1 q_i(1) + b_1 q_i(2) = \theta_i q_i(1). \tag{55}$$

Use (47) to eliminate a_1, then simplify using (53), noting that $c_1 = 1$. □

3.5 Lemma. *Fix any $x \in X$ and write $E_i^* = E_i^*(x)$ $(0 \leq i \leq D)$. Then for any integer i $(0 \leq i \leq D)$,*

$$|X| E_1^* E_i E_1^* = q_i(0) E_1^* + q_i(1) E_1^* A_1 E_1^* + q_i(2) E_1^* A_2 E_1^*. \tag{56}$$

In particular,

$$|X| E_1^* E_0 E_1^* = E_1^* + E_1^* A_1 E_1^* + E_1^* A_2 E_1^*. \tag{57}$$

Proof. By (22),

$$|X| E_1^* E_i E_1^* = E_1^* \left(\sum_{j=0}^{D} q_i(j) A_j \right) E_1^*. \tag{58}$$

To get (56), simplify (58) using (35) and (41). To obtain line (57), set $i = 0$ in (56) and apply (29). □

3.6 Lemma [8, p. 383]. *Fix any $x \in X$, and let W denote an irreducible $T(x)$-module. Then there exist unique integers r, d $(0 \leq r, d \leq D)$ such that*

$$WE_i^* \neq 0 \quad \textit{iff} \quad r \leq i \leq r + d \quad (0 \leq i \leq D). \tag{59}$$

The scalars r and d are known as the *endpoint* and the *diameter* of W, respectively.

4. The Q-Polynomial Property

Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be any scheme. We say that Y is *Q-polynomial* (with respect to an ordering $E_0, \ldots, E_D$ of the primitive idempotents) whenever $D \geq 1$, and for all integers h, i, j $(0 \leq h, i, j \leq D)$,

$$q_{ij}^h = 0 \text{ if one of } h, i, j \text{ is greater than the sum of the other two}, \tag{60}$$

$$q_{ij}^h \neq 0 \text{ if one of } h, i, j \text{ equals the sum of the other two}. \tag{61}$$

When Y is Q-polynomial, we set

$$\theta_i^* := q_1(i) \quad (0 \le i \le D), \tag{62}$$

$$c_i^* := q_{1i-1}^i \quad (1 \le i \le D), \tag{63}$$

$$a_i^* := q_{1i}^i \quad (0 \le i \le D), \tag{64}$$

$$b_i^* := q_{1i+1}^i \quad (0 \le i \le D-1), \tag{65}$$

and define $c_0^* = c_{D+1}^* = b_D^* = b_{-1}^* = 0$. We note $a_0^* = 0$ and $c_1^* = 1$ [1, p. 67]. Also,

$$a_i^* + b_i^* + c_i^* = m \quad (0 \le i \le D), \tag{66}$$

where $m := m_1$. Moreover, by (26) and (30),

$$\theta_0^* = m \tag{67}$$

$$\ge \theta_i^* \quad (0 \le i \le D). \tag{68}$$

For the remainder of this section, $Y = (X, \{R_i\}_{0 \le i \le D})$ will denote a scheme which is Q-polynomial with respect to the ordering $E_0, \dots, E_D$ of the primitive idempotents.

4.1 Lemma [1, p. 193].

$$\theta_i^* \ne \theta_j^* \quad \textit{if} \quad i \ne j \quad (0 \le i, j \le D). \tag{69}$$

4.2 Lemma [2, p. 49]. *For any integers* i, j $(0 \le i, j \le D)$,

$$c_{j+1}^* q_{j+1}(i) + a_j^* q_j(i) + b_{j-1}^* q_{j-1}(i) = \theta_i^* q_j(i), \tag{70}$$

where $q_{-1}(i)$, $q_{D+1}(i)$ *are indeterminants.*

4.3 Lemma [2, p. 49]. *For any integers* i, j $(0 \le i, j \le D)$,

$$c_j^* p_i(j-1) + a_j^* p_i(j) + b_j^* p_i(j+1) = \theta_i^* p_i(j), \tag{71}$$

where $p_i(-1)$, $p_i(D+1)$ *are indeterminants.*

4.4 Lemma. *For any integer* i $(0 \le i \le D)$,

(i)

$$\frac{\theta_i^*}{m} = \frac{p_i(1)}{p_i(0)}. \tag{72}$$

(ii) *Suppose* $D \ge 2$, *and that* $i \ne 0$. *Then* $p_i(0) > p_i(1)$. *Moreover,*

$$\frac{1 + \theta_i^*}{b_1^*} = \frac{p_i(1) - p_i(2)}{p_i(0) - p_i(1)}. \tag{73}$$

Proof. Similar to the proof of Lemma 3.4. □

4.5 Lemma. *Fix any $x \in X$, and write $A_i^* = A_i^*(x)$, $E_i^* = E_i^*(x)$ $(0 \leq i \leq D)$. Then for any integer i $(0 \leq i \leq D)$,*

$$|X|E_1 E_i^* E_1 = p_i(0)E_1 + p_i(1)E_1 A_1^* E_1 + p_i(2)E_1 A_2^* E_1. \tag{74}$$

In particular,

$$|X|E_1 E_0^* E_1 = E_1 + E_1 A_1^* E_1 + E_1 A_2^* E_1. \tag{75}$$

Proof. Similar to the proof of Lemma 3.5. □

4.6 Lemma [8, p. 385]. *Fix any $x \in X$, and let W denote an irreducible $T(x)$-module. Then there exist unique integers r^*, d^* $(0 \leq r^*, d^* \leq D)$ such that*

$$WE_i \neq 0 \quad \textit{iff} \quad r^* \leq i \leq r^* + d^* \quad (0 \leq i \leq D). \tag{76}$$

The scalars r^* and d^* are known as the *dual endpoint* and the *dual diameter* of W, respectively.

5. A Computation in $T(x)$ for P-Polynomial Schemes

In this section, $Y = (X, \{R_i\}_{0 \leq i \leq D})$ will denote a scheme with $D \geq 2$ which is P-polynomial with respect to the given ordering $A_0, \ldots, A_D$ of the associate matrices.

5.1 Definition. Fix any $x \in X$, write $E_1^* = E_1^*(x)$, and pick any nonzero $v \in VE_1^*$ such that $vE_0 = 0$. By the *type* of v, we mean the element $\psi \in \mathbb{R} \cup \{\infty\}$ such that

$$\frac{vA_2v^t}{\|v\|^2} = \frac{b_1}{\psi + 1}, \tag{77}$$

where we interpret $\psi = \infty$ whenever $vA_2v^t = 0$.

5.2 Theorem. *Fix any $x \in X$, write $E_1^* = E_1^*(x)$, and pick any non-zero $v \in VE_1^*$ such that $vE_0 = 0$. Then*

$$\frac{\|vE_i\|^2}{\|v\|^2} = \frac{m_i(k - \theta_i)(\psi - \theta_i)}{k|X|(\psi + 1)} \quad (0 \leq i \leq D), \tag{78}$$

where ψ denotes the type of v.

5.3 Remark. With the notation of Theorem 5.2, suppose $\psi = \infty$. Then we take limits to obtain

$$\frac{\|vE_i\|^2}{\|v\|^2} = \frac{m_i(k - \theta_i)}{k|X|} \quad (0 \leq i \leq D). \tag{79}$$

Proof of Theorem 5.2. Line (78) holds for $i = 0$, since $vE_0 = 0$ by assumption, and since $k = \theta_0$ by (48). Now assume $i > 0$. Observe $v = vE_1^*$ by construction, so

$$\|vE_i\|^2 = \langle vE_1^*E_i, vE_1^*E_i \rangle \tag{80}$$

$$= vE_1^*E_iE_1^*v^t, \tag{81}$$

by (7), (8), (13), and (37). Eliminate $E_1^*A_1E_1^*$ in (56) using (57) to obtain

$$|X|E_1^*E_iE_1^* = (q_i(0) - q_i(1))E_1^* - (q_i(1) - q_i(2))E_1^*A_2E_1^* + q_i(1)|X|E_1^*E_0E_1^*. \tag{82}$$

Eliminate $E_1^*E_iE_1^*$ in (81) using (82) and simplify using (77), Lemma 3.4(ii) and the assumption that $vE_0 = 0$ to obtain

$$\|vE_i\|^2 = |X|^{-1}(q_i(0) - q_i(1))\frac{\psi - \theta_i}{\psi + 1}\|v\|^2. \tag{83}$$

By (53) and (26),

$$q_i(0) - q_i(1) = \frac{m_i(k - \theta_i)}{k}. \tag{84}$$

Now simplify (83) using (84) to obtain the result. □

5.4 Theorem. *Let θ_{sec} and θ_{min} denote the second greatest and minimal of $\theta_0, \ldots, \theta_D$, respectively, and let E_{sec}, E_{min} denote the associated primitive idempotents. Fix $x \in X$, write $E_1^* = E_1^*(x)$, and pick any non-zero $v \in VE_1^*$ such that $vE_0 = 0$. Let ψ denote the type of v. Then*

(i) *Suppose $\psi \neq \infty$. Then $\psi \geq \theta_{sec}$ or $\psi \leq \theta_{min}$.*
(ii) *$vE_{sec} = 0$ iff $\psi = \theta_{sec}$.*
(iii) *$vE_{min} = 0$ iff $\psi = \theta_{min}$.*
(iv) *Let E denote a primitive idempotent of Y other than E_0, E_{sec}, or E_{min}. Then $vE \neq 0$.*

Proof of (i). For convenience, set $m_{sec} := m_i$, where $E_i = E_{sec}$, and define m_{min} similarly. Suppose $\theta_{sec} > \psi > \theta_{min}$. Applying Theorem 5.2, (49), (50),

$$0 \leq \frac{\|vE_{sec}\|^2}{\|v\|^2}\frac{\|vE_{min}\|^2}{\|v\|^2} \tag{85}$$

$$= \frac{m_{sec}}{|X|}\frac{m_{min}}{|X|}\frac{(k - \theta_{sec})(k - \theta_{min})}{k^2(\psi + 1)^2}(\psi - \theta_{sec})(\psi - \theta_{min}) \tag{86}$$

$$< 0, \tag{87}$$

a contradiction. We conclude that $\psi \geq \theta_{sec}$ or $\psi \leq \theta_{min}$. □

Proof of (ii). By Theorem 5.2, $\psi = \theta_{sec}$ iff $\|vE_{sec}\|^2 = 0$ iff $vE_{sec} = 0$. □

Proof of (iii). Similar. □

Proof of (iv). $\|vE\|^2$ is never zero, by Theorem 5.2 and part (i) above. □

5.5 Corollary. *Fix $x \in X$, write $E_1^* = E_1^*(x)$, and pick any non-zero $v \in VE_1^*$ such that $vE_0 = 0$. Then $vE_i = 0$ for at most one i $(1 \leq i \leq D)$.*

Proof. Immediate from Theorem 5.4(ii)–(iv). □

6. A Computation in $T(x)$ for Q-Polynomial Schemes

In this section, $Y = (X, \{R_i\}_{0 \leq i \leq D})$ will denote a scheme with $D \geq 2$ which is Q-polynomial with respect to the ordering $E_0, \ldots, E_D$ of the primitive idempotents.

6.1 Definition. Fix any $x \in X$, write $A_2^* = A_2^*(x)$, $E_0^* = E_0^*(x)$, and pick any non-zero $v \in VE_1$ such that $vE_0^* = 0$. By the *dual type* of v, we mean the element $\psi^* \in \mathbb{R} \cup \{\infty\}$ such that

$$\frac{vA_2^*v^t}{\|v\|^2} = \frac{b_1^*}{\psi^* + 1}, \tag{88}$$

where we interpret $\psi^* = \infty$ whenever $vA_2^*v^t = 0$.

6.2 Theorem. *Fix any $x \in X$, write $E_i^* = E_i^*(x)$ $(0 \leq i \leq D)$, and pick any non-zero $v \in VE_1$ such that $vE_0^* = 0$. Then*

$$\frac{\|vE_i^*\|^2}{\|v\|^2} = \frac{k_i(m - \theta_i^*)(\psi^* - \theta_i^*)}{m|X|(\psi^* + 1)} \quad (0 \leq i \leq D), \tag{89}$$

where ψ^* denotes the dual type of v.

6.3 Remark. With the notation of Theorem 6.2, suppose $\psi^* = \infty$. Then we take limits to obtain

$$\frac{\|vE_i^*\|^2}{\|v\|^2} = \frac{k_i(m - \theta_i^*)}{m|X|} \quad (0 \leq i \leq D). \tag{90}$$

Proof of Theorem 6.2. Similar to the proof of Theorem 5.2. □

6.4 Theorem. *Fix any $x \in X$, and write $E_i^* = E_i^*(x)$ $(0 \leq i \leq D)$. Let θ_{sec}^* and θ_{min}^* denote the second greatest and minimal of $\theta_0^*, \ldots, \theta_D^*$, respectively, and let E_{sec}^* and E_{min}^* denote the associated dual idempotents. Pick any non-zero $v \in VE_1$ such that $vE_0^* = 0$. Let ψ^* denote the dual type of v. Then*

(i) *Suppose $\psi^* \neq \infty$. Then $\psi^* \geq \theta_{sec}^*$ or $\psi^* \leq \theta_{min}^*$.*
(ii) *$vE_{sec}^* = 0$ iff $\psi^* = \theta_{sec}^*$.*
(iii) *$vE_{min}^* = 0$ iff $\psi^* = \theta_{min}^*$.*
(iv) *Let E^* denote a dual idempotent of Y other than E_0^*, E_{sec}^* or E_{min}^*. Then $vE^* \neq 0$.*

Proof. Similar to the proof of Theorem 5.4. □

6.5 Corollary. *Fix* $x \in X$, *write* $E_i^* = E_i^*(x)$ $(0 \le i \le D)$, *and pick any non-zero* $v \in VE_1$ *such that* $vE_0^* = 0$. *Then* $vE_i^* = 0$ *for at most one* i $(1 \le i \le D)$.

Proof. Immediate from Theorem 6.4(ii)–(iv). □

7. A Matrix Result

Let $v = (v_0, \ldots, v_D)$ be a finite sequence of real numbers. First assume the terms of v are non-zero. By the *number of sign changes* of v, we mean the number of indices i $(0 \le i \le D-1)$ such that $v_i v_{i+1} < 0$. If v has one or more terms which are zero, we count sign changes by first deleting the zero terms of v and then counting the sign changes of the resulting sequence. The following is a reworking of a result found in [5, p. 143].

7.1 Theorem. *Let* D *denote a nonnegative integer, and suppose* B *is any real* $(D+1) \times (D+1)$ *matrix of the form:*

$$\begin{pmatrix} a_0 & c_1 & & & \mathbf{0} \\ b_0 & a_1 & c_2 & & \\ & b_1 & \ddots & \ddots & \\ & & \ddots & a_{D-1} & c_D \\ \mathbf{0} & & & b_{D-1} & a_D \end{pmatrix}$$

where $b_i c_{i+1} > 0$ $(0 \le i \le D-1)$.

(i) *B has $D+1$ distinct eigenvalues. In particular, the maximal eigenspaces for B are 1-dimensional.*
(ii) *Fix any i $(1 \le i \le D+1)$, let θ denote the i^{th} greatest eigenvalue of B, and let v denote an associated (left) eigenvector. Then v has exactly $i-1$ sign changes.*

Proof of (i). We first show that B is diagonalizable. To this end, set

$$K := diag(1, k_1^{-1/2}, k_2^{-1/2}, \ldots, k_D^{-1/2}), \tag{91}$$

where

$$k_i := \frac{b_0 \ldots b_{i-1}}{c_1 \ldots c_i} \quad (0 \le i \le D). \tag{92}$$

One readily checks that KBK^{-1} is real and symmetric; it follows by elementary linear algebra that B is diagonalizable. It remains to show that the minimal polynomial of B has degree $D+1$. But this is immediate, since the tridiagonal form of B implies that I, B, $B^2, \ldots, B^D$ are linearly independent. □

Proof of (ii). Set

$$L := diag(1, b_0, b_0 b_1, \ldots, b_0 \ldots b_{D-1}), \tag{93}$$

and observe

$$L^{-1}BL = \begin{pmatrix} a_0 & b_0c_1 & & & & \mathbf{0} \\ 1 & a_1 & b_1c_2 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & a_{D-1} & b_{D-1}c_D \\ \mathbf{0} & & & 1 & a_D \end{pmatrix}.$$

Note that B and $L^{-1}BL$ have the same eigenvalues, and in particular, θ is the i^{th} greatest eigenvalue for $L^{-1}BL$. Let S denote the maximal (left) eigenspace of $L^{-1}BL$ associated with θ and note that S is 1-dimensional. Observe that $vL \in S$, and so vL must span S. By [5, p. 143], there exists a vector in S which has exactly $i-1$ sign changes, so vL must have $i-1$ sign changes. Observe that by (93), the i^{th} coordinate of vL equals the i^{th} coordinate of v times a positive scalar, so v and vL have the same number of sign changes. □

8. *P*- and *Q*-Polynomial Schemes

In this section, let $Y = (X, \{R_i\}_{0 \le i \le D})$ denote a scheme which is P-polynomial with respect to the given ordering $A_0, \ldots, A_D$ of the associate matrices and Q-polynomial with respect to the ordering $E_0, \ldots, E_D$ of the primitive idempotents. We begin with a slight modification of a result of Godsil [5, p. 264].

8.1 Theorem. *The following are equivalent*:

(i) $\theta_1^* = \theta_{sec}^*$.
(ii) $\theta_0 > \theta_1 > \theta_2 > \theta_3 > \ldots > \theta_{D-3} > \theta_{D-2} > \theta_{D-1} > \theta_D$.
(iii) $\theta_1 = \theta_{sec}$.
(iv) $\theta_0^* > \theta_1^* > \theta_2^* > \theta_3^* > \ldots > \theta_{D-3}^* > \theta_{D-2}^* > \theta_{D-1}^* > \theta_D^*$.

Proof. (i) $\Rightarrow$ (ii). Consider the vector $v := (\theta_0 - \theta_1,\ \theta_1 - \theta_2, \ldots, \theta_{D-1} - \theta_D)$. Observe the first coordinate of v is positive by (48)–(50), and no coordinate is zero by (50), so it remains to show v has no sign changes. To this end, consider the matrix

$$C = \begin{pmatrix} \theta_0^* - b_0^* - c_1^* & c_1^* & & & & \mathbf{0} \\ b_1^* & \theta_0^* - b_1^* - c_2^* & c_2^* & & & \\ & b_2^* & \ddots & \ddots & & \\ & & \ddots & \ddots & c_{D-1}^* \\ \mathbf{0} & & & b_{D-1}^* & \theta_0^* - b_{D-1}^* - c_D^* \end{pmatrix}.$$

Observe by (11), (61), (63), and (65),

$$b_i^* c_i^* > 0 \quad (0 \le i \le D-1), \tag{94}$$

so C satisfies the assumptions of Theorem 7.1. By [2, p. 130], the eigenvalues of C are $\theta_1^*, \theta_2^*, \ldots, \theta_D^*$; in particular, θ_1^* is the maximal eigenvalue of C. Setting $i = 1$ in (71),

$$c_j^* \theta_{j-1} + a_j^* \theta_j + b_j^* \theta_{j+1} = \theta_1^* \theta_j \quad (0 \leq j \leq D). \tag{95}$$

Using this and (66), (67), one readily shows

$$vC = \theta_1^* v. \tag{96}$$

Now v has no sign changes by Theorem 7.1, and we are done.

(ii) ⇒ (iii). Immediate.

(iii) ⇒ (iv). Similar to the argument for (i) ⇒ (ii), replacing $(a_i, b_i, c_i, \theta_i^*, \theta_i)$ by $(a_i^*, b_i^*, c_i^*, \theta_i, \theta_i^*)$.

(iv) ⇒ (i). Immediate. □

8.2 Lemma. *Let $\alpha, \beta, \gamma, \delta$ denote integers such that*

$$0 \leq \alpha \pm \gamma, \alpha \pm \delta, \beta \pm \gamma, \beta \pm \delta \leq D. \tag{97}$$

Then

$$(\theta_{\alpha-\gamma} - \theta_{\beta+\delta})(\theta_{\alpha+\gamma} - \theta_{\beta-\delta}) = (\theta_{\alpha-\delta} - \theta_{\beta+\gamma})(\theta_{\alpha+\delta} - \theta_{\beta-\gamma}), \tag{98}$$

$$(\theta^*_{\alpha-\gamma} - \theta^*_{\beta+\delta})(\theta^*_{\alpha+\gamma} - \theta^*_{\beta-\delta}) = (\theta^*_{\alpha-\delta} - \theta^*_{\beta+\gamma})(\theta^*_{\alpha+\delta} - \theta^*_{\beta-\gamma}), \tag{99}$$

$$(\theta_{\alpha-\gamma} - \theta_{\beta+\delta})(\theta^*_{\alpha+\gamma} - \theta^*_{\beta-\delta}) = (\theta^*_{\alpha-\delta} - \theta^*_{\beta+\gamma})(\theta_{\alpha+\delta} - \theta_{\beta-\gamma}). \tag{100}$$

Proof. Multiply out using the formulas given in [8, pp. 370–372]. □

9. The Bipartite Case

9.1 Definition. Let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ be any scheme. Suppose Y is P-polynomial with respect to the given ordering $A_0, \ldots, A_D$ of the associate matrices. We say that Y is *bipartite* if there exists a bipartition

$$X = X^+ \cup X^- \tag{101}$$

such that the restrictions of R_1 to X^+ and X^- are empty.

Observe that if Y is bipartite, then there can be no cycles of odd length, from which it follows that

$$a_i = 0 \quad (0 \leq i \leq D). \tag{102}$$

For the remainder of this section, let $Y = (X, \{R_i\}_{0 \leq i \leq D})$ denote a scheme which is bipartite P-polynomial with respect to the given ordering $A_0, \ldots, A_D$ of the associate matrices.

9.2 Lemma [2, p. 82]. *Suppose the primitive idempotents of Y are ordered so that* $\theta_0 > \theta_1 > \cdots > \theta_D$*. Then*

$$\theta_i = -\theta_{D-i} \quad (0 \le i \le D), \tag{103}$$

$$m_i = m_{D-i} \quad (0 \le i \le D). \tag{104}$$

We next show that any Q-polynomial ordering of the primitive idempotents of Y also has the above property.

9.3 Definition. Suppose Y is Q-polynomial with respect to the ordering $E_0, \ldots, E_D$ of the primitive idempotents. Then for any integer i $(0 \le i \le D)$, we define i' to be the unique integer $(0 \le i' \le D)$ such that $\theta_{i'} = -\theta_i$. Note that $m_{i'} = m_i$ by Lemma 9.2.

9.4 Lemma. *Suppose that Y is Q-polynomial with respect to the ordering* $E_0, \ldots, E_D$ *of the primitive idempotents. Then*

$$p_j(i') = (-1)^j p_j(i) \quad (0 \le i, j \le D). \tag{105}$$

Proof. Consider the polynomials $P_j(x)$ $(0 \le j \le D)$ defined recursively such that $P_0(x) = 1$, $P_1(x) = x$, and

$$xP_j(x) = b_{j-1}P_{j-1}(x) + c_{j+1}P_{j+1}(x) \quad (1 \le j \le D-1). \tag{106}$$

Observe $P_j(x)$ is an even function when j is even, and an odd function when j is odd. So it follows that

$$P_j(\theta_{i'}) = (-1)^j P_j(\theta_i) \quad (0 \le i, j \le D) \tag{107}$$

since $\theta_{i'} = -\theta_i$. But by (51), (102), and (106), we see that

$$p_j(i) = P_j(\theta_i) \quad (0 \le i, j \le D), \tag{108}$$

and so the result now follows. □

9.5 Lemma. *Suppose that Y is Q-polynomial with respect to the ordering* $E_0, \ldots, E_D$ *of the primitive idempotents. Then*

$$q^{h'}_{ij'} = q^h_{ij} \quad (0 \le h, i, j \le D). \tag{109}$$

Proof. For any integers h, i, j $(0 \le h, i, j \le D)$, we have

$$q^{h'}_{ij'} = \frac{m_i m_{j'}}{|X|} \sum_{r=0}^{D} \frac{1}{k_r^2} p_r(i) p_r(j') p_r(h'), \tag{110}$$

$$= \frac{m_i m_j}{|X|} \sum_{r=0}^{D} \frac{1}{k_r^2} p_r(i)(-1)^r p_r(j)(-1)^r p_r(h), \tag{111}$$

$$= q^h_{ij}, \tag{112}$$

by (33) and (105). □

9.6 Theorem. *Suppose that* Y *is* Q*-polynomial with respect to the ordering* $E_0, \ldots, E_D$ *of the primitive idempotents. Then*

$$\theta_i = -\theta_{D-i} \quad (0 \le i \le D), \tag{113}$$

$$m_i = m_{D-i} \quad (0 \le i \le D). \tag{114}$$

Proof. In view of Definition 9.3, it suffices to show

$$i' = D - i \quad (0 \le i \le D). \tag{115}$$

Let D_1 denote the (undirected) graph with vertex set $\mathscr{I} = \{0, 1, \ldots, D\}$, where for any distinct $i, j \in \mathscr{I}$, i is adjacent to j if and only if $q^i_{1j} \neq 0$. By the definition of Q-polynomial, it follows that for any $i, j \in \mathscr{I}$, i is adjacent to j precisely when $|i - j| = 1$. In particular, D_1 is simply a path. By Lemma 9.5, the map $i \mapsto i'$ induces a nontrivial automorphism of D_1. But the only nontrivial automorphism of D_1 is the map $i \mapsto D - i$, and (115) follows. Line (114) follows in view of (104). □

For the remainder of this section, we note some important properties of bipartite graphs which will be useful in the proof of our main theorem.

9.7 Lemma. *Suppose that* Y *is* Q*-polynomial with respect to the ordering* $E_0, \ldots, E_D$ *of the primitive idempotents. Fix* $x \in X$*, write* $T = T(x)$*, and let* W_0 *denote the trivial* T*-module, as in Lemma* 2.2*. Then* $m_D = 1$*. In particular,*

$$W_0^{\perp} E_D = 0. \tag{116}$$

Proof. The first statement is immediate from (114), (26) and (29). Now (116) follows by Lemma 2.4(i). □

9.8 Lemma. *Suppose that* Y *is* Q*-polynomial with respect to the ordering* $E_0, \ldots, E_D$ *of the primitive idempotents. Then*

(i) [7, p. 301] $m \ge k$.
(ii) $\theta^*_0 \ge \theta_0$.

Proof of (ii). Immediate from (48), (67), and (i). □

9.9 Lemma [1, p. 316]. *Suppose* $D \ge 2$*, and set* $t := \lfloor D/2 \rfloor$ *(i.e., the greatest integer less than or equal to* $D/2$*). Then* $\frac{1}{2} Y := (X^+, \{\mathscr{R}_i\}_{0 \le i \le t})$ *is a* P*-polynomial scheme, where* X^+ *is from* (101)*, and where*

$$\mathscr{R}_i = \{yz | y, z \in X^+, yz \in R_{2i}\} \quad (0 \le i \le t). \tag{117}$$

We refer to $\frac{1}{2} Y$ *as a halved scheme of* Y.

9.10 Lemma. *Suppose that* Y *is* Q*-polynomial with respect to the ordering* $E_0, \ldots, E_D$ *of the primitive idempotents. Then*

(i) [2, p. 142] *The eigenvalues of* $\frac{1}{2}Y$ *are* $\phi_0, \ldots, \phi_t$, *where*

$$\phi_i := \frac{\theta_i^2 - k}{c_2} \quad (0 \le i \le t). \tag{118}$$

(ii) [1, p. 328] $\frac{1}{2}Y$ *is Q-polynomial with respect to the ordering* $\mathscr{E}_0, \ldots, \mathscr{E}_t$, *where* $\mathscr{E}_i$ *denotes the primitive idempotent for* $\frac{1}{2}Y$ *associated with* $\phi_i (0 \le i \le t)$.

(iii) [2, p. 241] *With respect to the Q-polynomial structure in* (ii), *the dual eigenvalues are* $\phi_0^*, \phi_1^*, \ldots, \phi_t^*$, *where*

$$\phi_i^* := \theta_{2i}^* \quad (0 \le i \le t). \tag{119}$$

10. The Proof of the Main Theorem

In this section, $Y = (X, \{R_i\}_{0 \le i \le D})$ will denote a scheme with $D \ge 3$ which is not a cycle. We also assume Y is bipartite P-polynomial with respect to the given ordering $A_0, \ldots, A_D$ of the associate matrices and Q-polynomial with respect to the ordering $E_0, \ldots, E_D$ of the primitive idempotents.

10.1 Lemma. [9, Th. 1], [6, Th. 5.1]. *Suppose* $\theta_0^* = \theta_0$. *Then* Y *is an antipodal* 2-*cover, and exactly one of the following must occur*:

(i) $D = 3$, *and for some integer* $k \ge 3$,

$$(c_0, c_1, c_2, c_3) = (0, 1, k-1, k). \tag{120}$$

(ii) $D = 4$, *and for some positive real number* γ,

$$(c_0, c_1, c_2, c_3, c_4) = (0, 1, 2\gamma, 4\gamma - 1, 4\gamma). \tag{121}$$

(iii) $D = 5$, *and for some positive real number* γ,

$$(c_0, c_1, c_2, c_3, c_4, c_5) = (0, 1, \gamma^2 + \gamma, k - \gamma^2 - \gamma, k - 1, k), \tag{122}$$

where $k = \gamma^3 + 3\gamma^2 + \gamma$.

(iv) D *is arbitrary, and*

$$c_i = i \quad (0 \le i \le D). \tag{123}$$

Proof. By (48) and (67), $m = k$. By a result of Yamazaki [9, Th. 1], Y is 2-homogeneous in the sense of Nomura [6, Def. 3.1]. Nomura's classification of these schemes [6, Th. 5.1] provides the intersection arrays. From these arrays, one easily computes that $k_D = 1$, which implies that Y is an antipodal 2-cover. □

10.2 Lemma. *Suppose* $\theta_0^* = \theta_0$. *Then*

$$\theta_i^* = \theta_i \quad (0 \le i \le D). \tag{124}$$

Proof. Setting $\alpha = i$, $\beta = D - i$, $\gamma = -1$, $\delta = 0$ in (100),

$$(\theta_{i+1} - \theta_{D-i})(\theta_{i-1}^* - \theta_{D-i}^*) = (\theta_i^* - \theta_{D-i-1}^*)(\theta_i - \theta_{D-i+1}), \tag{125}$$

for $(1 \le i \le D-1)$. By Lemma 10.1, Y is an antipodal 2-cover. So by [2, p. 243],

$$\theta_i^* = -\theta_{D-i}^* \quad (0 \le i \le D). \tag{126}$$

Now by (113) and (126), line (125) becomes

$$(\theta_{i+1} + \theta_i)(\theta_{i-1}^* + \theta_i^*) = (\theta_i^* + \theta_{i+1}^*)(\theta_i + \theta_{i-1}), \tag{127}$$

for $(1 \le i \le D-1)$. By assumption, $\theta_0^* = \theta_0$. So by (62) and (53) with $i = 1$, $\theta_1^* = \theta_1$. Observe that for any i $(0 \le i \le \lfloor \frac{D}{2} \rfloor)$, the coefficient of θ_{i+1}^* is $\theta_i + \theta_{i-1}$, which is nonzero by (50), (113). So by a simple induction on (127), we see that

$$\theta_i = \theta_i^* \quad \left(0 \le i \le \left\lfloor \frac{D}{2} \right\rfloor\right). \tag{128}$$

Line (124) follows by (113) and (126). □

10.3 Lemma. *Suppose* $\theta_0^* = \theta_0$. *Then one of the following must occur*:

(i) $D = 3$, *and for some integer* $k \ge 3$,

$$(\theta_0, \theta_1, \theta_2, \theta_3) = (k, 1, -1, -k). \tag{129}$$

(ii) $D = 4$, *and for some positive real number* γ,

$$(\theta_0, \theta_1, \theta_2, \theta_3, \theta_4) = (4\gamma, 2\sqrt{\gamma}, 0, -2\sqrt{\gamma}, -4\gamma). \tag{130}$$

(iii) $D = 4$, *and for some positive real number* γ,

$$(\theta_0, \theta_1, \theta_2, \theta_3, \theta_4) = (4\gamma, -2\sqrt{\gamma}, 0, 2\sqrt{\gamma}, -4\gamma). \tag{131}$$

(iv) $D = 5$, *and for some positive real number* γ,

$$(\theta_0, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5) = (k, \gamma^2 + 2\gamma, \gamma, -\gamma, -\gamma^2 - 2\gamma, -k), \tag{132}$$

where $k = \gamma^3 + 3\gamma^2 + \gamma$.

(v) D *is arbitrary, and*

$$\theta_i = D - 2i \quad (0 \le i \le D). \tag{133}$$

(vi) D *is even, and*

$$\theta_i = (-1)^i (D - 2i) \quad (0 \le i \le D). \tag{134}$$

In particular, one of (i), (ii) *holds in Theorem* 1.1.

Proof. From the possible intersection arrays given in Lemma 10.1, the eigenvalues (unordered) are readily computed. To compute the possible Q-polynomial orderings, observe that by (124), and (52) with $i = 1$,

$$c_i \theta_{i-1} + a_i \theta_i + b_i \theta_{i+1} = \theta_1 \theta_i \quad 0 \le i \le D. \tag{135}$$

Certainly $\theta_0 = k$, and by [3, Th. 9.6], $\theta_1 \in \{\theta_{sec}, -\theta_{sec}\}$ if D is even, and $\theta_1 = \theta_{sec}$ if D is odd. Equation (135) can now be used inductively to solve for the remainder of the Q-polynomial ordering. □

10.4 Lemma. *Fix $x \in X$ and write $E_i^* = E_i^*(x)$ $(0 \le i \le D)$.*

(i) *There exists a nonzero $v \in VE_1$ such that $vE_0^* = 0$ and $vE_D^* = 0$.*
(ii) $\theta_D^* = \theta_{sec}^*$ *or* $\theta_D^* = \theta_{min}^*$.

Proof of (i). By [4, Th. 8.7], there exists an irreducible $T(x)$-module W with endpoint 1 and diameter $D-2$. We pick any nonzero $u \in WE_1^*$ and show that $v := uE_1$ has the desired properties. Certainly $v \in VE_1$. Observe

$$vE_0^* \in WE_0^*, \tag{136}$$

which is 0, since W has endpoint 1, and

$$vE_D^* \in WE_D^*, \tag{137}$$

which is 0, since W has diameter $D-2$. It remains to show $v \ne 0$. Observe $W \ne W_0$, so $u \in W \subseteq W_0^\perp$ by Lemma 2.3(ii). Now $uE_0 = 0$ by Lemma 2.4(ii), and $uE_D = 0$ by Lemma 9.7. We conclude $v = uE_1 \ne 0$ by Corollary 5.5, as desired. □

Proof of (ii). By (i) and Theorem 6.4(iv), $E_D^* = E_{sec}^*$ or $E_D^* = E_{min}^*$. □

10.5 Lemma. *Fix $x \in X$, write $E_i^* = E_i^*(x)$ $(0 \le i \le D)$, and suppose $\theta_0^* > \theta_0$.*

(i) *There exists a nonzero $v \in VE_1$ such that $vE_0^* = 0$ and $vE_1^* = 0$.*
(ii) $\theta_1^* = \theta_{sec}^*$ *or* $\theta_1^* = \theta_{min}^*$.

Proof of (i). Observe by Lemma 2.4(i), (iii) and lines (67), (48),

$$dim(W_0^\perp E_1) = \theta_0^* - 1 \tag{138}$$

$$> \theta_0 - 1 \tag{139}$$

$$= dim(W_0^\perp E_1^*). \tag{140}$$

It follows that the linear transformation

$$W_0^\perp E_1 \to W_0^\perp E_1^* \tag{141}$$

$$v \to vE_1^* \tag{142}$$

has a nontrivial kernel. This means there exists a nonzero $v \in W_0^\perp E_1$ such that $vE_1^* = 0$. Observe $v \in W_0^\perp$, so $vE_0^* = 0$ by Lemma 2.4(iv). □

Proof of (ii). By (i) and Theorem 6.4(iv), $E_1^* = E_{sec}^*$ or $E_1^* = E_{min}^*$. □

10.6 Lemma. *Suppose $\theta_0^* > \theta_0$ and $\theta_1^* \ne \theta_{sec}^*$. Then*

(i) $(-1)^i\theta_i > 0 \quad (0 \le i \le D)$.
(ii) *D is odd.*

Proof of (i). Consider the vector $v = (\theta_0, \theta_1, \ldots, \theta_D)$. Recall $\theta_0 > 0$ by (48), so it suffices to show v has D sign changes. By Lemma 4.3, v is a left eigenvector for the tridiagonal matrix

$$B^* := \begin{pmatrix} a_0^* & c_1^* & & & \mathbf{0} \\ b_0^* & a_1^* & c_2^* & & \\ & b_1^* & \ddots & \ddots & \\ & & \ddots & a_{D-1}^* & c_D^* \\ \mathbf{0} & & & b_{D-1}^* & a_D^* \end{pmatrix},$$

with associated eigenvalue θ_1^*. Observe B^* satisfies the assumptions of Theorem 7.1, so we will be done by part (ii) of that theorem if we can show θ_1^* is the minimal eigenvalue of B^*. But this is the case, since the eigenvalues of B^* are $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$ by [1, p. 193], and θ_1^* is the minimum of these scalars by Lemma 10.5(ii) and our assumptions. □

Proof of (ii). Recall $\theta_D = -\theta_0$ by (113), so $\theta_D < 0$. But $(-1)^D\theta_D > 0$ by (i) above, so D must be odd. □

10.7 Lemma. *Suppose $\theta_0^* > \theta_0$. Then θ_2^* is the third greatest of $\theta_0^*, \theta_1^*, \ldots, \theta_D^*$.*

Proof. We may assume $\theta_1^* \neq \theta_{sec}^*$; otherwise we are done by Theorem 8.1. Now $\theta_1^* = \theta_{min}^*$ by Lemma 10.5(ii), so $\theta_D^* = \theta_{sec}^*$ by Lemma 10.4(ii). It now suffices to show

$$\theta_2^* > \theta_i^* \quad (3 \leq i \leq D-1). \tag{143}$$

Setting $\alpha = 1$, $\beta = i$, $\gamma = 1$, $\delta = 0$ in (99),

$$(\theta_0^* - \theta_i^*)(\theta_2^* - \theta_i^*) = (\theta_1^* - \theta_{i+1}^*)(\theta_1^* - \theta_{i-1}^*). \tag{144}$$

Both factors on the right side of (144) are negative since $\theta_1^* = \theta_{min}^*$. The first factor on the left in (144) is positive by (67), (68), so the second factor in (144) is positive as well. Line (143) follows. □

10.8 Lemma. *Suppose $\theta_0^* > \theta_0$ and $\theta_1^* \neq \theta_{sec}^*$. Then with reference to Lemmas* 9.9 *and* 9.10,

(i) *ϕ_1^* is the second largest of $\phi_0^*, \phi_1^*, \ldots, \phi_t^*$,*
(ii) *$\phi_0 > \phi_1 > \cdots > \phi_t$.*

Proof of (i). By Lemma 9.10(iii) it suffices to show

$$\theta_2^* \text{ is the second largest of} \{\theta_i^* | 0 \leq i \leq D, i \text{ even}\}. \tag{145}$$

To this end, observe $\theta_1^* = \theta_{min}^*$ by Lemma 10.5(ii) and our assumptions. Now $\theta_D^* \neq \theta_{min}^*$ since $D \neq 1$, so $\theta_D^* = \theta_{sec}^*$ by Lemma 10.4(ii). By this, line (68), and Lemma 10.7,

$$\theta_2^* \geq \theta_i^* \quad (1 \leq i \leq D-1). \tag{146}$$

Line (145) follows from this, line (68), and the fact that D is odd. □

Proof of (ii). Apply Theorem 8.1 to $\frac{1}{2}Y$. □

10.9 Lemma. *Suppose* $\theta_0^* > \theta_0$ *and* $\theta_1^* \neq \theta_{sec}^*$. *Then*

$$\theta_0 > \theta_{D-1} > \theta_2 > \theta_{D-3} > \cdots > \theta_3 > \theta_{D-2} > \theta_1 > \theta_D. \tag{147}$$

Proof. By Lemma 9.10(i), Lemma 10.8(ii), and since D is odd,

$$\theta_0^2 > \theta_1^2 > \cdots > \theta_{(D-1)/2}^2. \tag{148}$$

The result now follows from this, Lemma 10.6(i), and (113). □

10.10 Lemma. *Suppose* $\theta_0^* > \theta_0$ *and* $\theta_1^* \neq \theta_{sec}^*$. *Then*

$$\theta_0^* > \theta_D^* > \theta_2^* > \theta_{D-2}^* > \cdots > \theta_{D-3}^* > \theta_3^* > \theta_{D-1}^* > \theta_1^*. \tag{149}$$

Proof. Recall D is odd by Lemma 10.6(ii), and $\theta_1^* = \theta_{min}^*$ by Lemma 10.5 (ii), so it suffices to show

$$\theta_{2i}^* > \theta_{D-2i}^* > \theta_{2i+2}^* \tag{150}$$

for $0 \leq i \leq \frac{D-3}{2}$. We proceed by induction on i. First assume $i = 0$. Then (150) holds by Lemmas 10.7 and 10.4 (ii). Next assume $i > 0$. Setting $\alpha = 2i - 1$, $\beta = D - 2i + 1$, $\gamma = -1$, $\delta = -1$ in (100),

$$(\theta_{2i} - \theta_{D-2i})(\theta_{2i-2}^* - \theta_{D-2i+2}^*) = (\theta_{2i}^* - \theta_{D-2i}^*)(\theta_{2i-2} - \theta_{D-2i+2}). \tag{151}$$

The second factor in (151) is positive by the induction hypothesis, and the first and fourth factors in (151) are positive by Lemma 10.6(i). We conclude the third factor in (151) is positive, so the left inequality in (150) holds. Setting $\alpha = 2i + 1$, $\beta = D - 2i + 1$, $\gamma = -1$, $\delta = -1$ in (100),

$$(\theta_{2i+2} - \theta_{D-2i})(\theta_{2i}^* - \theta_{D-2i+2}^*) = (\theta_{2i+2}^* - \theta_{D-2i}^*)(\theta_{2i} - \theta_{D-2i+2}). \tag{152}$$

The second factor in (152) is negative by the induction hypothesis, and the first and fourth factors in (152) are positive by Lemma 10.6(i). We conclude the third factor in (152) is negative, so the right inequality in (150) holds, and the induction is complete. □

Proof of Theorem 1.1. By Lemma 9.8(ii), $\theta_0^* \geq \theta_0$. If $\theta_0^* = \theta_0$, then we are done by Lemma 10.3. If $\theta_0^* > \theta_0$, and $\theta_1^* = \theta_{sec}^*$, then we are done by Theorem 8.1. If $\theta_0^* > \theta_0$, and $\theta_1^* \neq \theta_{sec}^*$, then we are done by Lemmas 10.9 and 10.10. In any case, we are done. □

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