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On Tournaments Free of Large Transitive Subtournaments

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Abstract. Erdös and Moser posed the problem of determining, for each integer $n > 0$, the greatest integer $v(n)$ such that all tournaments of order n contain the transitive subtournament of order $v(n)$ (denoted $TT_{v(n)}$). It is known that $v(n) = 3$ for $4 \le n \le 7$, $v(n) = 4$ for $8 \le n \le 13$, $v(n) = 5$ for $14 \le n \le 27$, and $v(n) \ge 6$ for $n > 27$. Moreover, the uniqueness of the tournaments free of TT_4 of orders 6 and 7, and free of TT_5 and TT_6 of orders 13 and 27, respectively, has been established. Here we prove that the tournaments of orders 12 and 26, free of TT_5 and TT_6, respectively, are also unique. Then, we see that all tournaments of order 54 contain TT_7 (improving the best lower bound known for $v(n)$). Finally, with the aid of a computer, we obtain the orders $cv(r)$ and $gv(s)$ of the biggest transitive tournaments contained, respectively, in all circulant tournaments of order $r \le 55$ and in each Galois tournament of order $s < 1000$, *i.e.*, in the tournament with set of vertices the Galois field of order s (whenever it exists) and edge directions induced by the quadratic residues. We get better upper bounds of $v(n)$, for $n \le 991$.

1. Introduction and Notation

Along this paper, the graphs we consider are directed and without loops or multiple edges. Given a digraph G, let $V(G)$ – or G, if there is no confusion – and $E(G)$ denote the sets of its vertices and edges, respectively. A succession of three vertices e_1, e_2, e_3, of G forms a *directed triangle* if $e_1e_2, e_2e_3, e_3e_1 \in E(G)$. A *tournament* T is a digraph such that each couple of vertices, u and v, is joined by exactly one edge, either uv or vu; T' is a *subtournament of* T if T' is a tournament with $V(T') \subset V(T)$ and $E(T') \subset E(T)$. A tournament is *transitive* if it is free of directed triangles; since there exists a unique transitive tournament of order n, isomorphisms excepted, we denote it by TT_n.

Let n be an odd integer and let A be a set of nonzero elements of the ring Z_n of integers mod. n such that $|A| = (n-1)/2$ and $-x \notin A$, for all $x \in A$. Then, the digraph T defined as $V(T) = Z_n$ and $xy \in E(T)$ if and only if $y - x \in A$, for all $x, y \in V(T)$, is a tournament called *the circulant tournament of order n induced by A*. Similarly, since there exists a unique field F of order m for any integer m of the form p^r, with p prime and r a positive integer, let G be the digraph such that $V(G) = F$ and, for all $x, y \in V(G)$, $xy \in E(G)$ if and only if there exists a nonzero

$z \in F$ with $y - x = z^2$. It is easy to see that G is a tournament (that we call *Galois tournament*) if and only if $m \equiv 3$ (mod. 4).

Erdös and Moser [2] posed the problem of determining, for each positive integer n, the greatest integer $v(n)$ such that all tournaments of order n contain $TT_{v(n)}$. It is well known that $v(1) = 1$, $v(2) = v(3) = 2$, $v(n) = 3$ for $4 \le n \le 7$, $v(n) = 4$ for $8 \le n \le 13$, $v(n) = 5$ for $14 \le n \le 27$, and $\lfloor \log_2(n/55) \rfloor \le v(n) \le 2\lfloor \log_2(n) \rfloor + 1$ for $n \ge 28$ [2, 6, 7].

The uniqueness of the tournaments of orders 7, 13, and 27, free of TT_4, TT_5, and TT_6, respectively, has been proved [6, 7]. Then, for $k = 4$, 5, and 6, the largest tournament free of TT_k is unique. Moreover, it has also been proved that there exists a unique tournament of order 6 free of TT_4.

In this paper, after presenting some properties of the tournaments free of TT_4 (Section 2), we prove that the tournaments of orders 12 and 26, free of TT_5 and TT_6, respectively, are also unique. We show a tournament of order 31 free of TT_7 and see that all tournaments of order 54 contain TT_7, implying $v(n) = 6$ for $28 \le n \le 31$, and $v(n) \ge \lfloor \log_2(n/54) \rfloor + 7$ for $n \ge 32$; this improves the best lower bound known for $v(n)$ (Sections 3 and 4). Finally, since for each integer $n \le 31$, the exact values of $v(n)$ are induced from circulant or Galois tournaments, in the last Section (5) we obtain, using a computer, the orders $cv(r)$ $(r \le 55)$ and $gv(s)$ $(s < 1000)$ of the biggest transitive tournaments contained, respectively, in all circulant tournaments of order r and in each Galois tournament of order s. We get better upper bounds of $v(n)$, for $n \le 991$.

Let us introduce some concepts. Suppose that G is a tournament.

Let $x, x_1, x_2, \ldots, x_r \in V(G)$ and $W = \{x_1, x_2, \ldots, x_r\}$. Define $N_G^+(x) = \{y \in V(G) | xy \in E(G)\}$, $N_G^-(x) = \{y \in V(G) | yx \in E(G)\}$, $N_G^+(x_1, x_2, \ldots, x_r) = N_G^+(W) = \bigcap_{i=1}^r N_G^+(x_i)$, and $N_G^-(x_1, x_2, \ldots, x_r) = N_G^-(W) = \bigcap_{i=1}^r N_G^-(x_i)$; $N_G^+(x)$ and $N_G^-(x)$ are called the *outset of* x and the *inset of* x, respectively. For the sake of clarity, we will omit the subscript G when the tournament G considered in a given context is clear. Given $R, S \subset V(G)$, we say that R *covers* S or S *is covered by* R whenever $S \subset N_G^+(R)$. The set of directed triangles of G is represented by $DT(G)$. If $x, y, z \in V(G)$ form a directed triangle with $xy, yz, zx \in E(G)$, let xyz denote such a triangle. For a subtournament G' of G and $abc \in DT(G')$, a *center of abc in* G' is any vertex of G' that covers or is covered by $\{a, b, c\}$.

Given the integer n, let $V_n(G) = \{u \in V(G) || N^+(u)| = n\}$; G is an *n-regular* (or *regular*) tournament if $V(G) = V_n(G)$. If $H \subset V(G)$, denote by $\langle H \rangle$ – or simply H, if there is no confusion – the subtournament of G induced by H. The tournament G^c is defined as: $V(G^c) = V(G)$ and $xy \in E(G^c)$ if and only if $yx \in E(G)$. Let *Aut(G)* denote the group of automorphisms of G. Finally, $P \approx Q$ means that the tournaments P and Q are mutually isomorphic.

2. Tournaments Free of TT_4

In this section we present some properties of the tournaments free of TT_4 of orders 5, 6, and 7.

Let ST_7 be the circulant tournament of order 7 induced by the set of quadratic residues (mod. 7), that is, $\{1, 2, 4\}$. It is well known that ST_7 is the unique tournament of order 7 free of TT_4 [6]. We denote by ST_6 the tournament obtained when a vertex x is eliminated from ST_7; note that ST_6 is independent of the vertex x selected. For $i = 0, 1, 2, 3$, let $\mathscr{T}_i$ be the set of the directed triangles in ST_6 which have exactly i vertices with outsets of order 3 (Fig. 1). Clearly, $|\mathscr{T}_0| = 1$ and $|\mathscr{T}_3| = 1$; let T_0 (resp., T_3) denote the unique element of $\mathscr{T}_0$ (resp., $\mathscr{T}_3$). It is easy to verify that:

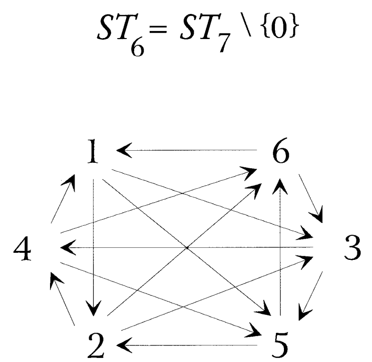

Fig. 1. $\mathscr{T}_0 = \{356\}$, $\mathscr{T}_1 = \{156, 235, 463\}$, $\mathscr{T}_2 = \{126, 245, 134\}$, and $\mathscr{T}_3 = \{124\}$

P2.1. *ST_6 is the unique tournament of order 6 free of TT_4* [6].

P2.2. i) *$V(ST_6) = V_2(ST_6) \cup V_3(ST_6)$ with $|V_2(ST_6)| = |V_3(ST_6)| = 3$. Moreover*, ii) *if $x \in V_3(ST_6)$ (resp., $x \in V_2(ST_6)$), then $\langle N^+(x)\rangle \in \mathscr{T}_1(ST_6)$ (resp., $\langle N^-(x)\rangle \in \mathscr{T}_2(ST_6)$).*

P2.3. *ST_6 contains exactly eight directed triangles*: $|\mathscr{T}_0| = 1$, $|\mathscr{T}_1| = 3$, $|\mathscr{T}_2| = 3$, *and* $|\mathscr{T}_3| = 1$.

P2.4. *$\{T_0, T_3\}$ is the unique bipartition of $V(ST_6)$ such that each component forms a directed triangle.*

P2.5. *Let $T \in DT(ST_6)$. Then* i) *T covers (resp., is covered by) a vertex t if and only if $T \in \mathscr{T}_2$ (resp., $T \in \mathscr{T}_1$); moreover, when it exists, t is unique. In particular*, ii) *every $T \in DT(ST_6)$ has at most one center in ST_6.*

P2.6. *Let $ab \in E(T_0)$ (resp., $ab \in E(T_3)$). Then, there exists $c \in V(T_3)$ (resp., $c \in V(T_0)$) such that $abc \in DT(ST_6)$.*

Given a tournament R isomorphic to ST_6, let $\mathscr{T}_i(R)$ denote the family of directed triangles of R corresponding to $\mathscr{T}_i$ in ST_6, for $i = 0, 1, 2, 3$, and let $T_j(R)$ denote the triangle of R corresponding to T_j, for $j = 0, 3$.

Let ST_5 be the tournament obtained when a vertex y is eliminated from ST_6; it is easy to see that ST_5 is independent of the vertex y selected. Also, let QT_5 and RT_5 be the tournaments shown in Fig. 2. It is not difficult to prove that:

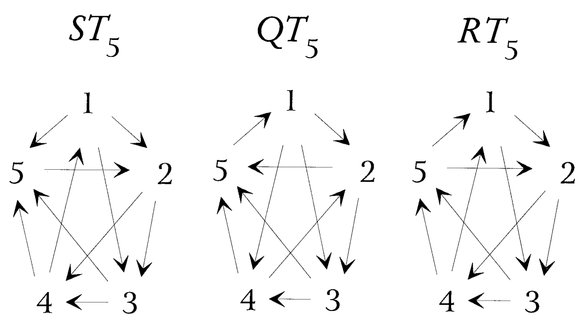

Fig. 2. The tournaments of order 5 free of TT_4

Lemma 2.1. *The tournaments ST_5, QT_5, and RT_5 are not isomorphic to each other and they are the unique tournaments of order 5 free of TT_4.*

P2.7. *All subtournaments of ST_6 of order 5 are isomorphic to ST_5.*

P2.8. *If $H \in \{ST_5, QT_5, RT_5\}$ then $H \approx H^c$.*

P2.9. *The tournament RT_5 is 2-regular.*

P2.10. *Let $H \in \{ST_5, QT_5\}$. Then, $|V_1(H)| = 1$, $|V_2(H)| = 3$, and $|V_3(H)| = 1$. Moreover, if $x \in V_1(H)$ (resp., $x \in V_3(H)$), we have $\langle N^-(x)\rangle \in DT(H)$ (resp., $\langle N^+(x)\rangle \in DT(H)$).*

3. Tournaments Free of TT_5

The uniqueness of the tournament of order 13 not containing the transitive subtournament of order 5 is well known [6]. In this section we prove that there also exists only one tournament of order 12 free of this sobtournament.

Let ST_{13} be the circulant tournament of order 13 induced by the set $\{1, 2, 3, 5, 6, 9\}$. This tournament satisfies [6]:

P3.1. *ST_{13} is the unique tournament of order 13 free of TT_5.*

P3.2. *For $x \in V(ST_{13})$ we have $\langle N^+(x)\rangle \approx ST_6$ and $\langle N^-(x)\rangle \approx ST_6$.*

Moreover,

P3.3. *If $x, y \in V(ST_{13})$ with $y \in N^+(x)$ then $|N^-(x) \cap N^+(y)| \leq 4$* (since by *P*2.2-*i* and *P*3.2 $\langle N^+(x)\rangle \approx ST_6$, $|N^+(y)| = 6$, and $|N^+(x) \cap N^+(y)| = 2$ *or* 3).

Now let us consider the tournaments of order 12 free of TT_5.

Lemma 3.1. *Let G be a tournament of order* 12 *free of* TT_5 *and let* $x \in V(G)$. *Then,* $\langle N^+(x)\rangle$ (*and* $\langle N^-(x)\rangle$) *is isomorphic to* ST_6, ST_5, QT_5 *or* RT_5.

Proof. By P2.1 and the Lemma 2.1 we only need to see that $|N^+(x)| = 5$ or 6. Clearly, $|N^+(x)| \leq 7$ and $|N^-(x)| \leq 7$. If $|N^+(x)| = 7$, then $\langle N^+(x)\rangle \approx ST_7$, $|N^-(x)| = 4$, and $\langle N^-(x)\rangle$ contains TT_3; in this case, in [5, 6] it is proved that G contains TT_5. Hence, $|N^+(x)| \leq 6$. We similarly deduce $|N^+_{G^c}(x)| \leq 6$, that is, $|N^-(x)| \leq 6$. Thus, $|N^+(x)| = 5$ or 6.

Lemma 3.2. *Let G be a tournament of order* 12 *free of* TT_5, *and let* $x \in V_5(G)$. *Then,* $\langle N^+(x)\rangle \not\approx QT_5$.

Proof. Let us proceed by contradiction. Suppose that $\langle N^+(x)\rangle \approx QT_5$. Let $N^+(x) = \{a, b, c, d, e\}$ with $N^+(d, x) = N^-(e) \cap N^+(x) = \{a, b, c\}$ and $abc \in DT(G)$ (Fig. 3,*a*).

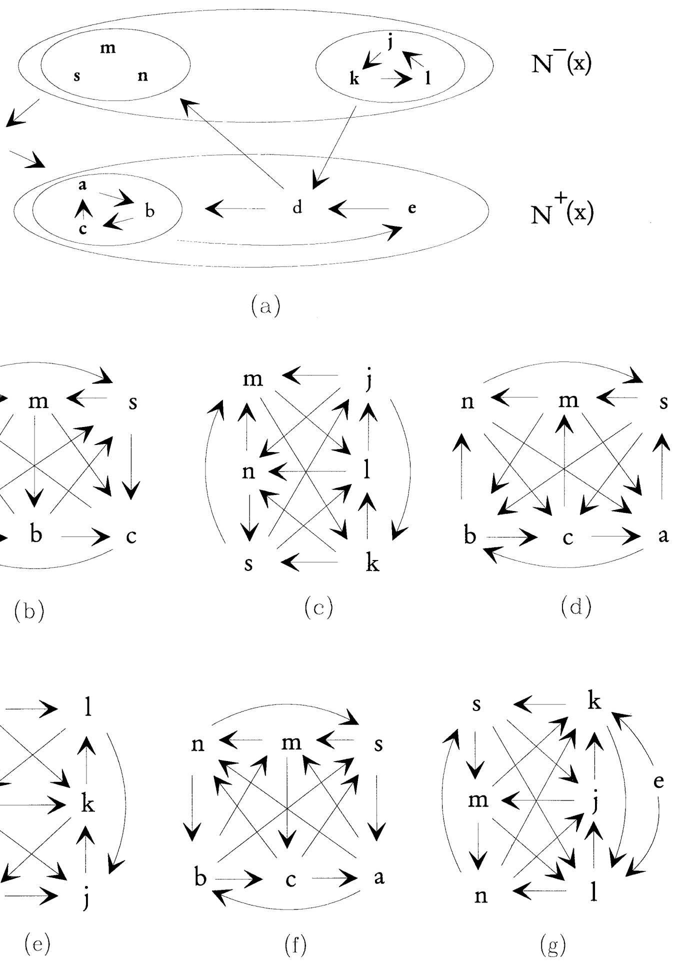

Fig. 3. There exists $x \in V(G)$ with $\langle N^+(x)\rangle \approx QT_5$. **a**: $N^+(x)$ and $N^-(x)$; **b, c**: Case $abc \in \mathcal{T}_1(N^+(d))$; **d, e**: Case $abc = T_0(N^+(d))$; **f, g**: Case $abc = T_3(N^+(d))$

Since $|N^-(d)| = 5$ or 6 (Lemma 3.1), we have $|N^-(d,x)| = 3$ or 4; but $\langle N^-(d,x)\rangle$ is free of TT_3 (as G is free of TT_5), implying $|N^-(d,x)| = 3$ and $\langle N^-(d,x)\rangle \in DT(G)$. Then, let $N^-(x) = \{j,k,l,m,n,s\}$ with $\langle N^-(d,x)\rangle = jkl$ and $N^+(d) \cap N^-(x) = \{m,n,s\}$. Thus, $\langle N^+(d)\rangle \approx ST_6$ and $abc \in DT(N^+(d))$. By P2.3 there are four cases.

Case I. $abc \in \mathscr{T}_2(N^+(d))$.

Assume that m is the center of abc in $\langle N^+(d)\rangle$ (P2.5-*i*), that is, $a,b,c \in N^-(m)$. Applying P2.2-*i* in $\langle N^+(d)\rangle$ we have $n,s \in N^+(m)$. Also by P2.2-*i* $|N^-(m,x)| = 2$ or 3 (as $\langle N^-(x)\rangle \approx ST_6$); because $|N^-(m)| = 5$ or 6 (Lemma 3.1) and a,b,c, $d \in N^-(m) \cap N^+(x)$, we have $|N^-(m,x)| = 2$ and $|N^-(m)| = 6$. Hence, $e \in N^+(m)$, and since $\{a,b,c,m,x\} \subset N^-(e)$ and $\langle\{a,b,c,m,x\}\rangle \approx QT_5$, by P2.7 $|N^-(e)| = 5$, $\langle N^+(e)\rangle \approx ST_6$, and $N^+(e) = \{d,j,k,l,n,s\}$. Finally, as $mns \notin DT(N^-(x))$, by P2.4 and P2.5-*i* there exists the center r of jkl in $\langle N^-(x)\rangle$, that is not m (because n, $s \in N^+(m)$ and $|N^-(m,x)| = 2$); this means $r = s$ or $r = n$. Then, r and d are centers of jkl in $\langle N^+(e)\rangle$, contradicting P2.5-*ii*.

Case II. $abc \in \mathscr{T}_1(N^+(d))$.

Assume that m is the center of abc in $\langle N^+(d)\rangle$ (by P2.5-*i* $a,b,c \in N^+(m)$) and $s \in N^+(n)$. From P2.2-*i* we get n, $s \in N^-(m)$. Since $|N^-(n) \cap \{a,b,c\}| = |N^-(n) \cap N^+(d)| \geq 2$, by symmetry we can also assume $b,c \in N^-(n)$; as m is the unique center of abc in $\langle N^+(d)\rangle$ (P2.5-*ii*), we have $a \in N^+(n)$ (Fig. 3,*b*). By P2.2-*ii* $\langle N^+(n,d)\rangle \in DT(N^+(d))$, that is, $s \in N^+(a)$. Also P2.2-*ii* implies $s \in N^-(c)$ (as $a,n \in N^+(c)$ but $\langle\{a,n,s\}\rangle \notin DT(N^+(d))$) and $s \in N^+(b)$ (as c, $m \in N^+(s)$ but $\langle\{b,c,m\}\rangle \notin DT(N^+(d))$).

On the other hand, since $\langle\{m,n,s\}\rangle \notin DT(N^-(x))$, by P2.4 and P2.5-*i* there exists the center r of jkl in $\langle N^-(x)\rangle$, different from s, because $s \in N^+(n) \cap N^-(m)$. If $r = m$, then j, k, $l \in N^+(m)$ (as n, $s \in N^-(m)$) and $|N^+(m)| \geq |\{a,b,c,j,k,l,x\}| = 7$, which is not possible. Thus, $r = n$ and $j,k,l \in N^-(n)$ (Fig. 3,*c*). Since $|N^+(m) \cap N^-(x)| \geq 2$, by symmetry we can assume $k,l \in N^+(m)$; then, $j \in N^-(m)$, as n is the unique center of jkl. Hence, $\langle N^-(m,x)\rangle = \langle\{n,s,j\}\rangle \in DT(N^-(x))$, by P2.2, and $j \in N^+(s)$. From P2.2-*ii* $k \in N^-(s)$ (as $j,m \in N^+(s)$ but $\langle\{j,k,m\}\rangle \notin DT(N^-(x))$) and $l \in N^+(s)$ (as $l,n,s \in N^+(k)$).

Note that $e \in N^+(n) \cap N^-(m) \cap N^+(s)$, because $|N^-(n)| \geq |\{b,c,d,j,k,l\}| = 6$, $|N^+(m)| \geq |\{a,b,c,k,l,x\}| = 6$, and, if $e \in N^-(s)$, we would have $\langle N^-(s)\rangle = \langle\{a,b,d,e,k,n\}\rangle \approx ST_6$ with d and e centers of abn, contradicting P2.5-*ii*.

Finally, since $\langle N^+(s)\rangle = \langle\{c,e,j,l,m,x\}\rangle \approx ST_6$ and x is the unique center of mlj in $\langle N^+(s)\rangle$, then there exists $t \in \{m,l,j\} \cap N^-(e)$, implying $|N^-(e)| \geq |\{a,b,c,n,s,x,t\}| = 7$, which is a contradiction.

Case III. $abc = T_0(N^+(d))$.

By P2.4 $\langle\{m,n,s\}\rangle = T_3(N^+(d))$; suppose, without loss of generality, that $mns = T_3(N^+(d))$ and $\langle N^+(d)\rangle$ ($\approx ST_6$) is as in Fig. 3,*d*. Since $|N^+(e) \cap \{j,k,l\}| \geq 2$ (as otherwise $\langle\{d,e,x\} \cup [N^-(e) \cap \{j,k,l\}]\rangle$ would contain TT_5), by symmetry we can assume $k,l \in N^+(e)$. Because $\{mns, jkl\} = \{T_0(N^-(x)),\ T_3(N^-(x))\}$ (P2.4), from P2.6 there exists $r \in \{m,n,s\}$ with $rkl \in DT(N^-(x))$; again by symmetry, assume $r = m$. Note that $m, n, s \in N^+(e)$, because $t \in \{m, n, s\} \cap N^-(e)$ implies $\langle \{t, x, e\} \cup [N^+(t) \cap \{a, b, c\}] \rangle \approx TT_5$.

Hence, $N^+(e) = \{d, k, l, m, n, s\}$. Since $\langle \{d, n, s\} \rangle \notin DT(N^+(e))$, by P2.4 and P2.5-*i* there exists the center $p \in \{d, n, s\}$ of *mkl* in $\langle N^+(e) \rangle$, that is not *d*, as $d \in N^+(k) \cap N^-(m)$. If $p = n$, since $n \in N^+(m)$, $n \in N^+(k, l)$ as well; so, $|N^-(n)| = 4$ in $\langle N^+(e) \rangle \approx ST_6$, contradicting P2.2-*i*. Thus, $p = s$, and (as $m \in N^+(s)$) *k*, *l*, $m \in N^+(s)$, implying $s \in V_3(N^-(x))$. From P2.4 $mns = T_3(N^-(x))$ and $jkl = T_0(N^-(x))$; then, $m \in V_3(N^-(x))$, *n*, *k*, $j \in N^+(m)$, and $njk \in DT(N^-(x))$ (Fig. 3,*e*).

We have $N^+(m) = \{a, b, j, k, n, x\}$ with $n \in V_3(N^+(m))$ and $x \in V_2(N^+(m))$, and by P2.5-*i* there exists $t \in \{a, j, k\}$ center of *bnx* in $\langle N^+(m) \rangle$. Since $a \in N^+(n) \cap N^-(b)$ and $j \in N^+(n) \cap N^-(x)$, then $t = k$ and $b \in N^+(k)$. Hence, $\langle \{s, m, k, x, b\} \rangle \approx TT_5$, a contradiction.

Case IV. $abc = T_3(N^+(d))$.

By P2.4 $\langle \{m, n, s\} \rangle = T_0(N^+(d))$; assume, without loss of generality, that $\langle N^+(d) \rangle$ is as in Fig. 3,*f*. Since $jkl, mns \in DT(N^-(x))$, then $mns = T_3(N^-(x))$ or $mns = T_0(N^-(x))$. In the latter, for $t \in \{m, n, s\}$ we have $5 \leq |N^+(t)| = |N^+(t) \cap N^-(x)| + |\{x\}| + |N^+(t) \cap \{a, b, c\}| + |N^+(t) \cap \{e\}| = 4 + |N^+(t) \cap \{e\}|$, that is, $e \in N^+(t)$ and $|N^-(e)| \geq |\{a, b, c, m, n, s, x\}| = 7$, which is impossible. Hence, $mns = T_3(N^-(x))$ and $jkl = T_0(N^-(x))$; by symmetry of ST_6 we can assume that $\langle N^-(x) \rangle$ is as in Fig. 3,*g*.

First let us see that if $h \in \{m, n, s\}$ with $N^+(h) \cap \{j, k, l\} \subset N^+(e)$, then $e \in N^-(h)$. Suppose, without loss of generality, that $h = m$ and $N^+(m) \cap \{j, k, l\} = \{k, l\} \subset N^+(e)$ with $e \in N^+(m)$. Then, $\langle N^+(m) \rangle = \langle \{c, e, k, l, n, x\} \rangle \approx ST_6$. Because $\langle \{c, l, n\} \rangle \notin DT(G)$, by P2.4 and P2.5-*i* there exists the center $t \in \{c, l, n\}$ of *xek* in $\langle N^+(m) \rangle$. Since $c \in N^+(x) \cap N^-(e)$ and $l \in N^+(e) \cap N^-(x)$, then $t = n$ and $e \in N^+(n)$. Thus, $N^-(e) = \{a, b, c, m, n, x\}$, that is, $N^+(n) \cap \{j, k, l\} = \{j, k\} \subset N^+(e)$ with $e \in N^+(n)$. Repeating the previous arguments in *n*, *s*, and *xej* (instead of *m*, *n*, and *xek*, resp.), we obtain $e \in N^+(s)$, that is not possible.

Note that $|N^+(e) \cap \{j, k, l\}| \geq 2$, as otherwise $\langle \{d, e, x\} \cup [N^-(e) \cap \{j, k, l\}] \rangle$ would contain TT_5; by symmetry suppose, without loss of generality, that *k*, $l \in N^+(e)$. From the previous paragraph we have $e \in N^-(m)$. Also by that paragraph we must have $j \in N^-(e)$, as otherwise, applying it in *n* and *s*, instead of *m*, we would deduce $n, s \in N^+(e)$, implying $|N^+(e)| = |\{d, j, k, l, m, n, s\}| = 7$. Then, $\langle N^-(m) \rangle = \langle \{a, b, d, e, j, s\} \rangle \approx ST_6$; since $\langle \{a, e, j\} \rangle \notin DT(N^-(m))$ and *a*, $j \in N^+(s)$, by P2.2-*ii* $e \in N^-(s)$. Note that $b \in N^+(k)$, as otherwise, $\langle \{b, j, e, m, k\} \rangle \approx TT_5$. Hence, $\langle \{k, d, b, c, s\} \rangle \approx TT_5$ or $\langle \{c, e, n, k, s\} \rangle \approx TT_5$, if $c \in N^+(k)$ or $c \in N^-(k)$, respectively. □

Lemma 3.3. *Let G be a tournament of order* 12 *free of* TT_5. *Then, there exists* $x \in V(G)$ *with* $\langle N^+(x) \rangle \approx ST_5$ *or* $\langle N^-(x) \rangle \approx ST_5$.

Proof. Let us proceed by contradiction. Since G^c is free of TT_5, by P2.8 and Lemmas 3.1 and 3.2 we can assume that $\langle N^+(r) \rangle \approx RT_5$ and $\langle N^-(s) \rangle \approx RT_5$, for all $r \in V_5(G)$ and for all $s \in V_6(G)$. Clearly, $|V_5(G)| = |V_6(G)| = 6$. Let $x \in V_5(G)$, that is, $\langle N^+(x) \rangle \approx RT_5$. We have:

l1. *If* $j \in N^+(x)$ *with* $|N^-(j,x)| \geq 3$, *then* $\langle N^-(j)\rangle \approx ST_6$ (since $\langle N^-(j)\rangle \approx RT_5$ implies $|N^-(j,x)| = 2$) *and* $\langle N^-(j,x)\rangle \in DT(N^-(x))$.

l2. *If* $y_0 \in V_3(N^-(x))$ *then* $\langle N^+(y_0)\rangle \approx ST_6$ (because $\langle N^+(y_0)\rangle \approx RT_5$ implies $2 = |N^-(x) \cap N^+(y_0)|$).

l3. *If* $j \in N^+(x)$ *and* $y_1, y_2, y_3 \in N^-(x,j)$, *then* $|\{y_1, y_2, y_3\} \cap V_6(G)| \geq 2$ (since by (l1) $\langle N^-(j)\rangle \approx ST_6$ and $\langle\{y_1, y_2, y_3\}\rangle$ is a directed triangle in $\langle N^-(j)\rangle$ that covers x, that is, $j \in V_5(G)$ and $\langle\{y_1, y_2, y_3\}\rangle \in \mathscr{T}_2(N^-(j))$ (P2.5), following the stated from (l2) applied in $\langle N^-(j)\rangle$).

Let $N^+(x) = \{a, b, c, d, e\}$, $N^-(x) = \{m, n, s, u, v, w\}$, and $mns = T_3(N^-(x))$, as in Fig. 4. We consider two cases.

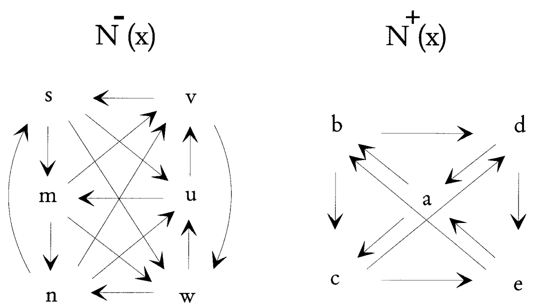

Fig. 4. $\langle N^+(x)\rangle = \langle\{a, b, c, d, e\}\rangle \approx RT_5$ and $\langle N^-(x)\rangle = \langle\{m, n, s, u, v, w\}\rangle \approx ST_6$

Case I. There exists $p \in N^+(x)$ with $\langle N^-(p,x)\rangle = T_3(N^-(x))$.

Assume $p = a$; by (l1) $|N^-(a)| = 6$. Since x is the center of mns in $\langle N^-(a)\rangle$ $(= \langle\{d, e, m, n, s, x\}\rangle)$, then $mns = \mathscr{T}_2(N^-(a))$ and $|\{m, n, s\} \cap V_3(N^-(a))| = 2$, say $m, n \in V_3(N^-(a))$, implying $s, x \in V_2(N^-(a))$. From P2.5-*ii* $|N^+(e) \cap \{m, n, s\}| \leq 2$, that is, $e \in V_2(N^-(a))$. Thus, $d \in V_3(N^-(a))$ and $mnd = T_3(N^-(a))$. Since $\langle N^+(n)\rangle = \langle\{a, d, s, u, v, x\}\rangle \approx ST_6$ and $\langle\{a, d, s\} \notin DT(N^+(n))$ with $a, s \in N^-(u)$, by P2.2-*ii* $d \in N^+(u)$. Finally, by ($l2$) $6 = |N^+(m)| \geq |N^+(m) \cap N^-(a)| + |\{a, v, w\}| = 6$; then, $b, c, d, u \in N^-(m)$ with $b, c, u \in N^-(d)$, implying $m \in V_6(G)$ with $\langle N^-(m)\rangle \not\approx RT_5$, a contradiction.

Case II. If there exists $p \in N^+(x)$ with $\langle N^-(p,x)\rangle \in DT(N^-(x))$, then $\langle N^-(p,x)\rangle \in \mathscr{T}_0(N^-(x)) \cup \mathscr{T}_1(N^-(x)) \cup \mathscr{T}_2(N^-(x))$.

Let $B = \langle V_5(G)\rangle$. Since $|V(B)| = 6$, there exists $g \in V(B)$ such that $|N_B^-(g)| \geq 3$, and by ($l2$) $|N_B^-(g)| = 3$ with $N_B^-(g) = V_2(N^-(g))$; we can assume $x = g$. Then, $|N^+(k,x)| = 2$, for all $k \in N^-(x)$, and there are 12 edges from $N^-(x)$ to $N^+(x)$; by ($l1$) there exist $j, l \in N^+(x)$ with $\langle N^-(j,x)\rangle, \langle N^-(l,x)\rangle \in DT(N^-(x))$, say $a = j$ and $b = l$. Let $T_a = \langle N^-(a,x)\rangle$ and $T_b = \langle N^-(b,x)\rangle$; since G is free of TT_5 then $T_a \neq T_b$, and by case I we get $T_a \neq T_3(N^-(x))$ and $T_b \neq T_3(N^-(x))$. Because $u, v, w \in V_5(G)$ (as $\{u, v, w\} = V_2(N^-(x))$), then ($l3$) implies $T_a, T_b \in \mathscr{T}_2(N^-(x))$; hence, there exists $h_x \in V_3(N^-(x))$ with $h_x \in V(T_a) \cap V(T_b)$, say $h_x = m$.

Since $a, b \in V_5(G)$ – by ($l1$) –, then $N^+(m) = \{a, b, n, v, w, x\}$ with $a, b, v, w, x \in V_5(G)$, and there exists a unique vertex $f_x \in V_5(G)$ with $f_x \in N^-(m)$, that is, $f_x = u$.

Note that $m \in V_5(G^c)$ with inset of order 1 in $\langle V_5(G^c)\rangle$. Since Lemma 3.3 holds in G if and only if it holds in G^c, we can assume that there exists $j \in V_5(G)$ with inset of order 1 in $\langle V_5(G)\rangle$, that is, $|N_B^-(j)| = 1$. Then, because $\sum_{r \in B} |N_B^-(r)| = |E(B)| = 15$ and by ($l2$) $|N_B^-(t)| \le 3$ (for all $t \in V(B)$), besides x, there exist three vertices $y, z, q \in V_5(G)$ with inset of order 3 in $\langle V_5(G)\rangle$.

Finally, applying in y the same arguments used for x, if $h_y \ne h_x$ then, as $|N^+(h_x) \cap V_5(G)| = |N^+(h_y) \cap V_5(G)| = 5$ and $|V_5(G)| = 6$, we must have $|N^+(h_x, h_y)| \ge 4$, implying that $\langle \{h_x, h_y\} \cup N^+(h_x, h_y)\rangle$ contains TT_5. Hence, $h_y = h_x$, that implies $f_y = f_x$. Similarly we deduce $h_z = h_q = h_x$ and $f_z = f_q = f_x$. Then, $\{x, y, z, q\} \subset N^+(h_x, f_x)$ and $\langle \{h_x, f_x, x, y, z, q\}\rangle$ contains TT_5. □

We denote by ST_{12} the tournament obtained when a vertex z is eliminated from ST_{13}. Since ST_{13} is a circulant tournament with $ST_{13}^c \approx ST_{13}$, then ST_{12} is independent of the vertex z selected and $ST_{12}^c \approx ST_{12}$.

Theorem 3.4. *Let G be a tournament of order 12 free of TT_5. Then, G is isomorphic to ST_{12}.*

Proof. By lemmas 3.2 and 3.3 we know that there exists $x \in V(G)$ with $\langle N^+(x)\rangle \approx ST_5$ or $\langle N^-(x)\rangle \approx ST_5$. Clearly, we can assume that $\langle N^+(x)\rangle \approx ST_5$; then, $\langle N^-(x)\rangle \approx ST_6$. Let $N^+(x) = \{a, b, c, d, e\}$, as in Fig. 5.

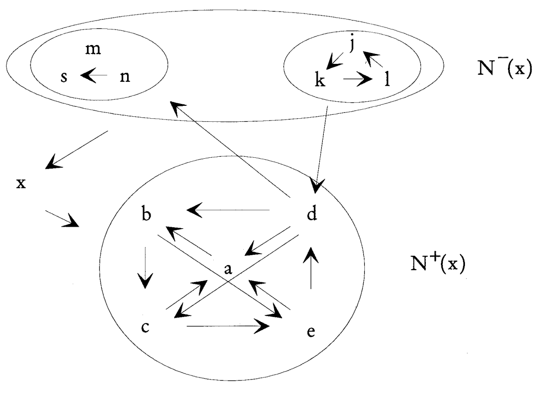

Fig. 5. There exists $x \in V(G)$ with $\langle N^+(x)\rangle \approx ST_5$ and, hence, $\langle N^-(x)\rangle \approx ST_6$

Since $|N^-(d)| \ge 5$, $x \in N^-(d)$, and $\{e\} = N^-(d) \cap N^+(x)$, we have $|N^-(d, x)| \ge 3$, that implies $\langle N^-(d, x)\rangle \in DT(G)$ (as G is free of TT_5) and $|N^-(d)| = 5$. Let $N^-(x) = \{m, n, s, j, k, l\}$ with $jkl = \langle N^-(d, x)\rangle$, $\{m, n, s\} = N^+(d) \cap N^-(x)$, and $s \in N^+(n)$. Note that:

t1. $\langle N^-(d)\rangle \approx ST_5$ (because $\langle N^-(d)\rangle \not\approx QT_5$ – Lemma 3.2 – and $N^-(d) = \{e, j, k, l, x\}$ with $j, k, l \in N^-(x)$).
t2. $|N^+(e) \cap \{j, k, l\}| = 2$ – by $(t1)$.

Since $\langle N^+(d)\rangle \approx ST_6$ with $abc \in DT(N^+(d))$, by P2.3 there are four cases.

Case I. $abc \in \mathcal{T}_2(N^+(d))$.
Assume that m is the center of abc in $N^+(d)$ –P2.5–; then, $a, b, c \in N^-(m)$ and $n, s \in N^+(m)$ (Fig. 6,a). Since $6 \geq |N^-(m)| \geq |\{a, b, c, d\}| + |N^-(m, x)| \geq 6$, we have $\langle N^-(m)\rangle \approx ST_6$ and $|N^-(m, x)| = 2$.

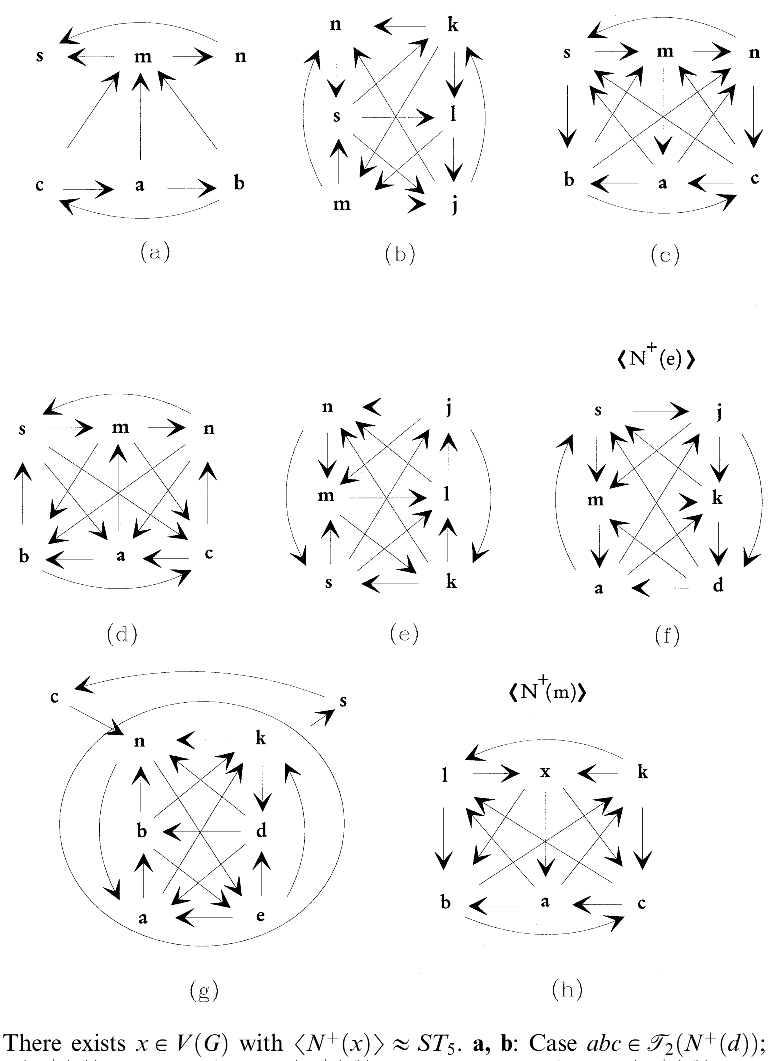

Fig. 6. There exists $x \in V(G)$ with $\langle N^+(x)\rangle \approx ST_5$. **a, b**: Case $abc \in \mathcal{T}_2(N^+(d))$; **c**: Case $abc = T_3(N^+(d))$; **d**: Case $abc = T_0(N^+(d))$; **e, f, g, h**: Case $abc \in \mathcal{T}_1(N^+(d))$

Let us see the edge directions in $\langle N^-(x)\rangle$. Since $jkl \in DT(N^-(x))$ and $mns \notin DT(N^-(x))$, by P2.4 and P2.5 there exists the center $y \in \{m, n, s\}$ of jkl in $\langle N^-(x)\rangle$, which is neither n (as $n \in N^+(m) \cap N^-(s)$) nor m (because $|N^-(m, x)| = 2$ and $s, n \in N^+(m)$), that is, $s = y$. Then, $j, k, l \in N^+(s)$ and from P2.5-*ii* we get $N^+(m) \cap \{j, k, l\} \neq \emptyset$; by symmetry, we can assume $j \in N^+(m)$ (Fig. 6,*b*). Hence, P2.2 implies $k, l \in N^-(m)$ and $jns = \langle N^+(m) \cap N^-(x)\rangle \in DT(N^-(x))$, and since $j, s \in N^-(k)$, also by P2.5-*ii* $n \in N^+(k)$.

Now, consider the set $N^+(e)$. Inasmuch as $\langle N^+(d)\rangle \approx ST_5$ in G^c – by $(t1)$ and P2.8 – and $j, k, l \in N^-_{G^c}(s)$, then we obtain a case similar to the present one if we take G^c and d instead of G and x, respectively, in which e plays the same role. Thus, we can assume that $|N^+(e)| = 6$ in G. Note that $m \in N^-(e)$, since $N^-(m) = \{a, b, c, d, k, l\}$. Then, because $6 = |N^+(e)| = |\{a, d\}| + |N^+(e) \cap \{j, k, l\}| + |N^+(e) \cap \{n, s\}|$, by $(t2)$ we get $n, s \in N^+(e)$. Also, $j \in N^+(e)$, as otherwise $\langle\{j, e, k, d, n\}\rangle \approx TT_5$.

Finally, since $N^+(m) = \{e, j, n, s, x\}$ with $n, s, j \in N^-(x) \cap N^+(e)$, then $\langle N^+(m)\rangle \not\approx RT_5$ (P2.9) and $\langle N^+(m)\rangle \not\approx ST_5$ (P2.5-*ii*), a contradiction.

Case II. $abc = T_3(N^+(d))$.

Here we have $\langle\{m, n, s\}\rangle = T_0(N^+(d))$. Assume that $\langle N^+(d)\rangle$ is as in Fig. 6,*c*. Note that $e \in N^+(m)$, as otherwise, since $c, d, e, m, x \in N^-(a)$ with $c, d, e \in N^+(x) \cap N^-(m)$, by P2.5-*ii* and P2.9 $\langle N^-(a)\rangle \approx QT_5$, contradicting Lemma 3.2. Also we must have $|N^+(e)| = 5$, because $|N^+(e)| = 6$ implies by $(t2)$ that $n, s \in N^+(e)$ (as $b, c, m, x \in N^-(e)$) and $\langle\{e, d, a, n, s\}\rangle \approx TT_5$.

By P2.4 the triangle $mns \in \{T_0(N^-(x)),\ T_3(N^-(x))\}$. If $mns = T_0(N^-(x))$, then, for all $z \in \{m, n, s\}$, $5 \leq |N^+(z)| = |N^+(z) \cap N^-(x)| + |\{x\}| + |N^+(z) \cap \{a, b, c, d\}| + |N^+(z) \cap \{e\}| = 4 + |N^+(z) \cap \{e\}|$, that is, $e \in N^+(z)$, implying $|N^-(e)| = |\{b, c, m, n, s, x\} \cup [N^-(e) \cap \{j, k, l\}]| = 7$ – by $(t2)$ –, contradicting Lemma 3.1.

Hence, $mns = T_3(N^-(x))$ and $jkl = T_0(N^-(x))$. Since $jlk = T_3(N^+_{G^c}(x))$, if now we consider G^c, d, x, jlk, msn, and acb instead of G, x, d, abc, mns, and jkl, respectively, by $(t1)$ we obtain a case similar to the present one in which e plays the same role. This implies $|N^+(e)| = |N^+_{G^c}(e)| = 5$, a contradiction.

Case III. $abc = T_0(N^+(d))$.

In this case, $mns = T_3(N^+(d))$. Assume that $\langle N^+(d)\rangle$ is as in Fig. 6,*d*. Since by $(t1)$ $\langle N^-(d)\rangle \approx ST_5$, taking G^c, d, x, jlk, and msn, instead of G, x, d, abc, and mns, resp., we get a case similar to the present one or to case II. Hence, we must have $jlk = T_0(N^+_{G^c}(x))$, that is, $jkl = T_3(N^-(x))$, implying $mns = T_0(N^-(x))$. Because e and the set $\{m, n, s\}$ play the same role in both situations, we can assume $|N^+(e) \cap \{m, n, s\}| \geq 2$.

Since $\langle N^-(a)\rangle = \langle\{c, d, e, n, s, x\}\rangle \approx ST_6$ with x the center of ced and $c, d \in N^-(n)$, by P2.5-*ii* we have $e \in N^+(n)$, implying $m, s \in N^+(e)$ and $\langle\{e, d, s, a, m\}\rangle \approx TT_5$, a contradiction.

Case IV. $abc \in \mathcal{T}_1(N^+(d))$.

Here we see $G \approx ST_{12}$. Assume that m is the center of abc in $\langle N^+(d)\rangle$; then, $a, b, c \in N^+(m)$ – by P2.5 – and $n, s \in N^-(m)$.

Let us begin proving that the edge directions in $\langle N^-(x)\rangle$ are as in Fig. 6,*e*. Since $\langle\{m,n,s\}\rangle \notin DT(N^-(x))$, by P2.4 and P2.5-*i* there exists the center $r \in \{m,n,s\}$ of *jkl* in $\langle N^-(x)\rangle$, different from *s*, as $s \in N^-(m)\cap N^+(n)$. We have $|N^+(m)\cap\{j,k,l\}| \le 2$ (because $a,b,c,x \in N^+(m)$) and $n,s \in N^-(m)$, then $r = n$ with $j,k,l \in N^-(n)$ and there exists $t \in \{j,k,l\}\cap N^-(m)$; by symmetry, we can assume $t = j$. Thus, $s,n,j \in N^-(m,x)$, and by P2.2 the triangle $sjn \in DT(N^-(x))$ and $k,l \in N^+(m)$. Finally, because $\langle\{m,k,j\}\rangle \notin DT(N^-(x))$ and $m,j \in N^+(s)$, by P2.2-*ii* we get $k \in N^-(s)$, that implies $s,n,l \in N^+(k)$ and $sln \in DT(N^-(x))$.

Now, we will prove that the set $N^+(e)$ and its edge directions are as in Fig. 6,*f*. Considering G^c with *d* and *x*, instead of G, x, and *d*, resp., we obtain a case similar to the present one, by $(t1)$, in which *e* plays the same role. Hence, we can assume $|N^+(e)| = 6$ in *G*.

Since $N^+(m) = \{a,b,c,k,l,x\}$, then $N^-(m) = \{d,e,j,n,s\}$; because $\langle\{j,d,e\}\rangle \notin DT(N^-(m))$ and $j,d \in N^-(n)$, by P2.5-*ii* and P2.9 the vertex $e \in N^+(n)$. Then, as $6 = |N^+(e)| = |\{a,d,m\}| + |N^+(e)\cap\{j,k,l\}| + |N^+(e)\cap\{s\}| = 5 + |N^+(e)\cap\{s\}|$, it follows that $s \in N^+(e)$. Also $j \in N^+(e)$, because *s* is the center of *ned* in $\langle N^-(m)\rangle$ and $n,d \in N^+(j)$. Finally, let us see that $k \in N^+(e)$, that implies – by $(t2)$ – $l \in N^-(e)$. Since $\langle N^+(m)\rangle = \langle\{a,b,c,k,l,x\}\rangle \approx ST_6$, and *x* is the center of *abc* in $\langle N^+(m)\rangle$, by P2.5-*ii* there exists $r \in N^+(k)\cap\{a,b,c\}$; then, $N^+(k) = \{r,d,l,n,s,x\}$, implying $e \in N^-(k)$.

We have $N^+(e) = \{a,d,j,k,m,s\}$. By P2.2-*ii* the triangle $asm \in DT(N^+(e))$, as $a,s,m \in N^+(d)$ and $a \in N^+(m)$. Therefore, $akd = \langle N^-(s)\cap N^+(e)\rangle \in DT(N^+(e))$, and hence $ajm = \langle N^-(k)\cap N^+(e)\rangle \in DT(N^+(e))$.

Next, we will see that $\langle N^-(s)\rangle$ is as in Fig. 6,*g*. Note that $s \in V_2(N^+(d))$, because $s \in V_3(N^+(d))$ implies $b,c,m \in N^+(s)$ (as $a,n \in N^-(s)$) and $\langle\{b,c,m\}\rangle \in DT(N^+(d))$, contradicting $b,c \in N^+(m)$. Then, there exists $h \in \{b,c\}$ with $h \in N^-(s)$, that implies $\langle N^-(s)\rangle = \langle\{a,h,d,e,k,n\}\rangle \approx ST_6$. Inasmuch as *a*, $h \in N^+(d)$, by P2.4 and P2.5 there exists the center of *ekn* in $\langle N^-(s)\rangle$, that must be *h*, as $d \in N^+(e)\cap N^-(n)$ and $a \in N^+(e)\cap N^-(k)$. Thus, $e,k,n \in N^+(h)$ (because $e \in N^+(h)$), and $a,d \in N^-(h)$; this means $h = b$ (and $c \in N^+(s)$). Since a,b, $n \in N^+(d)$, we have $abn \in DT(N^-(s))$. Moreover, $n \in N^+(c)$, as $\langle\{c,m,n\}\rangle = \langle N^-(a)\cap N^+(d)\rangle \in DT(N^+(d))$.

Hence, $N^+(a) = \{b,j,k,l,s\}$ (because $N^-(a) = \{c,d,e,m,n,x\}$), and it remains to know the directions of some edges between $\{b,c\}$ and $\{j,k,l\}$. Since $N^+(m) = \{a,b,c,k,l,x\}$ with $b,k,l \in N^+(a)$, then $bkl \in DT(N^+(m))$. Because $\langle\{b,c,x\}\rangle \notin DT(N^+(m))$ and $b,x \in N^+(l)$, by P2.2-*ii* the vertex $c \in N^-(l)$; thus, $akc = \langle N^-(l) \cap N^+(m)\rangle \in DT(N^+(m))$ (Fig. 6,*h*). Finally, since $\langle N^-(n)\rangle = \langle\{b,c,d,j,k,l\}\rangle \approx ST_6$, then $j \in N^+(b,c)$, as $bjd = \langle N^+(l)\cap N^-(n)\rangle \in DT(N^-(n))$ and *b*, *d*, $k \in N^-(c)$.

We have seen that the edge directions in *G* or G^c are fixed, up to isomorphisms, and since ST_{12} is free of TT_5 with $ST_{12}^c \approx ST_{12}$, we must have $G \approx ST_{12}$. □

Now we show a tournament of order 11 free of TT_5, that is not contained in ST_{13}. Let ST_{11} be the circulant tournament of order 11 induced by the set of quadratic residues (mod. 11), that is, $\{1,3,4,5,9\}$.

Theorem 3.5. *The tournament ST_{11} is free of TT_5 and it is not contained in ST_{13}.*

Proof. In [4] it is proved that ST_{11} is free of TT_5. In order to see that ST_{11} is not contained in ST_{13}, since ST_{11} is regular, we need only prove that no subtournament of order 11 in ST_{13} is regular. Let $x, y \in V(ST_{13})$ with $y \in N^+(x)$ and let $H = ST_{13} \setminus \{x, y\}$. Take any vertex $z \in N^+(x, y)$. Then, $|N_H^+(z)| = 6$ and $|N_H^-(z)| = 4$, that is, H is not regular. □

4. Tournaments Free of TT_6 and TT_7

The uniqueness of the tournament of order 27 which does not contain transitive subtournaments of order 6 has been established [7]. Here we see that there also exists a unique tournament of order 26 free of these subtournaments. Using this result, we prove that all tournaments of order 54 contain TT_7, and present a tournament of order 31 free of TT_7.

Tournaments free of TT_6.

Let ST_{27} be the Galois tournament of order 27; this is the unique tournament of order 27 free of TT_6 [7]. It is not difficult to prove the following properties:

P4.1. *If $x \in V(ST_{27})$ then $\langle N^+(x)\rangle \approx ST_{13}$ and $\langle N^-(x)\rangle \approx ST_{13}$.*

P4.2. *If $x, y \in V(ST_{27})$ with $y \in N^+(x)$, then $|N^-(x) \cap N^+(y)| = 7$* (since by P3.2 and P4.1 $|N^+(y)| = 13$ and $|N^+(x, y)| = 6$).

P4.3. *Let G be a tournament of order 26 free of TT_6. Then, $V(G) = V_{12}(G) \cup V_{13}(G)$, $|V_{12}(G)| = 13$, and $|V_{13}(G)| = 13$.*

P4.4. *Let G be a tournament of order 26 free of TT_6, and let $x \in V_{12}(G)$ (resp., $x \in V_{13}(G)$). Then, $\langle N^+(x)\rangle \approx ST_{12}$ and $\langle N^-(x)\rangle \approx ST_{13}$ (resp., $\langle N^+(x)\rangle \approx ST_{13}$ and $\langle N^-(x)\rangle \approx ST_{12}$).*

Lemma 4.1. *Let G be a tournament of order 26 free of TT_6 and let $x, y \in V(G)$ with $y \in N^+(x) \cap V_{13}(G)$. Then, $|N^+(x, y)| = 6$.*

Proof. By P4.4 we get $\langle N^+(x)\rangle \approx ST_{12}$ or ST_{13}. If $\langle N^+(x)\rangle \approx ST_{13}$, the result is immediate from P3.2. Suppose $\langle N^+(x)\rangle \approx ST_{12}$ with $|N^+(x, y)| \neq 6$; by P2.7, P3.2, and Theorem 3.4 the tournament $\langle N^+(x, y)\rangle \approx ST_5$. Then, there exists $z \in N^+(x, y)$ with $|N^-(z) \cap N^+(x, y)| = 1$. Since $\langle N^+(y)\rangle \approx ST_{13}$, P3.2 implies $|N^-(z) \cap N^+(y)| = 6$; hence, $|N^-(z, x) \cap N^+(y)| = 5$, that is, $|N^-(x) \cap N^+(y)| > 4$ in $\langle N^-(z)\rangle$ (which is isomorphic to ST_{12} or ST_{13}), contradicting P3.3. □

Lemma 4.2. *Let G be a tournament of order 26 free of TT_6 and let $x, y \in V_{12}(G)$. Then, $|N^+(x, y)| = 5$.*

Proof. If $v \in V_{12}(G)$ and $z \in N^+(v)$ with $z \in V_5(N^+(v))$, from Lemma 4.1 $z \in V_{12}(G)$. Then, for all $v \in V_{12}(G)$, since $\langle N^+(v)\rangle \approx ST_{12}$ and $|V_5(N^+(v))| = 6$, we have $|N^+(v) \cap V_{12}(G)| \geq 6$. But $|V_{12}(G)| = 13$ and $\sum_{v \in V_{12}(G)} |N^+(v) \cap V_{12}(G)| = |E(\langle V_{12}(G)\rangle)| = 13(6)$, that implies $|N^+(v) \cap V_{12}(G)| = 6$, for all $v \in V_{12}(G)$. Hence, $y \in V_5(N^+(x))$ if $x, y \in V_{12}(G)$ with $y \in N^+(x)$, that is, $|N^+(x,y)| = 5$. □

Theorem 4.3. *Let G be a tournament of order* 26 *free of* TT_6. *Then, G is contained in* ST_{27}. *In particular, there exists a unique tournament, which we denote by* ST_{26}, *of order* 26 *free of* TT_6, *up to isomorphisms.*

Proof. Let h be a vertex not contained in $V(G)$, and let H be the tournament that contains G such that $V(H) = V(G) \cup \{h\}$ and, for each $x \in V(G)$, $h \in N^+(x)$ or $h \in N^-(x)$ if, respectively, $x \in V_{12}(G)$ or $x \in V_{13}(G)$ (P4.3).

Let us see that H is free of TT_6. Let $x \in V_{12}(G)$ (the case $x \in V_{13}(G)$ is similarly analyzed in H^c). Since, by definition, $h \in N_H^+(x)$, then it is enough to prove that $\langle N^+(x) \cup \{h\}\rangle \approx ST_{13}$, that is, $y \in N_H^-(h)$ or $y \in N_H^+(h)$ if $y \in V_5(N^+(x))$ or $y \in V_6(N^+(x))$, respectively. If $y \in V_5(N^+(x))$, from Lemma 4.1 $y \in V_{12}(G)$, and hence $y \in N_H^-(h)$. And $y \in V_6(N^+(x))$ implies, by Lemma 4.2, $y \in V_{13}(G)$, that is, $y \in N_H^+(h)$.

Since ST_{27} is the unique tournament of order 27 free of TT_6, we have $H \approx ST_{27}$. It is also known that $Aut(ST_{27})$ is transitive in edges [7], hence, the theorem follows. □

For tournaments of order 24 and 25, we state the following:

Conjecture. *All tournaments free of* TT_6 *of order* 24 *and* 25 *are contained in* ST_{27}.

For tournaments of order 23 the corresponding conjecture is false. Let ST_{23} be the circulant tournament of order 23 induced by the set of quadratic residues (mod. 23), that is, by the set $QR_{23} = \{1, 2, 3, 4, 6, 8, 9, 12, 13, 16, 18\}$.

Theorem 4.4. *The tournament* ST_{23} *is free of* TT_6 *and is not contained in* ST_{27}.

Proof. Let us see first that ST_{23} is not contained in ST_{27}. It is enough to see that if $A = \{a, b, c, d\} \subset V(ST_{27})$, then $H = ST_{27} \backslash A$ is not regular. We can assume $b, c \in N^+(a)$ and $c \in N^+(b)$. Since $\langle N^+(a)\rangle \approx ST_{13}$ (by P4.1) and $\langle N^+(a,b)\rangle \approx ST_6$ (by P3.2), there exists $x \in N^+(a,b,c)$ with $x \neq d$. Then, $|N_H^-(x)| = 9$ or 10, and $|N_H^+(x)| = 13$ or 12, that is, H is not regular.

It remains to see that ST_{23} is free of TT_6. Since ST_{23} is the Galois tournament of order 23, the function $F_{r,s}(z) = rz + s$ is an automorphism of ST_{23}, for all $r \in QR_{23}$ and for all $s \in Z_{23}$, that is, $Aut(ST_{23})$ is transitive in edges. Then, if W is one of the biggest transitive subtournaments of ST_{23}, we can assume that $0, 1 \in W$ with $[W \backslash \{0, 1\}] \subset N^+(0,1)$. In this case, it is easy to verify that $\langle N^+(0,1)\rangle = \langle \{2, 3, 4, 9, 13\}\rangle \approx ST_5$, implying that ST_{23} is free of TT_6.

Tournaments free of TT_7

It is clear that the following properties hold:

P4.5. *Let G be a tournament of order* 54 *free of TT_7. Then,* $V(G) = V_{26}(G) \cup V_{27}(G)$, $|V_{26}(G)| = 27$, *and* $|V_{27}(G)| = 27$.

P4.6. *Let G be a tournament of order* 54 *free of TT_7, and let $x \in V_{26}(G)$ (resp., $x \in V_{27}(G)$). Then, $\langle N^+(x) \rangle \approx ST_{26}$ and $\langle N^-(x) \rangle \approx ST_{27}$ (resp., $\langle N^+(x) \rangle \approx ST_{27}$ and $\langle N^-(x) \rangle \approx ST_{26}$).*

Lemma 4.5. *Let G be a tournament of order* 54 *free of TT_7 and let $x, y \in V(G)$ with $y \in N^+(x) \cap V_{27}(G)$. Then, $|N^+(x, y)| = 13$.*

Proof. By P4.6 $\langle N^+(x) \rangle \approx ST_{26}$ or ST_{27}. If $\langle N^+(x) \rangle \approx ST_{27}$, the result is immediate. Suppose $\langle N^+(x) \rangle \approx ST_{26}$ with $|N^+(x, y)| \neq 13$; by P4.4 $\langle N^+(x, y) \rangle \approx ST_{12}$. Let $z \in V_6(N^+(x, y))$, that is, $|N^-(z) \cap N^+(x, y)| = 5$. Since $\langle N^+(y) \rangle \approx ST_{27}$, we have $|N^-(z) \cap N^+(y)| = 13$ and $|N^-(z, x) \cap N^+(y)| = 8$; this means that in $\langle N^-(z) \rangle$ (which is isomorphic to ST_{26} or ST_{27}) $|N^-(x) \cap N^+(y)| > 7$, contradicting P4.2. □

Lemma 4.6. *Let G be a tournament of order* 54 *free of TT_7, and let $x, y \in V_{26}(G)$. Then, $|N^+(x, y)| = 12$.*

Proof. If $v \in V_{26}(G)$ and $z \in N^+(v)$ with $z \in V_{12}(N^+(v))$, from Lemma 4.5 $z \in V_{26}(G)$. Then, for each $v \in V_{26}(G)$, since $\langle N^+(v) \rangle \approx ST_{26}$ and $|V_{12}(N^+(v))| = 13$ (P4.3), we have $|N^+(v) \cap V_{26}(G)| \geq 13$. But $|V_{26}(G)| = 27$ (P4.5) and $\sum_{v \in V_{26}(G)} |N^+(v) \cap V_{26}(G)| = |E(\langle V_{26}(G) \rangle)| = 27(13)$, that implies $|N^+(v) \cap V_{26}(G)| = 13$, for all $v \in V_{26}(G)$. Hence, $y \in V_{12}(N^+(x))$ if $x, y \in V_{26}(G)$ with $y \in N^+(x)$, that is, $|N^+(x, y)| = 12$. □

Theorem 4.7. *Every tournament of order* 54 *contains TT_7.*

Proof. Proceed by contradiction. Suppose that there exists a tournament G of order 54 free of TT_7. Let h be a vertex not contained in $V(G)$, and let H be the tournament that contains G such that $V(H) = V(G) \cup \{h\}$ and, for each $x \in V(G)$, $h \in N^+(x)$ or $h \in N^-(x)$ if, respectively, $x \in V_{26}(G)$ or $x \in V_{27}(G)$ (P4.5).

Let us see that H is free of TT_7. Let $x \in V_{26}(G)$ (the case $x \in V_{27}(G)$ is similarly analyzed in H^c). Since, by definition, $h \in N_H^+(x)$, then it is enough to prove that $\langle N^+(x) \cup \{h\} \rangle \approx ST_{27}$, that is, $y \in N_H^-(h)$ or $y \in N_H^+(h)$ if $y \in V_{12}(N^+(x))$ or $y \in V_{13}(N^+(x))$, respectively. If $y \in V_{12}(N^+(x))$, from Lemma 4.5 $y \in V_{26}(G)$, and hence $y \in N_H^-(h)$. And $y \in V_{13}(N^+(x))$ implies, by Lemma 4.6, $y \in V_{27}(G)$, that is, $y \in N_H^+(h)$.

Finally, since there is no tournament of order 55 free of TT_7 [7], we obtain a contradiction.

Using this theorem, we could easily prove:

Theorem 4.8. $v(n) \geq \lfloor \log_2(n/54) \rfloor + 7$, *for* $n \geq 28$.

Now we present a tournament of order 31 free of TT_7. Let ST_{31} be the circulant tournament of order 31 induced by $R = \{1, 2, 3, 4, 5, 7, 8, 9, 10, 13, 14, 15, 19, 20, 25\}$; ST_{31} is not the tournament induced by the set of quadratic residues (mod. 31).

Theorem 4.9. *The tournament ST_{31} is free of TT_7.*

Proof. Let $\langle\{u_0, u_1, \dots, u_r\}\rangle$ be one of the biggest transitive subtournaments of ST_{31}, with $u_{i+1}, \dots, u_r \in N^+(u_i)$, for all $i \in \{0, \dots, r-1\}$. Since $Aut(ST_{31})$ is transitive in vertices, we can assume that $u_0 = 0$. For $x \in Z_{31}$, let $f_x\colon Z_{31} \mapsto Z_{31}$ be the function $f_x(z) = zx$ (mod. 31). Because $R = 5R$, that is, $R = \{5r(\text{mod.}31) | r \in R\}$, we have $f_5, f_{25} \in Aut(ST_{31})$. Moreover, as each $r \in R$ is in $B = \{1, 2, 3, 4, 8\}$, $5B$, or $25B$, we also can assume that $u_1 \in B$. It is easy verify that the sets

$$N^+(0,2) = \{3,4,5,7,9\}, \quad N^+(0,3) = \{4,5,7,8,10,13\},$$

$$N^+(0,4) = \{5,7,8,9,13,14,19\}, \quad \text{and} \quad N^+(0,8) = \{2,9,10,13,15\}$$

are free of TT_5. Finally, for $u_1 = 1$, we have

$$\begin{aligned}
N^+(0,1) &= \{2,3,4,5,8,9,10,14,15,20\},\\
\langle N^+(0,1,2)\rangle &= \langle\{3,4,5,9,10,15\}\rangle \approx ST_6,\\
\langle N^+(0,1,3)\rangle &= \langle\{4,5,8,10\}\rangle \not\approx TT_4,\\
\langle N^+(0,1,4)\rangle &= \langle\{5,8,9,14\}\rangle \not\approx TT_4,\\
\langle N^+(0,1,5)\rangle &= \langle\{8,9,10,14,15,20\}\rangle \approx ST_6,\\
\langle N^+(0,1,8)\rangle &= \langle\{2,9,10,15\}\rangle \not\approx TT_4,\\
N^+(0,1,9) &= \{3,10,14\},\\
\langle N^+(0,1,10)\rangle &= \langle\{4,14,15,20\}\rangle \not\approx TT_4,\\
\langle N^+(0,1,14)\rangle &= \langle\{2,3,8,15\}\rangle \not\approx TT_4,\\
\langle N^+(0,1,15)\rangle &= \langle\{3,4,9,20\}\rangle \not\approx TT_4,
\end{aligned}$$

and

$$\langle N^+(0,1,20)\rangle = \langle\{2,3,4,8,9,14\}\rangle \approx ST_6.$$

Then, $r \leq 5$ and ST_{31} is free of TT_7. □

5. Computational Results

For small integers n (≤ 31), the exact values for $v(n)$ are induced by circulant or Galois tournaments. In this final section we obtain, using a computer, the values for $cv(r)$ ($r \leq 55$) and $gv(s)$ ($s < 1000$). In particular, we improve on the best upper bounds known of $v(n)$, for $n \leq 991$.

Circulant Tournaments

For an odd integer $n > 0$, we have defined $cv(n)$ as the biggest integer such that every circulant tournament of order n contains $TT_{cv(n)}$. In order to calculate $cv(n)$, let T be any circulant tournament of order n induced by a set A, and let H be one of the biggest transitive subtournaments in T; remember that $|A| = (n-1)/2$. Since a nonzero element x is in A if and only if $-x \notin A$, then there always exists a circulant tournament T' isomomorphic to T with $01 \in E(T')$; thus, we can assume $01 \in E(T)$ and essentially there are at most $2^{(n-3)/2}$ circulant tournaments of order n. Also we can take $0 \in H$ with $[H \backslash 0] \subset N^+(0)$, because $Aut(T)$ is transitive in vertices. Using these ideas and a computer (ACER 486 at 50MHZ), we have obtained the values of $cv(n)$ presented in Table 1.

Table 1. Values of $cv(n)$ and $\alpha(n)$, $n \leq 55$. The entries in boldface correspond to $\alpha'(n)$, instead of $\alpha(n)$

n	5	7	9	11	13	15	17	19	21	23	25	27	29
$cv(n)$	3	3	4	4	4	6	5	5	5	5	6	6	6
$\alpha(n)$	1	1	**3**	2	1	14	1	2	1	1	**16**	**9**	4

n	31	33	35	37	39	41	43	45	47	49	51	53	55
$cv(n)$	6	7	7	7	7	7	7	8	7	8	8	8	8
$\alpha(n)$	1	>40	>40	17	14	1	4	>**40**	1	>**40**	>40	>40	>40

Given two circulant tournaments T_1 and T_2 of order n induced by the sets A_1 and A_2, respectively, they are *Ádám-isomorphic* [1] if there exists a unit u of Z_n such that $A_2 = \{u \cdot t | t \in A_1\}$; clearly, T_1 and T_2 are isomorphic if they are Ádám-isomorphic. Let $\alpha(n)$ and $\alpha'(n)$ be the number of circulant tournaments of order n free of $TT_{cv(n)+1}$ that are not isomorphic and are not Ádám-isomorphic, respectively; then, $\alpha(n) \leq \alpha'(n)$. Also in Table 1 we show the values $\alpha(n)(n \leq 55)$. The entries in boldface correspond to $\alpha'(n)$, instead of $\alpha(n)$. With the possible exception of these entries, in general we have $\alpha(n) = \alpha'(n)$ by the following [3]:

Theorem 5.1. *Let n be an odd positive integer non divisible by a square. Also, let T_1 and T_2 be circulant tournaments of order n. Then, T_1 and T_2 are isomorphic to each other if and only if they are Ádám-isomorphic.*

Observe that $cv(n)$ is not an increasing function. Also note that the tournament ST_{31} presented in section 4 is the unique circulant tournament of order 31 free of TT_7; in general, for $k = 3, 4, 5, 6$ (and apparently too for $k = 7$), the largest circulant tournament free of TT_{k+1} is unique.

Galois Tournaments

It is well known that there exists a Galois tournament of order n if and only if $n \equiv 3$ (mod. 4) with $n = p^r$, for p prime and $r \in N$. In Table 2 we present our computational results (in a Silicon Graphics Power Series 4D/3105) of the order $gv(n)$ of the largest transitive subtournaments of the Galois tournament of order n, for $n < 1000$. Almost all these Galois tournaments are also circulant tournaments, since they are of prime order. The tournaments of order $27 = 3^3$, $243 = 3^5$, and $343 = 7^3$, are the unique Galois tournaments which are not of prime order, and hence, they are not circulant tournaments. From Tables 1 and 2 it is clear that $gv(n) = v(n)$ or $gv(n) \geq cv(n)$ in the Galois tournament of order $n \leq 47$ (excepting $n = 31$).

Table 2. The order $gv(n)$ of the largest transitive subtournaments of the Galois tournament of order $n < 1000$. The orders 27, 243, and 343 are the unique nonprime orders

$gv(n)$	n
3	7
4	11
5	19, 23, 27
7	31, 43, 47
8	67, 83
9	59, 71, 79, 107
11	103, 127, 131, 139, 151, 163, 167, 191, 199
12	179, 239, 251, 271
13	211, 223, 227, 263, 307, 311, 331, 343, 347, 367
14	243, 283, 443
15	379, 383, 419, 439, 463, 467, 479, 499, 547, 563, 587, 619
16	487, 571, 659
17	359, 431, 491, 503, 523, 599, 607, 631, 643, 647, 683, 691, 719, 727, 739, 743, 751, 787, 811, 827, 839, 859, 863, 883, 887, 947, 971
18	907, 967
19	823, 911, 919, 983, 991

On the other hand, the best upper bounds known for $v(n)$ are [2, 8]:

$$v(n) \leq \lfloor 2\,\log_2(n) \rfloor + 1 \tag{1}$$

and

$$v(n) \leq \left\lfloor -\tfrac{3}{2} + \sqrt{3n + \tfrac{13}{4}} \right\rfloor, \quad n \equiv 3 \pmod{4}, \tag{2}$$

where (2) is better than (1) only for $n \leq 59$. Since $v(n)$ is an increasing function with $v(n) \leq cv(n)$ and $v(n) \leq gv(n)$, in Table 3 we present the upper bounds C1 for $v(n)$, $n \leq 991$, induced from Tables 1 and 2, compared with the bounds C2 and C3 given by (1) and (2), respectively. In general, the bounds C1 are better than C2 and C3. However, as n increases, the difference between C1 and C2 becomes smaller.

Table 3. Upper bounds for $v(n)$, $n \leq 991$. C1: The best bounds induced from Tables 1 and 2. C2: $v(n) \leq \lfloor 2\log_2 n \rfloor + 1$.
C3: $v(n) \leq \left\lfloor -\frac{3}{2} + \sqrt{3n + \frac{13}{4}} \right\rfloor$, $n \equiv 3$ (mod. 4)

$n \leq$	31	45	47	63	83	90	107	127	181	199	255
C1	6	7	7	8	8	9	9	11	11	11	12
C2	10	11	12	12	13	13	14	14	15	16	16
C3	8	10	10	12	14	15	16	18	22	23	26
$n \leq$	271	362	367	443	511	619	659	724	971	991	
C1	12	13	13	14	15	15	16	17	17	19	
C2	17	17	18	18	18	19	19	19	20	20	
C3	27	31	31	35	37	41	43	45	52	53	

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